How well do rational functions approach more complicated functions?

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Our aim

We denote by $\mathcal{R}_{m,n}$ the set of rational functions p/q with polynomals p of degree $\leq m$ and q of degree $\leq n$, $q \neq 0$.
In particular $\mathcal{R}_{m,0}$ is the set of polynomials of degree $\leq m$.

For a compact set $E \subset \mathbb{C}$, give classes of functions f where we know more about

$$\eta_{m,n}(f, E) = \inf_{r \in \mathcal{R}_{m,n}} \|f - r\|_E, \quad \|g\|_E = \max_{z \in E} |g(z)|,$$

e.g., bounds, asymptotic behavior, construction of (near) optimal rational functions, etc.

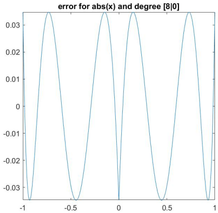
- Why? Applications in numerical linear algebra, see talk of Stefan Guettel.
- Examples
- Hint of theory for polynomial and rational approximation

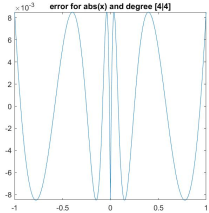
Alternation: six examples

Here E is a compact real interval and $f : E \mapsto \mathbb{R}$ continuous.
In this case we have existence and uniqueness of a best rational approximant, characterized by

Chebyshev alternation theorem: Let $r \in \mathcal{R}_{m,n}$ with defect d the largest integer such that $r \in \mathcal{R}_{m-d,n-d}$.
Then r is optimal for $\eta_{m,n}(f, E)$ iff there exists an alternant $x_0 < x_1 < \ldots < x_{m+n+1-d}$ of points in E such that $f(x_j) - r(x_j)$ is of constant modulus $\|f - r\|_E$ and alternating sign for $j = 0, 1, \ldots, m + n - d + 1$.

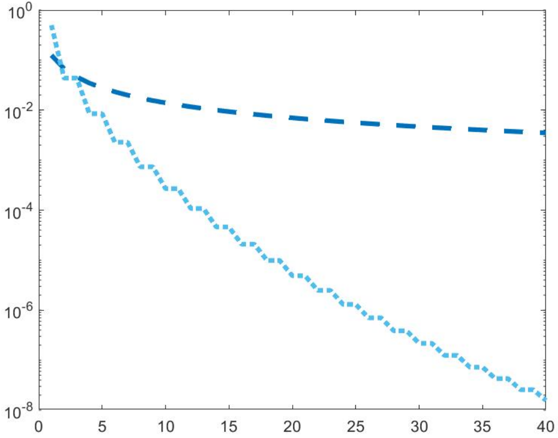
... and computable by Remez algorithm `chebfun/minimax`





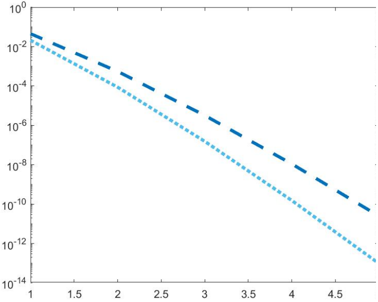
Example 1: $E = [-1, 1]$, $f(z) = |z|$

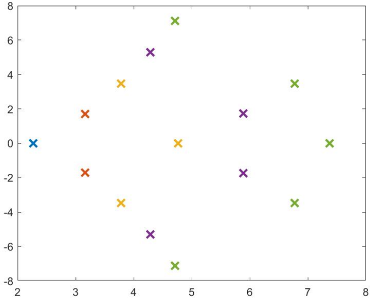
$\eta_{m,m}(f, E)$ (dotted) versus $\eta_{2m,0}(f, E)$ (dashed)



Sublinear convergence like $\exp(-\sqrt{m}c_0)$.

