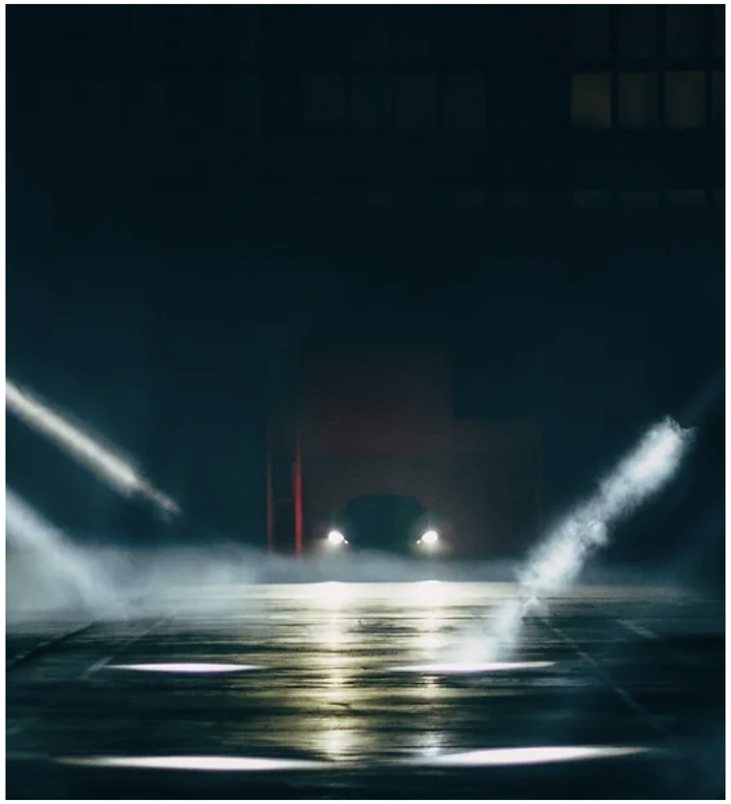
Numerical methods for the design of optical components, optimal transport and Generated Jacobian equations

Part 1: Non imaging optics & Optimal transport

Boris Thibert

Joint works with Quentin Mérigot, Jocelyn Meyron, Anatole Gallouet



Woudschoten conference, Sept. 25-27 2024

Nonimaging optics: motivations

Goal: design components that transfer a prescribed source light
to a prescribed target distribution

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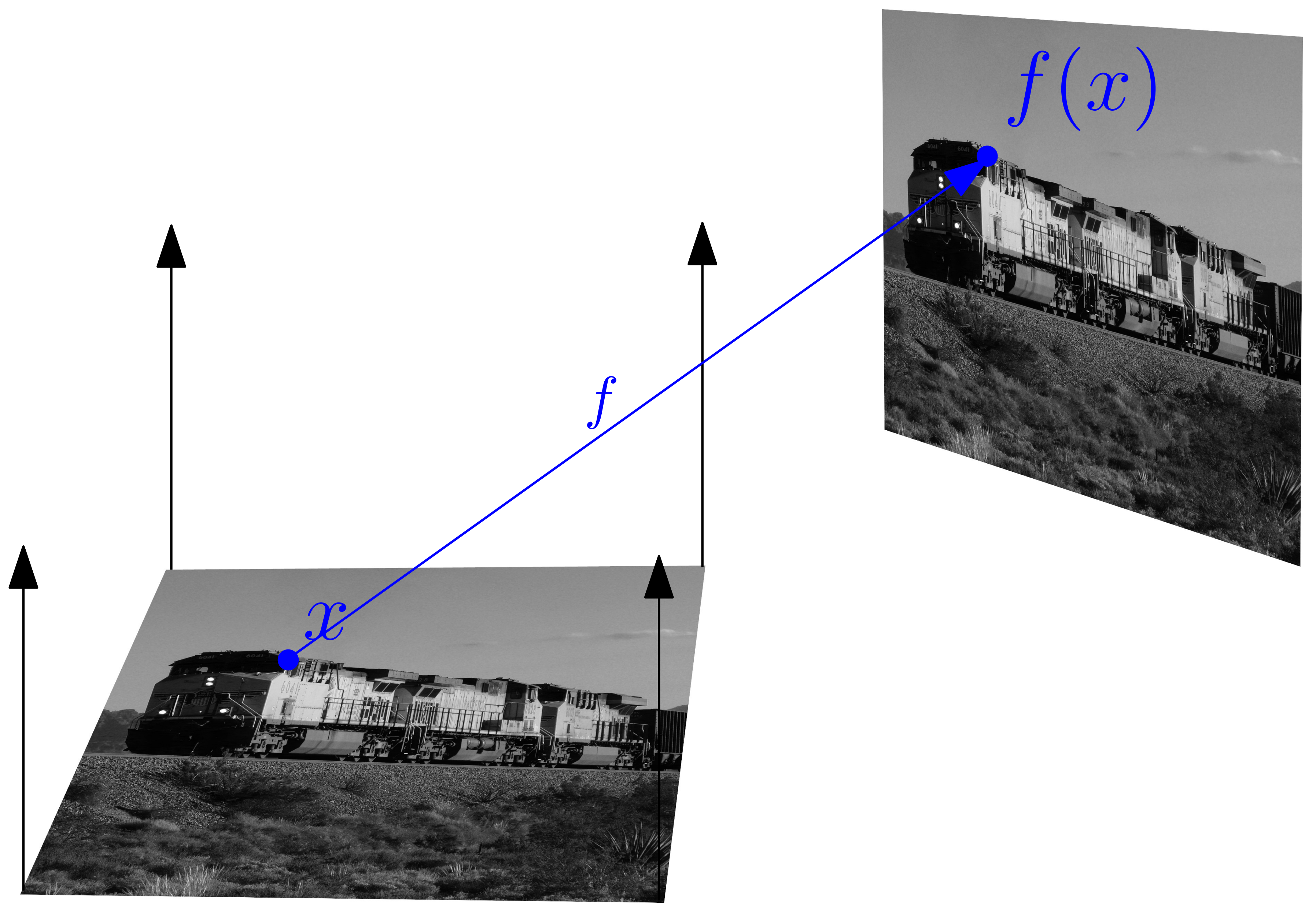
Motivations / applications

- Car beam design
- Public lighting
- Reduction of light pollution

Imaging optics: mirror case

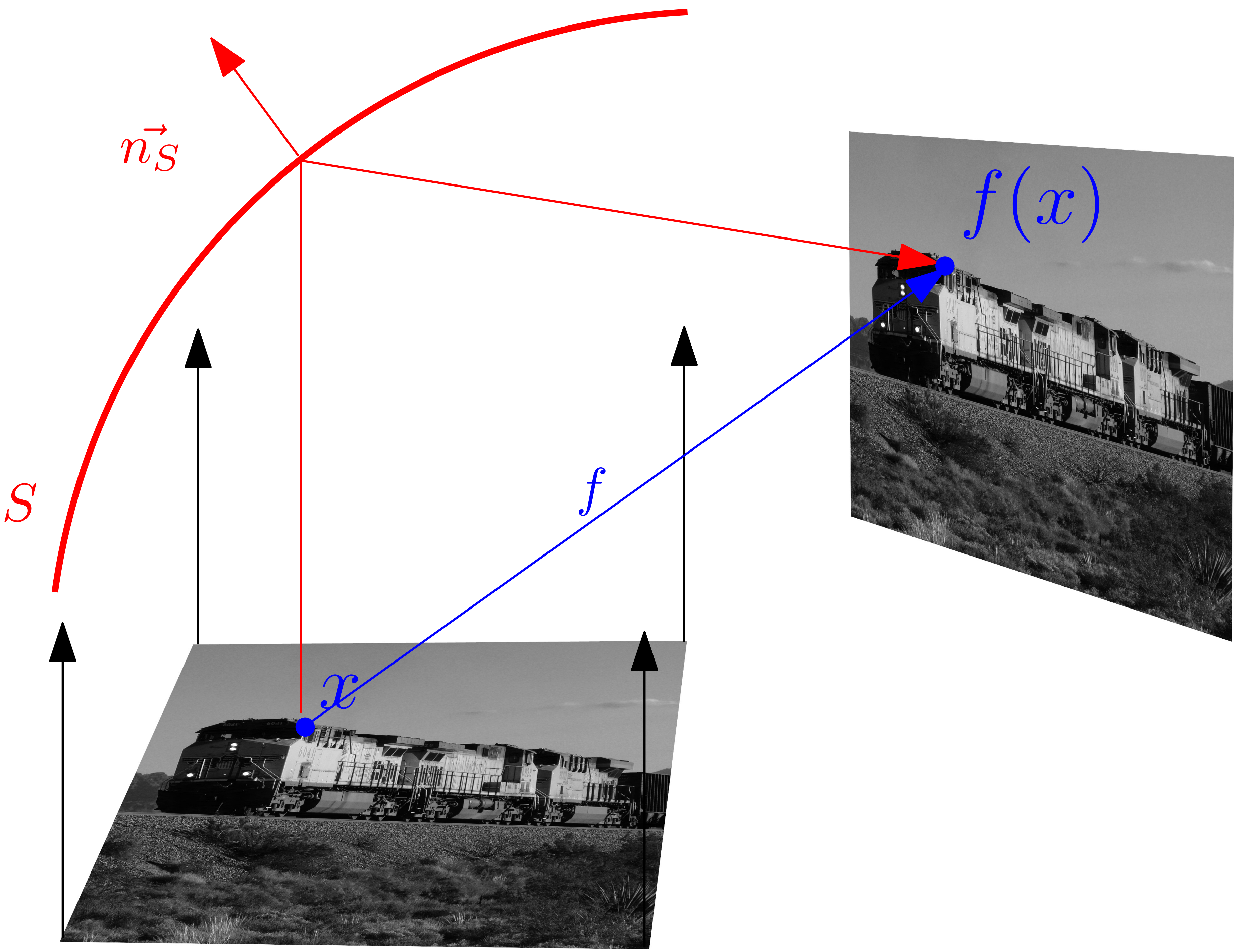
We are given a one-to-one map $f : X \to Y$.



Imaging optics: mirror case

We are given a one-to-one map $f : X \to Y$.

Goal: Find a surface S such that the reflection of X onto Y preserves f.

