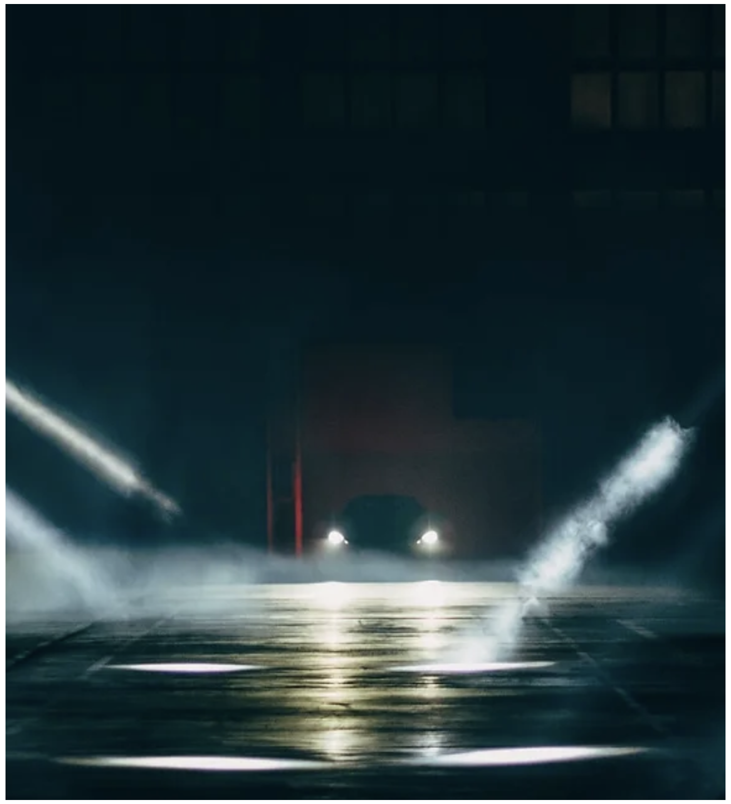
Numerical methods for the design of optical components, optimal transport and Generated Jacobian equations

Part 2: Stability in Optimal transport & Generated Jacobian Equations

Boris Thibert

Joint works with Quentin Mérigot, Jocelyn Meyron, Anatole Gallouet



Woudschoten conference, Sept. 25-27 2024

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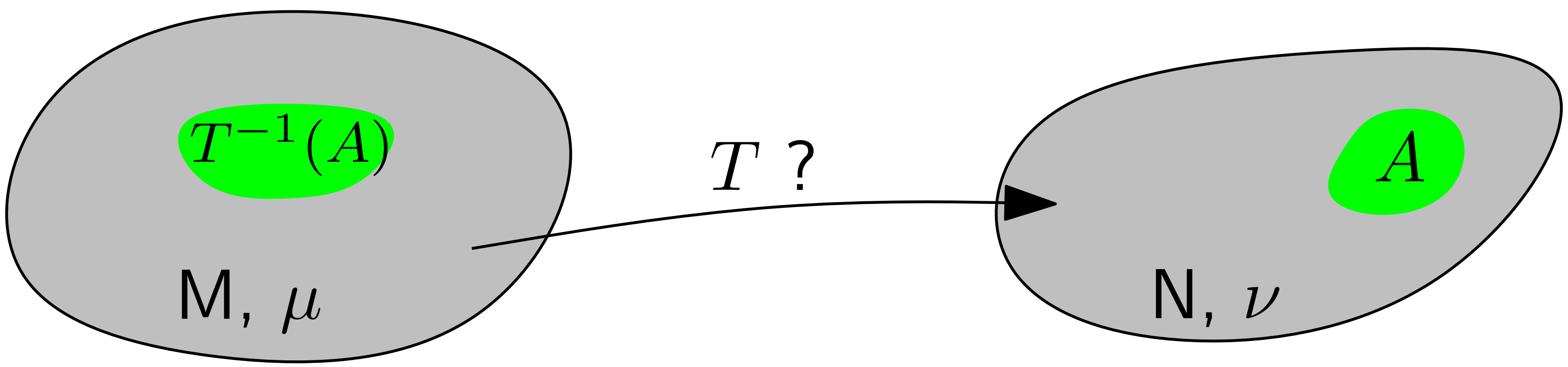

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under the constraint that $T_\sharp \mu = \nu$ (i.e $\mu(T^{-1}(A)) = \nu(A)$ for every A)

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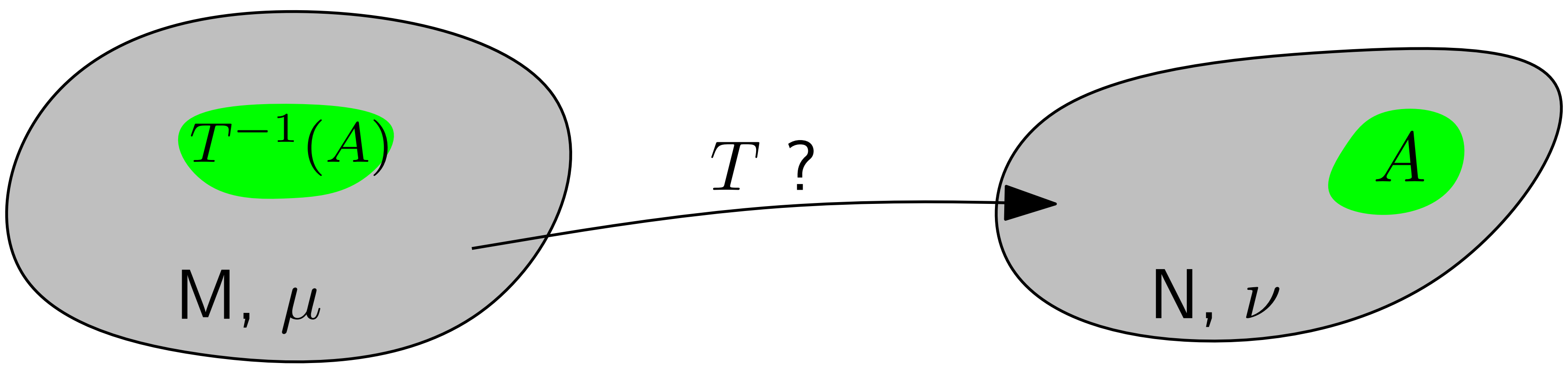

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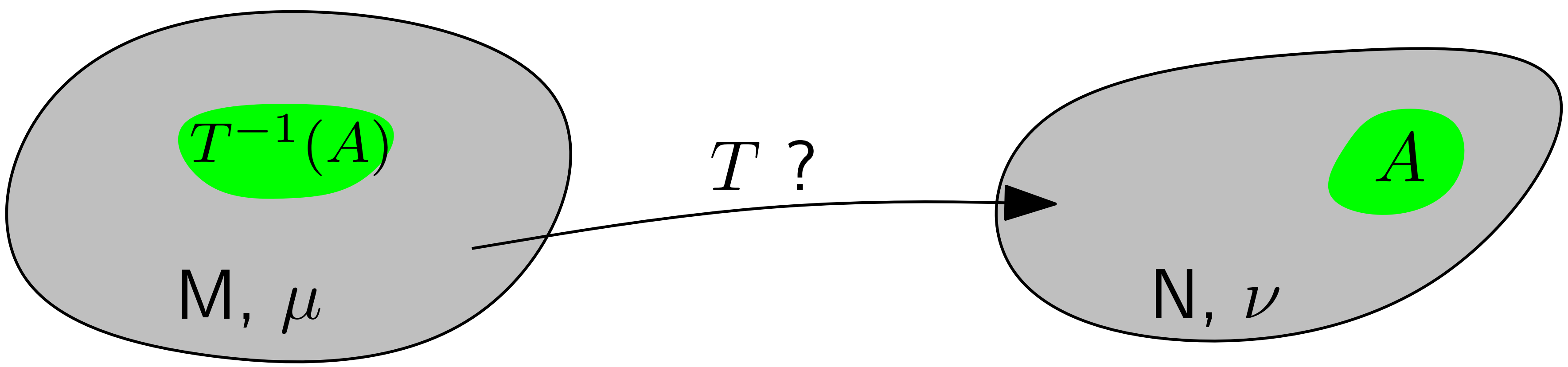

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Motivations:

- CV of numerical approaches: semi-discrete methods, discrete methods.

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Results hold for quadratic cost on $\mathbb{R}^d$ or manifold

$\rightsquigarrow$ Motivation: generalize (local) results to other costs and manifolds

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Theorem [Ambrosio-Gigli 11, Mérigot-Delalande Chazal 19] Let X, Y be compact domains of $\mathbb{R}^d$, $T_i : X \to Y$ be optimal transport maps between μ and ν_i $(i = 0, 1)$. If μ is absolutely continuous and T_0 is Lipschitz, then

$$\|T_1 - T_0\|_{L^2(\mu)} \leqslant \mathrm{C}\left(W_1(\nu_1, \nu_0)\right)^{\frac{1}{2}} .$$

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In order to generalize to any cost function $c : X \times Y \to \mathbb{R}$

$\rightsquigarrow$ notion of c-strongly concave functions.

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- Stability under strong c-concavity
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where $\Gamma(\mu, \nu)$ is the set of transport plans between μ and ν:

$$\Gamma(\mu,\nu) = \left\{ \gamma \in \mathcal{P}(M \times N), \quad \begin{array}{ll} \gamma(A \times N) = \mu(A) & \forall A \subset M \\ \gamma(M \times B) = \nu(B) & \forall B \subset N \end{array} \right\}$$

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Wasserstein distance $W_p(\mu,\nu) = \left(\inf_{\gamma \in \Gamma(\mu,\nu)} \int_M d_M(x,y)^p d\gamma(x,y) \right)^{1/p}$

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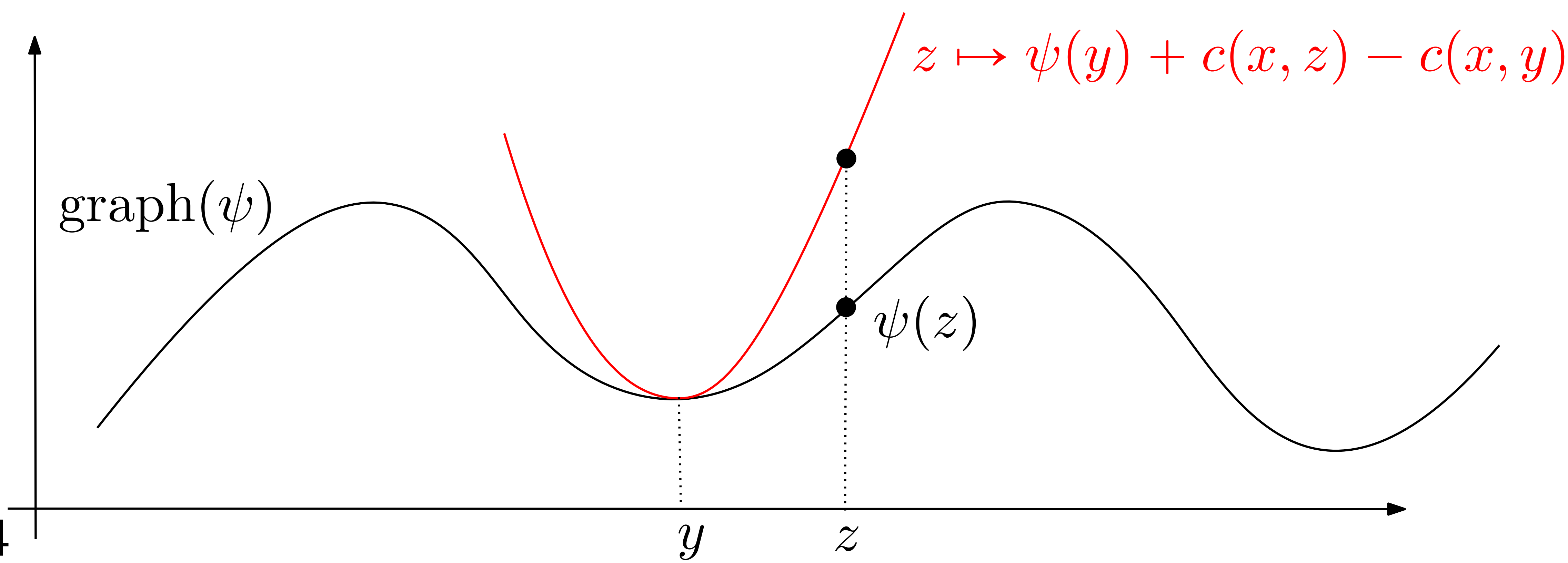

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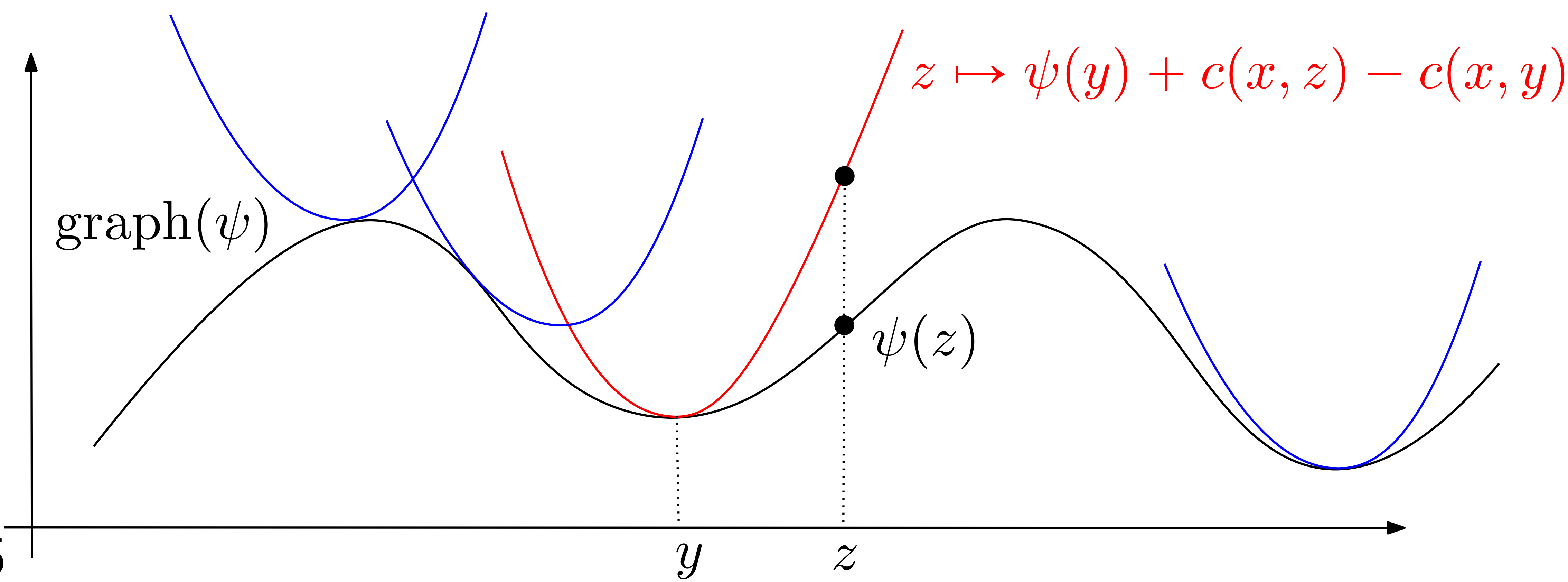

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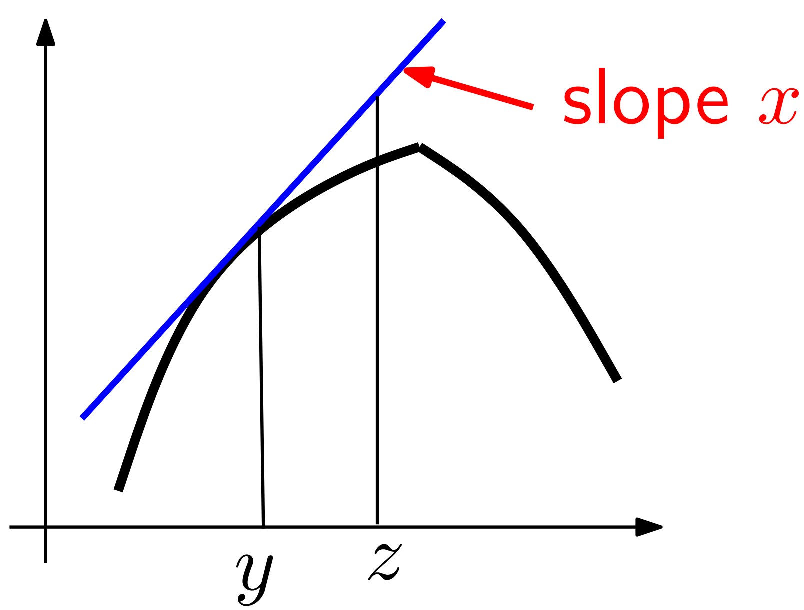
Particular case: When $c(x,y) = \langle x|y\rangle$, we have

- ψ is c-concave $\Longleftrightarrow$ ψ is concave



- $\partial^c\psi(y) = \partial^+\psi(y) := \{x \in M \mid \forall z \in N \quad \psi(z) \leqslant \psi(y) + \langle x|z-y\rangle\}$

Notion of strong c-concavity

Definition [Gallouet,Mérigot,T] A c-concave function ψ is c-strongly concave on a set $D \subset M \times N$ with modulus ω if for every x, y, z such that $(x, y) \in D$, $(x, z) \in D$ and $x \in \partial^c \psi(y)$, one has

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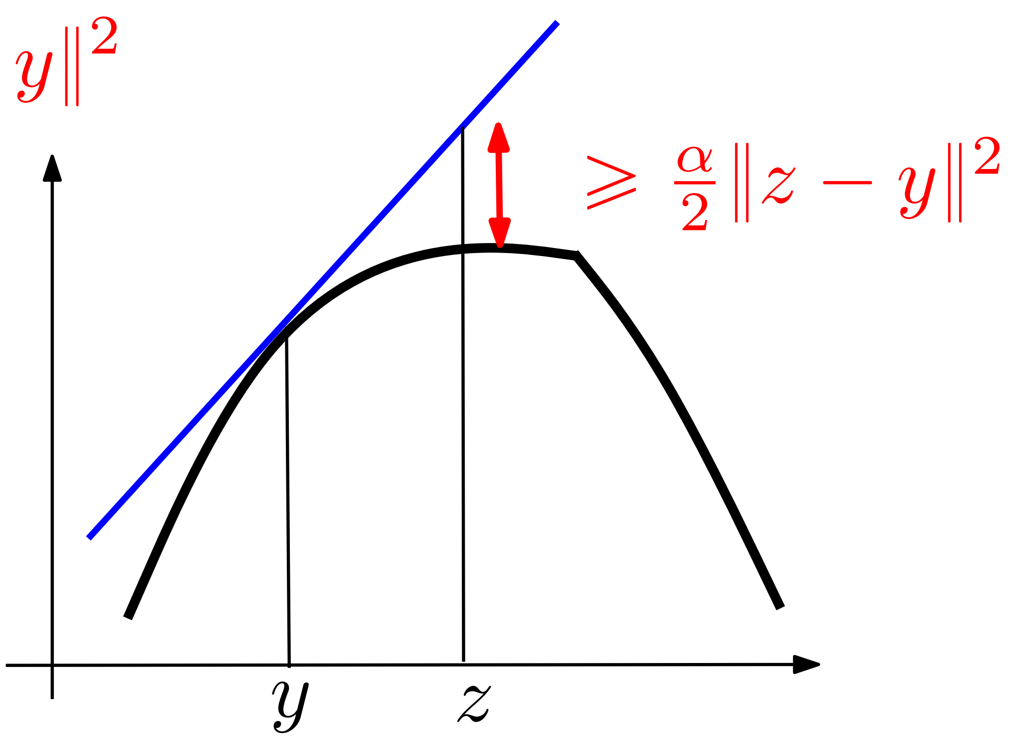
Particular case: When $c(x, y) = \langle x|y \rangle$ and $\omega(r) = \alpha r^2/2$, it coincides with the notion of strong concavity : A concave function $\psi : \mathbb{R}^d \to \mathbb{R}$ is α-strongly concave iff for every y

$$\forall x \in \partial\psi(y) \quad \psi(z) \leqslant \psi(y) + \langle x|z - y \rangle - \frac{\alpha}{2}\|z - y\|^2$$



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- ψ_0 is Lipschitz on N and c-concave on D.
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- The maps T_i satisfies for any $x \in M$, $(x, T_i(x)) \in D$.

Then

$$\int_M \omega(\mathrm{d}_N(T_0(x), T_1(x)))\mathrm{d}\mu(x) \leqslant (\mathrm{Lip}(\psi_0) + \mathrm{Lip}(\psi_1))W_1(\nu_0, \nu_1)$$

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- Justifies the semi-discrete approach
- Generalizes Ambrosio: replaces T_1 Lipschitz by ψ_1 strongly c-concave.

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$$\int_M \omega(\mathrm{d}_N(T_0(x), T_1(x)))\mathrm{d}\mu(x) \leqslant (\mathrm{Lip}(\psi_0) + \mathrm{Lip}(\psi_1))W_1(\nu_0, \nu_1)$$

Remark. If M and N are domains of $\mathbb{R}^d$ and $\omega(r) = r^2$ then

$$\|T_1 - T_0\|^2_{L^2(\mu)} \leqslant (\mathrm{Lip}(\psi_0) + \mathrm{Lip}(\psi_1))W_1(\nu_0, \nu_1)$$

- Justifies the semi-discrete approach
- Generalizes Ambrosio: replaces T_1 Lipschitz by ψ_1 strongly c-concave.
- If $D = M \times N$ blue assumptions disappear

Stability w.r.t target measure

Proof. Since ψ_0 and ψ_1 are Lipschitz, Kantorovich-Rubinstein theorem:

$$\int_N \left(\psi_1 - \psi_0\right) \mathrm{d}(\nu_1 - \nu_0) \leqslant (\mathrm{Lip}(\psi_0) + \mathrm{Lip}(\psi_1)) W_1(\nu_0, \nu_1)$$

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On the other hand

$$\int_N \Big(\psi_1 - \psi_0\Big)\mathrm{d}(\nu_1 - \nu_0) = \int_N \psi_1 \mathrm{d}(\nu_1 - \nu_0) + \int_N \psi_0 \mathrm{d}(\nu_0 - \nu_1)$$

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$$\begin{aligned} A &= \int_N \psi_1\,\mathrm{d}\,\nu_1 - \int_N \psi_1\,\mathrm{d}\,\nu_0 \\ &= \int_M \psi_1(T_1(x))\,\mathrm{d}\,\mu(x) - \int_M \psi_1(T_0(x))\,\mathrm{d}\,\mu(x) \qquad \text{(since } T_{i\sharp}\mu = \nu_i\text{)} \\ &= \int_M \Big(\psi_1(T_1(x)) - \psi_1(T_0(x))\Big)\,\mathrm{d}\,\mu(x) \end{aligned}$$

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&\geqslant \int_M c(x, T_1(x)) - c(x, T_0(x)) + \omega(\mathrm{d}_N(T_0(x), T_1(x)))\mathrm{d}\mu(x)
\end{aligned}$$

By strong c-concavity of ψ_1

Stability w.r.t target measure

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$$A \geqslant \int_M c(x, T_1(x)) - c(x, T_0(x)) + \omega(\mathrm{d}_N(T_0(x), T_1(x))) \mathrm{d}\mu(x)$$

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Similarly, since ψ_0 is c-concave

$$B \geqslant \int_M c(x, T_0(x)) - c(x, T_1(x))\mathrm{d}\mu(x)$$

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Summing A and B

$$\int_N \Big(\psi_1 - \psi_0\Big) \mathrm{d}(\nu_1 - \nu_0) \geqslant \int_M \omega(\mathrm{d}_N(T_0(x), T_1(x)))\mathrm{d}\mu(x) \qquad \square$$

Error bounds for OT problems

Theorem 2 [Gallouet,Mérigot,T]. Let $\mu \in \mathcal{P}(M)$, $\nu \in \mathcal{P}(N)$, $D \subseteq M \times N$ be a compact set. Let T be an optimal transport map from μ to ν with associated potential $\psi : N \to \mathbb{R}$ such that:

- ψ is strongly c-concave with modulus ω on D.
- For any $x \in M$, $(x, T(x)) \in D$.