Example 2: $E = [-1, 1]$, $f(z) = \exp(z)$

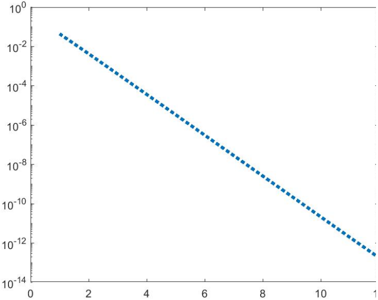


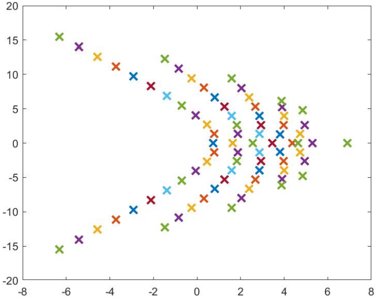


Entire function

Fast superlinear convergence like $\exp(-m(c_1 + \log(m)))$.

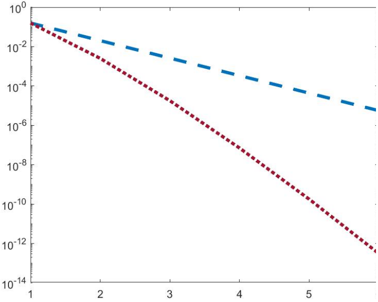
Example 3: $E = (-\infty, 0]$, $f(z) = \exp(z)$

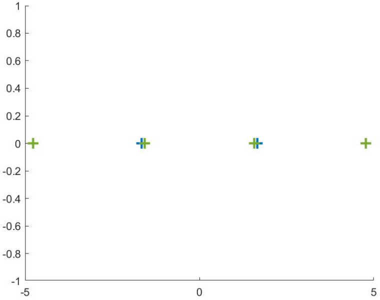




Entire function
Fast linear convergence like 9.81^{-m}.

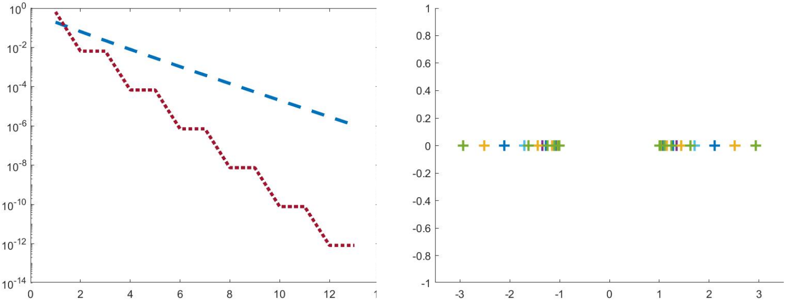
Example 4: $E = [-1, 1]$, $f(z) = \tan(z)$





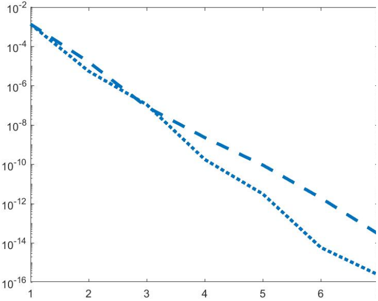
Meromorphic function
Fast superlinear convergence.

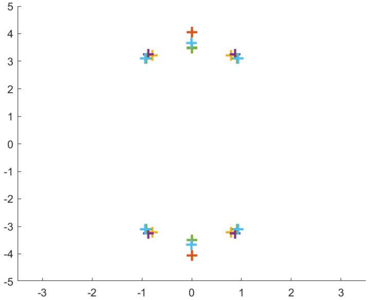
Example 5: $E = [-0.9, 0.9]$, $f(z) = 1/\sqrt{1-z^2}$



Algebraic function
Linear convergence like $\exp(-c_2 m)$.

Example 6: $E = [-0.9, 0.9]$, $f(z) = z/\sqrt{((z-1)^2+9)((z+1)^2+9)}$





Algebraic function

Linear convergence like $\exp(-c_3 m)$.