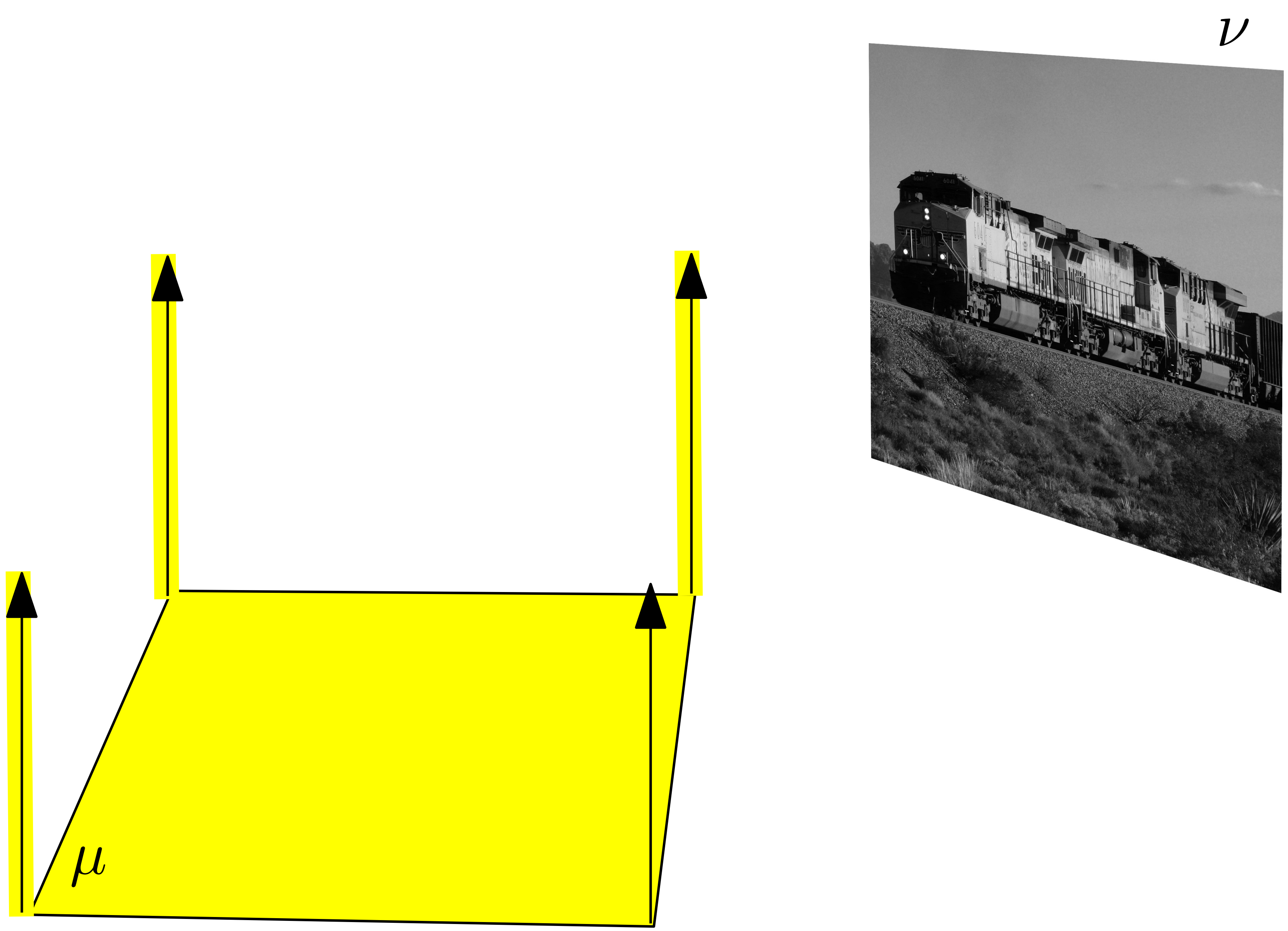


Non-imaging optics: mirror case

Input: Source light with intensity μ

Target light with intensity ν

No one-to-one map given



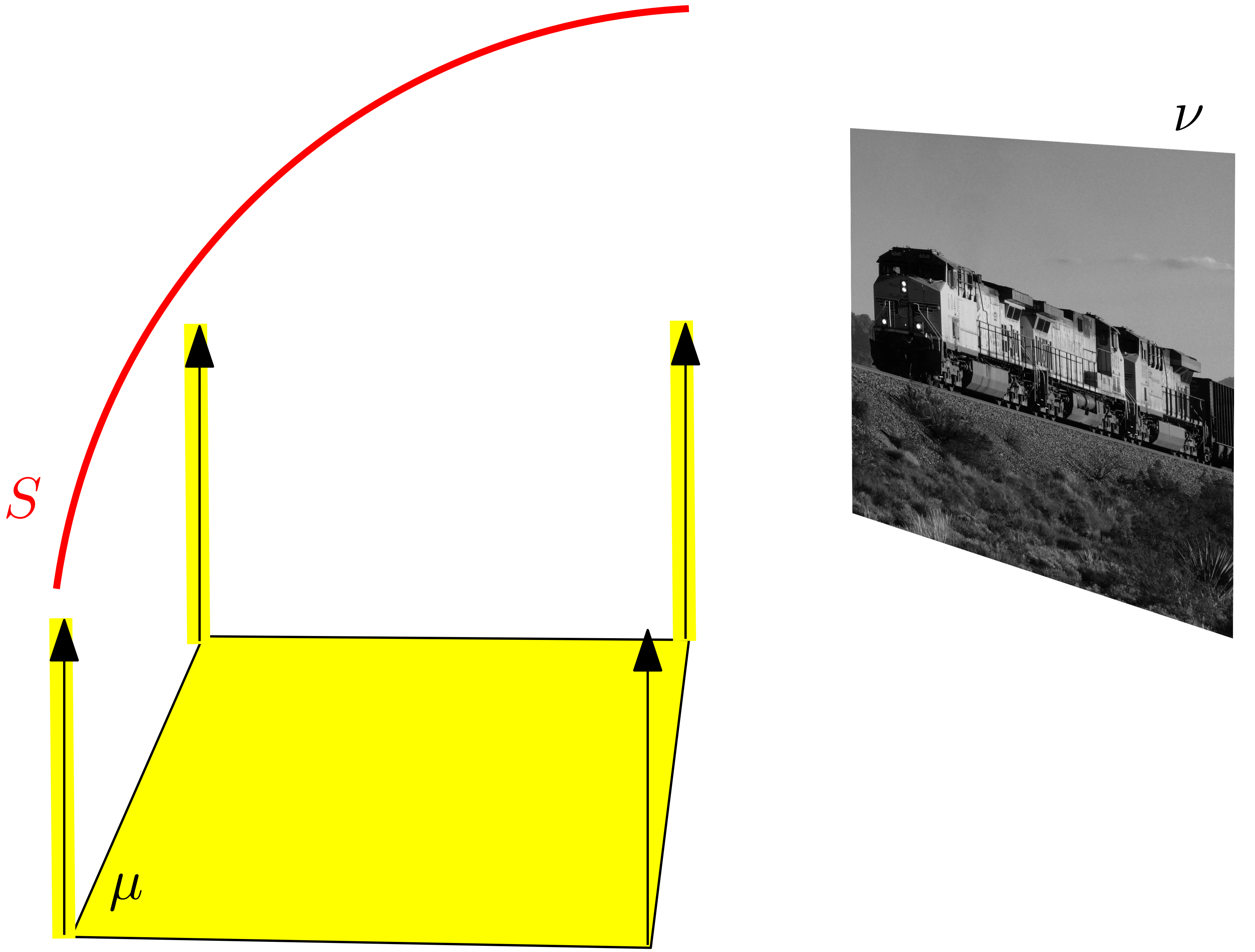
Non-imaging optics: mirror case

Input: Source light with intensity μ

Target light with intensity ν

No one-to-one map given

Goal: Find a surface S such that reflects μ to the ν by Snell's law



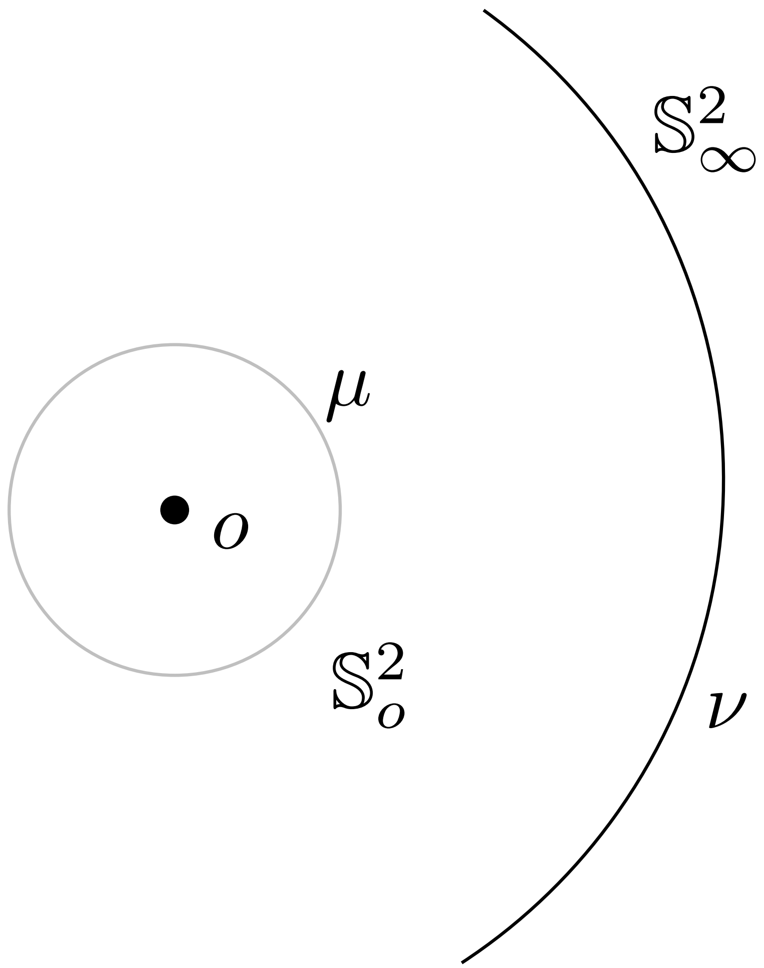
Outline

- Case 1: mirror for point light source
- Case 2: mirror for collimated source light
- Case 3: other cases

- Semi-discrete optimal transport
- Damped Newton algorithm

- Non-imaging optics: Far-Field target
- Non-imaging optics: Near-Field target

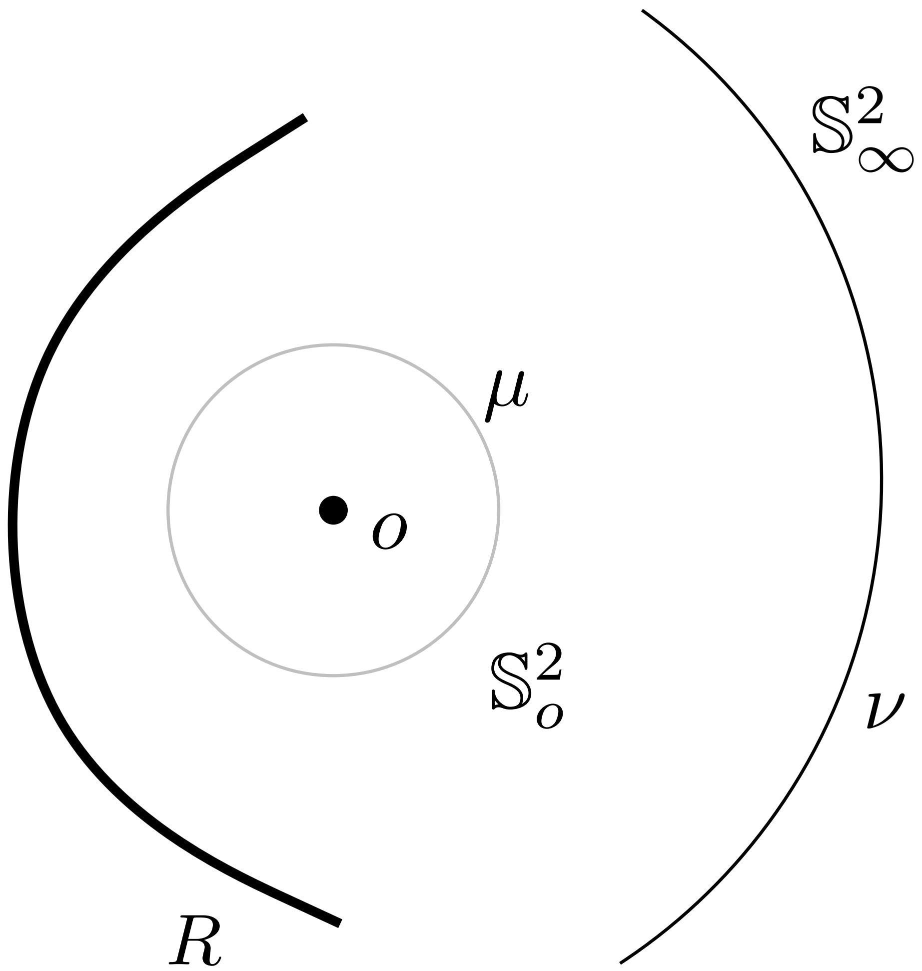
Mirror / Point light source



Punctual light at origin o, μ measure on $\mathbb{S}^2_o$

Prescribed far-field: ν on $\mathbb{S}^2_\infty$

Mirror / Point light source



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Goal: Find a surface R which sends $(\mathbb{S}^2_o, \mu)$ to $(\mathbb{S}_\infty, \nu)$ under reflection by Snell's law.