Then any transport plan $\gamma \in \Gamma(\mu, \nu)$ supported on D satisfies

$$\int_{M \times N} \omega(\mathrm{d}_N(T(x), y))\mathrm{d}\gamma(x, y) \leqslant \int_{M \times N} c(x, y)\mathrm{d}\gamma(x, y) - \int_M c(x, T(x))\mathrm{d}\mu(x)$$

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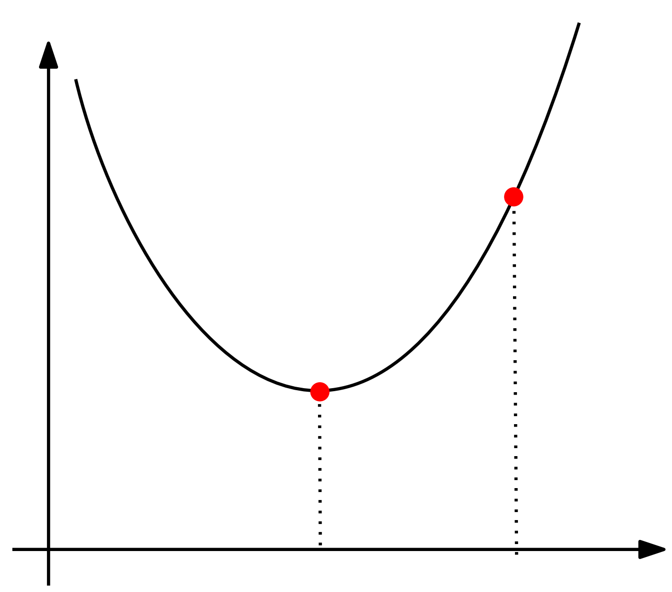
cost of γ Minimal cost



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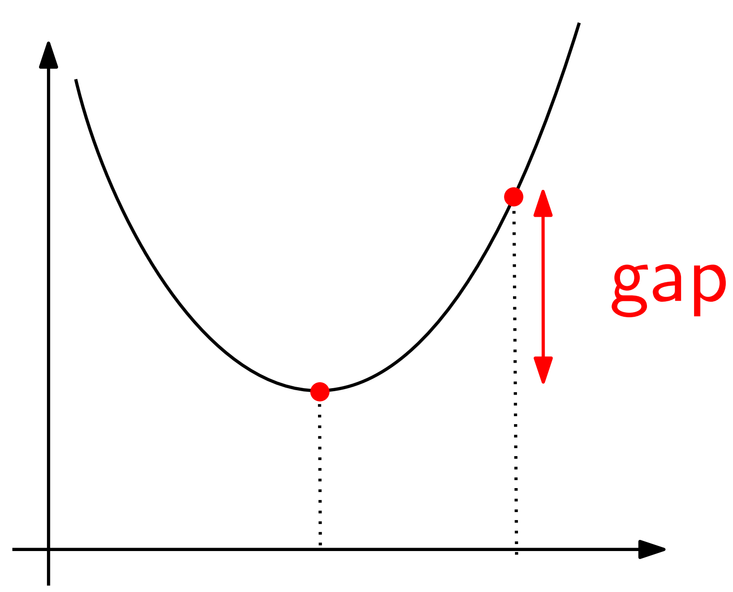
suboptimality gap $=$ cost of γ $-$ Minimal cost



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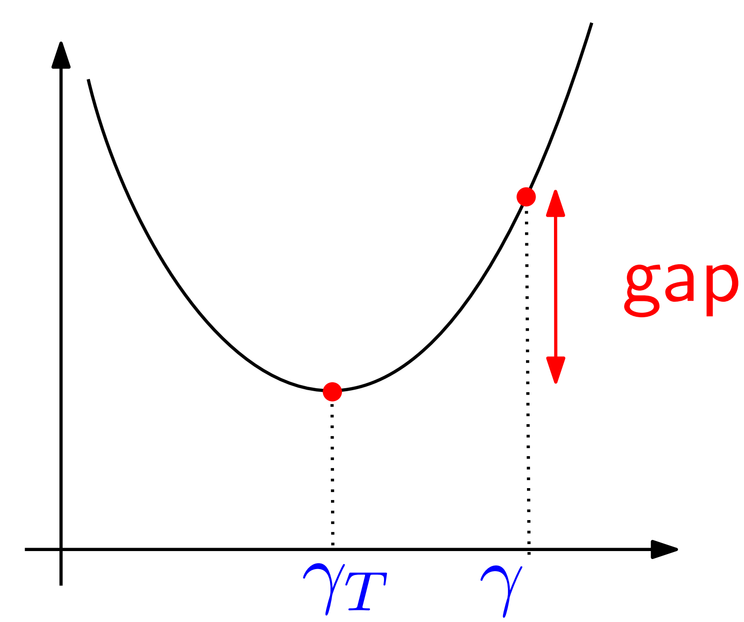
suboptimality gap = cost of γ − Minimal cost

Corollary. If $\omega(r) = Cr^2$, then

$$W_1(\gamma, \gamma_T) \leqslant \frac{1}{\sqrt{C}} (\text{suboptimality gap})^{1/2}$$

where $\gamma_T = (Id, T)_{\#}\mu$ and W_1 is for the distance $d_{M\times N} = d_M + d_N$.



$\rightsquigarrow$ kind of strong convexity of cost function

Stability w.r.t source and target measure

Theorem 3 [Gallouet,Mérigot,T]. Let $c : M \times N \to \mathbb{R}$ be Lipschitz, $\mu, \tilde{\mu} \in \mathcal{P}(M)$ and $\nu, \tilde{\nu} \in \mathcal{P}(N)$. Let $T : (M, \mu) \to (N, \nu)$ be an optimal transport map and $\tilde{\gamma}$ be an optimal transport plan between $\tilde{\mu}$ and $\tilde{\nu}$. We know that T is induced by a c-concave potential $\psi : N \to \mathbb{R}$. We assume that

- ψ is strongly c-concave potential with $\omega(r) = Cr^2$ on $D = M \times N$.

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$$\mathrm{W}_1(\gamma_T, \tilde{\gamma}) \leqslant \varepsilon + \sqrt{\frac{2\mathrm{Lip}(c)}{C}\varepsilon}, \quad \text{where } \varepsilon := \mathrm{W}_1(\tilde{\mu}, \mu) + \mathrm{W}_1(\nu, \tilde{\nu}).$$

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- It is a consequence of the previous “suboptimality gap inequality”

Outline

Part 1: Stability

- Optimal Transport and strong c concavity
- Stability under strong c-concavity
- Sufficient conditions for strong c-concavity
- Applications to the reflector problem

Part 2: Generated Jacobian Equation

- Case 1: Mirror for Point source light (Far Field and Near Field)
- Case 2: Mirror for Parallel source light (Far Field and Near Field)
- Semi-discrete Generated Jacobian equation

Ma-Trudinger-Wang tensor

Let $c : M \times N \to \mathbb{R} \cup \{+\infty\}$. We assume that it satisfies (Stwist).

(STwist): c is C^2, $\nabla_x c(x, \cdot)$ and $\nabla_y c(\cdot, y)$ are injective, $\nabla^2_{xy} c$ is non-singular.

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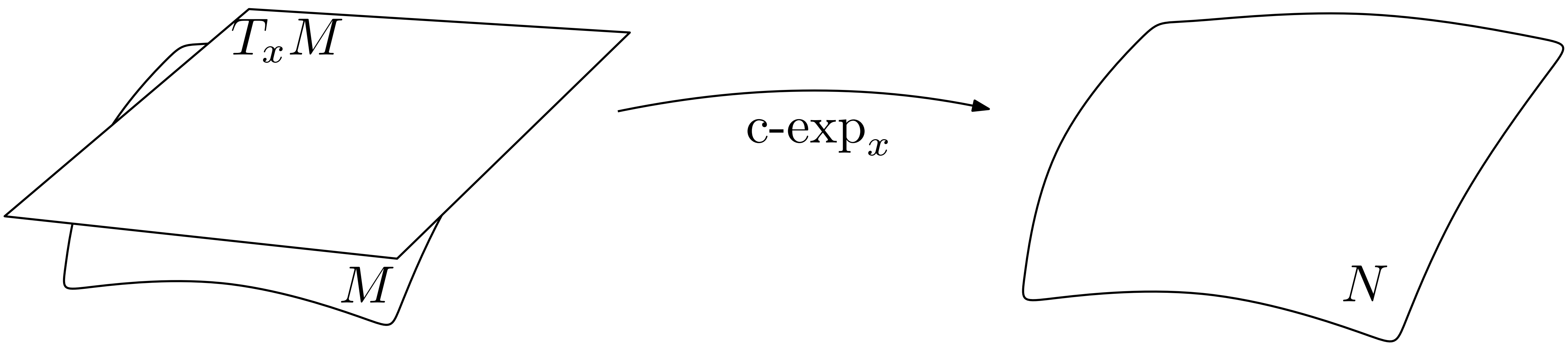

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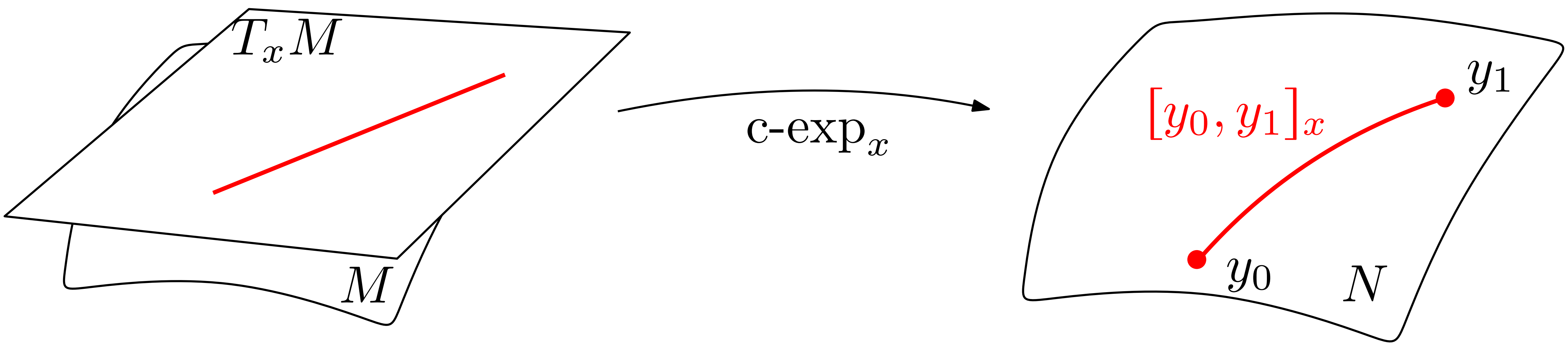

- A c-segment $(y_t)_{1 \leqslant t \leqslant 1} = [y_0, y_1]_x$ is the image by c-exp_x of a segment.

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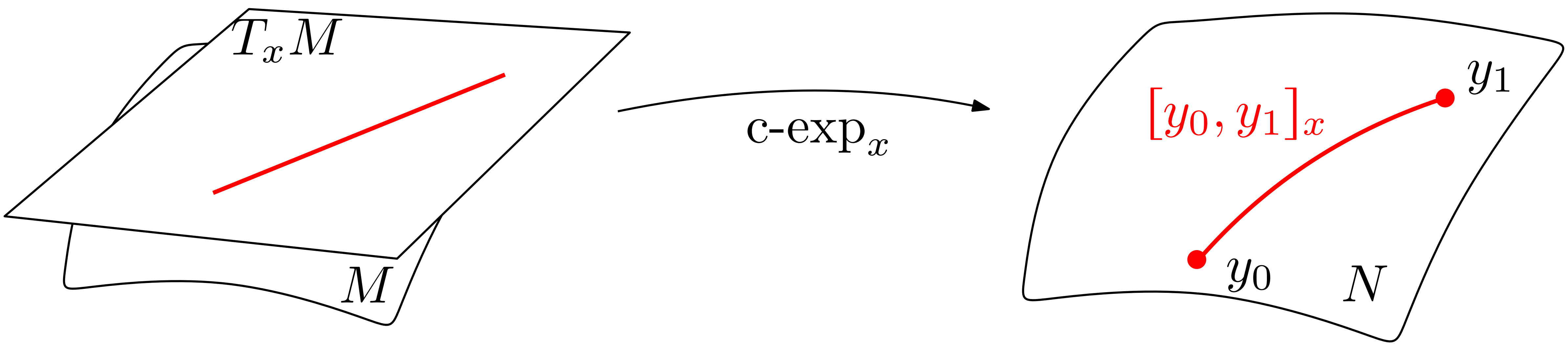

- A c-segment $(y_t)_{1 \leqslant t \leqslant 1} = [y_0, y_1]_x$ is the image by c-exp_x of a segment.
- $D \subset M \times N$ is symmetrically c-convex if

$$(x, y_0) \in D \text{ and } (x, y_1) \in D \quad \Longrightarrow \quad \forall t \in [0,1]\ (x, y_t) \in D$$

$$(x_0, y) \in D \text{ and } (x_1, y) \in D \quad \Longrightarrow \quad \forall t \in [0,1]\ (x_t, y) \in D$$

Ma-Trudinger-Wang tensor

Let $c : M \times N \to \mathbb{R} \cup \{+\infty\}$ of class C^4 that satisfies (Stwist).

Definition. The Ma-Trüdinger-Wang tensor is defined for $(x_0, y_0) \in D$ and $(\eta, \zeta) \in T_{x_0}M \times T_{y_0}N$ by

$$\mathfrak{S}_c(x_0, y_0)(\eta, \zeta) = -\frac{3}{2}\frac{\partial^2}{\partial q_{\tilde{\eta}}^2}\frac{\partial^2}{\partial y_{\zeta}^2}\left(c(\text{c-exp}_{y_0}(q), y)\right)\Big|_{y=y_0, q=-\nabla_y c(x_0,y_0)}$$

with $\tilde{\eta} = -\nabla^2_{xy} c(x_0, y_0)\eta \in T_{y_0}N$

Here $-\nabla^2_{xy} c(x_0, y_0) : T_{x_0}M \times T_{y_0}N \to \mathbb{R}$ is a non singular bilinear form.
the linear form $\tilde{\eta} : T_{y_0}N \to \mathbb{R}$ is identified with a vector.

Definition The weak MTW condition (MTWw) is satisfied on a compact set $D \subseteq M \times N$ if there exists a constant $C > 0$ such that for any $(x_0, y_0) \in D$ and $(\eta, \zeta) \in T_{x_0}M \times T_{y_0}N$ we have

$$\mathfrak{S}_c(x_0, y_0)(\eta, \zeta) \geqslant -C|\langle \zeta | \tilde{\eta} \rangle| \|\zeta\| \|\eta\|$$

- 4th order condition that appears in the regularity theory [MTW 2005]

Differential criterion for strong c-concavity

Theorem [Gallouet,Mérigot,T]. Let $D \subseteq M \times N$ be a closed symmetrically c-concave set and $c \in C^4(D, \mathbb{R})$ that satisfy (STwist) and (MTWw) on D. Let $\mathcal{Y} = \mathrm{proj}_N(D)$ and $\psi \in C^2(\mathcal{Y}, \mathbb{R})$ be c-concave on D. If there exists $\lambda > 0$ such that for any $y \in \mathcal{Y}$

$$\forall x \in \partial^c \psi(y) \quad D^2_{yy} c(x, y) - D^2 \psi(x) \geqslant \lambda I_d \qquad (*)$$

Then ψ is strongly c-concave on D with modulus $\omega(r) = Cr^2$, where $C > 0$ is a constant depending on c, λ and D.

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- We can replace $(*)$ by: The map $T : \mathcal{X} \to \mathcal{Y}$

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is of class $\mathcal{C}^1$ and satisfies for any $x \in \mathcal{X}$, $(x, T(x)) \in D$

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- One difficulty: to show that T is supported on a set D where c is smooth.

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Theorem [Gallouet,Mérigot,T]. Let $D \subseteq M \times N$ be a closed symmetrically c-concave set and $c \in C^4(D, \mathbb{R})$ that satisfy (STwist) and (MTWw) on D. Let $\mathcal{Y} = \mathrm{proj}_N(D)$ and $\psi \in C^2(\mathcal{Y}, \mathbb{R})$ be c-concave on D. If there exists $\lambda > 0$ such that for any $y \in \mathcal{Y}$

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Then ψ is strongly c-concave on D with modulus $\omega(r) = Cr^2$, where $C > 0$ is a constant depending on c, λ and D.

quantified version

- We can replace $(*)$ by: The map $T : \mathcal{X} \to \mathcal{Y}$

$$T(x) = \mathrm{argmin}_y c(x,y) - \psi(y)$$

is of class $\mathcal{C}^1$ and satisfies for any $x \in \mathcal{X}$, $(x, T(x)) \in D$

- Can apply to optimal transport maps to get stability results
- One difficulty: to show that T is supported on a set D where c is smooth.

$\rightsquigarrow$ has to be treated on each application

Outline

Part 1: Stability

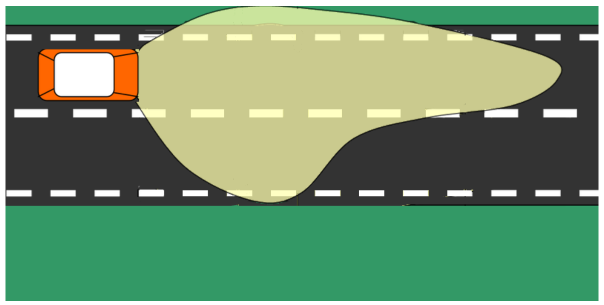
- Optimal Transport and strong c concavity
- Stability under strong c-concavity
- Sufficient conditions for strong c-concavity
- Applications to the reflector problem

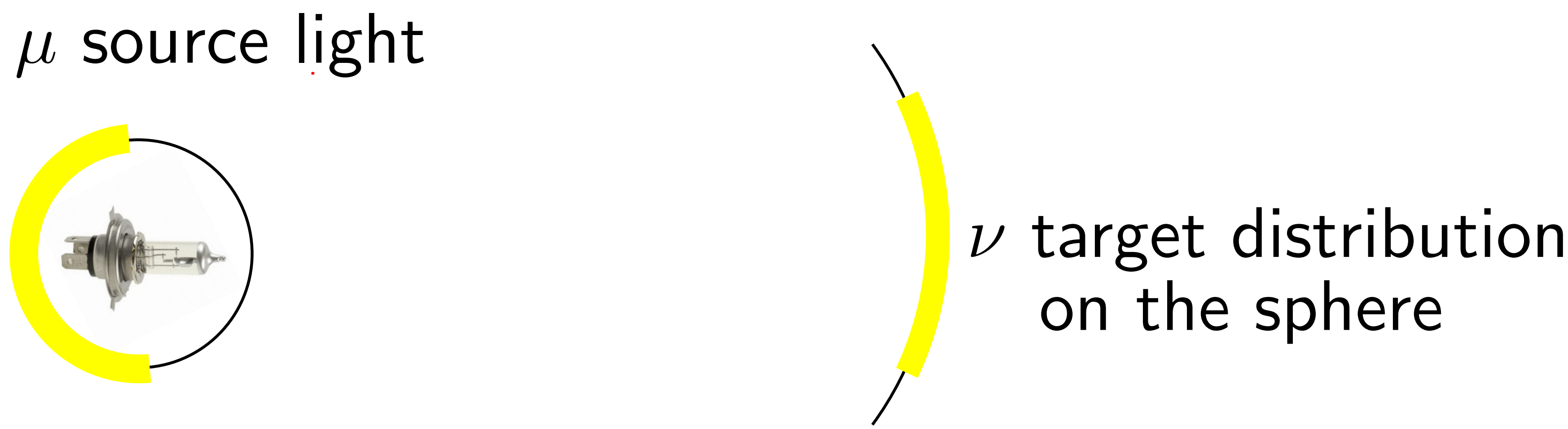
Part 2: Generated Jacobian Equation

- Case 1: Mirror for Point source light (Far Field and Near Field)
- Case 2: Mirror for Parallel source light (Near Field)
- Semi-discrete Generated Jacobian equation

Reflector problem: point source light

- The problem:





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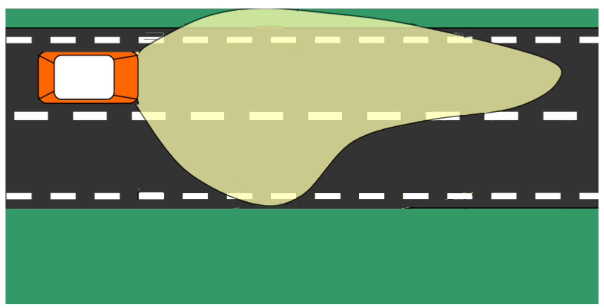

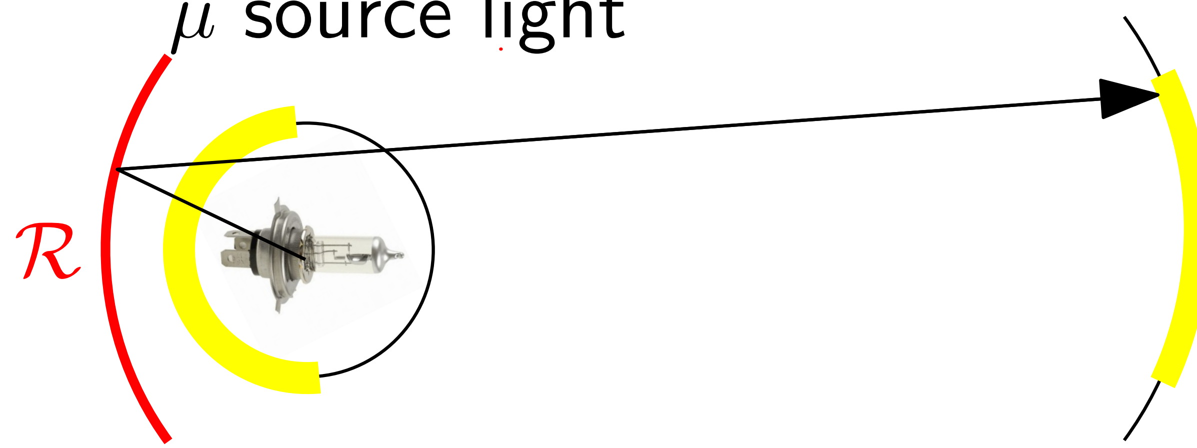

Pb: Find a reflector $\mathcal{R}$ that reflects μ to ν

[Caffarelli, Oliker 94]

Reflector problem: point source light

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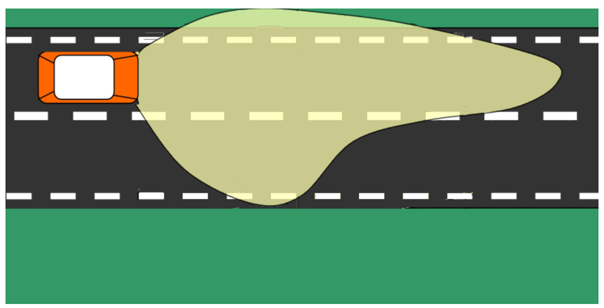

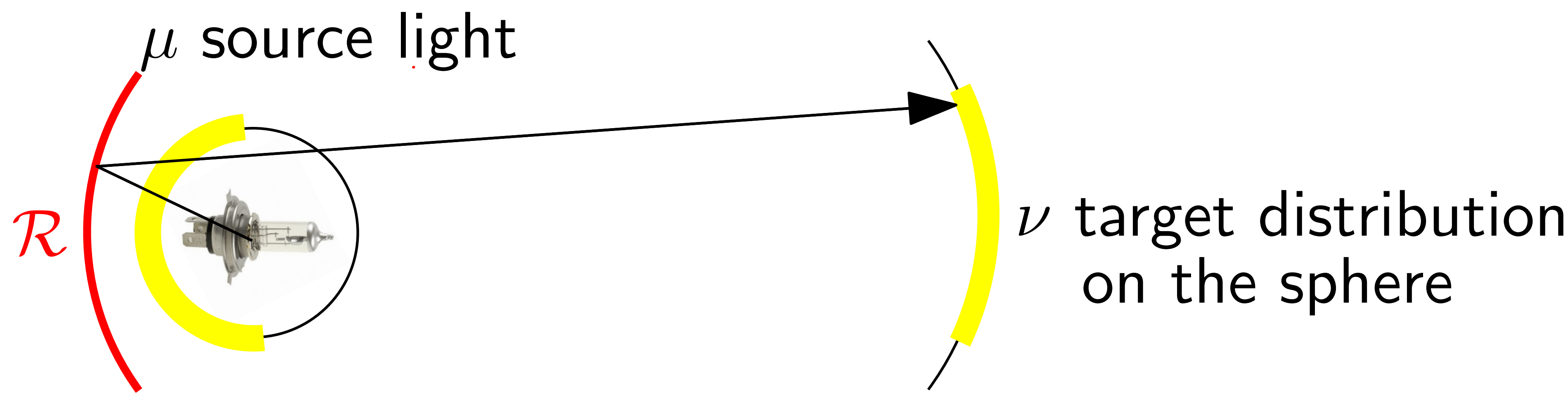

Pb: Find a reflector $\mathcal{R}$ that reflects μ to ν [Caffarelli, Oliker 94]

- Equivalent to optimal transport pb on $\mathbb{S}^2$ for $c(x,y) = -\ln(1 - \langle x|y \rangle)$.