Near best polynomial approximation via interpolation

Let $\omega_m(z) = \prod_{j=0}^{m}(z - z_j)$ with $z_0, ..., z_m \in E$ interpolation points, then we have the polynomial interpolant

$$\Pi_m(f)(z) = \sum_{j=0}^{m} f(z_j)\ell_j(z), \quad \ell_j(z) = \frac{\omega_m(z)}{(z - z_j)\omega'_m(z_j)},$$

and
$\Pi_m(f-q) = \Pi_m(f)-q$ for all polynomials q of degree $\leq m$. Thus

$$\eta_{m,0}(f, E) \leq \|f - \Pi_m(f)\|_E \leq (1 + \Lambda_m)\,\eta_{m,0}(f, E)$$

with the Lebesgue constant $\Lambda_m = \|\,|\ell_0| + \cdots + |\ell_m|\,\|_E$.

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with the Lebesgue constant $\Lambda_m = \|\, |\ell_0| + \cdots + |\ell_m| \,\|_E$.

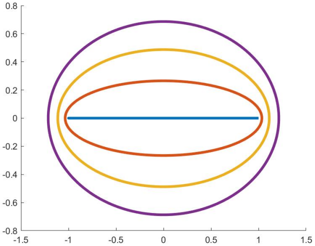
- $\Lambda_m = \mathcal{O}(\log(m))$ for $E = \mathbb{D}$ and $(m+1)$th roots of unity, and for $E = [-1, 1]$ and Chebyshev points $z_j = \cos(\pi j/m)$,
- $\Lambda_m \leq m + 1$ for Fekete points of E (difficult to compute),
- Λ_m "small" (??) for Leja points of E (easy to compute).

Geometric rate for polynomial approximation

We want to show that $\limsup_{m\to\infty} \eta_{m,0}(f,E)^{1/m} = 1/R < 1$ iff f is analytic in some neighborhood of E depending on R.

The Riemann map φ of a simply connected compact set E is the analytic bijction from $\overline{\mathbb{C}} \setminus E$ onto $\overline{\mathbb{C}} \setminus \mathbb{D}$, normalized at ∞ such that $\varphi(\infty) = \infty$, and $\varphi'(\infty) > 0$. We denote $\psi = \varphi^{-1}$ the inverse map.

The level set E_R for $R > 1$ is defined by $\overline{\mathbb{C}} \setminus E_R = \{z \notin E : |\varphi(z)| > R\}$



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Theorem 1

$\limsup_{m\to\infty} \eta_{m,0}(f, E)^{1/m} = 1/R < 1$ iff f is analytic in $Int(E_R)$ but not in any larger level set.

Theorem 2

If f is mermorphic in E_R, with at most n poles, then $\limsup_{m\to\infty} \eta_{m,n}(f, E)^{1/m} \leq 1/R$.

Proof of Theorem 1

Let $1 < r < \widetilde{r} < R$. Suppose that f is analytic in a neighborhood of $E_{\widetilde{r}}$. Then

$$\eta_{m,0}(f,E)^{1/m} \quad \leq \quad \|f - \Pi_m(f)\|_E^{1/m} \leq \max_{z\in E_r}\left|\frac{1}{2\pi i}\int_{\partial E_{\widetilde{r}}} \frac{\omega_m(z)}{\omega_m(x)}\frac{f(x)dx}{z-x}\right|^{1/m}$$

and $\limsup_m \eta_{m,0}(f,E)^{1/m} \leq r/\widetilde{r}$ follows by showing that, for Fekete points,

$$\lim_{m\to\infty}\max_{z\in\partial E_r, x\in\partial E_{\widetilde{r}}}\left|\frac{\omega_m(z)}{\omega_m(x)}\right|^{1/m} = r/\widetilde{r}.$$

Conversely, let $\eta_{m,0}(f,E) \leq c/(\widetilde{r})^m$ with extremal polynomial p_m, then

$$\|p_{m+1} - p_m\|_E \leq \|f - p_{m+1}\|_E + \|f - p_m\|_E \leq 2c/(\widetilde{r})^m,$$

and from the maximum principle applied to $(q_{m+1}(z) - q_m(z))/\varphi(z)^{m+1}$

$$\|p_{m+1} - p_m\|_{E_r} \leq r^{m+1}\|p_{m+1} - p_m\|_{E_r} \leq 2cr(r/\widetilde{r})^m.$$

Hence the series $p_0 + \sum_{m=0}^{\infty}(p_{m+1} - p_m)$ converges uniformly in E_r, and thus its limit f is analytic in $\mathrm{Int}(E_r)$.