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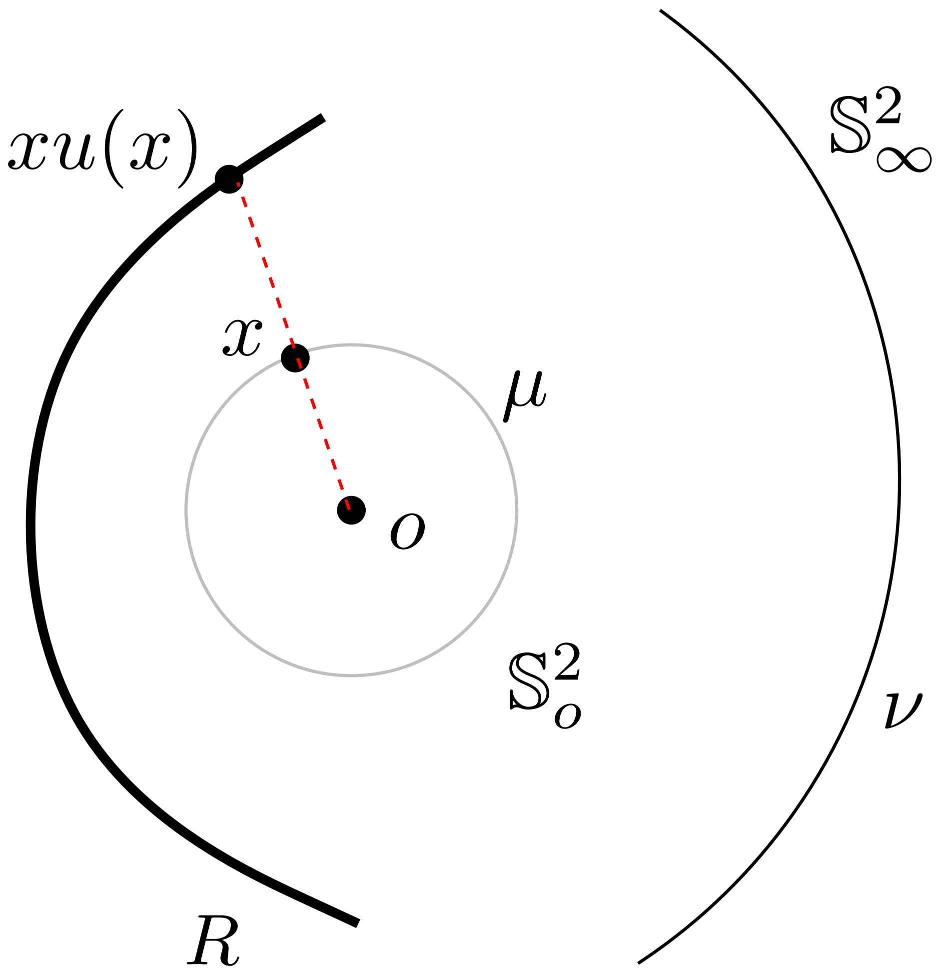

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- R is parameterized by $x \in \mathbb{S}^2_0 \mapsto xu(x)$ where $u : \mathbb{S}^2_0 \to \mathbb{R}^+$ radial distance

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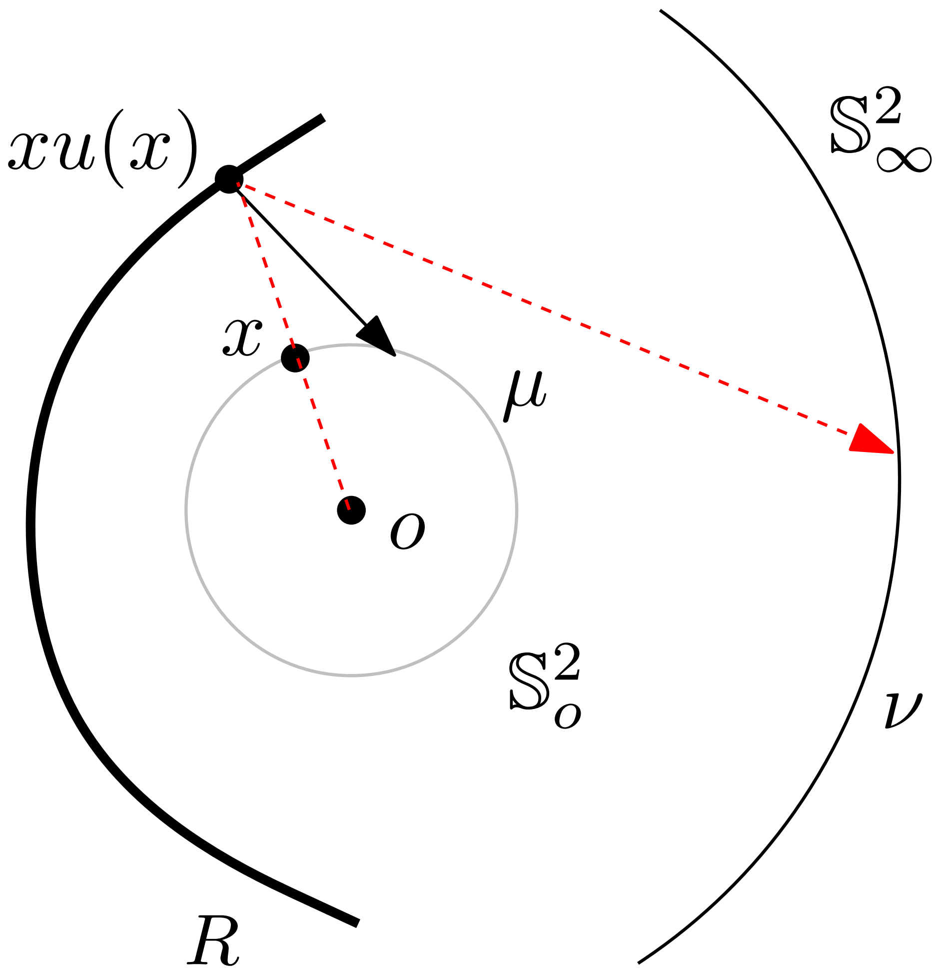

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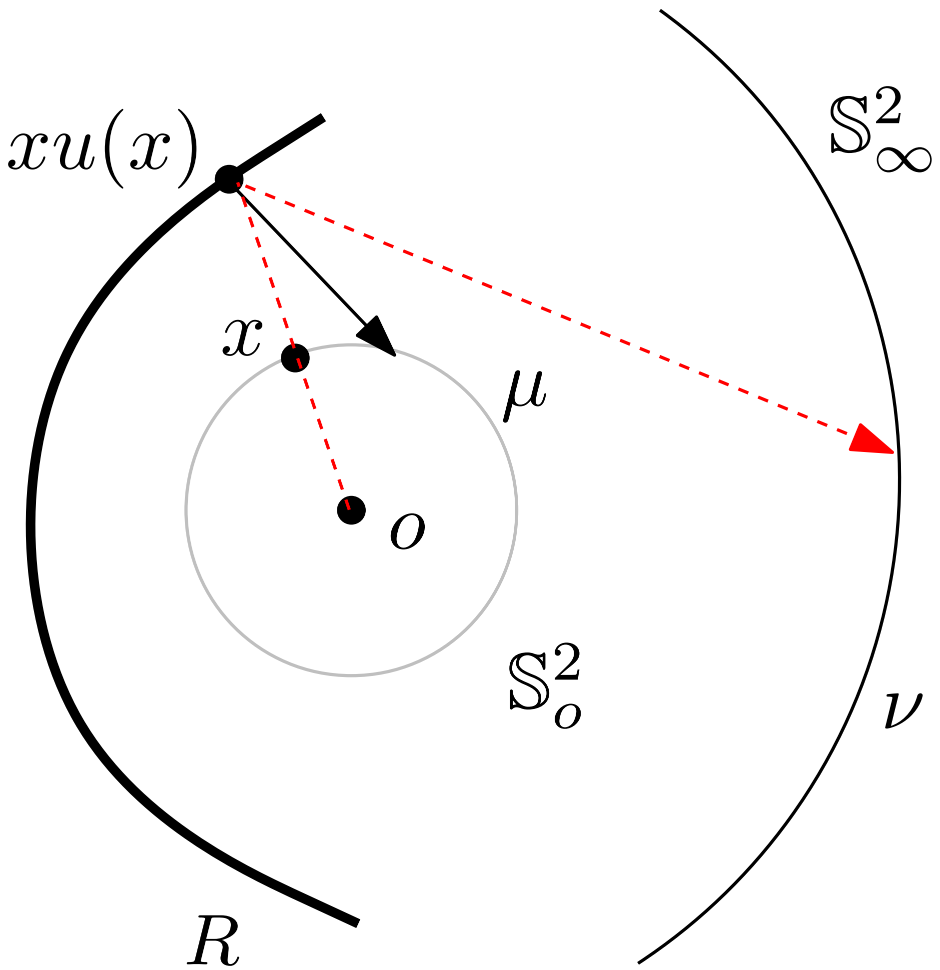

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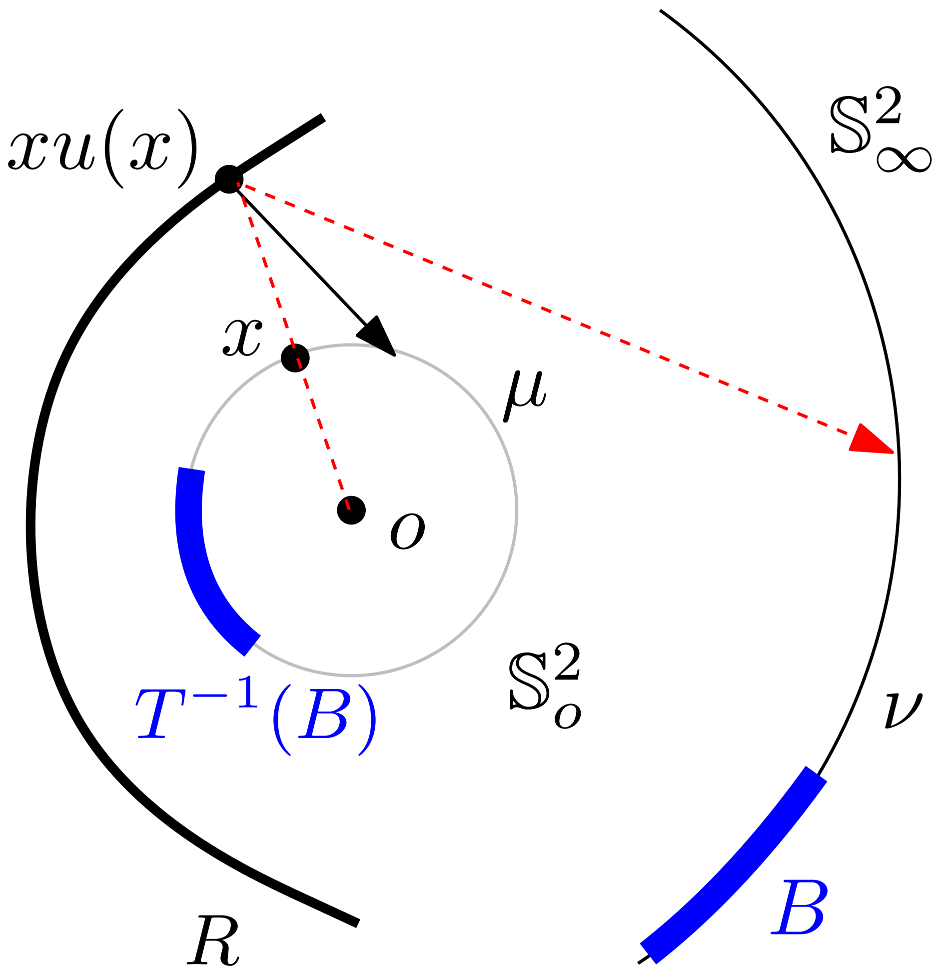