[Wang 2003, Oliker 2003]

Reflector problem: point source light

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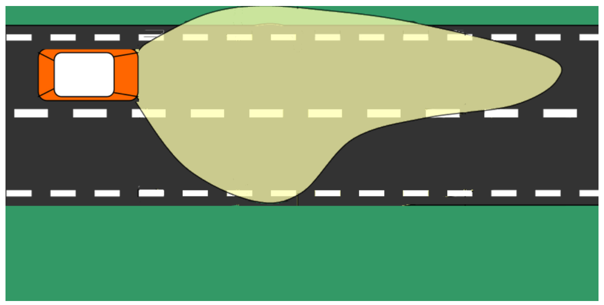
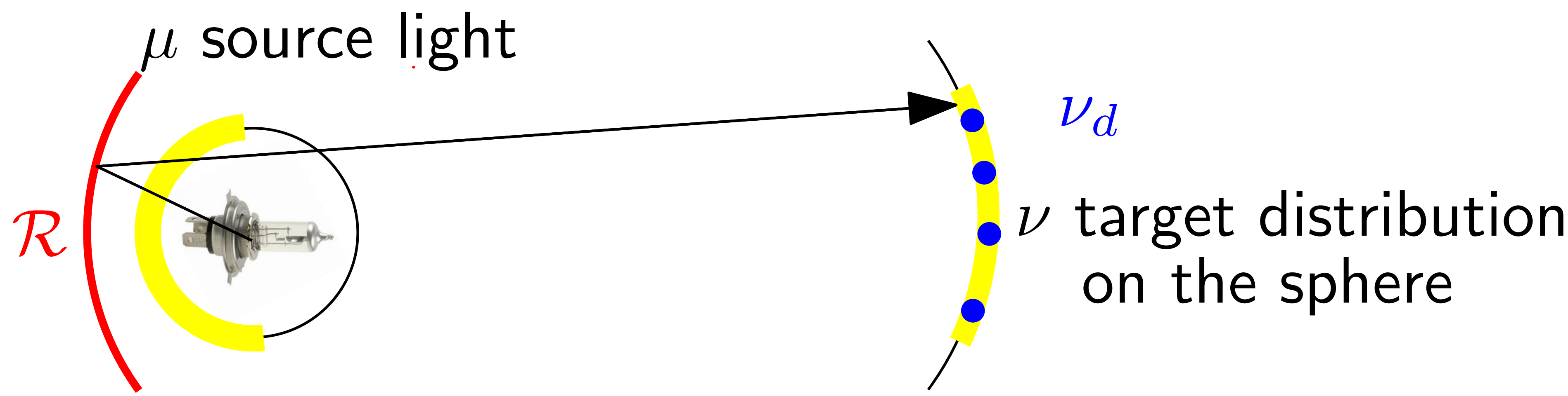
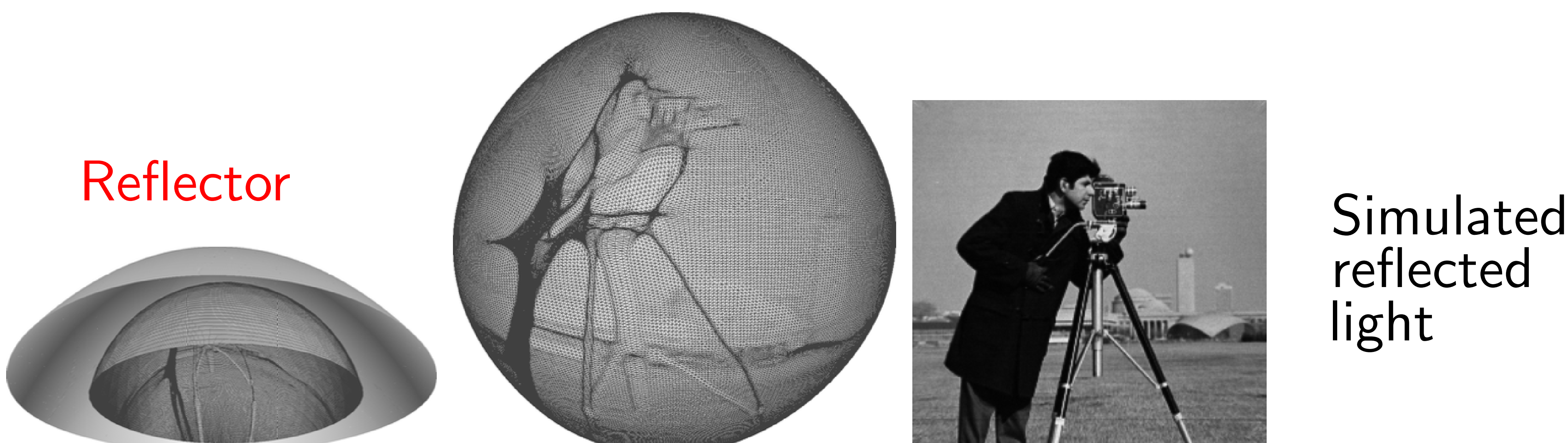




Pb: Find a reflector $\mathcal{R}$ that reflects μ to ν [Caffarelli, Oliker 94]

- Equivalent to optimal transport pb on $\mathbb{S}^2$ for $c(x,y) = -\ln(1 - \langle x|y \rangle)$.
[Wang 2003, Oliker 2003]
- Numerical methods: ν_d is a discretization of ν.

e.g. certified Newton algorithm [Mérigot,Meyron,T.]



Reflector problem: point source light

Theorem [Gallouet,Mérigot,T]. Let $c(x, y) = -\ln(1 - \langle x|y\rangle)$, μ and ν_0 be measures with $C^{1,1}$ densities. Then for all $r > 0$, there exists $C > 0$ such that for every measure ν_1 (e.g. ν_d) satisfying

$$\sup_{y \in \mathbb{S}^{d-1}} \nu_1(B(x, r)) < \frac{1}{8}$$

one has

$$\int_{\mathbb{S}^{d-1}} \mathrm{d}_M(T_0(x), T_1(x))^2 \mathrm{d}\mu(x) \leqslant C\ W_1(\nu_0, \nu_1)$$

where $T_i : \mu \to \nu_i$ are optimal transport maps.

Reflector problem: point source light

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ν_1 can be discrete

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- c is not differentiable on $\{x = y\}$.
- We therefore set $D_\varepsilon = \{(x,y) \in \mathbb{S}^{d-1} \times \mathbb{S}^{d-1} \mid \mathrm{d}_{\mathbb{S}^{d-1}}(x,y) \geqslant \varepsilon\}$

Reflector problem: point source light

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Sketch of proof

• We show any optimal $\gamma \in \Gamma(\mu,\nu)$ is supported on D_ε (similar results of [Gangbo, Oliker 2007], [Buttazzo 2018] and [Loeper 2011] for non discrete measures).

• We show that D_ε is symmetrically c-convex.
$\rightsquigarrow$ strong c-concavity and stability

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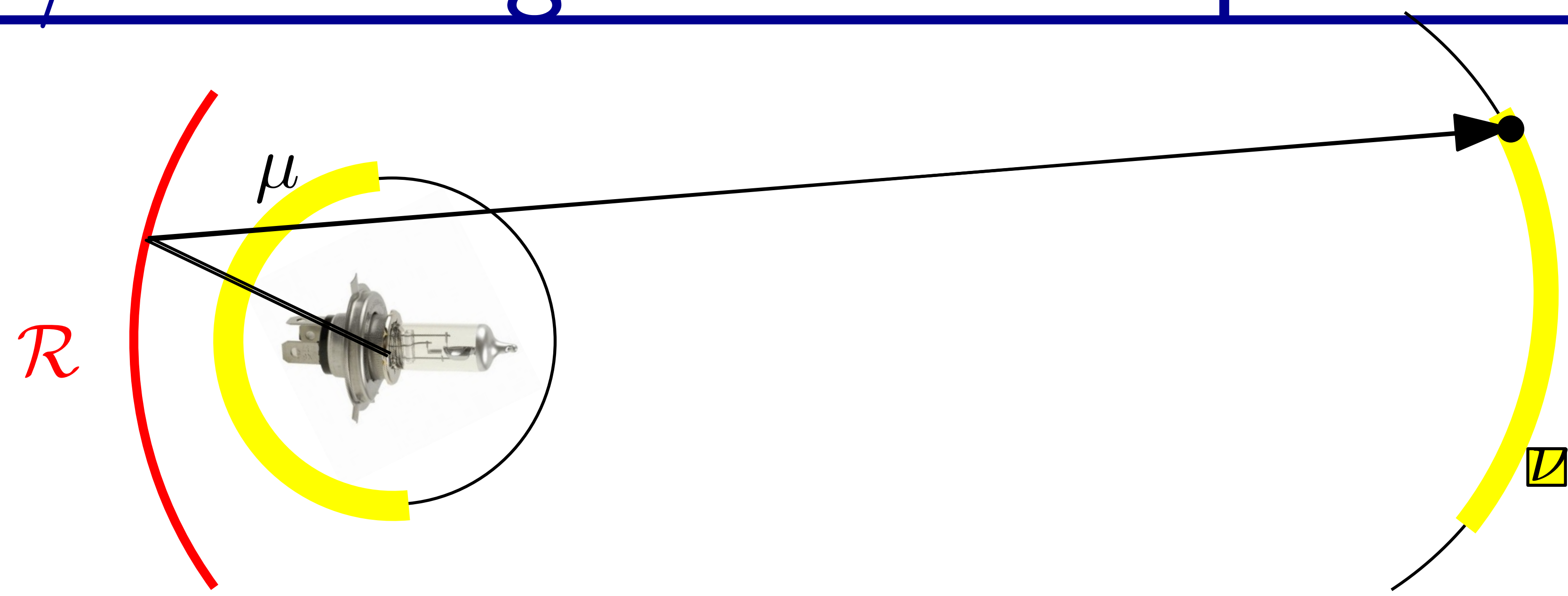
Mirror / Point light source: Optimal Transport



Mirror: Point light source / Far-Field $\rightsquigarrow$ OT

$\rightsquigarrow$ We have to solve an OT problem

Problem (FF): Find $\psi \in \mathbb{R}^N$ such that

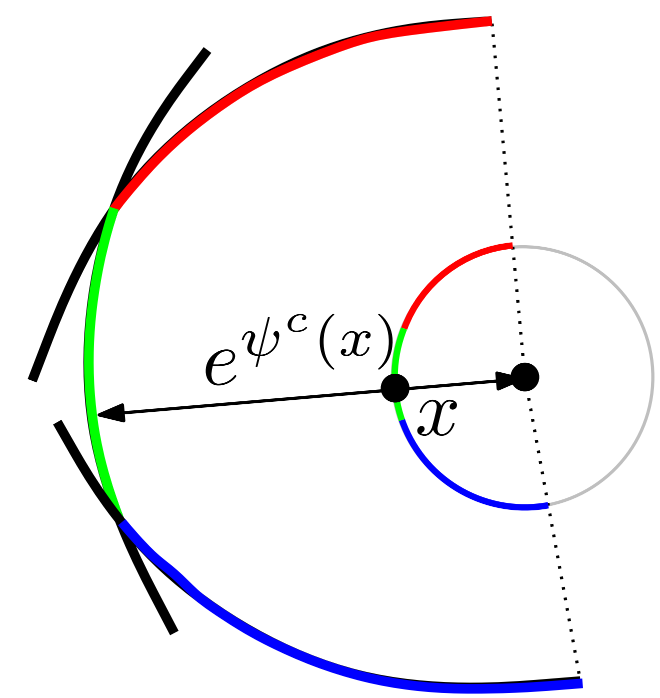
$$\forall i \in \{1, \cdots N\} \qquad \mu(Lag_i(\psi)) = \nu_i.$$

where $\mathrm{Lag}_i(\psi) = \{x \in \mathbb{S}_0^2,\ c(x, y_i) + \psi_i \leqslant c(x, y_j) + \psi_j \quad \forall j\}$, $\psi_i := \log(\kappa_i)$, and $c(x, y) = -\log(1 - \langle x|y\rangle)$.

$\rightsquigarrow$ The mirror is parametrized by

$$\begin{array}{ccc} \mathbb{S}^{d-1} & \rightarrow & \mathbb{R}^d \\ x & \mapsto & \left(\min_i \frac{e^{\psi_i}}{1-\langle x|y_i\rangle}\right) x \end{array}$$

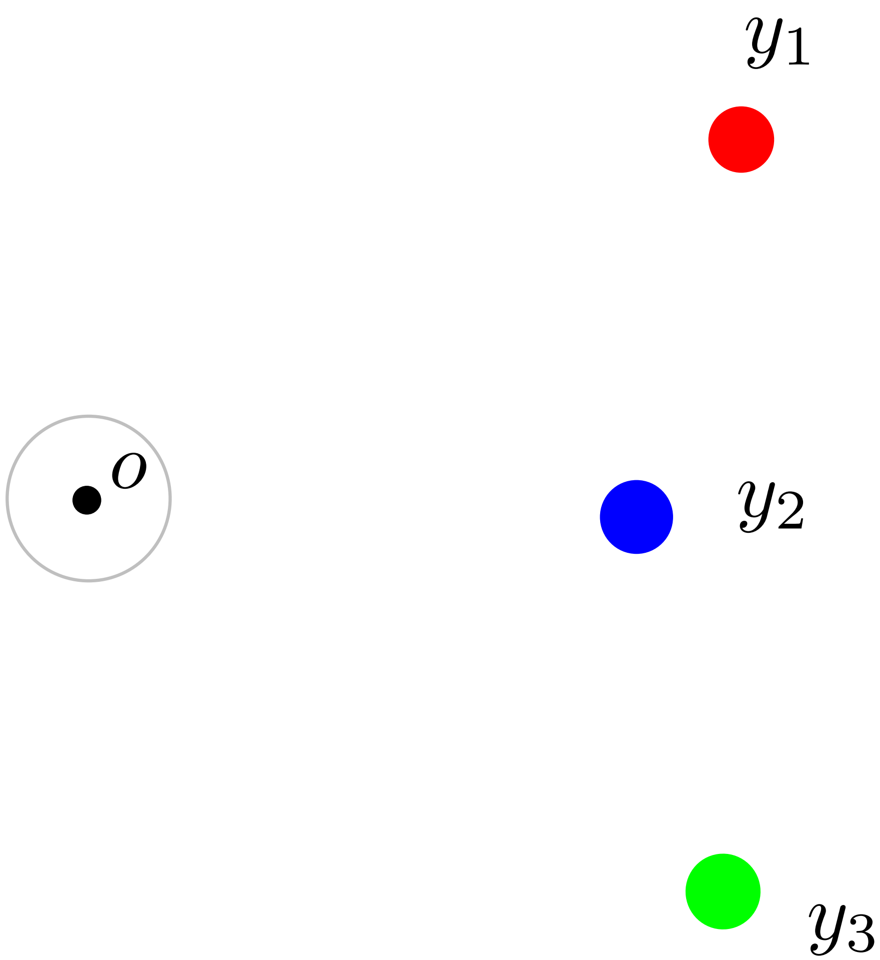
$$e^{\min_i c(x,y_i)+\psi_i} = e^{\psi^c(x)}$$



where $\psi^c(x) = \min_{y_i} c(x, y) - \psi(y_i)$ is the c-conjugate function of ψ.

ccl : $x \in \mathbb{S}_0^2 \mapsto e^{\psi^c(x)} x$ parametrizes the mirror.

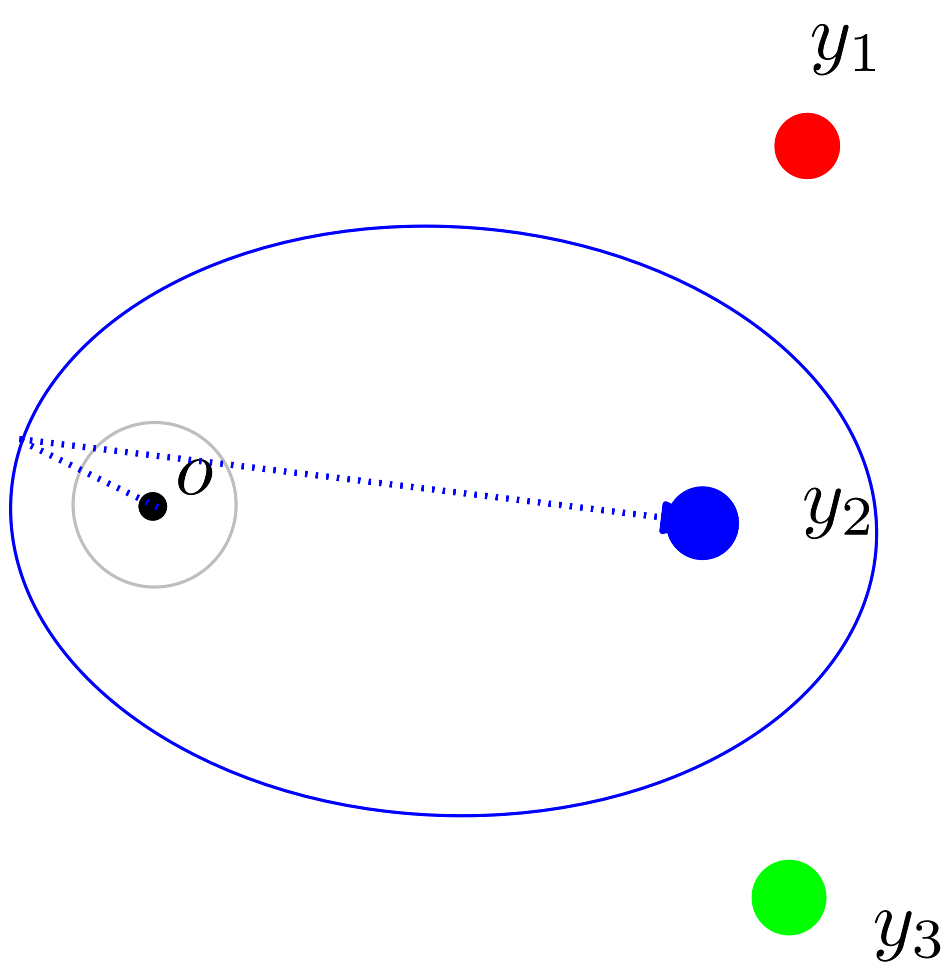
Mirror: Point light source / Near-Field $\rightsquigarrow$ JGE



Punctual light at origin o, μ measure on $\mathcal{S}_o^2$

Prescribed near-field: $\nu = \sum_i \nu_i \delta_{y_i}$ on $\mathbb{R}^3$

Mirror: Point light source / Near-Field $\rightsquigarrow$ JGE

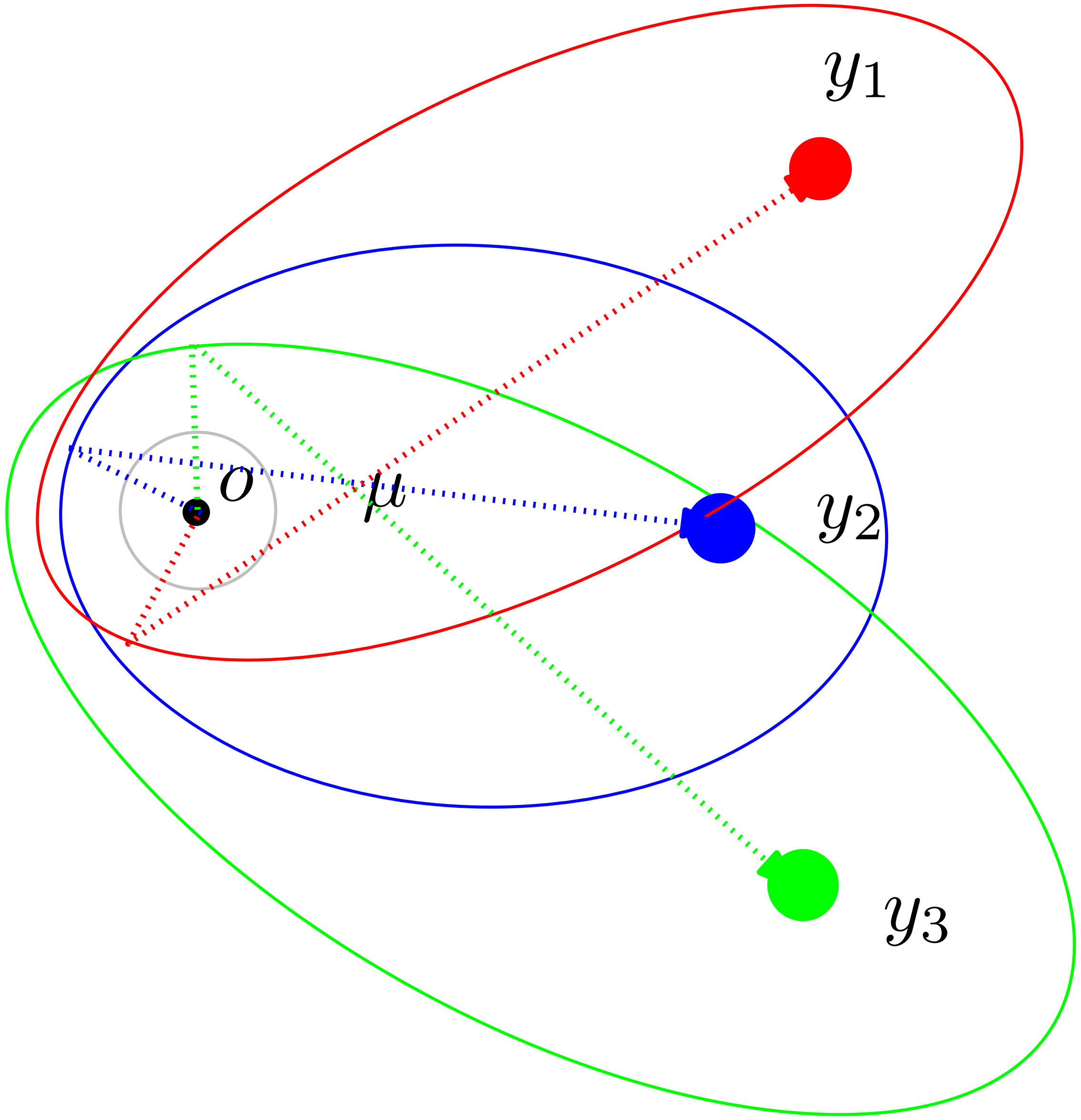


Punctual light at origin o, μ measure on $\mathcal{S}^2_o$

Prescribed near-field: $\nu = \sum_i \nu_i \delta_{y_i}$ on $\mathbb{R}^3$

$E_i(a_i) =$ convex hull of ellipsoid with focals o and y_i, and major axis length a_i