Faber polynomials and Faber operator

Finding good approximants for $E = \mathbb{D}$ is easy (e.g., interpolation at roots of unity). Can it help to construct good approximants for other classes of E? Here convex E.

Define the Faber polynomial $F_m(z) = \varphi(z)^m + \mathcal{O}(1/z)_{z\to\infty}$ and the Faber map (bijection from $\mathcal{R}_{m,0}$ onto $\mathcal{R}_{m,0}$) by

$$P(w) = \sum_{j=0}^{m} a_j w^j : \qquad \mathcal{F}(P)(z) = 2a_0 F_0(z) + \sum_{j=1}^{m} a_j F_j(z).$$

The residuum theorem shows for $z \in \mathrm{Int}(E)$ that

$$\frac{1}{2\pi i}\int_{|w|=1} \frac{\psi'(w)}{\psi(w)-z} w^j \frac{dw}{w} = \begin{cases} F_j(z) & \text{for } j \geq 0, \\ 0 & \text{for } j < 0, \end{cases}$$

and hence

$$\|\mathcal{F}(P)\|_E \leq 2\,\|P\|_{\mathbb{D})}, \quad \mathcal{F}(P)(z) = \frac{1}{\pi}\int_0^{2\pi} P(e^{it})\mathrm{Re}\left(\frac{e^{it}\psi'(e^{it})}{\psi(e^{it})-z}\right) dt.$$

Faber polynomials and Faber operator (2)

In particular $\|F_m\|_E \leq 2$, and we may extend the Faber operator and our inequality to functions P being analytic in a neighborhood of $\mathbb{D}$, and $\mathcal{F}(P)$ being analytic in a neighborhood of $\mathbb{E}$.
In particular for some $w_0 \notin \mathbb{D}$:

$$\mathcal{F}(\frac{1}{w - w_0})(z) = \frac{\psi'(w_0)}{z - \psi(w_0)},$$

showing that $\mathcal{F}(\mathcal{R}_{m,n}) = \mathcal{R}_{m,n}$ provided that $m \geq n - 1$.
Moreover,

Theorem 3
If E is a convex set and $f = \mathcal{F}(F)$ with F analytic in a neighborhood of $\mathbb{D}$ then for $m \geq n - 1$

$$\eta_{m,n}(f, E) \leq 2\eta_{m,n}(F, \mathbb{D}).$$

What are rational interpolants? [BGM96]

Definition 3.1

$r = p/q$ is called rational interpolant of type $[m|n]$ of f at interpolation points $z_0, ..., z_{m+n}$ if $p \in \mathcal{R}_{m,0}, q \in \mathcal{R}_{n,0} \setminus \{0\}$, and $fq - p$ vanishes at $z_0, ..., z_{m+n}$ counting multiplicity.

Example: if $z_0 = ... = z_{m+n}$ then r is called a Padé approximant of f at z_0, here $f(z)q(z) - p(z) = \mathcal{O}((z - z_0)^{m+n+1})_{z \to z_0}$.
Existence: write p, q in some polynomial basis and solve a homogeneous system with $m + n + 1$ equations and $m + n + 2$ unknowns.
Uniqueness: if p_1/q_1 and p_2/q_2 are rational interpolants then $p_1 q_2 - p_2 q_1 = (p_1 - fq_1)q_2 - (p_2 - fq_2)q_1$ is a polynomial of degree $\leq m + n$ which vanishes at $m + n + 1$ points.

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Caveat: There might be z_j called **unattainable interpolation points** with $f(z_j) \neq r(z_j)$ (after canceling).

What about near-optimal rational interpolants?

In what follows $F \subset \overline{\mathbb{C}}$ closed, and $E \subset \mathbb{C} \setminus F$ compact.

Given f being analytic in $\overline{\mathbb{C}} \setminus F$ we might get good candidates for $\eta_{m,n}(f, E)$ by choosing well distributed interpolation points in E. However, we have little control about the poles: ideally they should be in F simulating the singularities of f.