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i.e. for every borelian B

$$\mu(T^{-1}(B)) = \nu(B)$$

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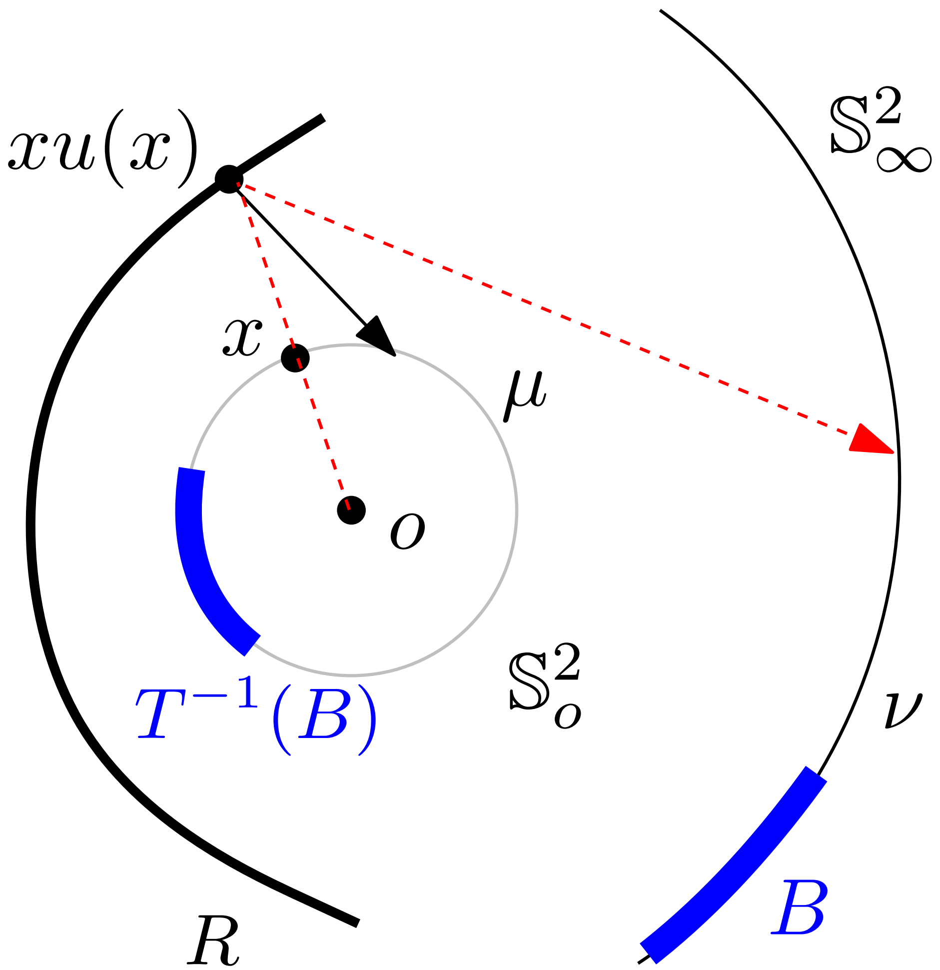

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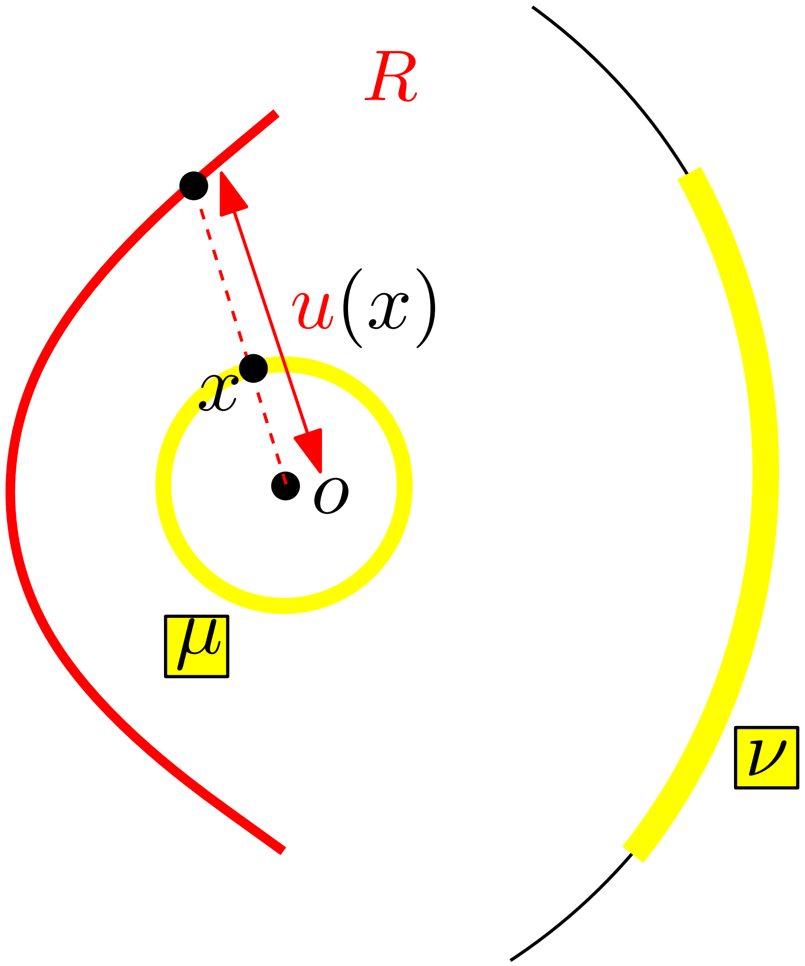
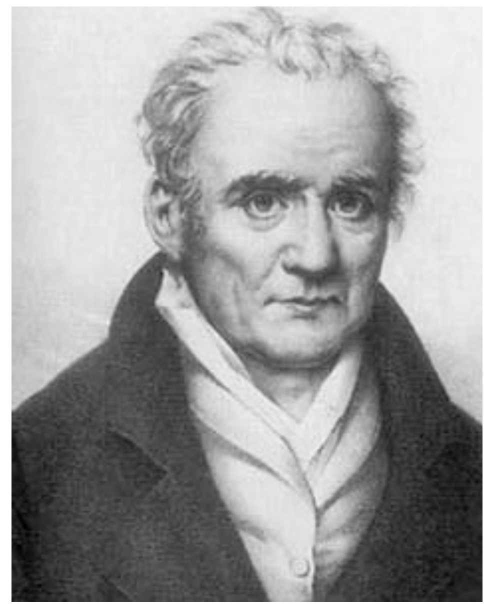
Change of variable

If $\mu(x) = f(x)dx$ and $\nu(y) = g(y)dy$

$$g(T(x)) \det(DT(x)) = f(x)$$

Mirror / Point light source





Monge

Ampère

Designing the mirror R amounts to solving

Monge-Ampère equation: Find $u : \mathbb{S}_0^2 \to \mathbb{R}^+$ vérifiant

$$\begin{cases} f_\nu(T(x))\det(DT(x)) = f_\mu(x) \\ T(x) = x - \langle x | n(x) \rangle n(x) \\ n(x) = \frac{\nabla u(x) - u(x) x}{\sqrt{\|\nabla u(x)\|^2 + u(x)^2}} \end{cases},$$

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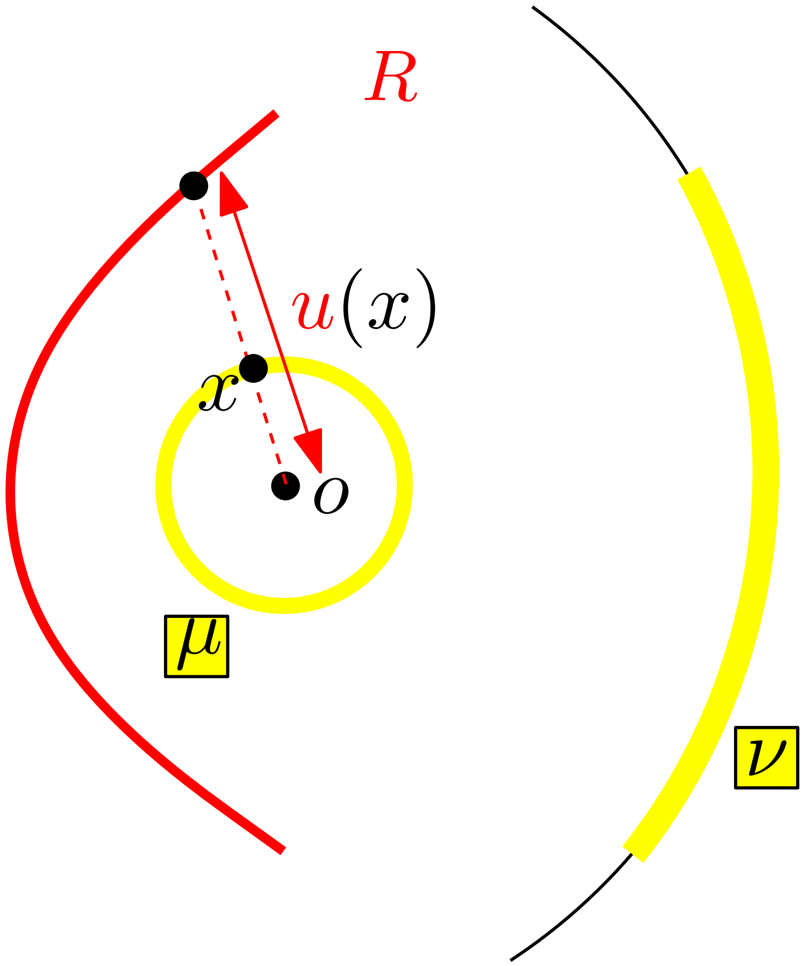