Mirror: Point light source / Near-Field ⇝ JGE



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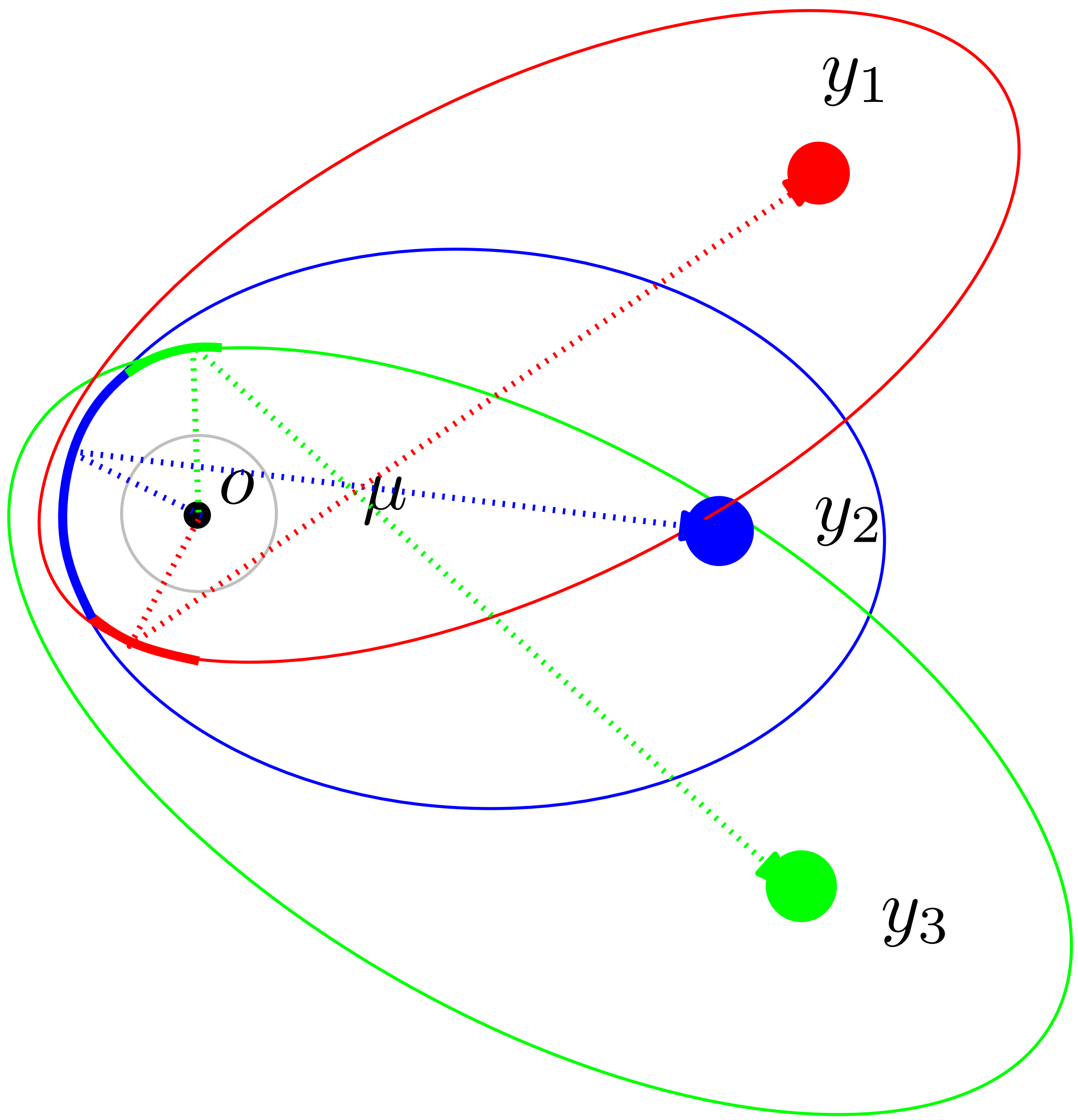
Mirror: Point light source / Near-Field $\rightsquigarrow$ JGE



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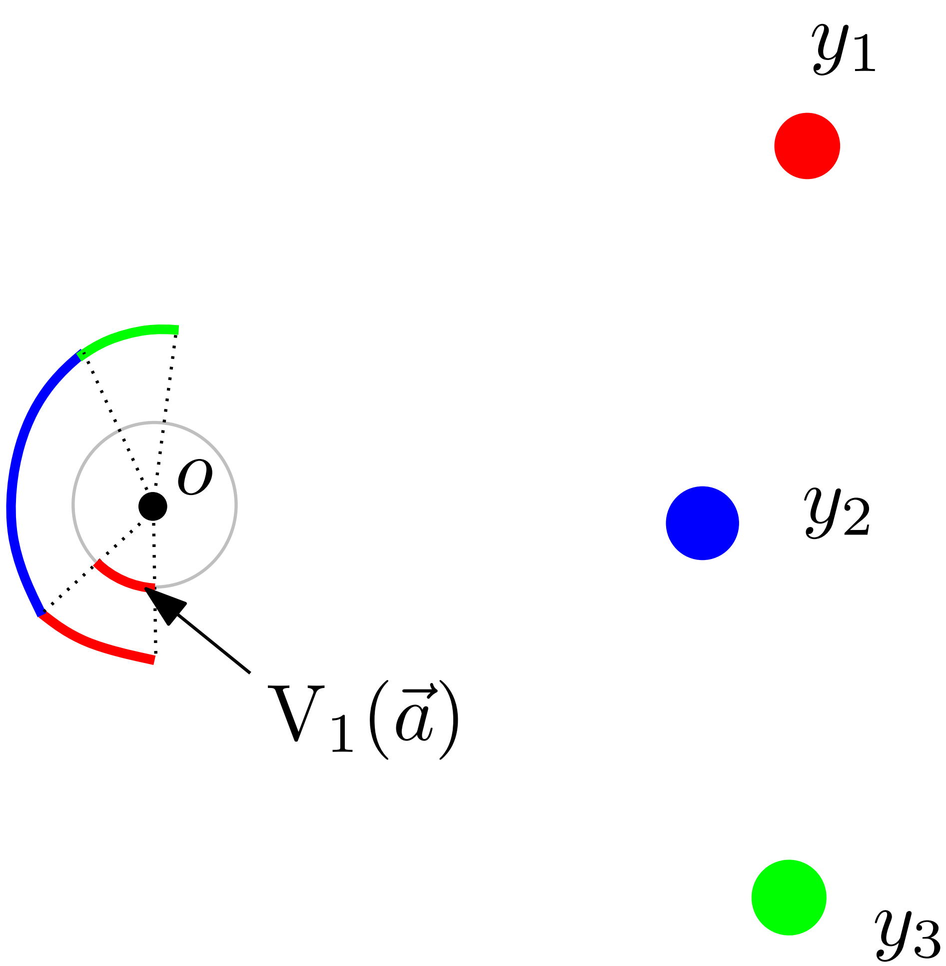
$E_i(a_i)$ = convex hull of ellipsoid with focals o and y_i, and major axis length a_i

$$R(\vec{a}) = \partial \left(\cap_{i=1}^N E_i(a_i) \right)$$

Mirror: Point light source / Near-Field $\rightsquigarrow$ JGE



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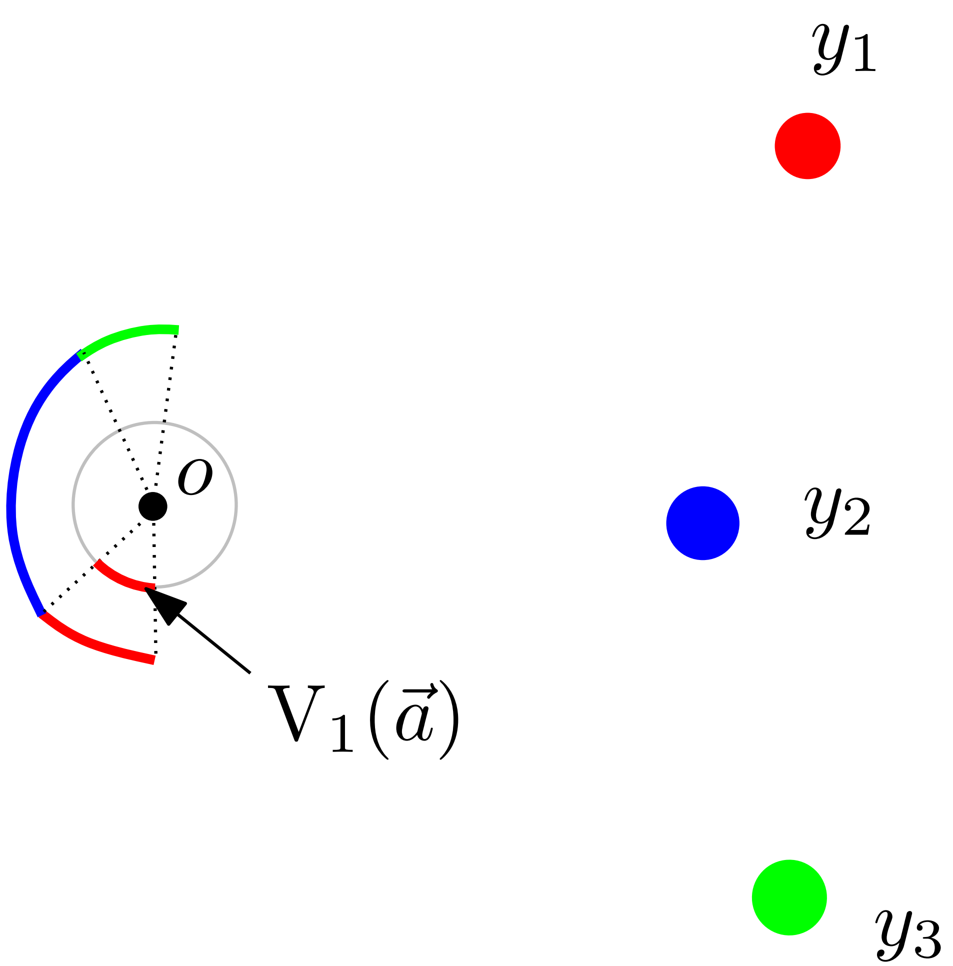
Mirror: Point light source / Near-Field ⇝ JGE



Punctual light at origin o, μ measure on $\mathcal{S}_o^2$

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Near-field reflector antenna problem:

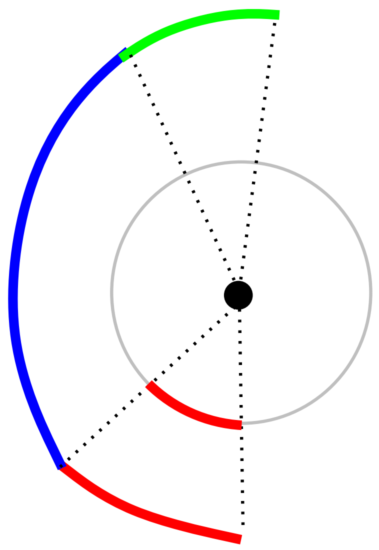
Oliker '04

Problem (NF): Find $a_1, \ldots, a_N$ such that for every i, $\mu(V_i(\vec{a})) = \nu_i$.

amount of light reflected to the point y_i.

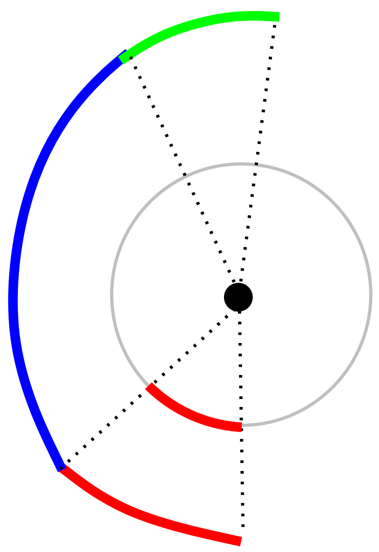
Mirror: Point light source / Near-Field ⇝ JGE

Computation of visibility cells:



Mirror: Point light source / Near-Field ⇝ JGE

Computation of visibility cells:



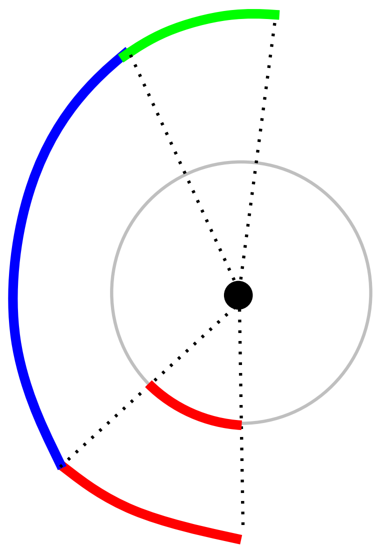
$\partial E_i(\kappa_i)$ is parameterized in radial coordinates by

$$\rho_i : x \in \mathbb{S}^2_o \mapsto \frac{a_i^2 - \|y_i\|^2/4}{a_i - \langle x | y_i \rangle / 2}$$

where $2a_i$ is the length of major axis

Mirror: Point light source / Near-Field $\rightsquigarrow$ JGE

Computation of visibility cells:



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$$x \in \mathrm{V}_i(\vec{\kappa}) \iff \frac{a_i^2 - \|y_i\|^2/4}{a_i - \langle x|y_i\rangle/2} \leqslant \frac{a_j^2 - \|y_j\|^2/4}{a_j - \langle x|y_j\rangle/2}$$

Mirror: Point light source / Near-Field $\rightsquigarrow$ JGE

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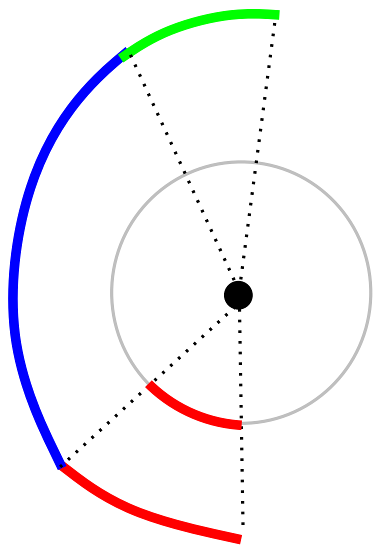

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Not linear in a_j

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Mirror: Point light source / Near-Field $\rightsquigarrow$ JGE

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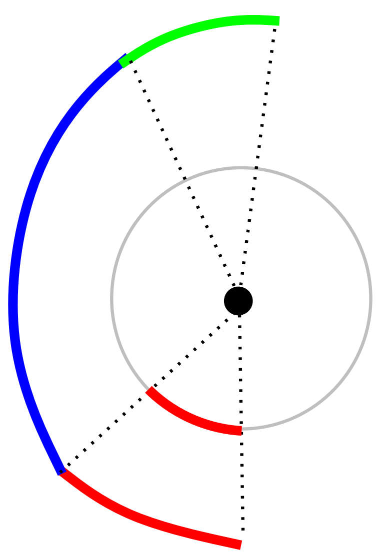

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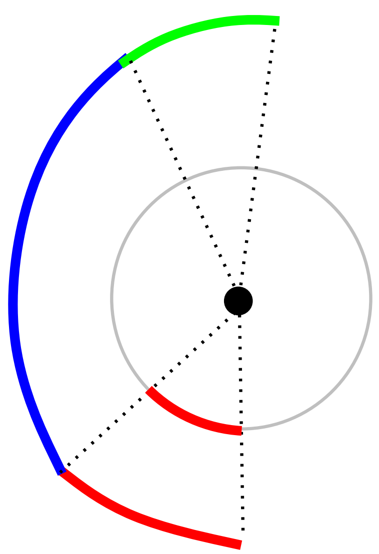
$$\iff G(x, y_i, 1/a_i) \geqslant G(x, y_j, 1/a_j)$$ with $\psi_i = 1/a_i$

Mirror: Point light source / Near-Field $\rightsquigarrow$ JGE

Computation of visibility cells:



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$$\iff G(x, y_i, 1/a_i) \geqslant G(x, y_j, 1/a_j) \quad \text{with } \psi_i = 1/a_i$$

$\rightsquigarrow$ Generated Jacobian Equation [Trudinger, 14]

Problem (FF): Find $\psi_1, \ldots, \psi_N$ such that

$$\forall i \in \{1, \cdots N\} \qquad \mu(Lag_i(\vec{\psi})) = \nu_i.$$

where $\mathrm{Lag}_i(\psi) = \{x \in \mathbb{S}^2_0,\ G(x, y_i, \psi_i) \geqslant G(x, y_j, \psi_j) \quad \forall j\}.$

Mirror: Point light source / Near-Field $\rightsquigarrow$ JGE

$\rightsquigarrow$ We have to solve the Generated Jacobian equation

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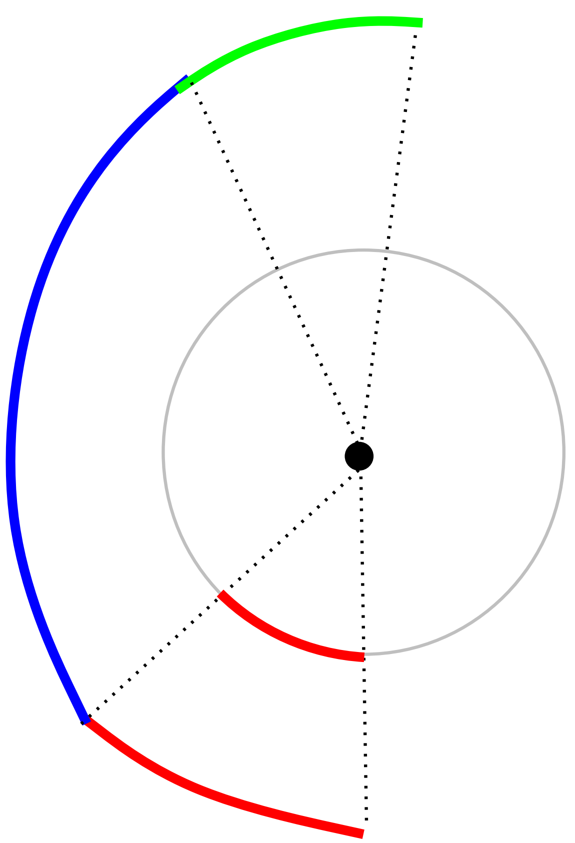
⇝ The mirror is parametrized by

$$\begin{array}{ccc} \mathbb{S}^{d-1} & \to & \mathbb{R}^d \\ x & \mapsto & (\max_i G(x, y_i, \psi_i))\, x \end{array}$$



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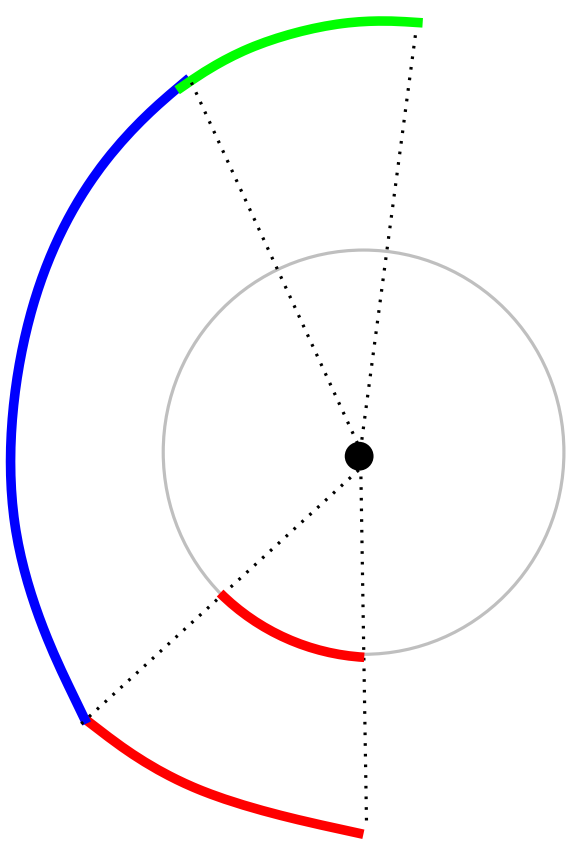

where $\psi^G(x) = \max_{y_i} G(x, y_i, \psi(y_i))$
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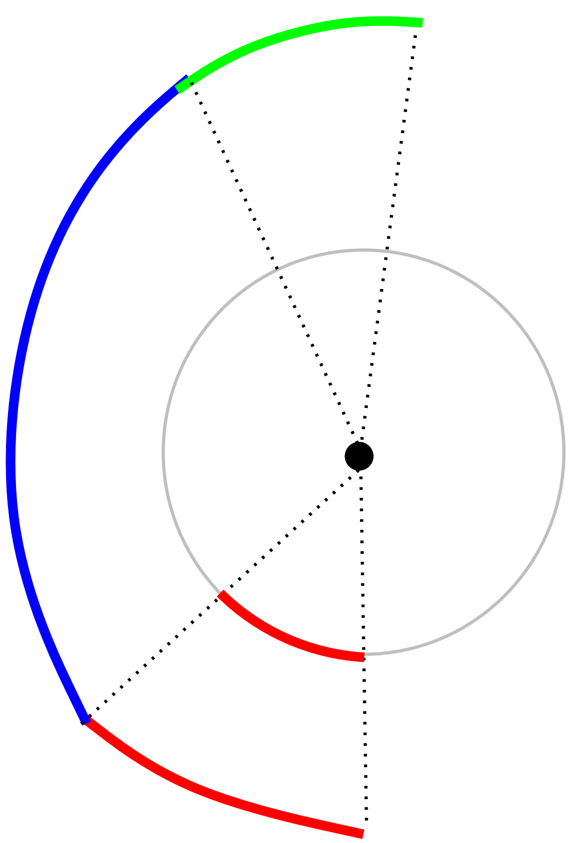
$\psi^G(x)$

where $\psi^G(x) = \max_{y_i} G(x, y_i, \psi(y_i))$
is the G-conjugate function of ψ.

ccl : $x \in \mathbb{S}^2_0 \mapsto \psi^G(x)x$ parametrizes the mirror.