It happens that a small number of poles of such rational interpolants can be in E, so-called **spurious poles** having a small residual. This does not allow to have uniform convergence in E, but maybe it is sufficient to drop small neighborhoods around these spurious poles? Not for all functions f !!!

Even worse, finite precision arithmetic tends to produce also spurious poles, and a numerical analysis for rational functions is lacking.

What to do next (second hour)

1. Energy, capacity, Green function, lemniscates (in order to describe convergence in capacity)
2. Convergence theorems (in capacity) Pommerenke/Gonchar/Stahl
3. Special case of Markov functions
4. Rational approximation with fixed poles, leads to balayage problem
5. Rational approximation with optimal fixed poles, leads to Zolotarev problem. Bagby points.

We will not speak about constrained energy problems and approximation on discrete sets (though interesting for applications).

Energy and capacity

We denote by $\mathcal{M}(E)$ the set of positive probability measures supported in E.
For $\mu_n \in \mathcal{M}(E)$ we say that $\mu_n \to \mu$ (weak star convergence) if

$$\forall f \in \mathcal{C}(E): \quad \lim_{n\to\infty} \int f d\mu_n = \int f d\mu.$$

For $\mu, \nu \in \mathcal{M}(E)$ we define the potential and mutual energy

$$U^\mu(z) = \int \log(\frac{1}{|z-x|}) d\mu(x), \quad I(\mu,\nu) = \int U^\nu(x) d\mu(x)$$

and the energy $I(\mu) = I(\mu,\mu)$ (electrostatics in the plane).

Theorem 4

There is a unique $\omega_E \in \mathcal{M}(E)$ called **equilibrium measure** *minimizing $\mathcal{M}(E) \ni \mu \mapsto I(\mu)$. ω_E is the unique measure with potential q.e. constant on E. This constant equals $I(\omega_E) =: \log(1/\operatorname{cap}(E))$.*

Proof uses compactness of $\mathcal{M}(E)$, and the facts that $\mu \mapsto I(\mu)$ is lower semi-continuous and strictly convex.

Capacity : some examples

If E is simply connected then link with Riemann map

$$\log(|\varphi(z)|) = \log(\frac{1}{\text{cap}(E)}) - U^{\omega_E}(z).$$

- For a disk $E = \{|z| \leq r\}$ we have $\varphi(z) = z/r$ and $\text{cap}(E) = r$, ω_E normalized arc length of ∂E, $U^{\omega_E}(z) = \log(\frac{1}{\max(r,|z|)})$.
- For a real interval $E = [a, b]$ we know φ and hence $\text{cap}(E) = (b-a)/4$, $d\omega_E/dx = \frac{1}{\pi\sqrt{(x-a)(b-x)}}$ on E.
- For level sets $\text{cap}(E_R) = R\,\text{cap}(E)$.
- For a lemniscate $E = \{z \in \mathbb{C} : |(z - z_1)|...|z - z_k| \leq r^k\}$ we know the Green function and thus $\text{cap}(E) = r$.

Examples of convergence in capacity

Theorem 5 (Pommerenke, 1973, [BGM96])

Let f be analytic at 0 and meromorphic in $\mathbb{C}$, and denote by r_m its Padé approximant at 0 of type $[m-1|m]$. Then for any compact E

$$\limsup_{m\to\infty}(\|f-r_m\|_{E\setminus E_m})^{1/m}=0$$

with exceptional sets E_m satisfying $\operatorname{cap}(E_m)\to 0$ for $m\to\infty$.

Theorem 6 (Stahl, 1997, [S97,BGM96])

Let f be an algebraic function with a finite number of branch points $\neq 0$, and let F be a union of arcs (cuts) connecting branch points such that f is single-valued (and analytic) in $\mathbb{C}\setminus F$. Denote by r_m its Padé approximant at 0 of type $[m-1|m]$. Then for any compact $E\subset\mathbb{C}\setminus F$

$$\limsup_{m\to\infty}(\|f-r_m\|_{E\setminus E_m})^{1/m}<1$$

with exceptional sets E_m satisfying $\operatorname{cap}(E_m)\to 0$. Similar Theorems are valid for other families of interpolation points, with a different rate.

Why logarithmic potential theory?