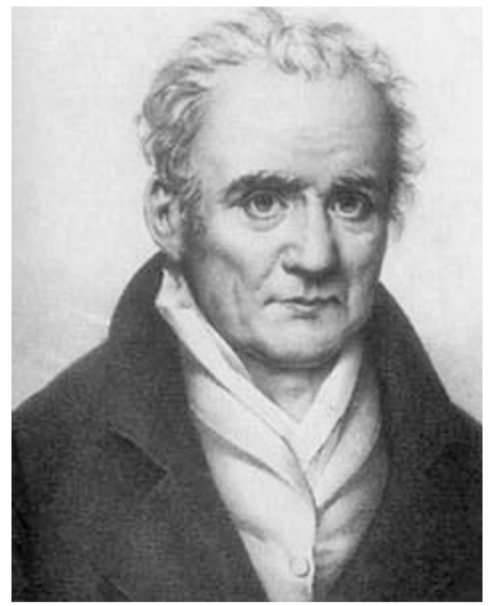

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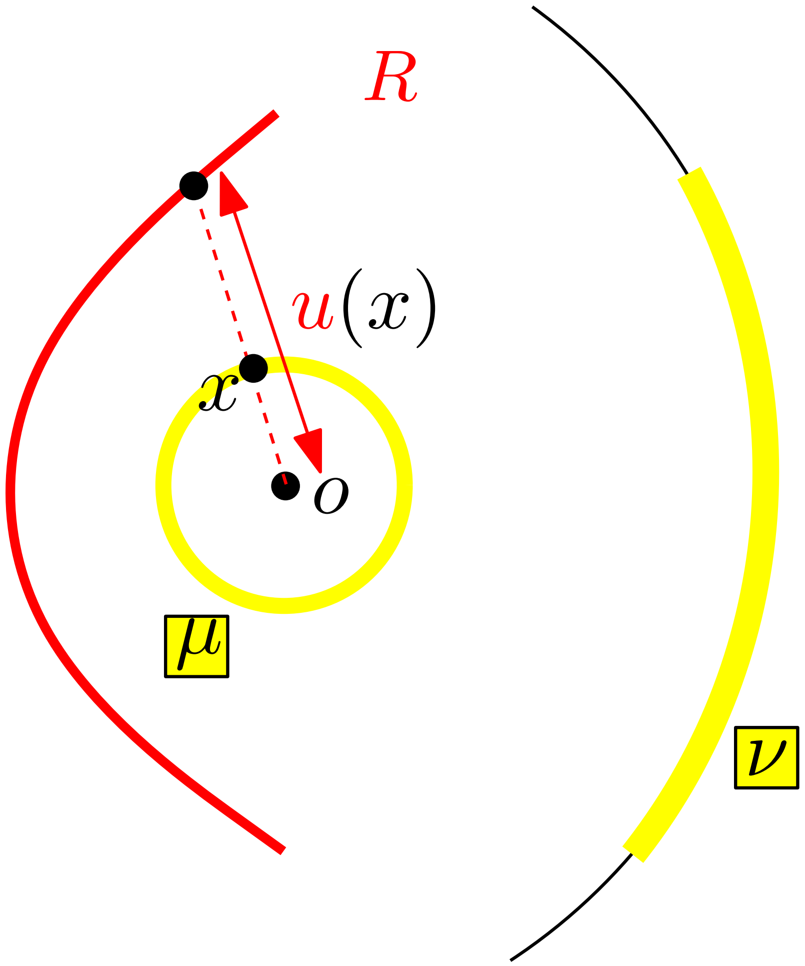

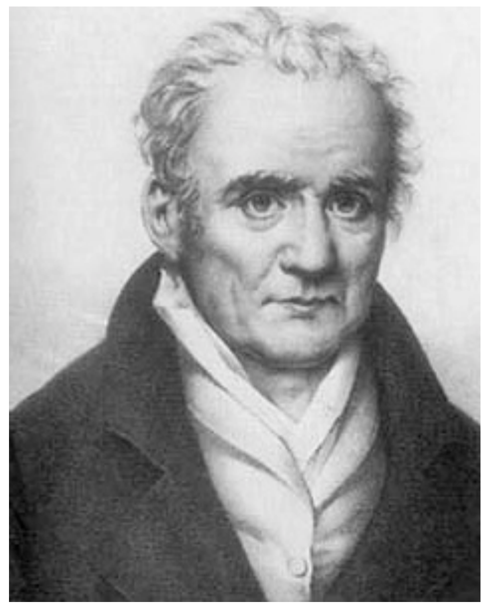
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Ampère


Caffarelli


Oliker

1994 : existence de solutions

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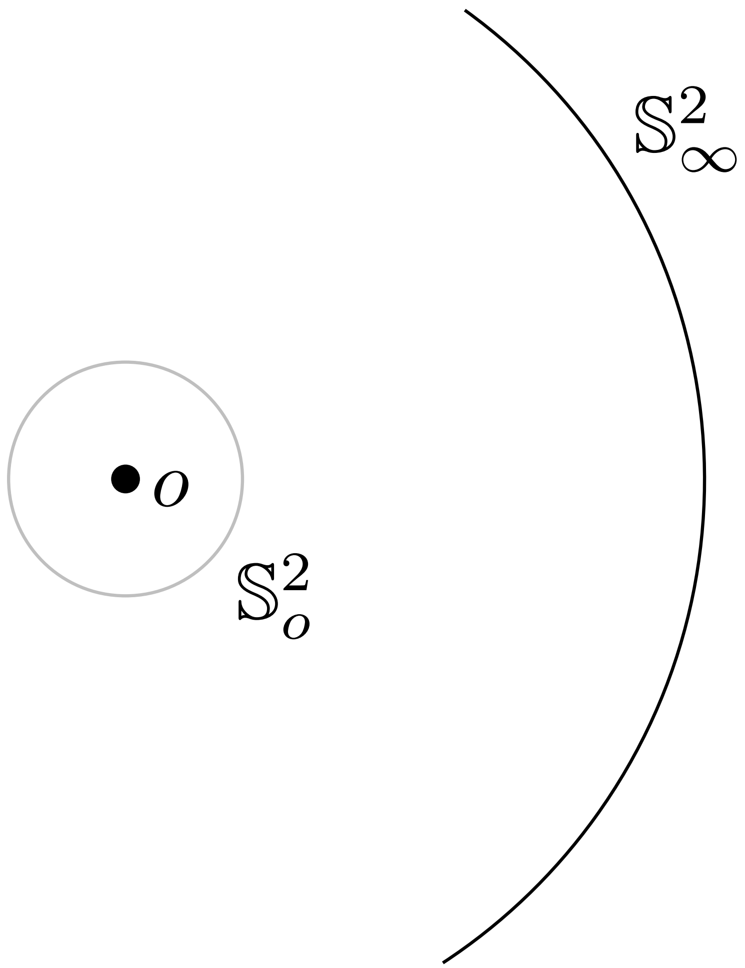
Mirror / Point light source: semi-discrete



Punctual light at origin o, μ measure on $\mathbb{S}^2_o$

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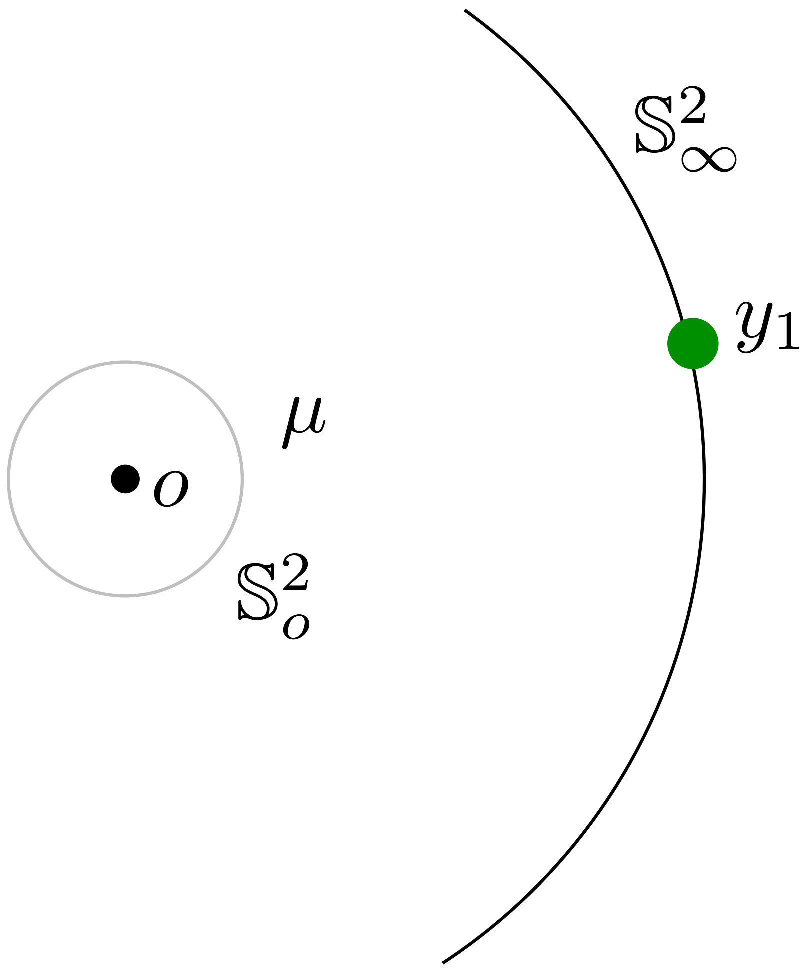

Punctual light at origin o, μ measure on $\mathbb{S}^2_o$

Prescribed far-field: $\nu = \nu_1 \delta_{y_1}$ on $\mathbb{S}^2_\infty$

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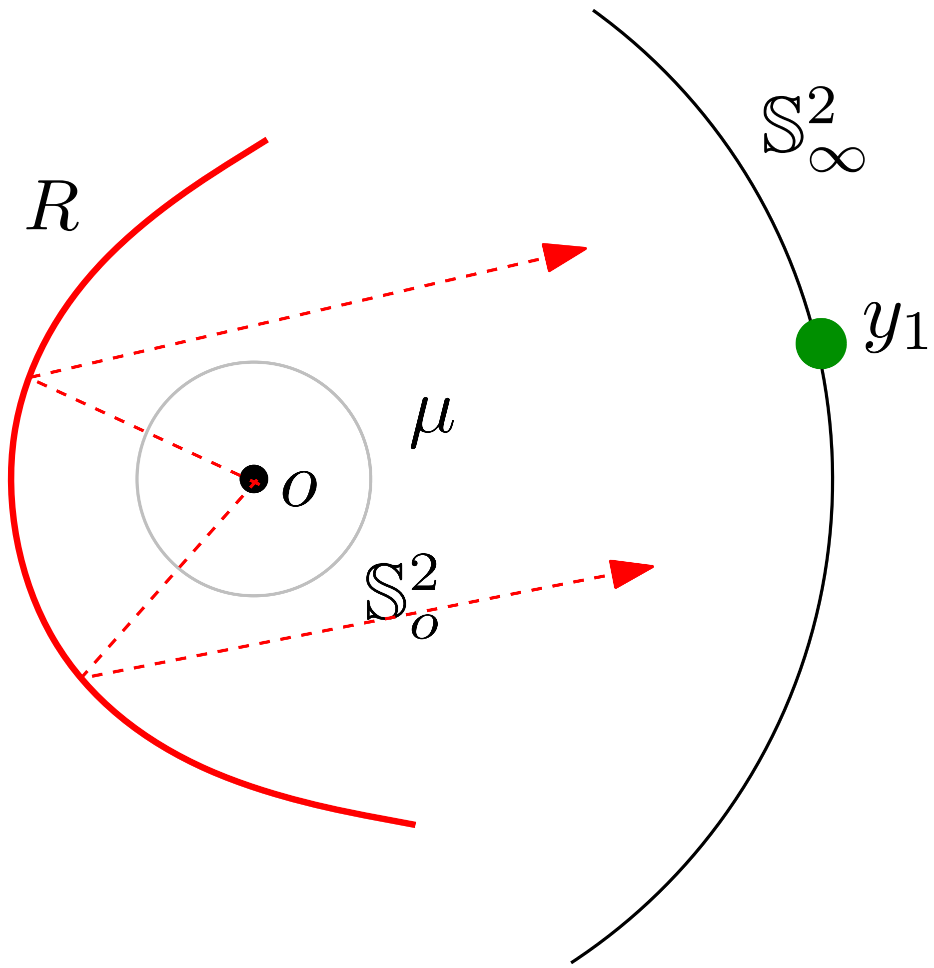