Outline

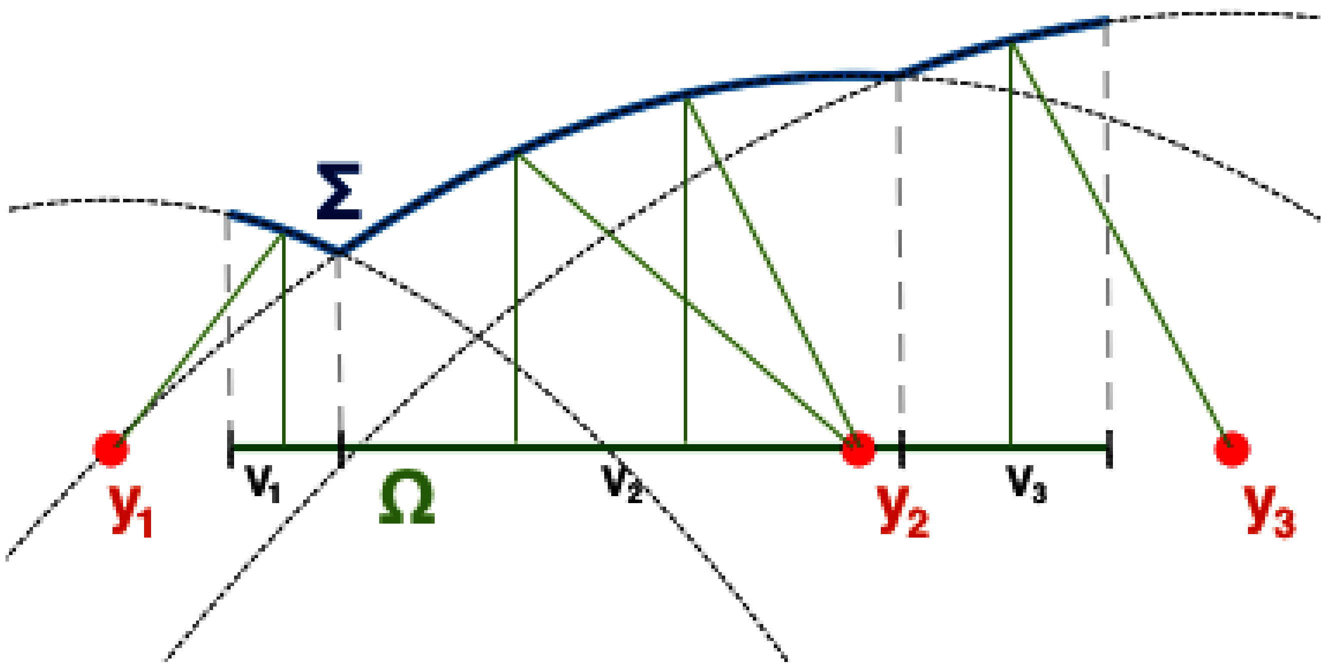
Part 1: Stability

- Optimal Transport and strong c concavity
- Stability under strong c-concavity
- Sufficient conditions for strong c-concavity
- Applications to the reflector problem

Part 2: Generated Jacobian Equation

- Case 1: Mirror for Point source light (Far Field and Near Field)
- Case 2: Mirror for Parallel source light (Near Field)
- Semi-discrete Generated Jacobian equation

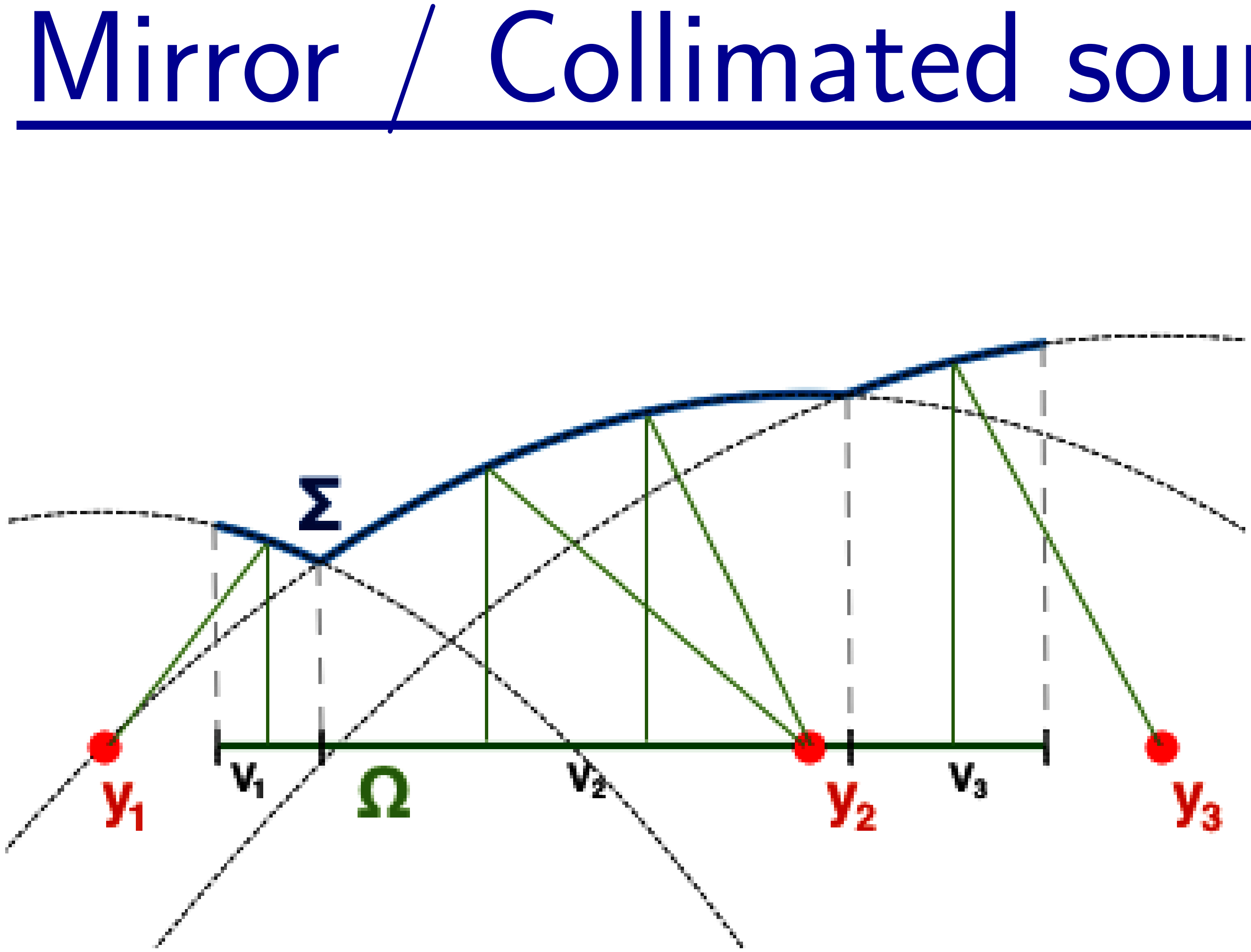
Mirror / Collimated source: Near Field



Here, Σ is a maximum of paraboloids of focus y_i and direction $(0, 0, -1)$.

$$u(x) = \max_{1 \leqslant i \leqslant N} \frac{1}{2\psi_i} - \frac{\psi_i}{2} ||x - y_i||^2$$

Mirror / Collimated source: Near Field



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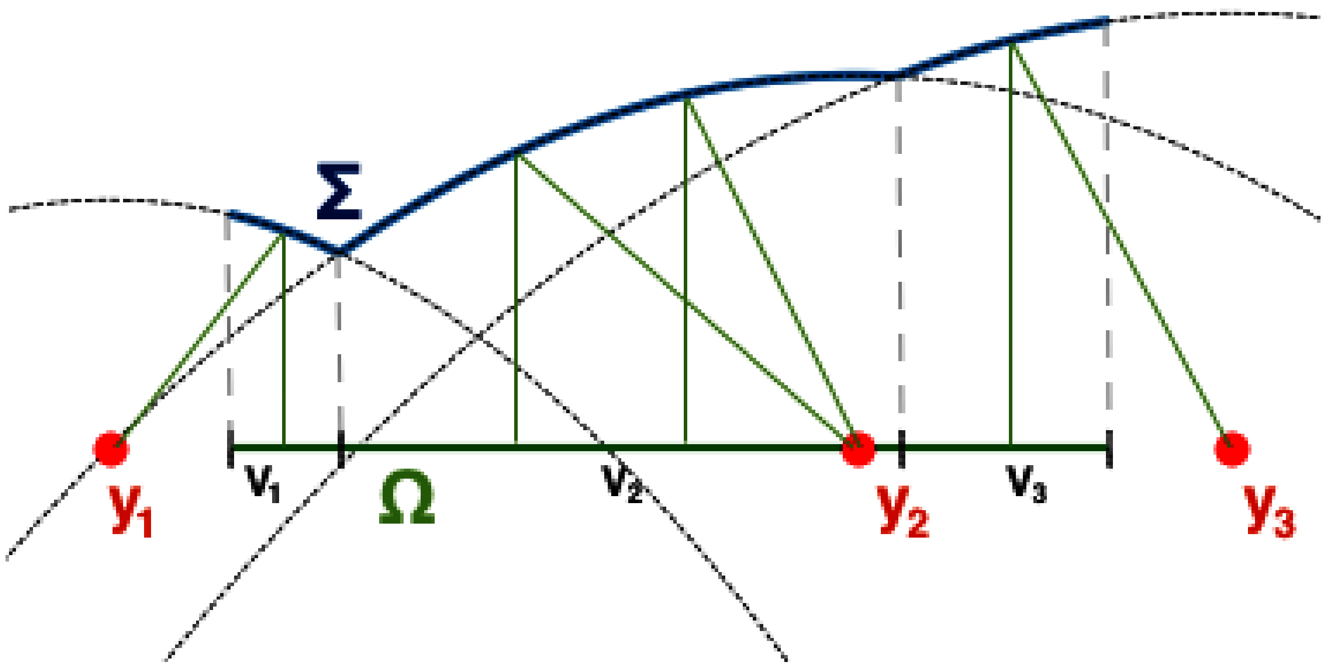

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$G(x, y_j, \psi_j)$

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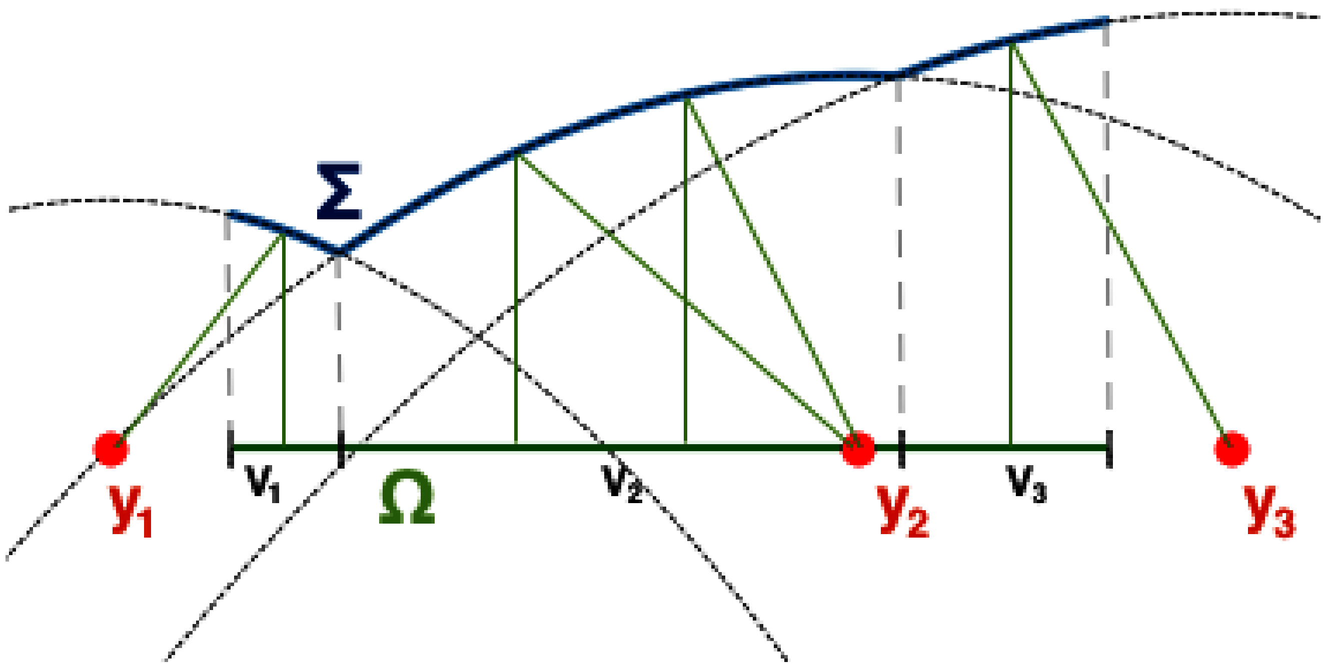

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$\rightsquigarrow$ Generated Jacobian Equation in $\mathbb{R}^2$

Problem (FF): Find $\psi_1, \ldots, \psi_N$ such that

$$\forall i \in \{1, \cdots N\} \qquad \mu(Lag_i(\vec{\psi})) = \nu_i.$$

where $\text{Lag}_i(\psi) = \{x \in \mathbb{S}^2_0,\ G(x, y_i, v_i) \geqslant G(x, y_j, v_j) \quad \forall j\}$.

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Generated Jacobian Equation

Definition: $G : \Omega \times Y \times \mathbb{R} \to \mathbb{R}$ is called a generating function if it satisfies :

(Reg) : $\forall \alpha \in \mathbb{R}, \sup_{(x,y,v)\in\Omega\times Y\times]-\infty,\alpha]} |\nabla_x G(x,y,v)| < +\infty$

(Mono) : $\forall (x,y,v) \in \Omega \times Y \times \mathbb{R} : \partial_v G(x,y,v) < 0$

(Twist) : $(y,v) \mapsto (G(x,y,v), \nabla_x G(x,y,v))$ is injective for any $x \in X$

(UC) $\forall y \in Y, \lim_{v\to-\infty} \inf_{x\in\Omega} G(x,y,v) = +\infty$

Definition: The Laguerre cells are :

$$\mathrm{Lag}_i(\psi) = \{x \in \Omega,\ G(x,y_i,v_i) \geqslant G(x,y_j,v_j) \quad \forall j\}.$$

The **Generated Jacobian equation** consists in finding $\psi \in \mathbb{R}^N$ such that

$$H(\psi) = \nu$$

where the function H is given by $H(\psi) = (\mu(Lag_i(\psi)))_{1\leqslant i\leqslant N}$.

Exemple:

Far field parallel reflector : $G(x,y,v) = \langle x, p\rangle - v$

Near field parallel reflector $G(x,y,v) = \frac{1}{2v} - \frac{v}{2}||x-y||^2$

Differential of H

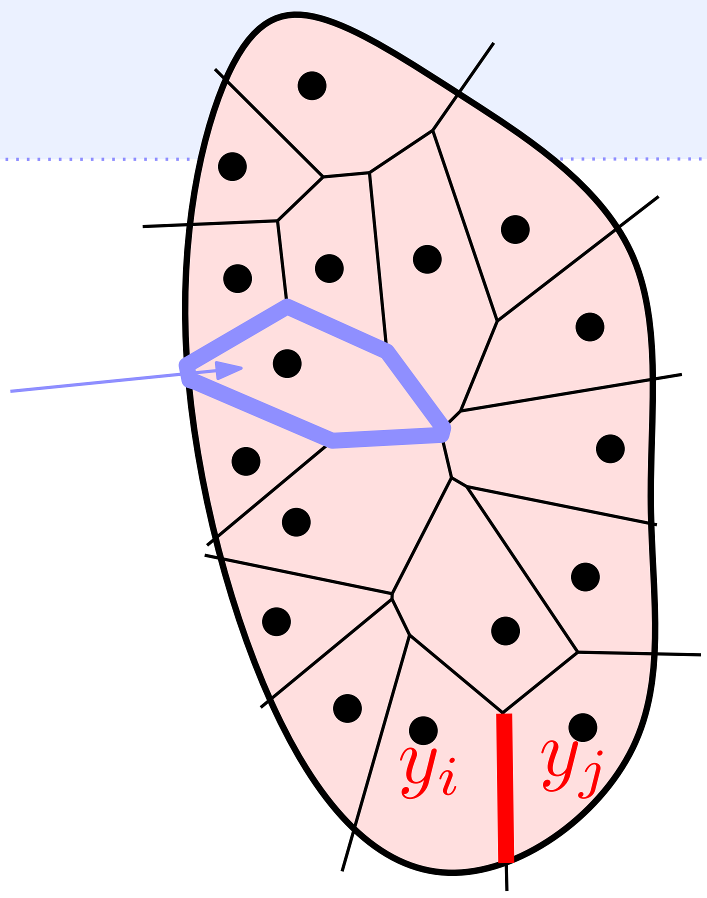
Recall: $H(\psi) = (\mu(Lag_i(\psi)))_{1 \leqslant i \leqslant N}$.

Proposition: Under an hypothesis of genericity of Y , H is of class $\mathcal{C}^1$

$$\frac{\partial H_j}{\partial \psi_i}(\psi) = \int_{Lag_{ij}(\psi)} \rho(x) \frac{|\partial_v G(x, y_i, \psi_i)|}{||\nabla_x G_j(x,\psi) - \nabla_x G_i(x,\psi)||} d\mathcal{H}^{d-1}(x) \geqslant 0 \text{ for } i \neq j$$

$$\frac{\partial H_i}{\partial \psi_i}(\psi) = - \sum_{j \neq i} \frac{\partial H_j}{\partial \psi_i}(\psi)$$

with $G_i(x,\psi) = G(x, y_i, \psi_i)$.



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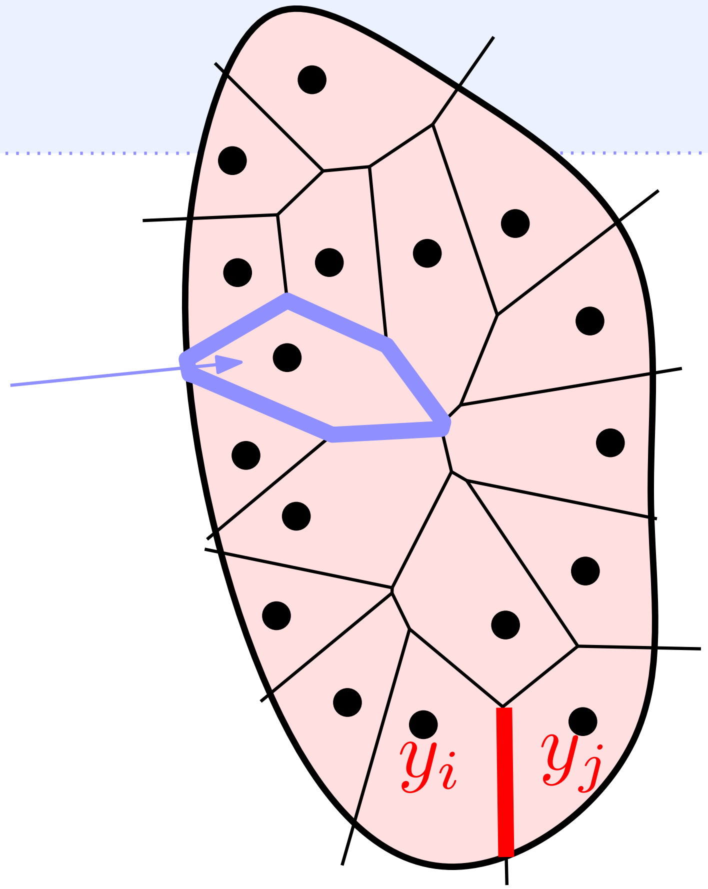


- ⇝ DH is not symmetric
- ⇝ $(1, \cdots, 1)$ is not in the Kernel of DH

Properties of DH

$$\mathcal{S}^+ = \{\psi \in \mathbb{R}^N | \forall i, H_i(\psi) > 0\}$$

Proposition:

- $DH(\psi)$ the differential of H is of rank $N-1$ on $\mathcal{S}^+$.
- The image of DH is $im(DH(\psi)) = 1^\perp$ where $1 = (1, \cdots, 1) \in \mathbb{R}^N$.
- $ker(DH(\psi)) = Span(w)$ with $w_i > 0$

Proposition: (Unique descent direction)
Let $\psi \in \mathcal{S}^+$, then the system:

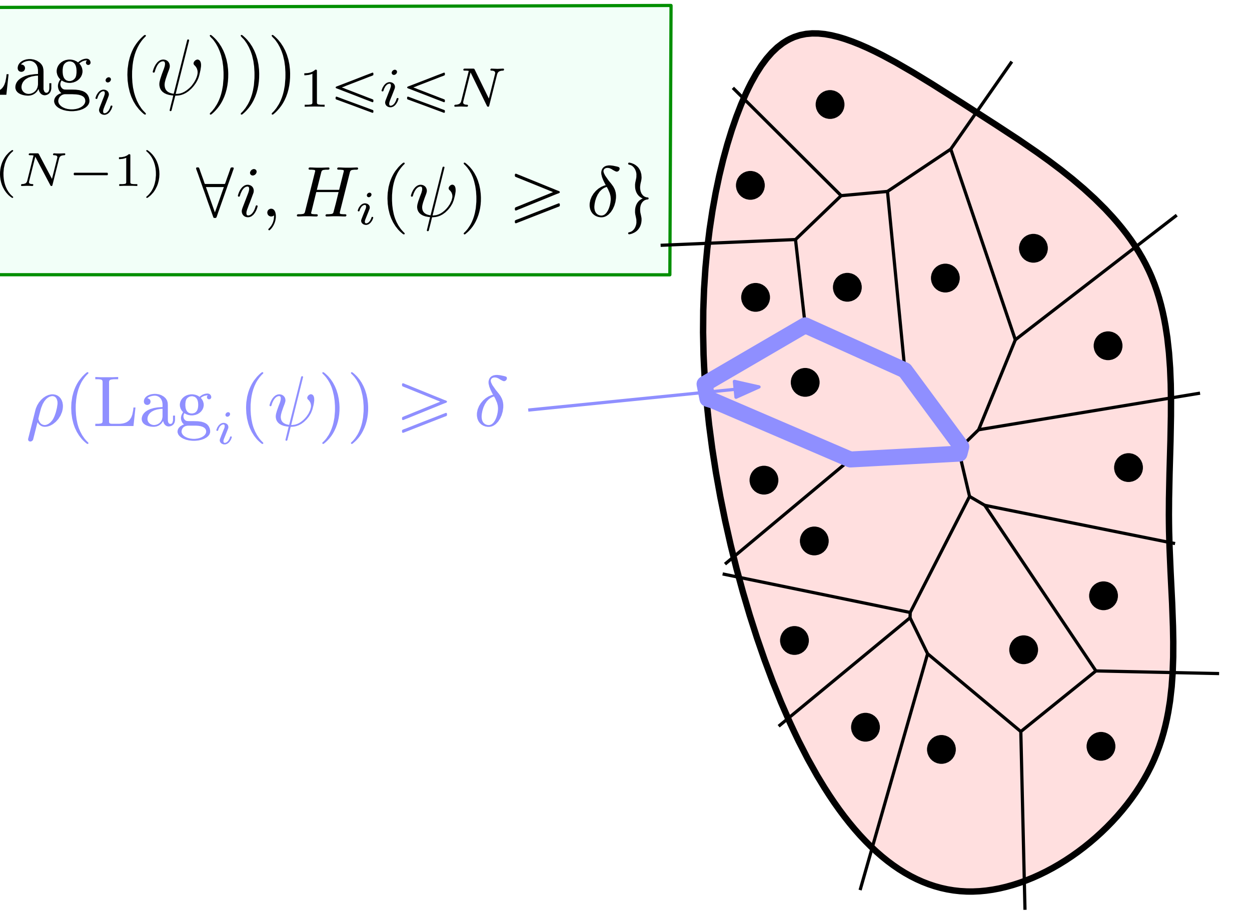
$$\begin{cases} DH(\psi)d = H(\psi) - \nu \\ d_1 = 0 \end{cases}$$

has a unique solution.