Given two monic polynomials of degree m

$$P_m(z) = \prod_{j=1}^{m}(z - a_{j,m}),\ Q_m(z) = \prod_{j=1}^{m}(z - b_{j,m}),$$

we have that with the counting measures $\mu_m := \frac{1}{m}\sum_{j=1}^{m}\delta_{a_{j,m}}$, $\nu_m := \frac{1}{m}\sum_{j=1}^{m}\delta_{b_{j,m}}$ that

$$\log(|P_m(z)|^{1/m}) = -U^{\mu_m}(z),\ \log(|\frac{P_m(z)}{Q_m(z)}|^{1/m}) = -U^{\mu_m}(z)+U^{\nu_m}(z),$$

having limits $-U^{\mu}(z)$ and $-U^{\mu}(z) + U^{\nu}(z)$ (for z far enough from the supports) provided that $\mu_m \to \mu$ and $\nu_m \to \nu$ for $m \to \infty$.

Condenser with two plates

Let E, F be as before ($F \subset \overline{\mathbb{C}}$ closed, $E \subset \overline{\mathbb{C}} \setminus F$ compact).

Theorem 7 (two conductors)

There is a unique $\omega_{E,F} \in \mathcal{M}(E)$, $\omega_{F,E} \in \mathcal{M}(F)$ *called* **equilibrium measure** *minimizing*

$$\mathcal{M}(E) \times \mathcal{M}(F) \ni (\mu, \nu) \mapsto I(\mu - \nu) = I(\mu) + I(\nu) - 2I(\mu, \nu),$$

uniquely characterized by the property that $U^{\mu-\nu}$ *equals constants* c_1 *and* c_2 *on* E *and* F*, respectively, with* $c_1 - c_2 =: \log(1/\operatorname{cap}(E, F))$.

Theorem 8 (One isolator, balayage)

Given $\nu \in \mathcal{M}(F)$ *there is a unique* $\omega_\nu \in \mathcal{M}(E)$ *minimizing*

$$\mathcal{M}(E) \ni \mu \mapsto I(\mu - \nu) = I(\mu) + I(\nu) - 2I(\mu, \nu),$$

uniquely characterized by the property that $U^{\mu-\nu}$ *equals the constant* c_ν *on* E*. If* E *simply connected,* $c_\nu - U^{\omega_\nu - \nu}(z) = \int \log \left| \frac{1 - \overline{\varphi(x)}\varphi(z)}{\varphi(z) - \varphi(x)} \right| d\nu(x)$.

The main theorem

Theorem 9 (Gonchar 1984, Parfenov 1986, [Go84,Pa86])

For f analytic in $\mathbb{C} \setminus F$ and $E \subset \mathbb{C} \setminus F$ compact

$$\limsup \eta_{m-1,m}(f, E)^{1/m} \leq \exp(-2/\operatorname{cap}(E, F)).$$

1. idea of simple proof for pre-assigned poles following Walsh (without factor 2)
2. discretizing extremal measures, Zolotarev problem
3. much sharper bounds for Markov functions.

Pre-assigned poles following Walsh

Define the polynomials

- q_m of degree m with roots pre-assigned poles $b_{j,m}$ in F, counting measure ν_m
- ω_m of degree m with roots pre-assigned interpolation points $a_{j,m} \in E$, counting measure μ_m
- $p_m \in \mathcal{R}_{m-1,0}$ interpolating fq_m in roots of ω_m.

Then by Cauchy formula for analytic $(fq_m - p_m)/\omega_m$ and $z \in E$

$$f(z) - \frac{p_m(z)}{q_m(z)} = \frac{1}{2\pi i} \int_{\partial F} \frac{\omega_m(z)}{q_m(z)} \frac{q_m(x)}{\omega_m(x)} \frac{f(x)}{x - z} \, dx$$

$$\limsup_{m\to\infty} \|f - \frac{p_m}{q_m}\|_E^{1/m} \leq \limsup_{m\to\infty} \exp\Big(\max_{z\in\partial E, x\in\partial F} -U^{\mu_m-\nu_m}(x) + U^{\mu_m-\nu_m}(z) \Big).$$

Gives $\exp(-1/\operatorname{cap}(E,F))$ if $\mu_m \to \omega_{E,F}$ and $\nu_m \to \omega_{F,E}$.
Other limits if $\nu_m \to \nu$, $\mu_m \to \omega_\nu$.