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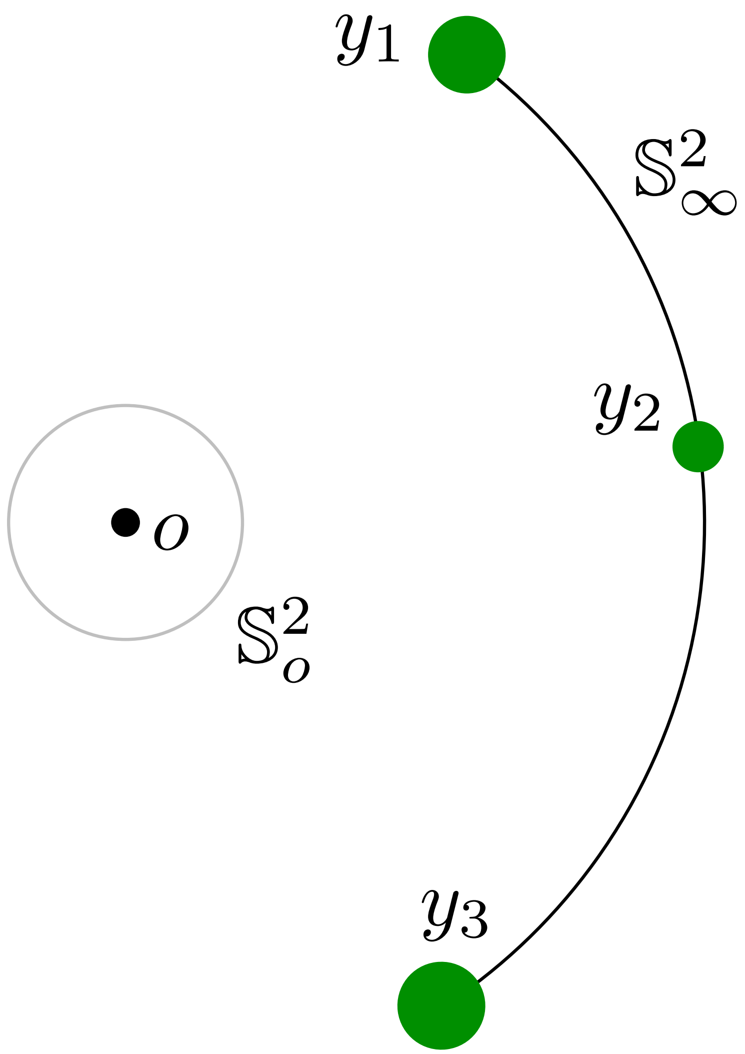
R : paraboloid of direction y_1 and focal O

Mirror / Point light source: semi-discrete



Punctual light at origin o, μ measure on $\mathbb{S}^2_o$

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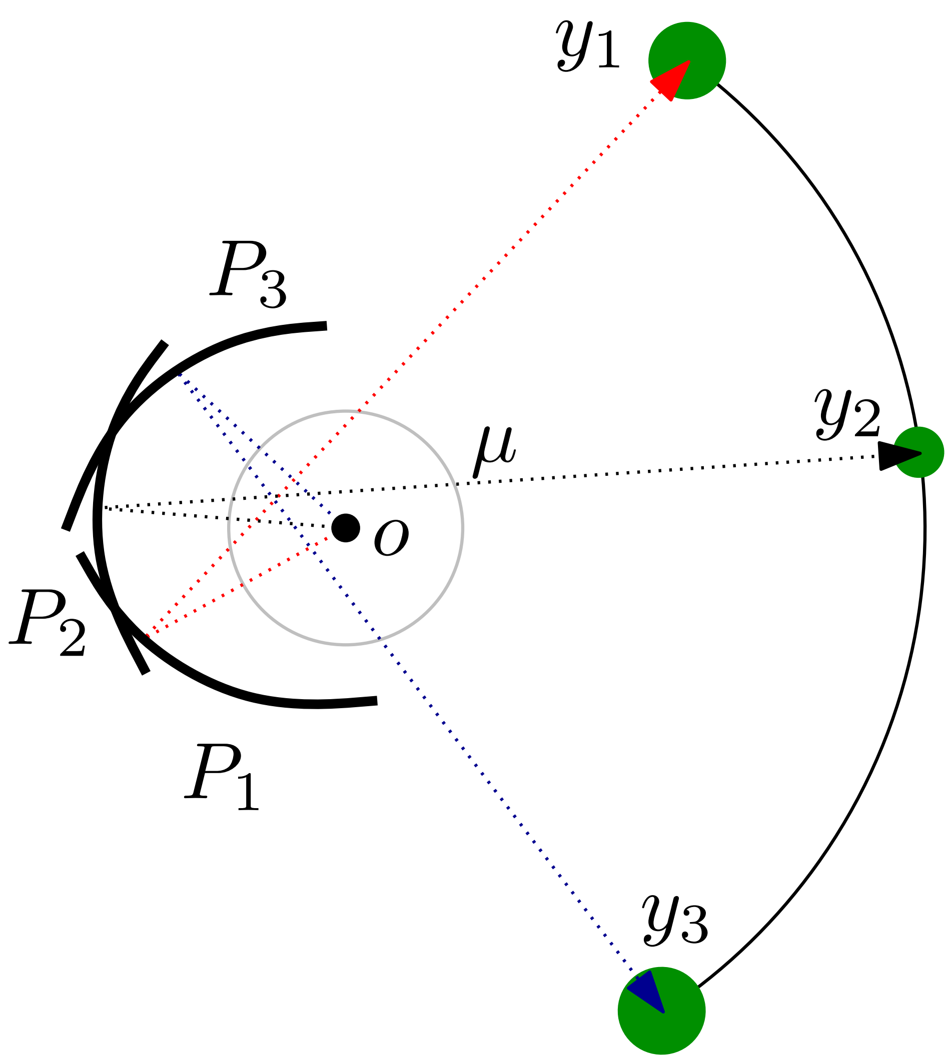

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$P_i(\kappa_i)$ = solid paraboloid of revolution with focal o, direction y_i and focal distance κ_i

$R(\vec{\kappa}) = \partial \left(\cap_{i=1}^N P_i(\kappa_i) \right)$

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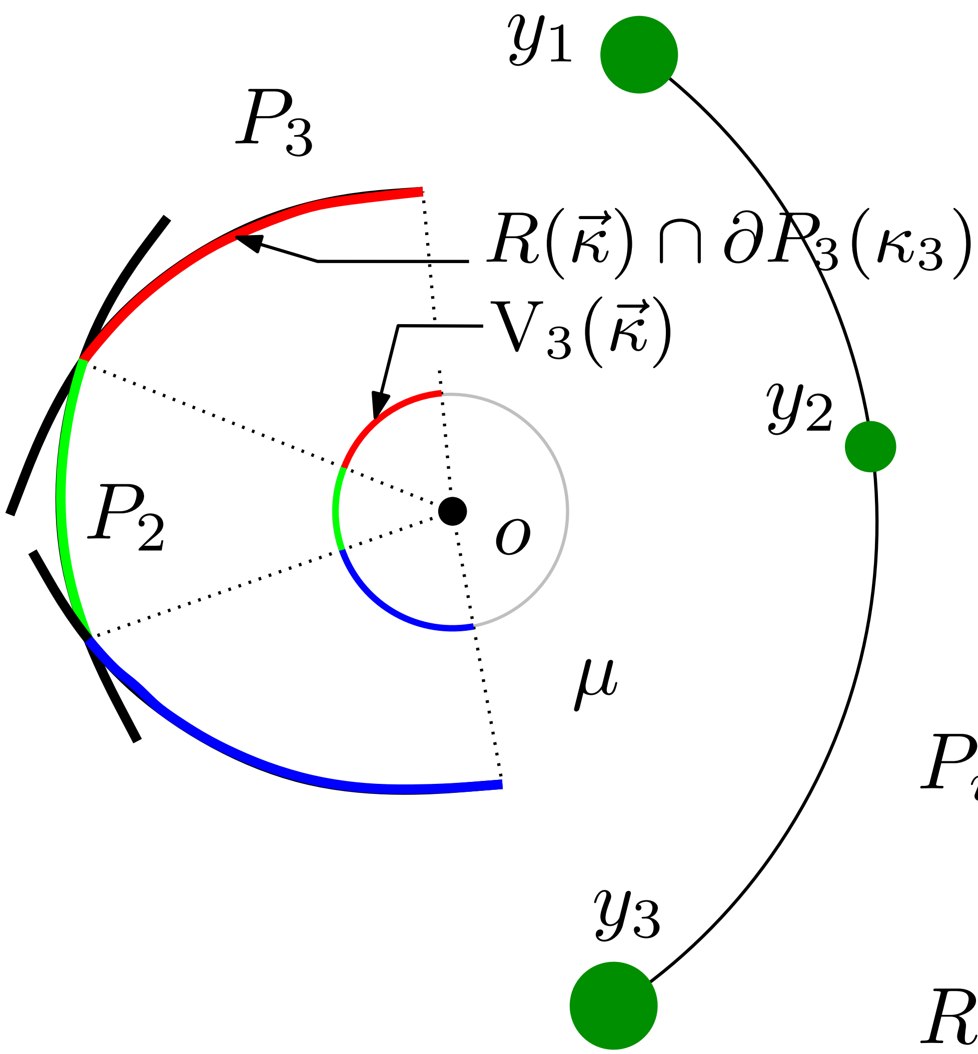

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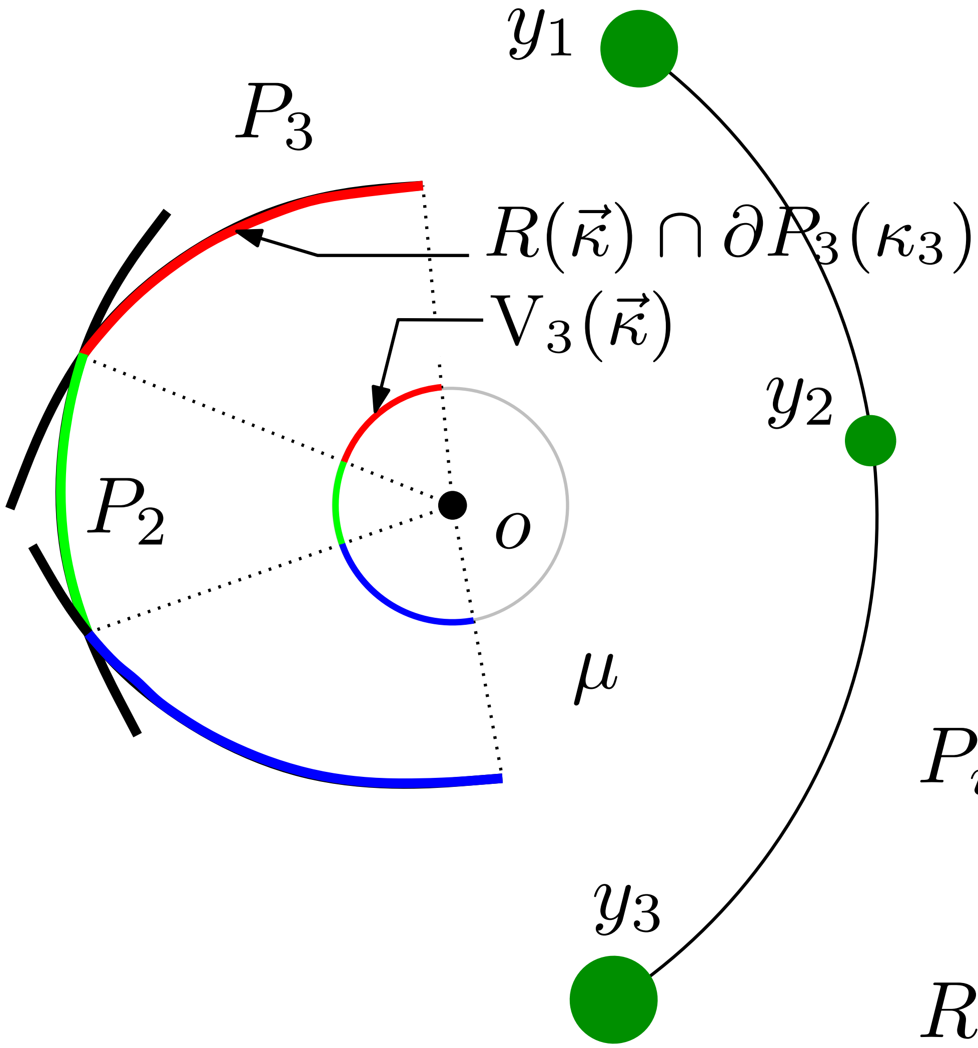
Decomposition of $\mathbb{S}^2_o$: $\mathrm{V}_i(\vec{\kappa}) = \pi_{\mathcal{S}^2_o}(R(\vec{\kappa}) \cap \partial P_i(\kappa_i))$
= directions that are reflected towards y_i.