Damped Newton Algorithm

Equation $H(\psi) = \nu$ where $H(\psi) = (\rho(\mathrm{Lag}_i(\psi)))_{1 \leqslant i \leqslant N}$

Admissible domain: $\mathcal{S}^\delta = \{\psi \in \{\alpha\} \times [\beta, \gamma]^{(N-1)} \; \forall i, H_i(\psi) \geqslant \delta\}$



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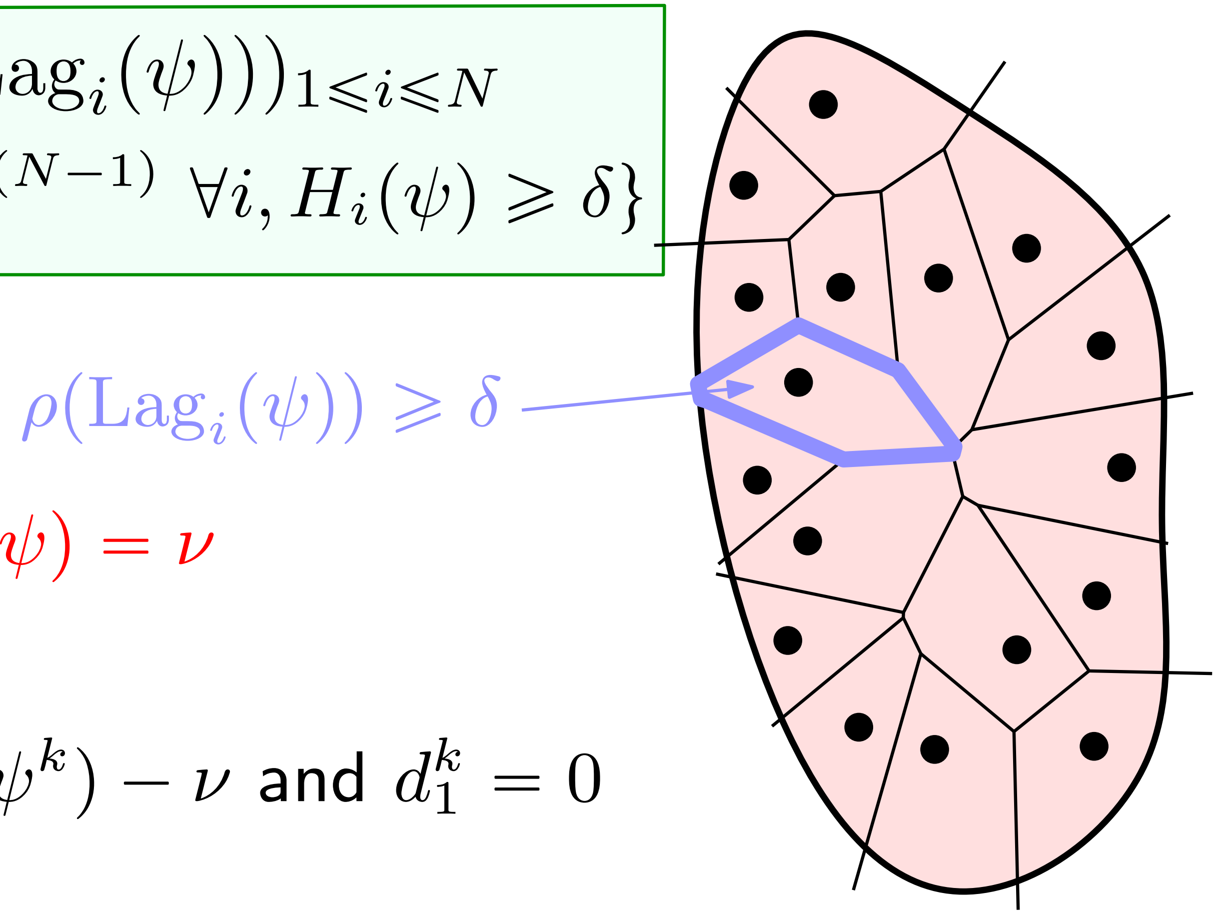

Damped Newton algorithm: for solving $H(\psi) = \nu$

Input: $\psi^0 \in \mathcal{S}^\delta$ and precision ε

Loop: $\rightarrow$ Calculate d^k s.t. $\mathrm{D}H(\psi^k)d^k = H(\psi^k) - \nu$ and $d_1^k = 0$

$\rightarrow$ Define $\psi^{k,\tau} = \psi^k - \tau d^k$

$\rightarrow$ $\tau^k = \max\{\tau \in 2^{-\mathbb{N}} \mid \psi^{k\tau} \in \mathcal{S}^\delta$ and $\|H(\psi^{k\tau}) - \nu\| \leqslant (1 - \frac{\tau}{2})\|H(\psi^k) - \nu\|\}$

$\rightarrow$ $\psi_{k+1} := \psi_k^{\tau_k}$

Damped Newton Algorithm

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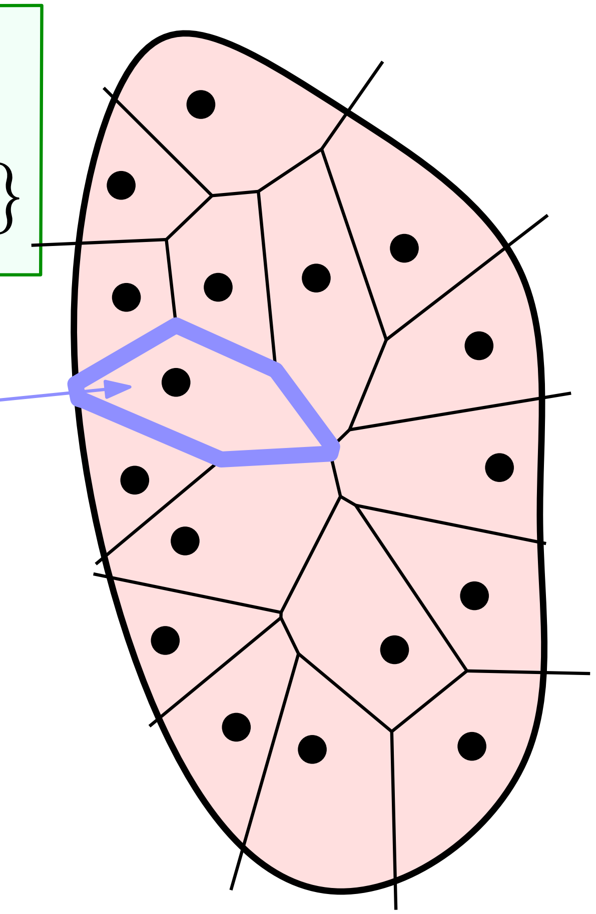

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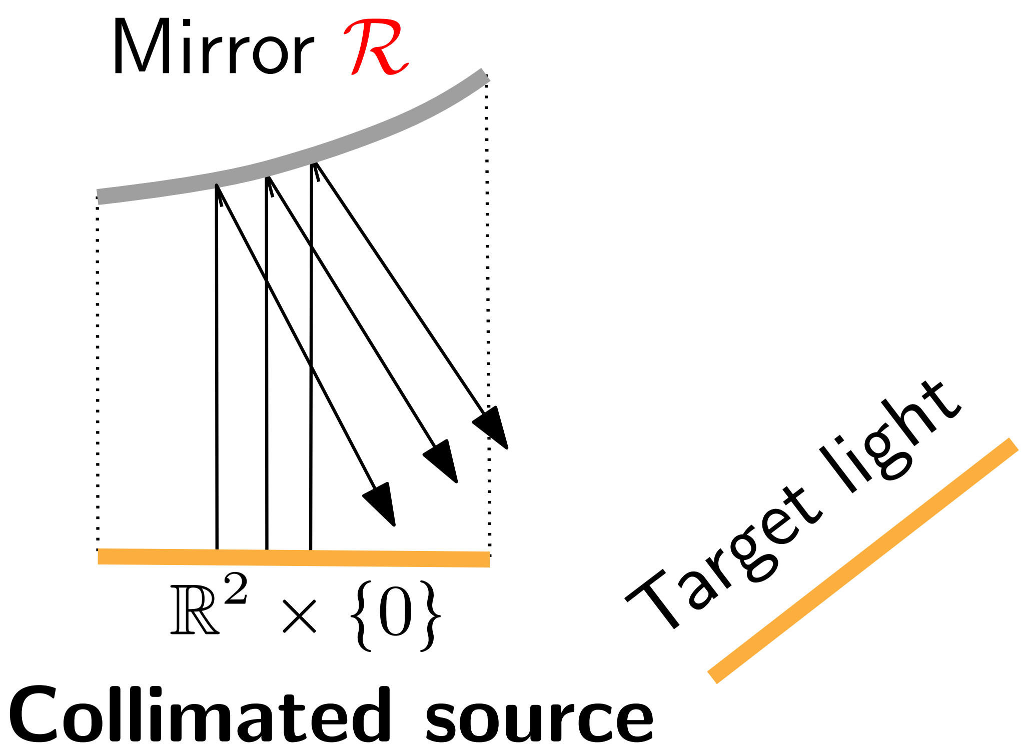
Theorem(Gallouet, Mérigot, T., CVPDE 2022): Let $X \subset \Omega$ be a compact set, $\rho \in C^0(X)$, $\{\rho > 0\} \cap \mathrm{int}(X)$ is path-connected, and Y satisfies generic assumptions w.r.t. ∂X, $\delta \leqslant \min_{1\leqslant i\leqslant N} \nu_i/2$. Then there is linear convergence:

$$\|H(\psi^{k+1}) - \nu\| \leqslant \left(1 - \frac{\tau^*}{2}\right)\|H(\psi^k) - \nu\|$$

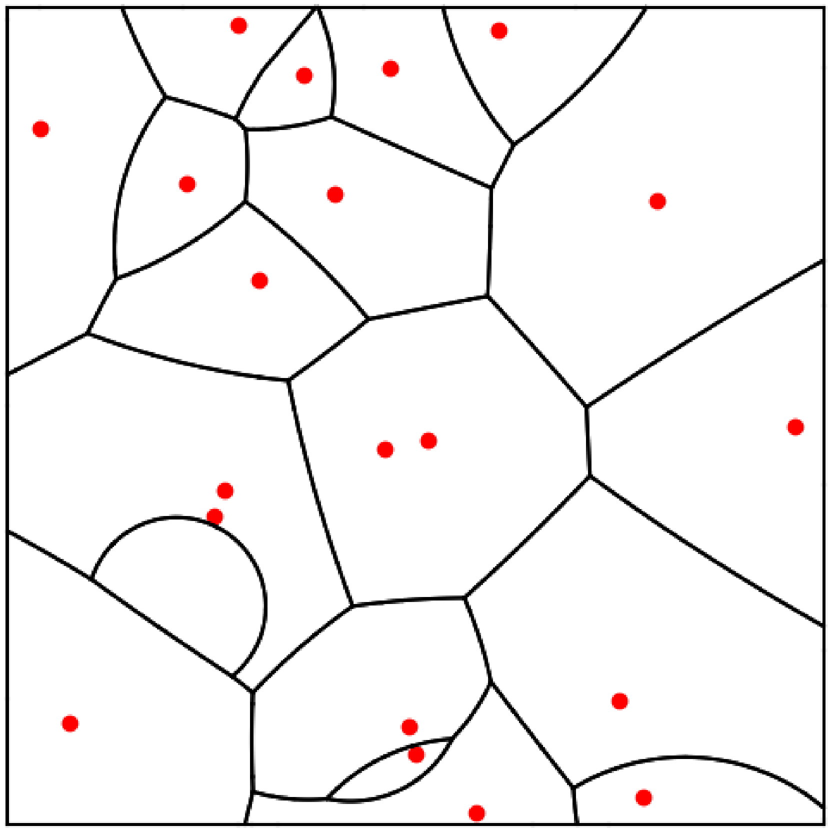
Mirror for parallel source light: algorithm



targeted image $N = 400 \times 480$

Mirror for parallel source light: algorithm

- **Computation of Laguerre cells:**



Laguerre cells

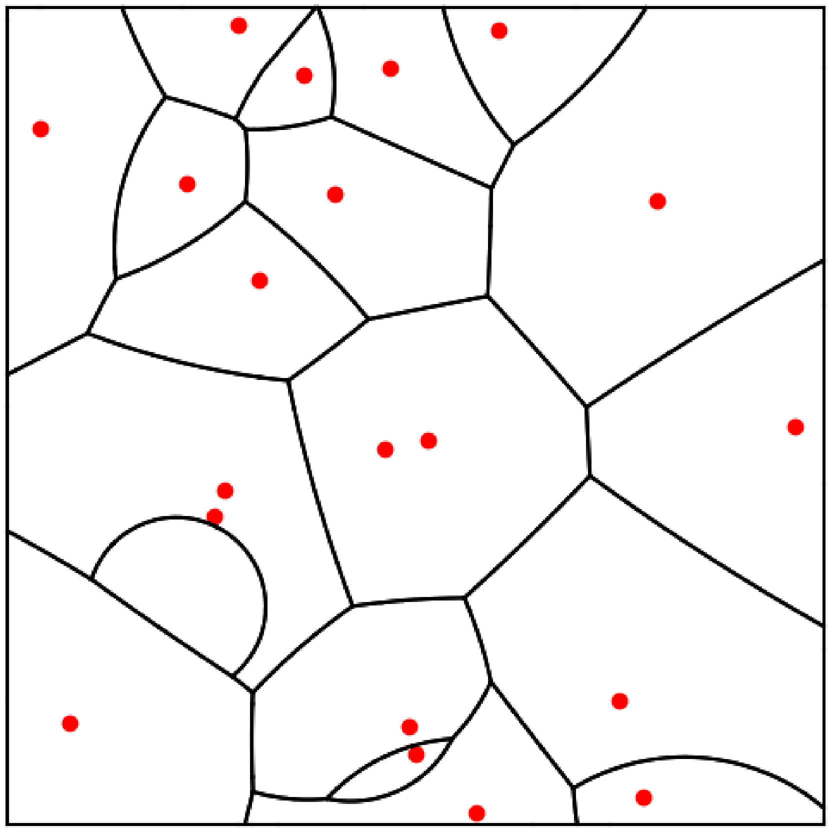
Mirror for parallel source light: algorithm

- **Computation of Laguerre cells:**

⇝ Mobius diagrams:

Definition: Given $P = \{p_i\} \subseteq \mathbb{R}^d$ and $(\omega_i) \in \mathbb{R}^N$

$$\mathrm{Mob}(p_i) := \{x \in \mathbb{R}^d; i = \arg\min_j \lambda_i \|x - p_j\|^2 + \omega_j\}$$



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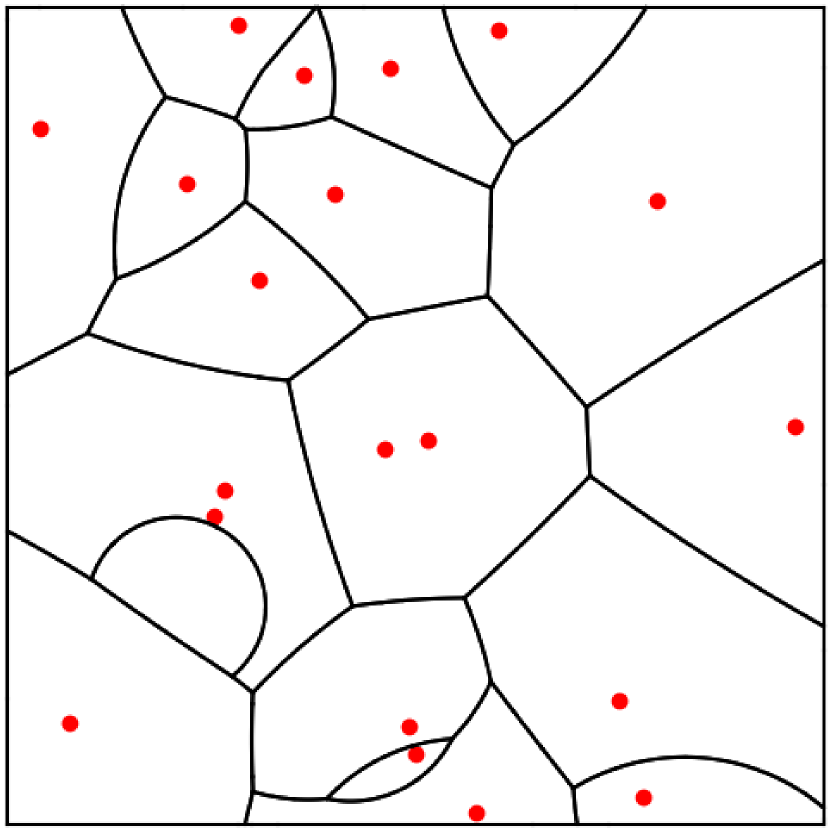


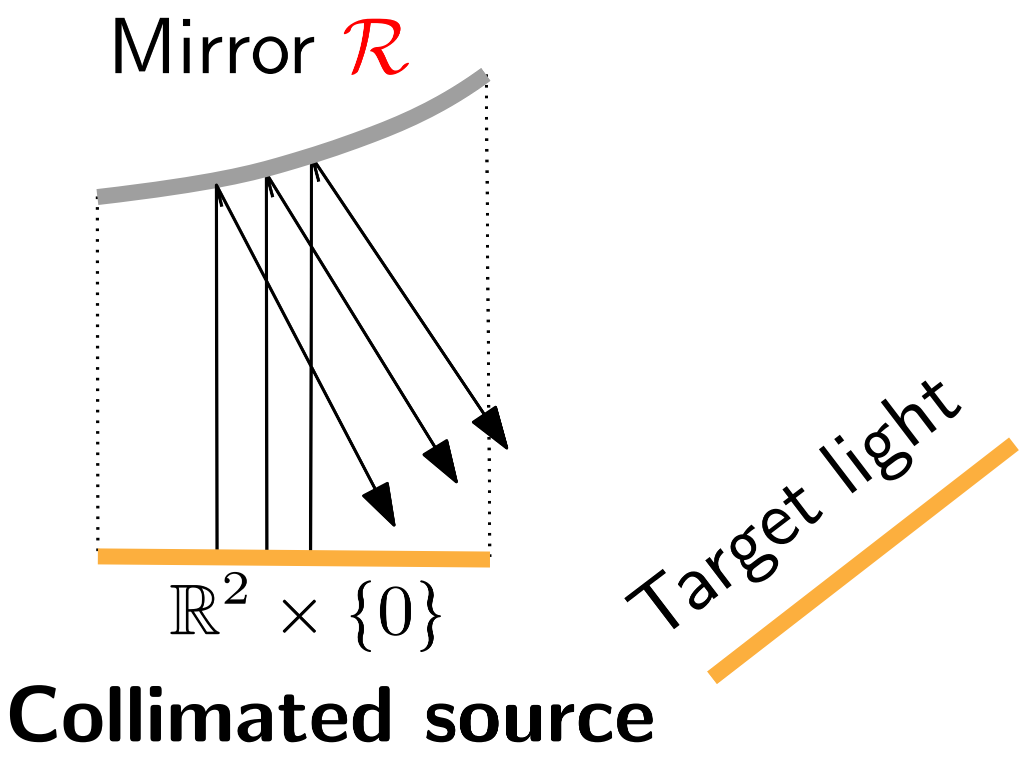
Laguerre cells

Lemma: (Boissonnat, Wormser, Yvinec, 07)

$$\mathrm{Lag}_i(\psi) = \mathrm{Mob}(p_i) = \mathrm{Proj}_{z=0}(Pow_i \cap \mathcal{P})$$

where $\mathrm{Proj}_{z=0}$ is the orthogonal projection.
and $\mathcal{P}$ is the paraboloid $z = x^2 + y^2$

Mirror for parallel source light: algorithm



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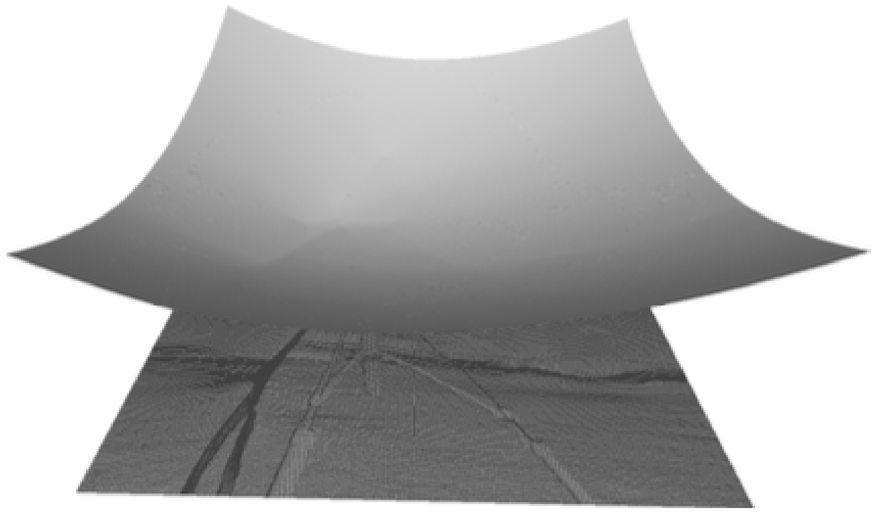
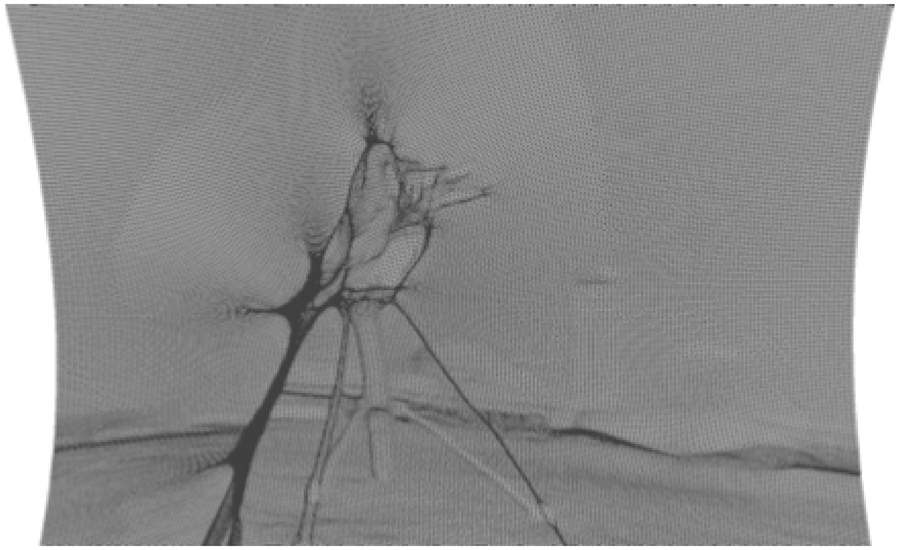
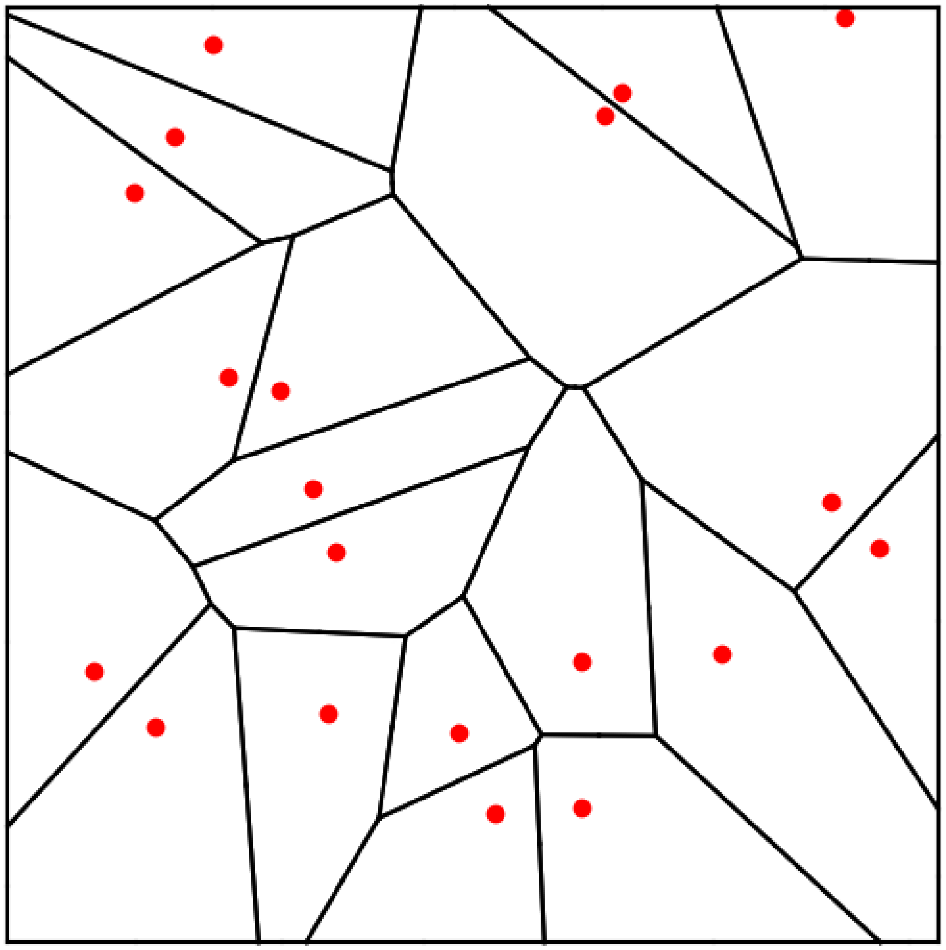
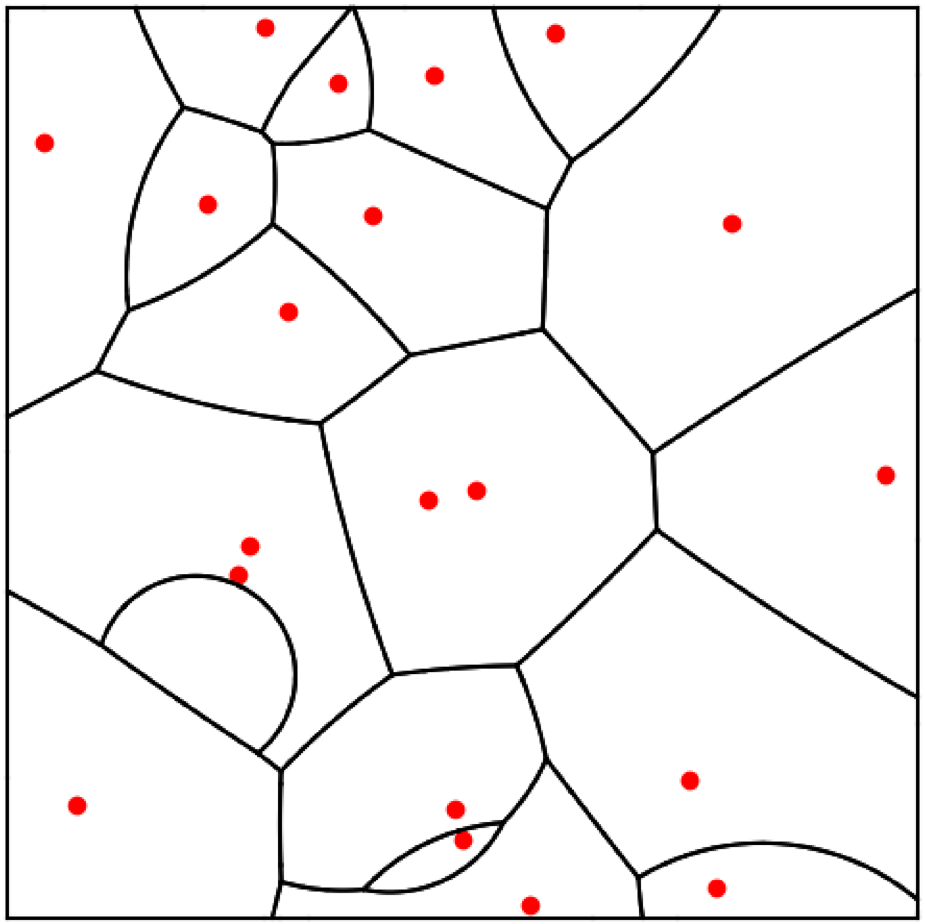




Comparison Far Field / Near Field



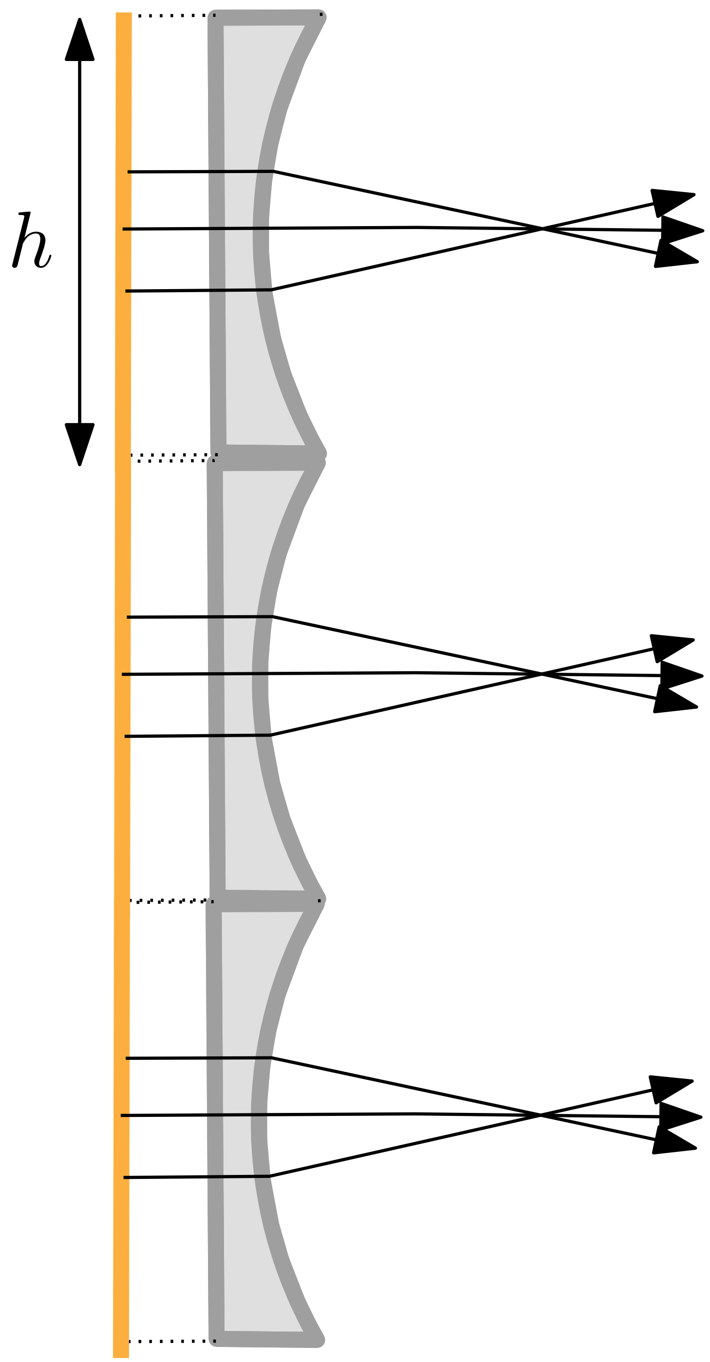
Visibility cells in Far Field



Visibility cells in Near Field

Problem with far-field assumption

Putting three copies of the same lens shifted by h...