Discretize equilibrium measure (1)

Consider the **Zolotarev problem**

$$Z_m(E,F) = \inf_{R \in \mathcal{R}_{m,m}} \|R\|_E \|1/R\|_F,$$

then

$$\lim_{m\to\infty} Z_m(E,F)^{1/m} = \exp(-1/\operatorname{cap}(E,F)),$$

and thus the counting measures of zeros/poles of extremal functions $\mu_m \to \omega_{E,F}$ and $\nu_m \to \omega_{F,E}$.

Caveat: some of the roots/poles might be outside E and F, respectively.

Discretize equilibrium measure (2)

Minimze the **discrete energy** (regularized counterpart of $I(\mu_n - \nu_n)$)

$$\frac{1}{n^2}\sum_{j,k=1,j\neq k}^{m}\left(\log\frac{1}{|a_{j,m}-a_{k,m}|}+\log\frac{1}{|b_{j,m}-b_{k,m}|}\right)-\frac{2}{n^2}\sum_{j,k=1}^{m}\log\frac{1}{|a_{j,m}-b_{k,m}|}$$

over $a_{1,m},\dots,a_{m,m}\in E$ and $b_{1,m},\dots,b_{m,m}\in F$.

Minimizer gives **weighted Fekete points**, with counting measures $\mu_m \to \omega_{E,F}$ and $\nu_m \to \omega_{F,E}$.

Special case of Markov functions

Cauchy transform of some positive measure μ

$$f^{[\mu]}(z) = \int \frac{d\mu(x)}{z-x}, \quad \text{supp}(\mu) \subset F$$

with $F \subset \mathbb{R}$ a closed interval.

- Examples for $\text{supp}(\mu) = (-\infty, 0]$ with $\gamma \in (-1, 0)$

$$f^{[\mu]}(z) = \frac{1}{\sqrt{z}}, \quad f^{[\mu]}(z) = \frac{\log(z)}{z-1}, \quad f^{[\mu]}(z) = z^{\gamma}.$$

- Example for $\text{supp}(\mu) = F = [a, b]$:

$$f^{[\omega_F]}(z) = \frac{1}{\sqrt{(z-a)(z-b)}}, \quad \frac{d\omega_F}{dx}(x) = \frac{1}{\pi\sqrt{(x-a)(b-x)}}.$$

Other elementary functions can be written as a rational function times a Markov function.

Sharp upper bounds for Markov functions

E, F two disjont real intervals, $\rho = \exp(-1/\operatorname{cap}(E,F))$,

Theorem 10 (Zolotarev 1877, [Akh90, Zol77])

For the particular Markov function $f = f^{[\omega_F]}$

$$\min_{r \in \mathcal{R}_{m-1,m}} \|\frac{f-r}{f}\|_E = Z_{2m}(E,F) \in [\frac{4\rho^{2m}}{(1+\rho^{4m})^2}, 4\rho^{2m}].$$

Theorem 11 (Beckermann 2024, [BBL22])

For any Markov function $f^{[\mu]}$ with $\operatorname{supp}(\mu) \subset F$

$$\max_{\operatorname{supp}(\mu) \subset F} \min_{r \in \mathcal{R}_{m-1,m}} \|\frac{f^{[\mu]} - r}{f^{[\mu]}}\|_E = \frac{2 Z_{2m}(E,F)}{1 + Z_{2m}(E,F)^2} \leq 8\rho^{2m}.$$

What we should bring home

1. We have seen various results of *m*th root asymptotics for best polynomial and rational approximation.
2. Natural tool is logarithmic potential theory.
3. The rate of best polynomial approximation on E is determined by the singularity "closest" to E (in terms of level sets)
4. The rate of best rational approximation on E is determined by the a condenser with plate E and second plate the set F of singularities.
5. Best rational appoximants are not always interpolants, problems with spourious poles.
6. Very sharp bounds for the special case of Markov functions.

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