Mirror / Point light source: semi-discrete



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$=$ directions that are reflected towards y_i.

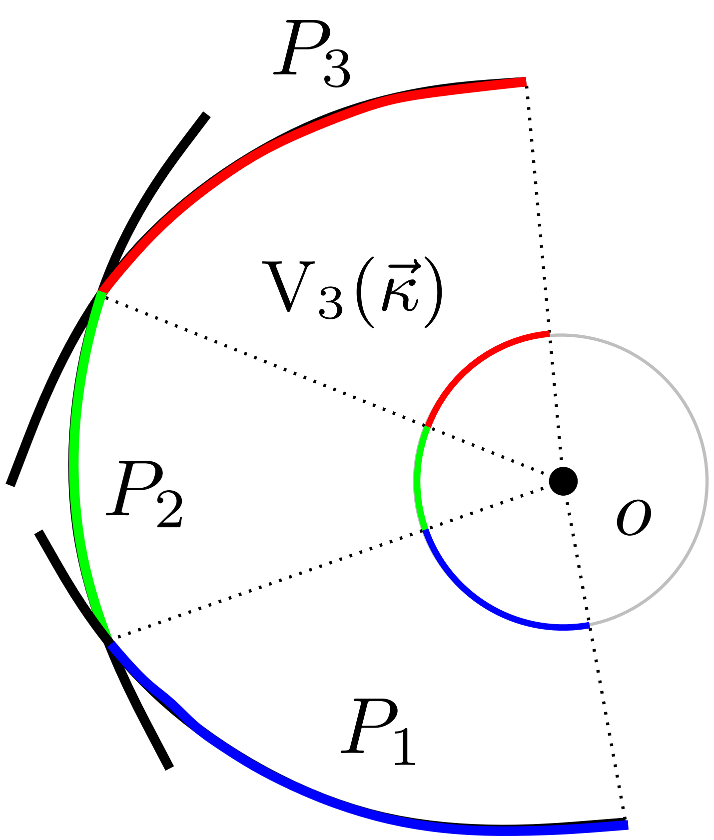
Problem (FF): Find $\kappa_1, \ldots, \kappa_N$ such that for every i, $\mu(V_i(\vec{\kappa})) = \nu_i$.

amount of light reflected in direction y_i.

Mirror / Point light source: Optimal Transport

Lemma: With $c(x,y) = -\log(1 - \langle x|y\rangle)$, and $\psi_i := \log(\kappa_i)$,
$\mathrm{Lag}_i(\psi) := \mathrm{V}_i(\vec{\kappa}) = \{x \in \mathbb{S}^2_0,\ c(x,y_i) + \psi_i \leq c(x,y_j) + \psi_j \quad \forall j\}$.

Caffarelli-Oliker '94



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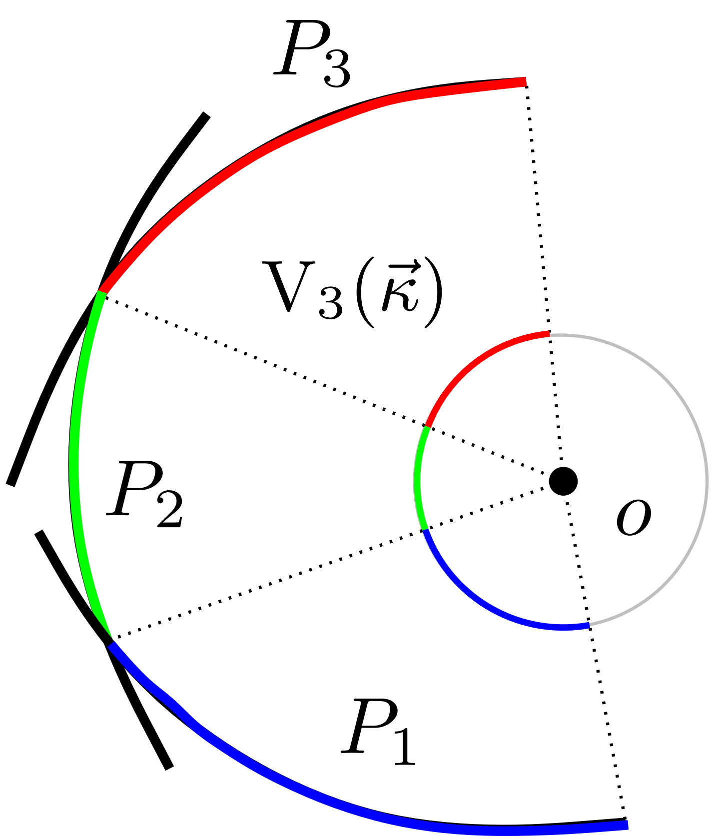

$\rightsquigarrow$ An optimal transport problem on $\mathbb{S}^2$

Mirror / Point light source: Optimal Transport

$\rightsquigarrow$ We have to solve an OT problem

Problem (FF): Find $\psi \in \mathbb{R}^N$ such that

$$\forall i \in \{1, \cdots N\} \qquad \mu(Lag_i(\psi)) = \nu_i.$$

where $\mathrm{Lag}_i(\psi) = \{x \in \mathbb{S}^2_0,\ c(x, y_i) + \psi_i \leq c(x, y_j) + \psi_j \quad \forall j\}$,
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$\rightsquigarrow$ The mirror is parametrized by

$$\begin{array}{ccc} \mathbb{S}^{d-1} & \to & \mathbb{R}^d \\ x & \mapsto & \left(\min_i \frac{e^{\psi_i}}{1-\langle x|y_i\rangle}\right) x \end{array}$$

Mirror / Point light source: Optimal Transport

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$e^{\min_i c(x,y_i)+\psi_i} = e^{\psi^c(x)}$

where $\psi^c(x) = \min_{y_i} c(x, y) - \psi(y_i)$
is the c-conjugate function of ψ.

Mirror / Point light source: Optimal Transport

⇝ We have to solve an OT problem

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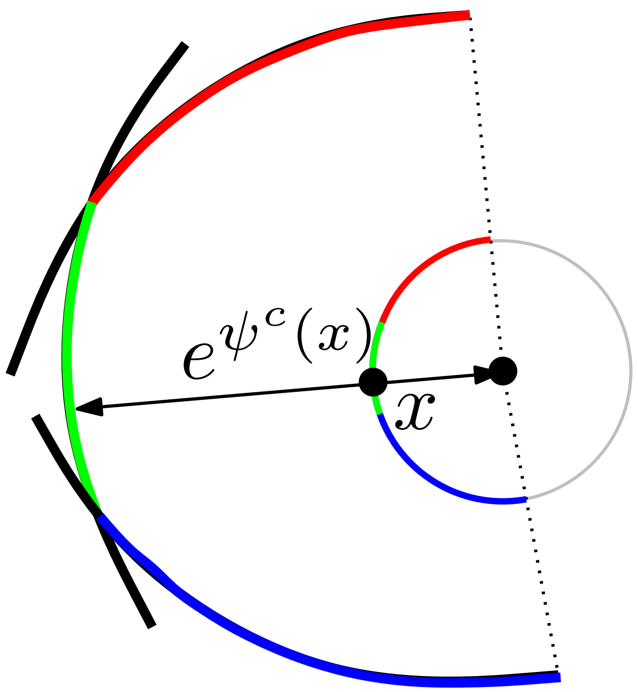

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ccl : $x \in \mathbb{S}^2_0 \mapsto e^{\psi^c(x)} x$ parametrizes the mirror.

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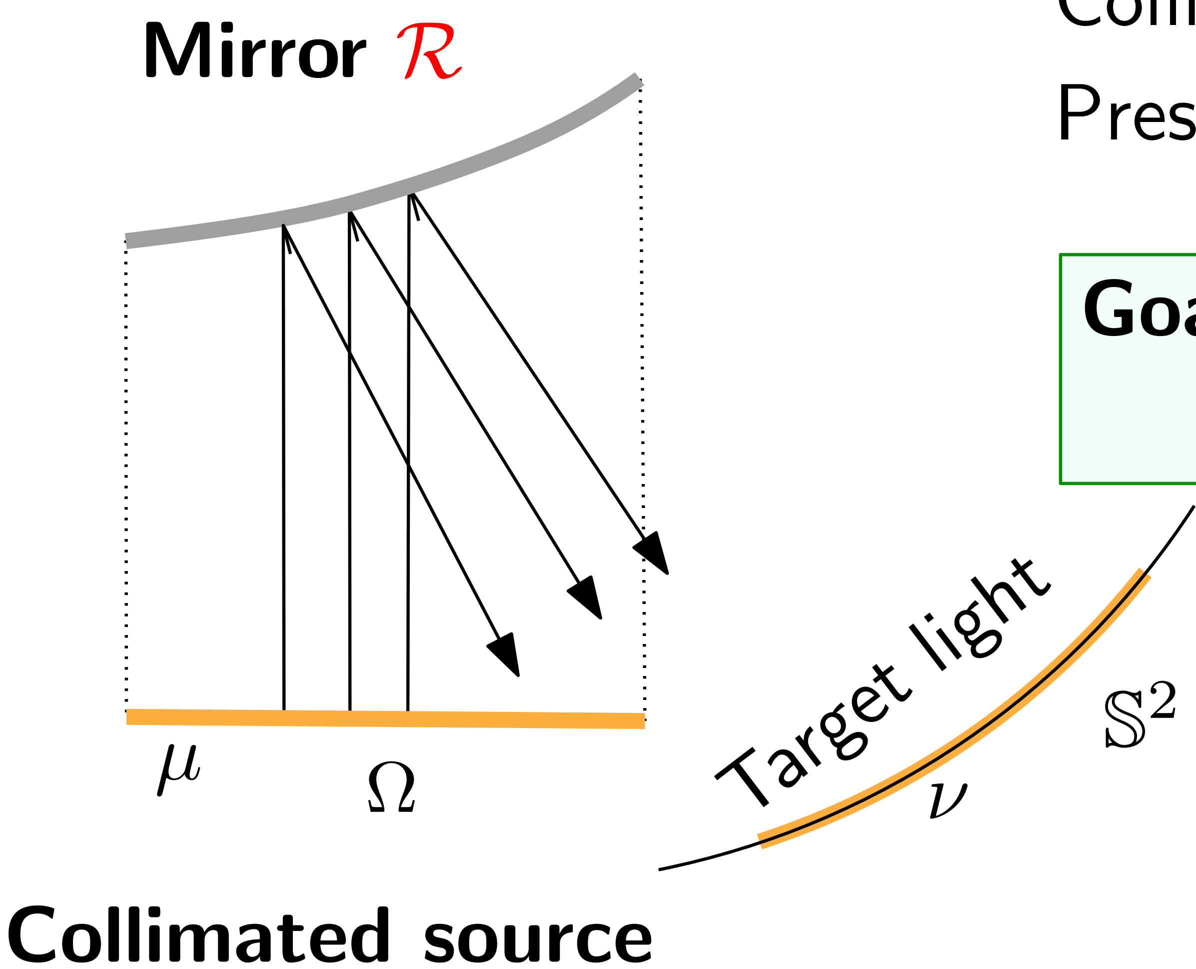
Mirror / Collimated source light