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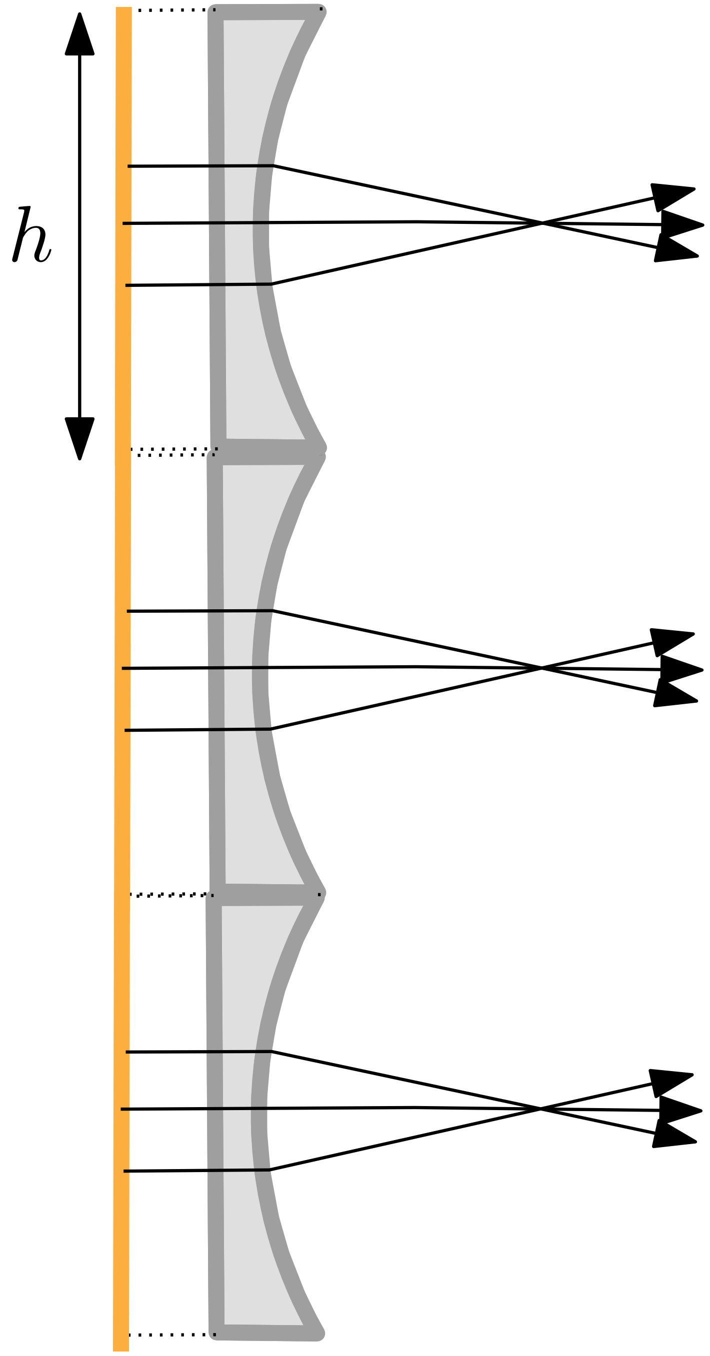

... produces a superposition of images shifted by h.

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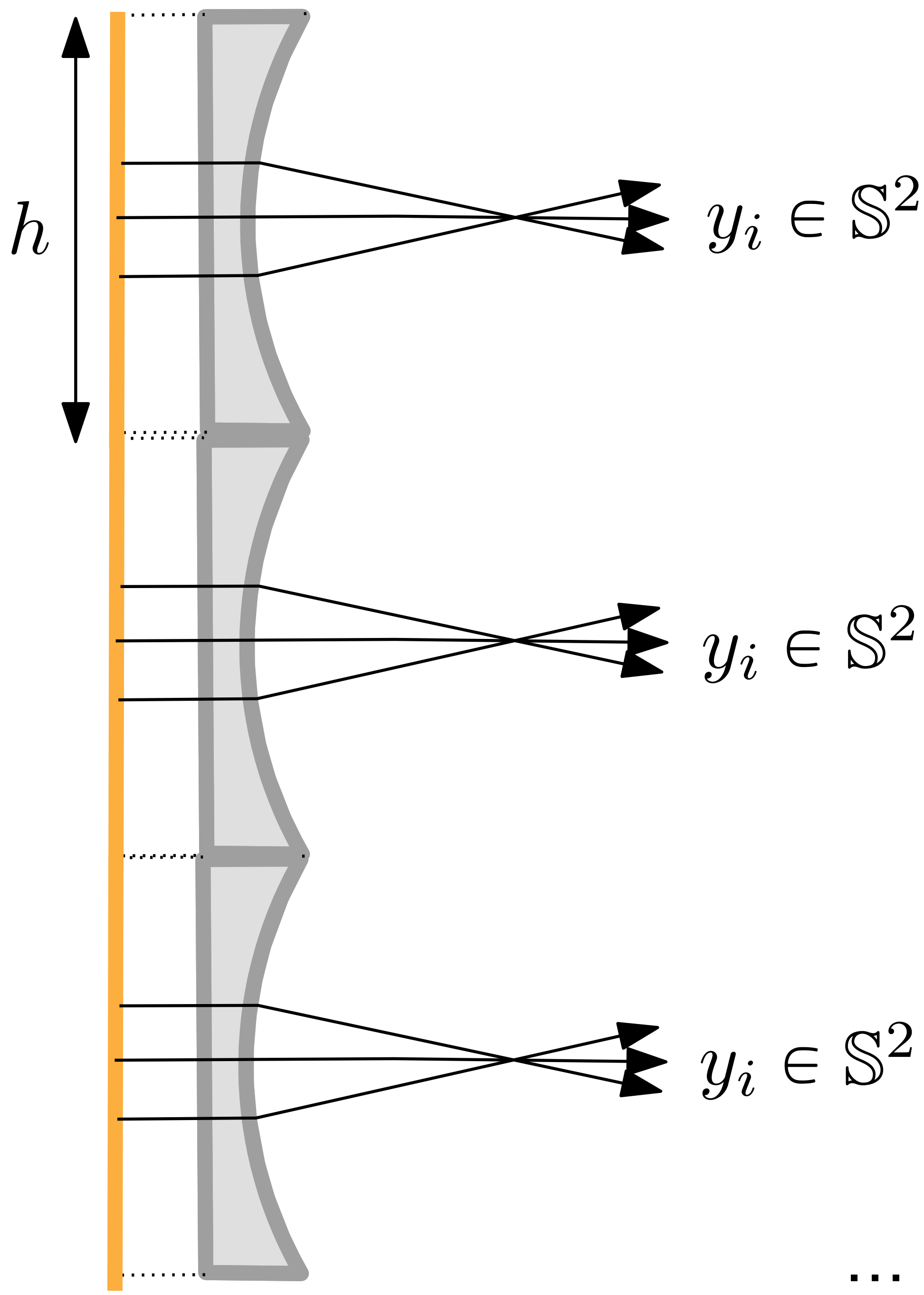

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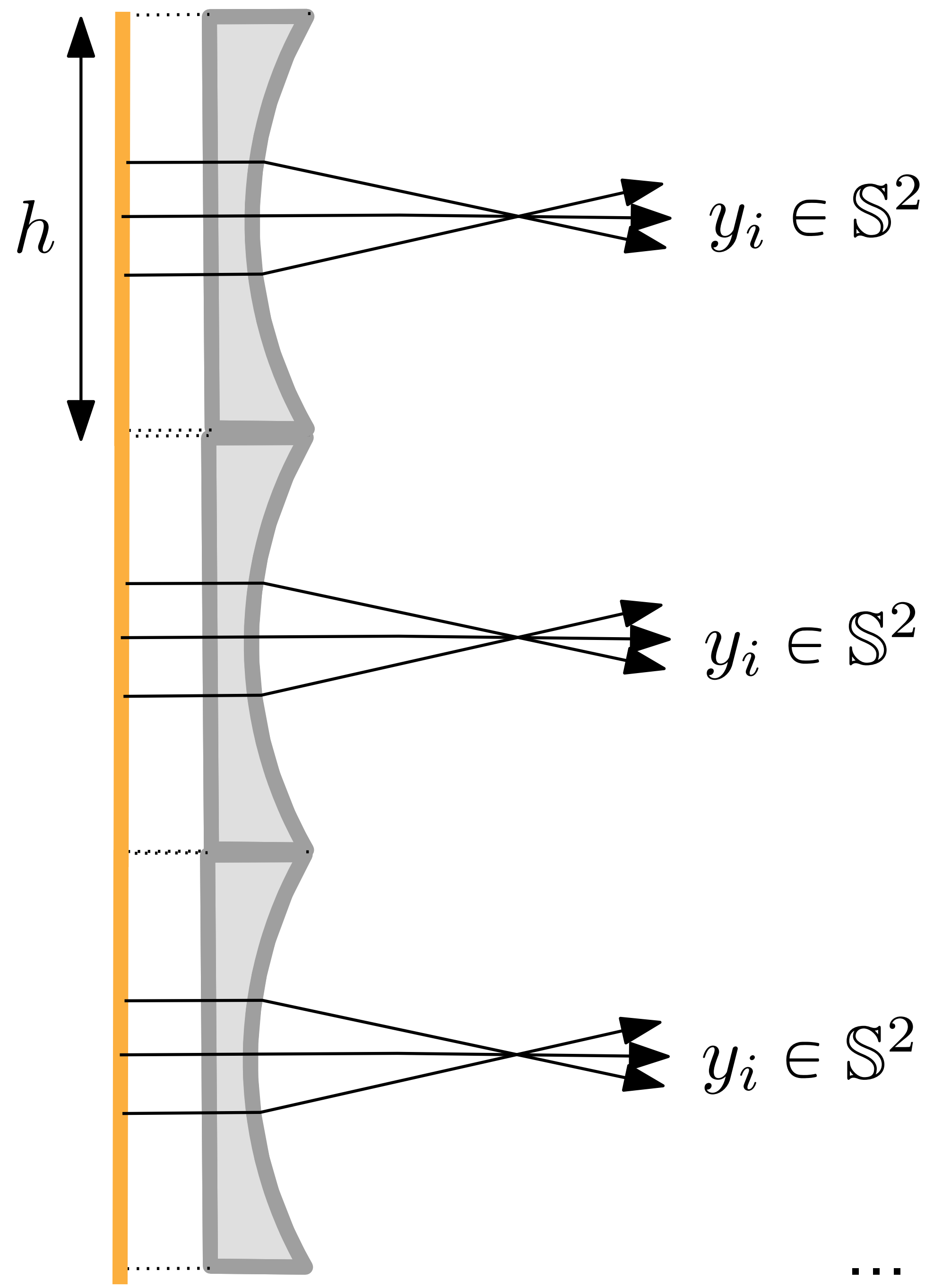

... produces a superposition of images shifted by h.

One wants to produce images at finite distance $\longrightarrow$ near-field problem.

Iterated FF problem

NF pb: Build a component $\mathcal{R}$ sending light towards $z_1, \ldots, z_N \in \{D\} \times \mathbb{R}^2$ (instead of $y_1, \ldots, y_N \in \mathbb{S}^2$))

Iterated FF problem

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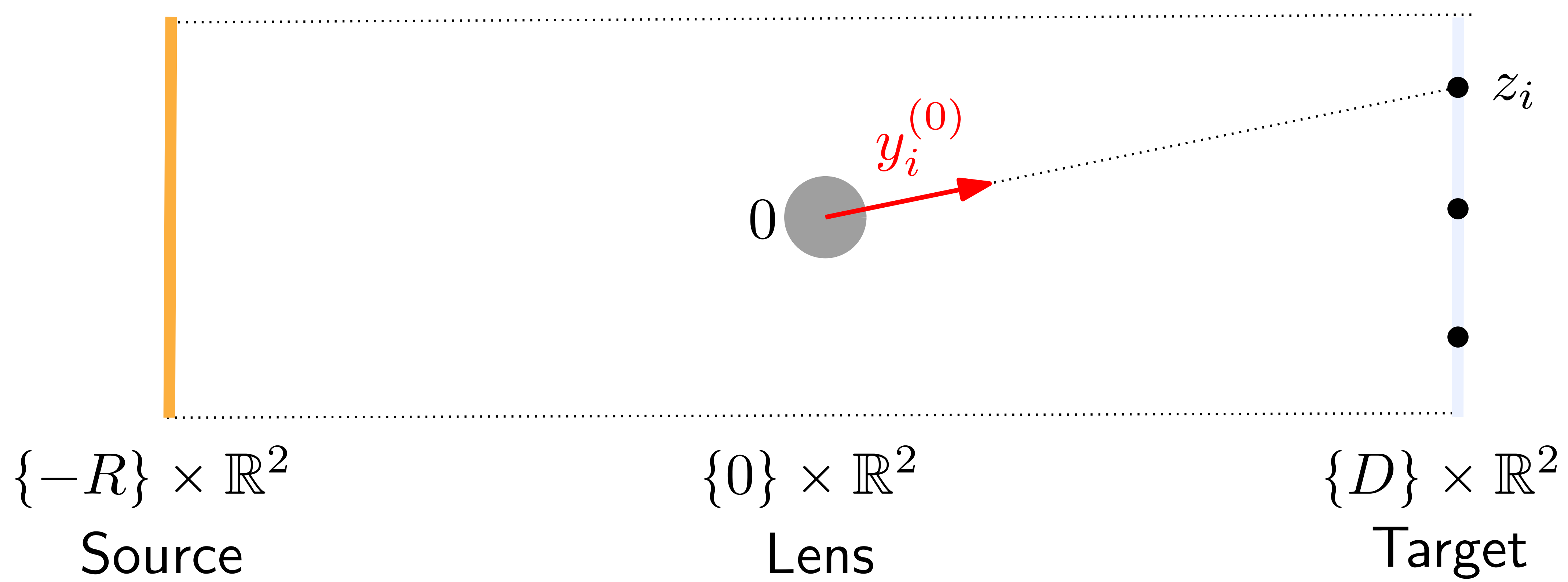
We approximate solutions to the NF problem using a sequence of FF pb.

Step 0: Solve far-field problem with target $y_i^{(0)} = z_i/\|z_i\|$



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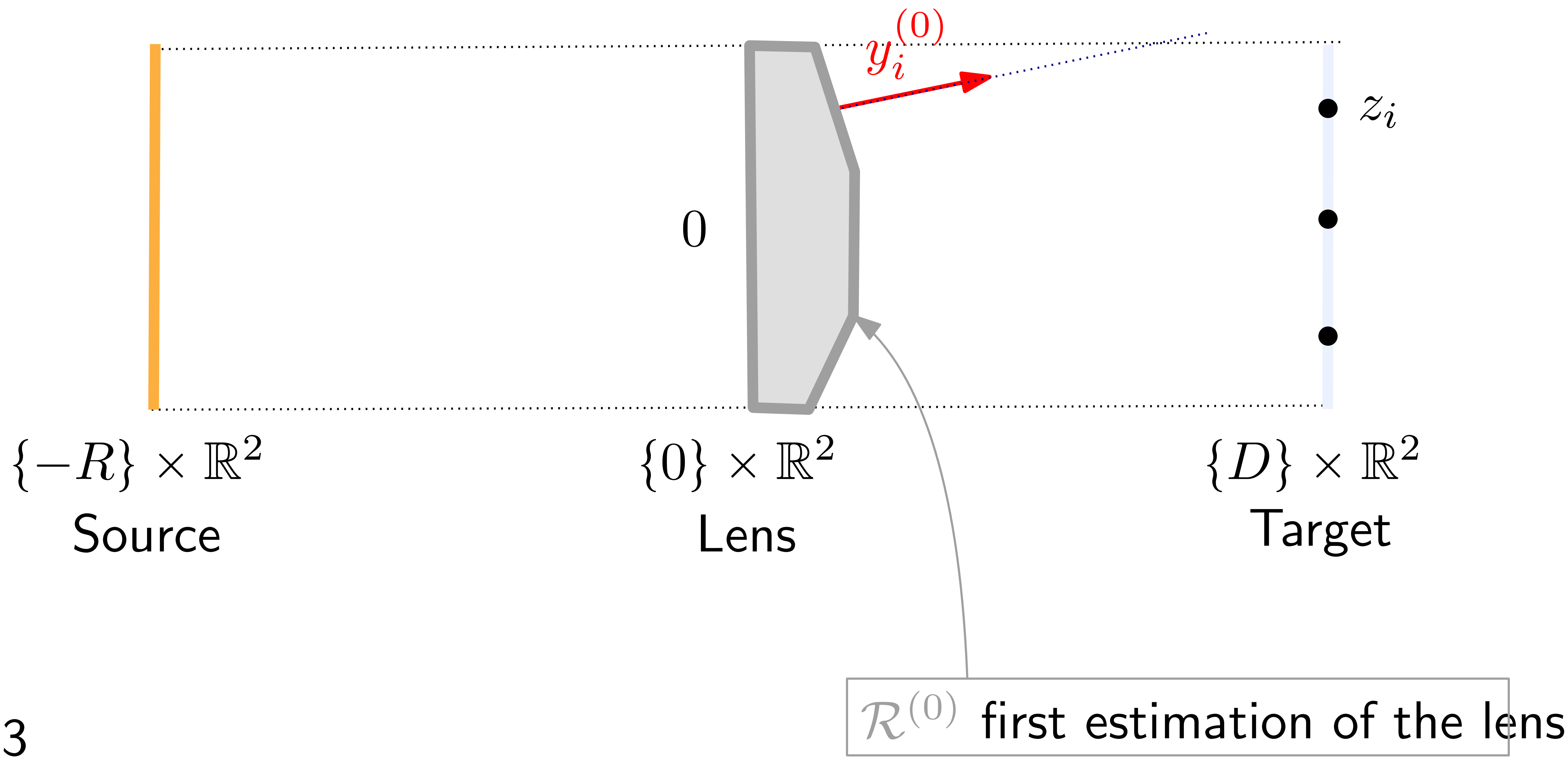

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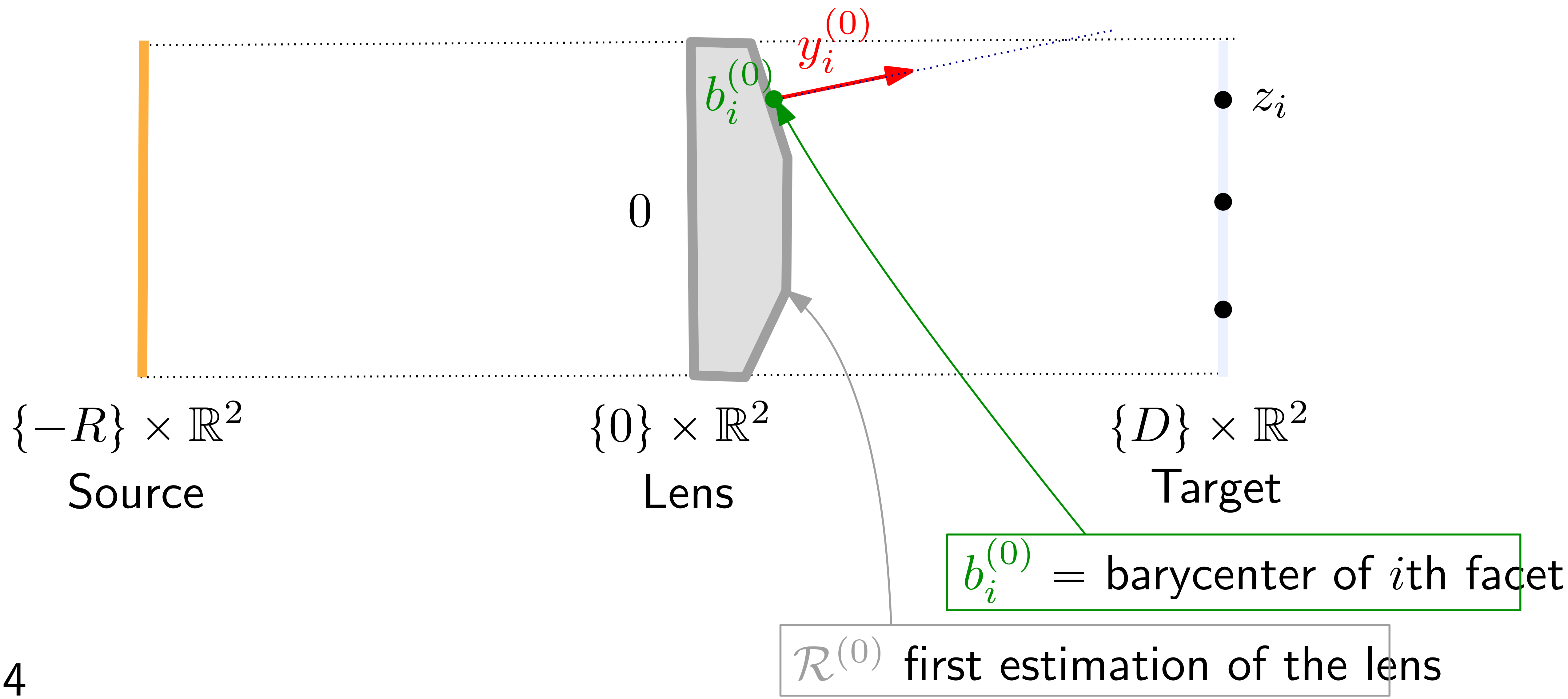

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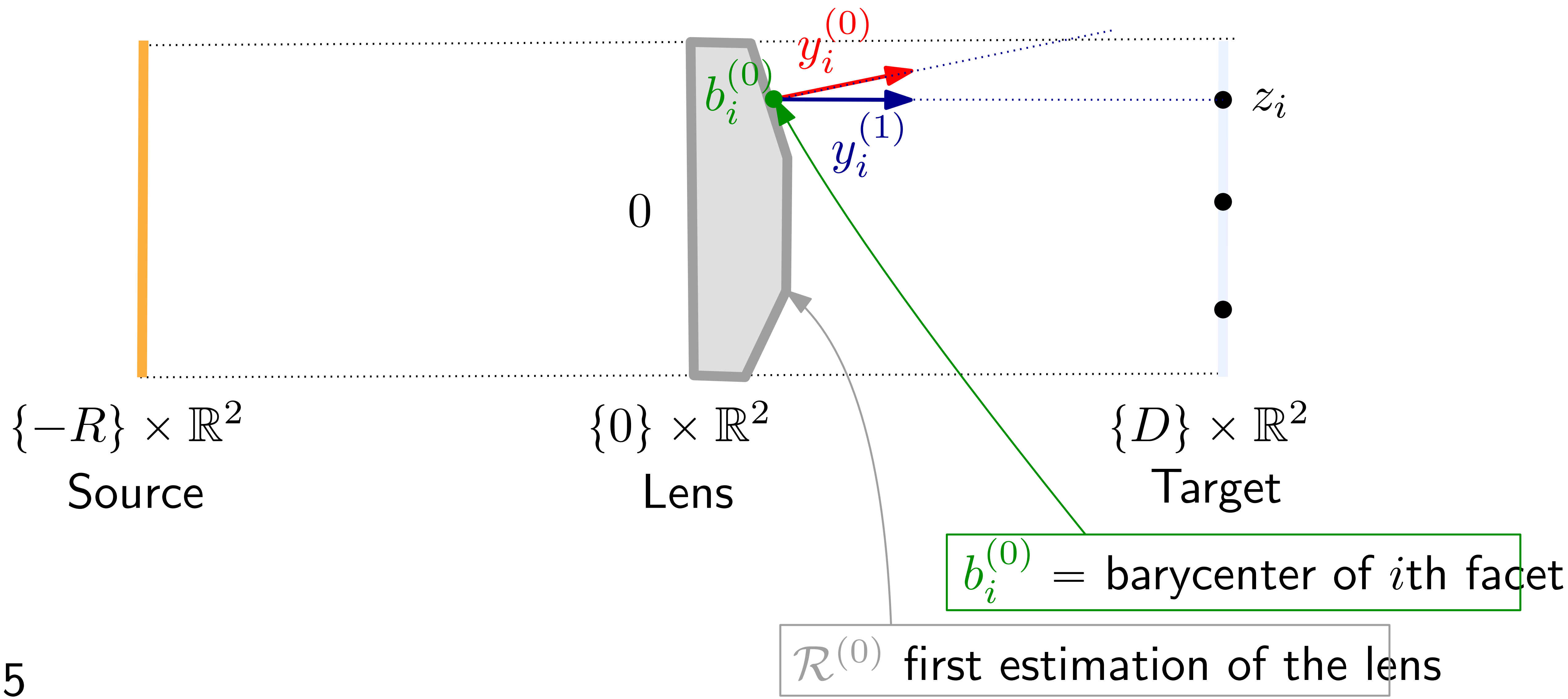

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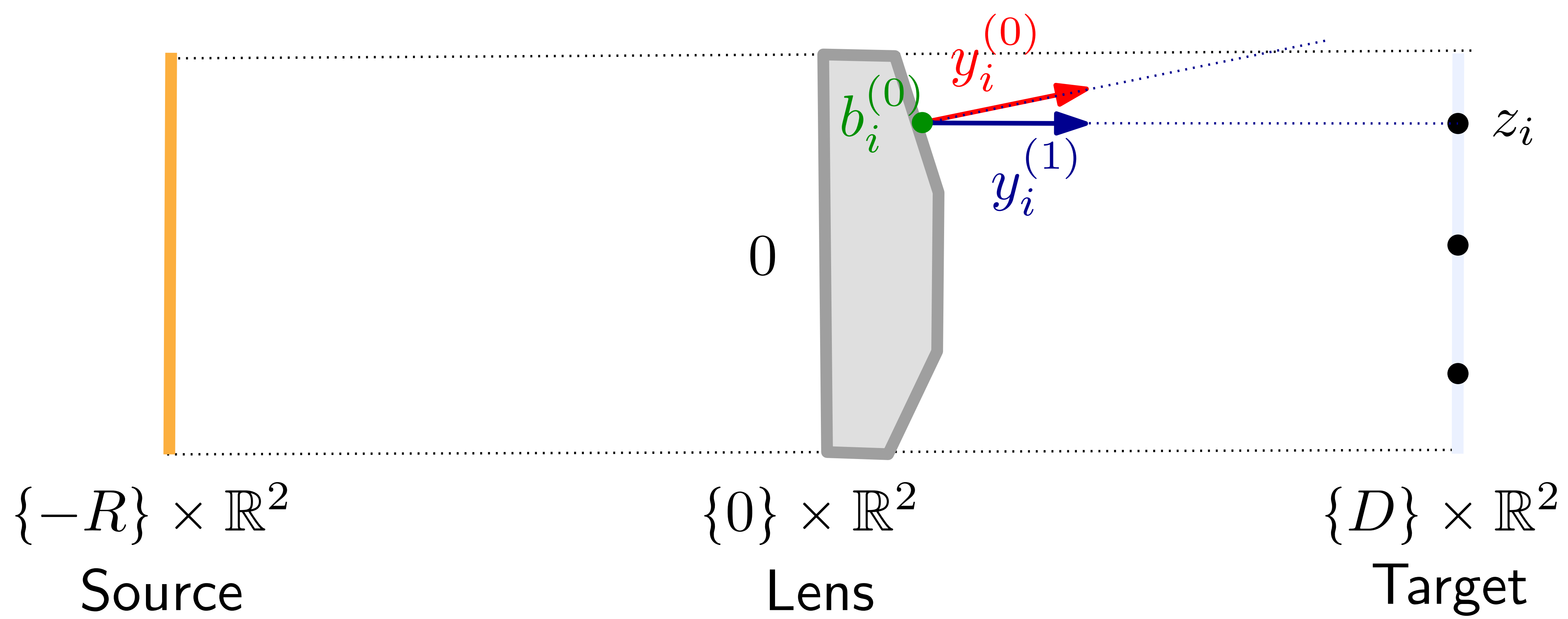


Step k+1: Solve far-field problem with target $y_i^{(k+1)} = (z_i - b_i^{(k)}) / \|z_i - b_i^{(k)}\|$,

Efficient heuristic to solve NF problem using a FF solver...

Convergence of the algorithm

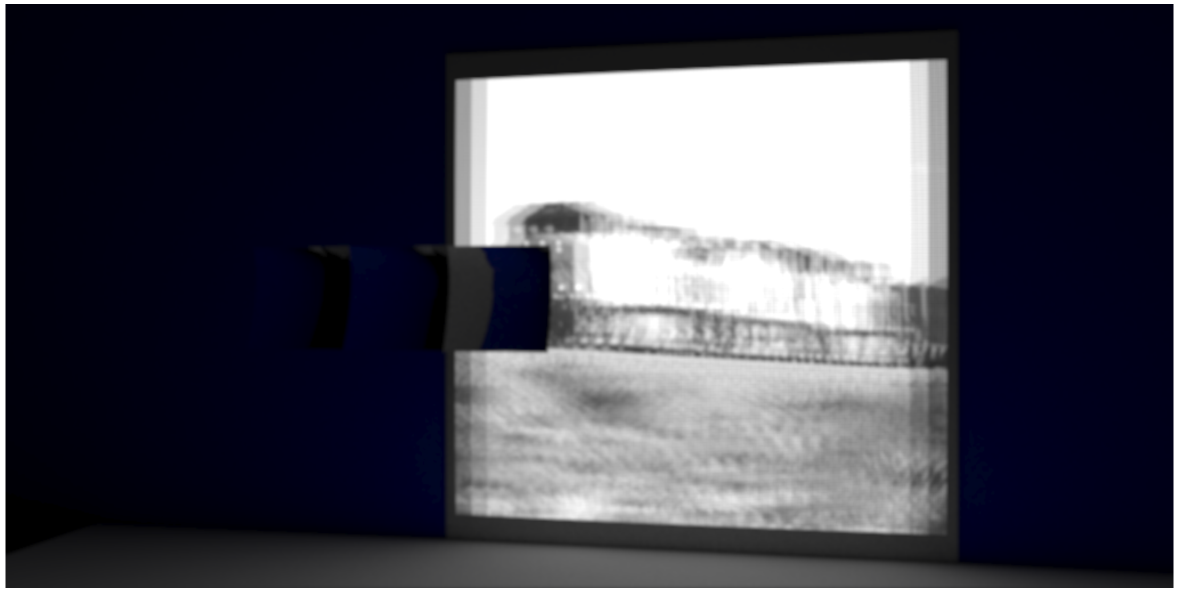
Target

1st iteration

2nd iteration

5th iteration

Pillows



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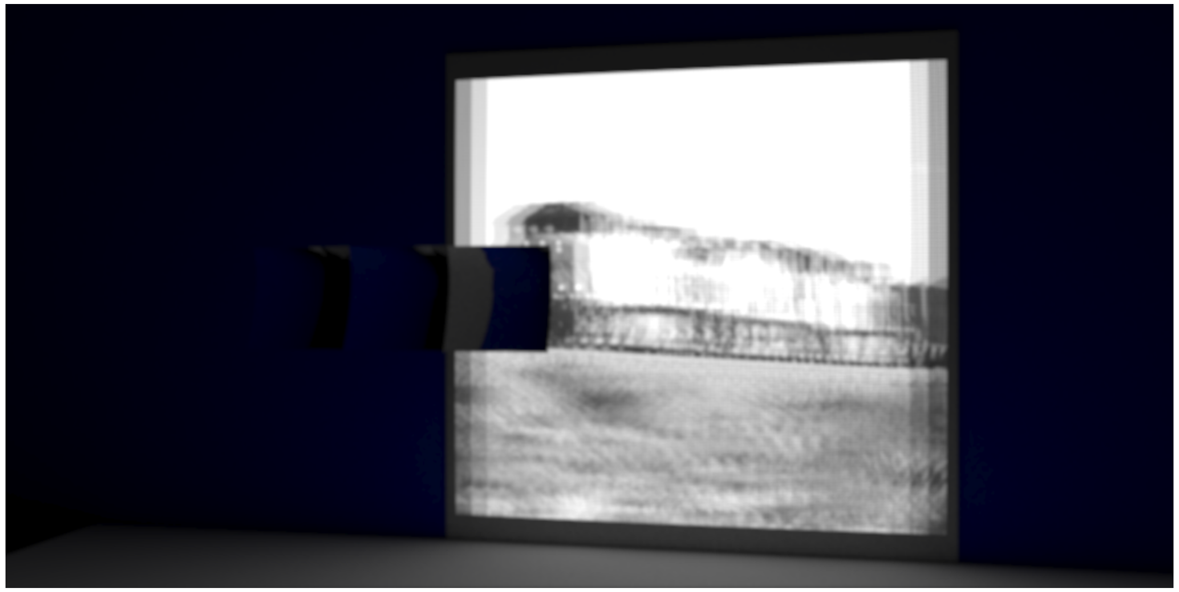

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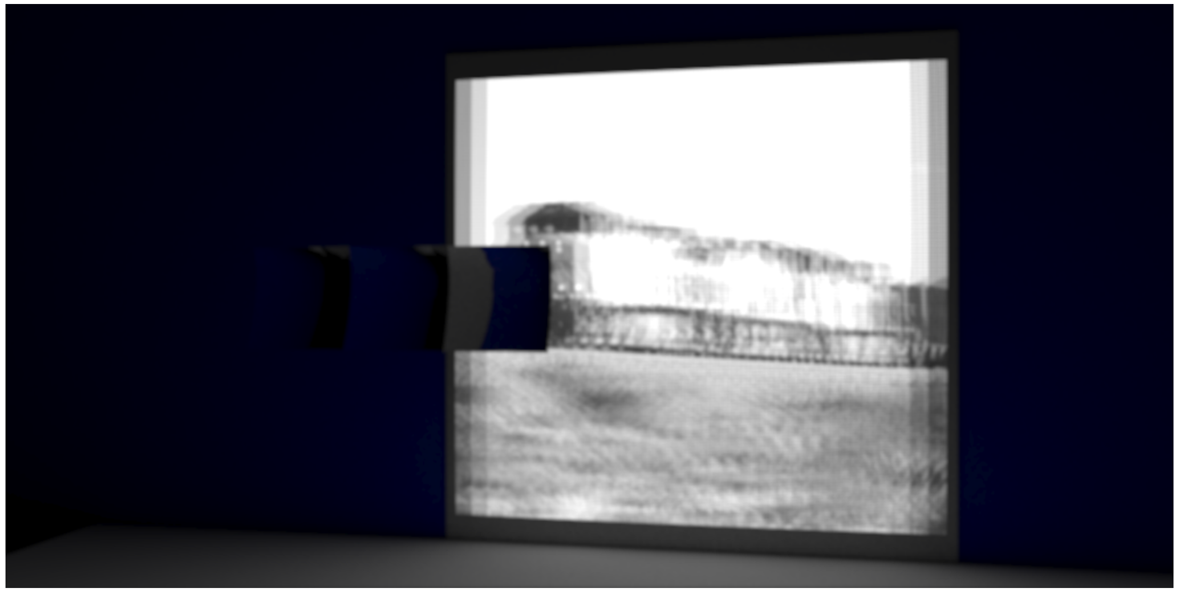
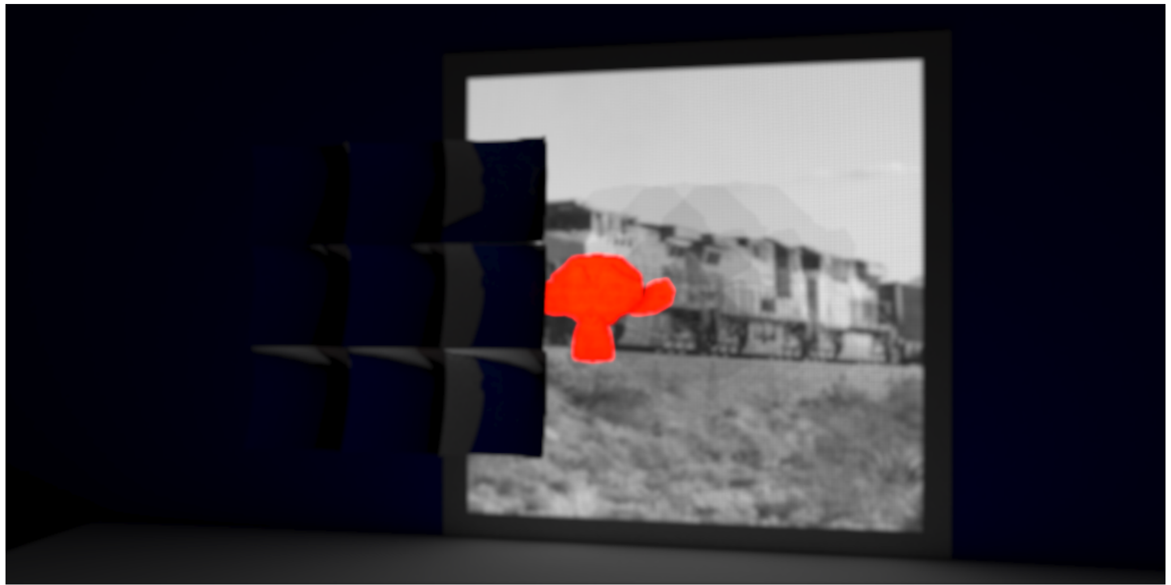
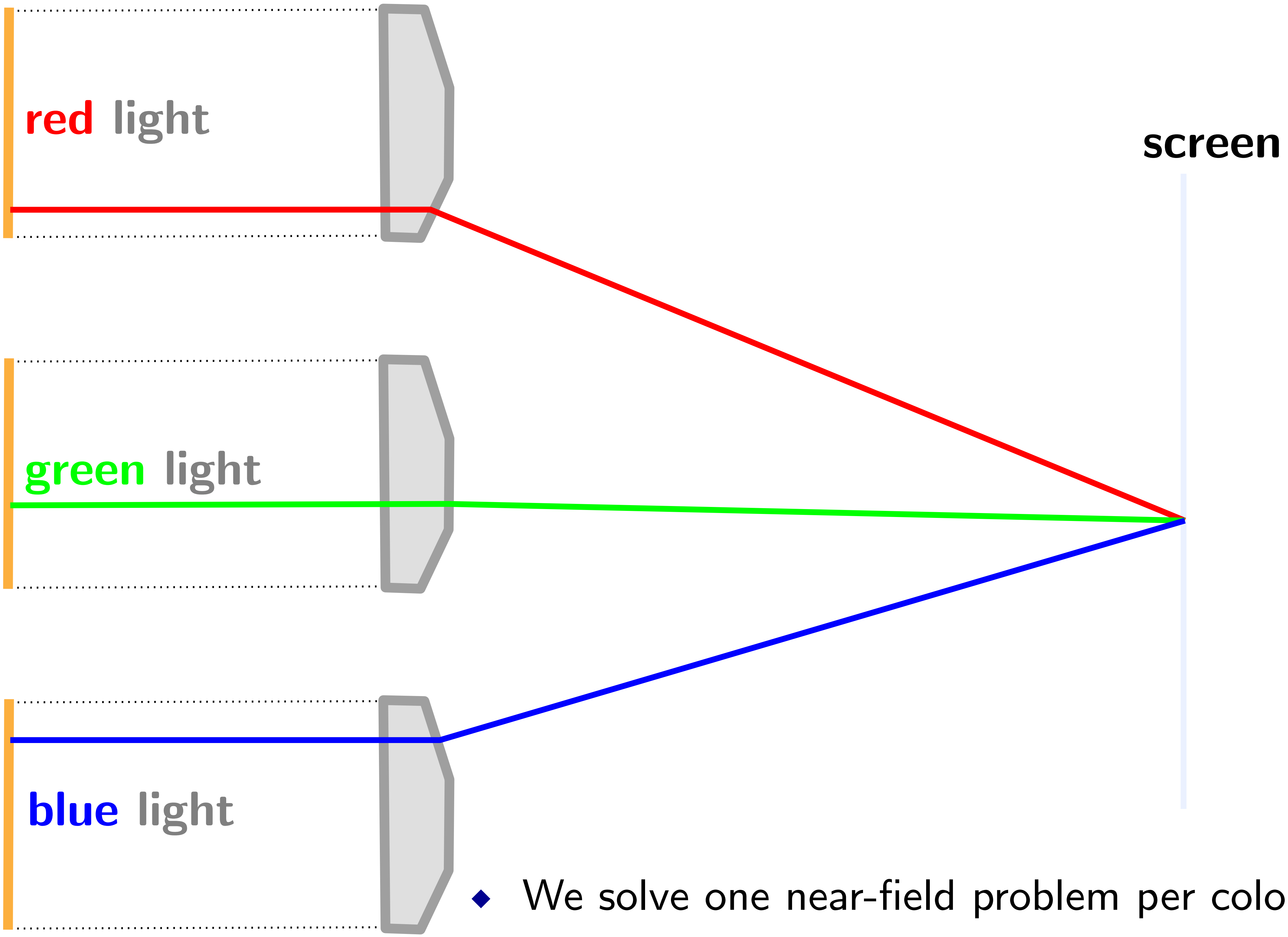


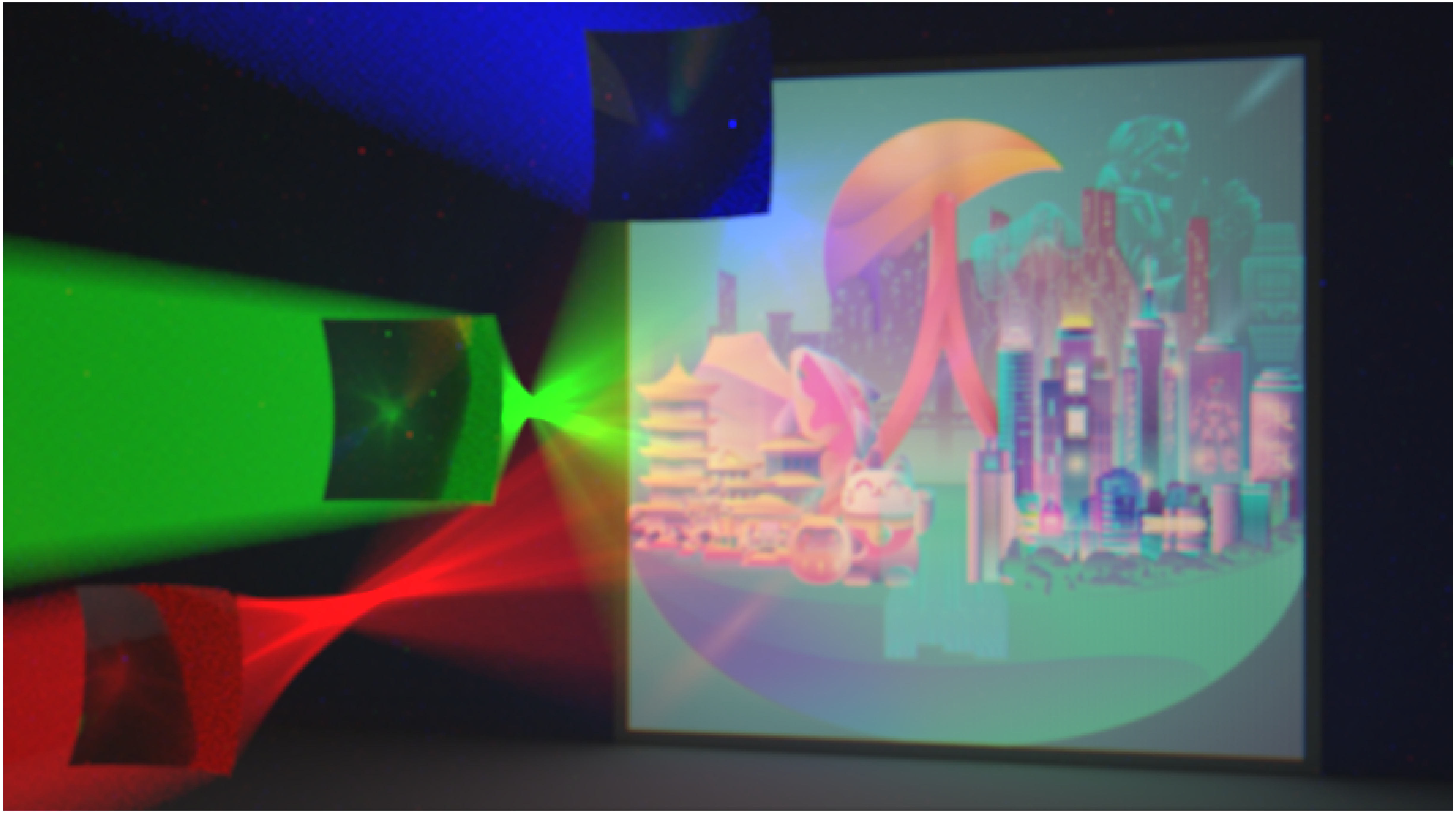


Color channels



- We solve one near-field problem per color channel.
- Near-field assumption needs to be taken into account for the image to be perfectly superimposed on the screen.

Color channels



Conclusion

Stabilily

- We propose a definition of strong c-concavity
- Several stability results under this assumption
- Provide a sufficient condition for strong concavity.
- Stability results in non imaging optics

Generated Jacobian Equation

- We extended an algorithm to Generated Jacobian Equation
- Each problem is a Monge-Ampère equation

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- Extended light
- Global stability results with general cost functions

Thank you !