Collimated light μ measure on $\Omega \subset \mathbb{R}^2 \times \{0\}$

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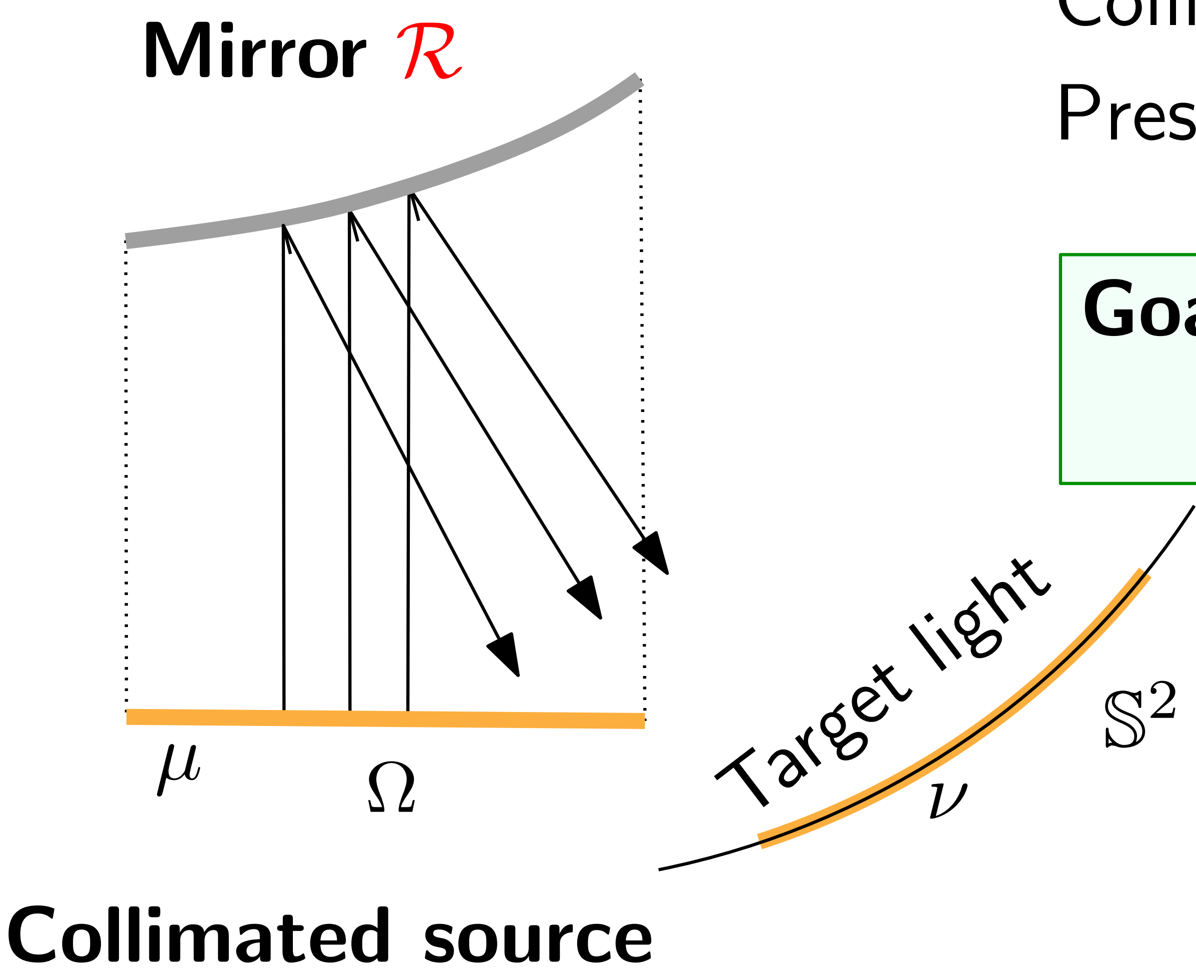
Goal: Find a surface R which sends (Ω, μ) to $(\mathbb{S}^2, \nu)$ under reflection by Snell's law.

Mirror / Collimated source light



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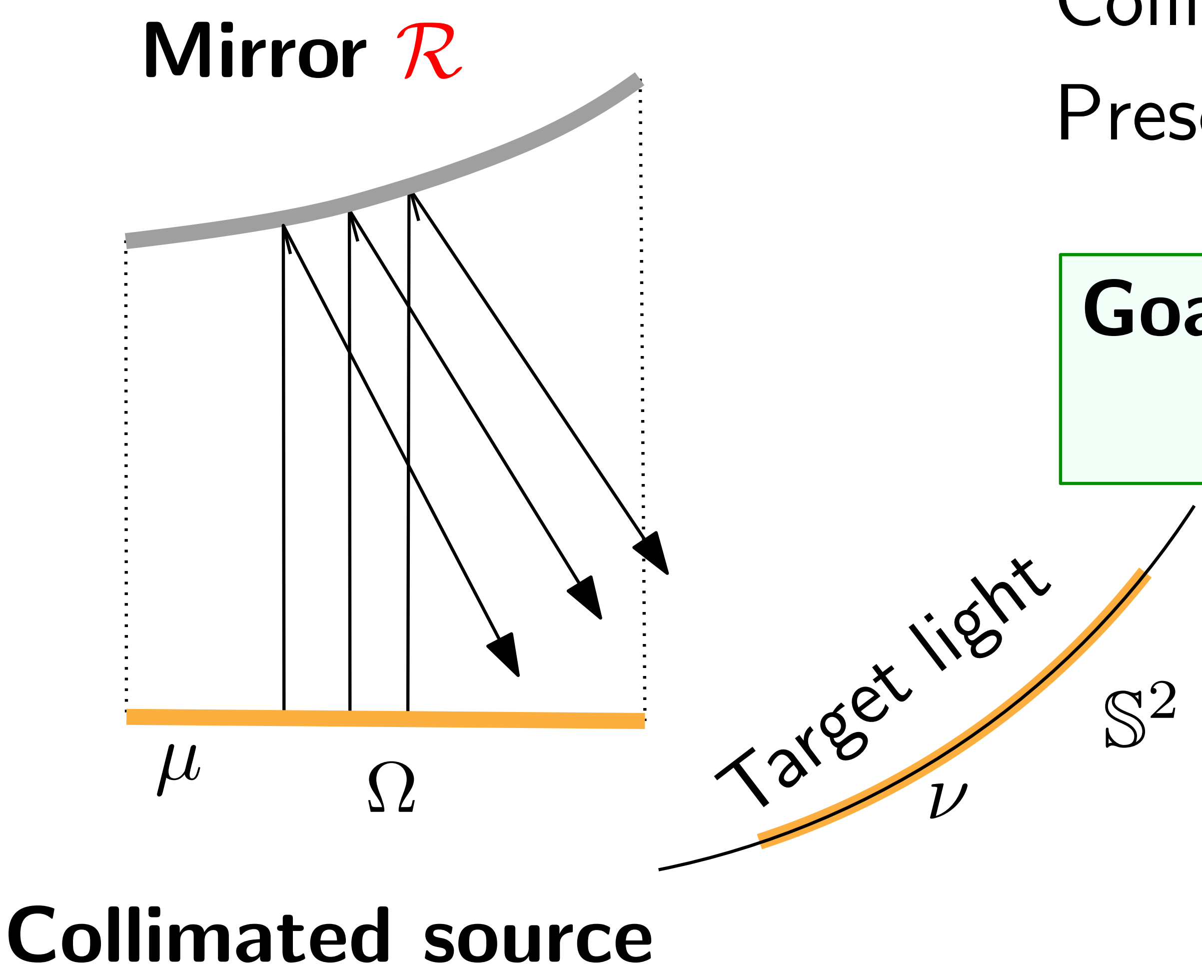

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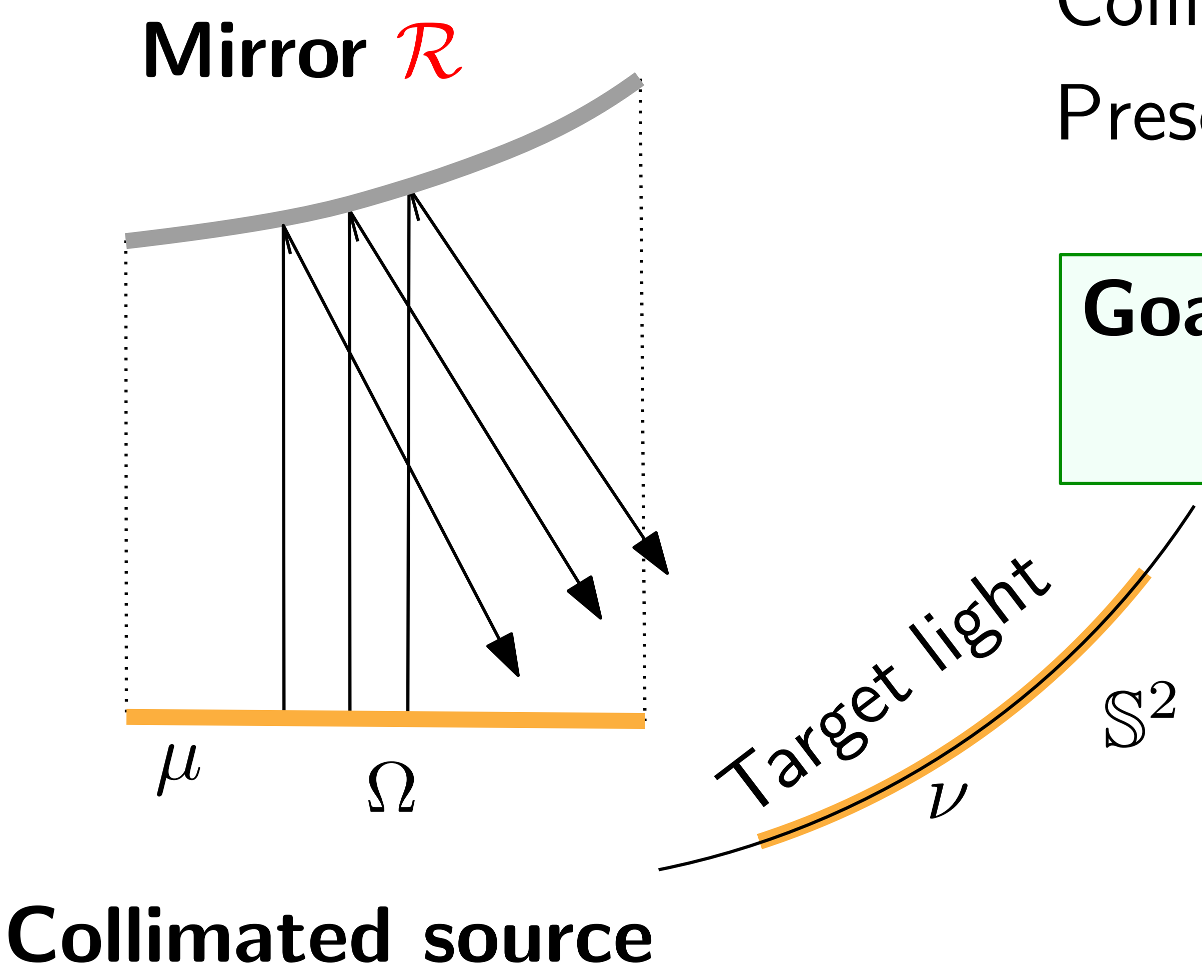
Brenier formulation $(F \circ \nabla u)_\sharp \mu = \nu$

$$\Leftrightarrow \forall A \; \mu((F \circ \nabla u)^{-1}(A)) = \nu(A)$$

Mirror / Collimated source light



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- R param. by $x \in \Omega \mapsto (x, u(x))$ where $u : \Omega \to \mathbb{R}$ height function
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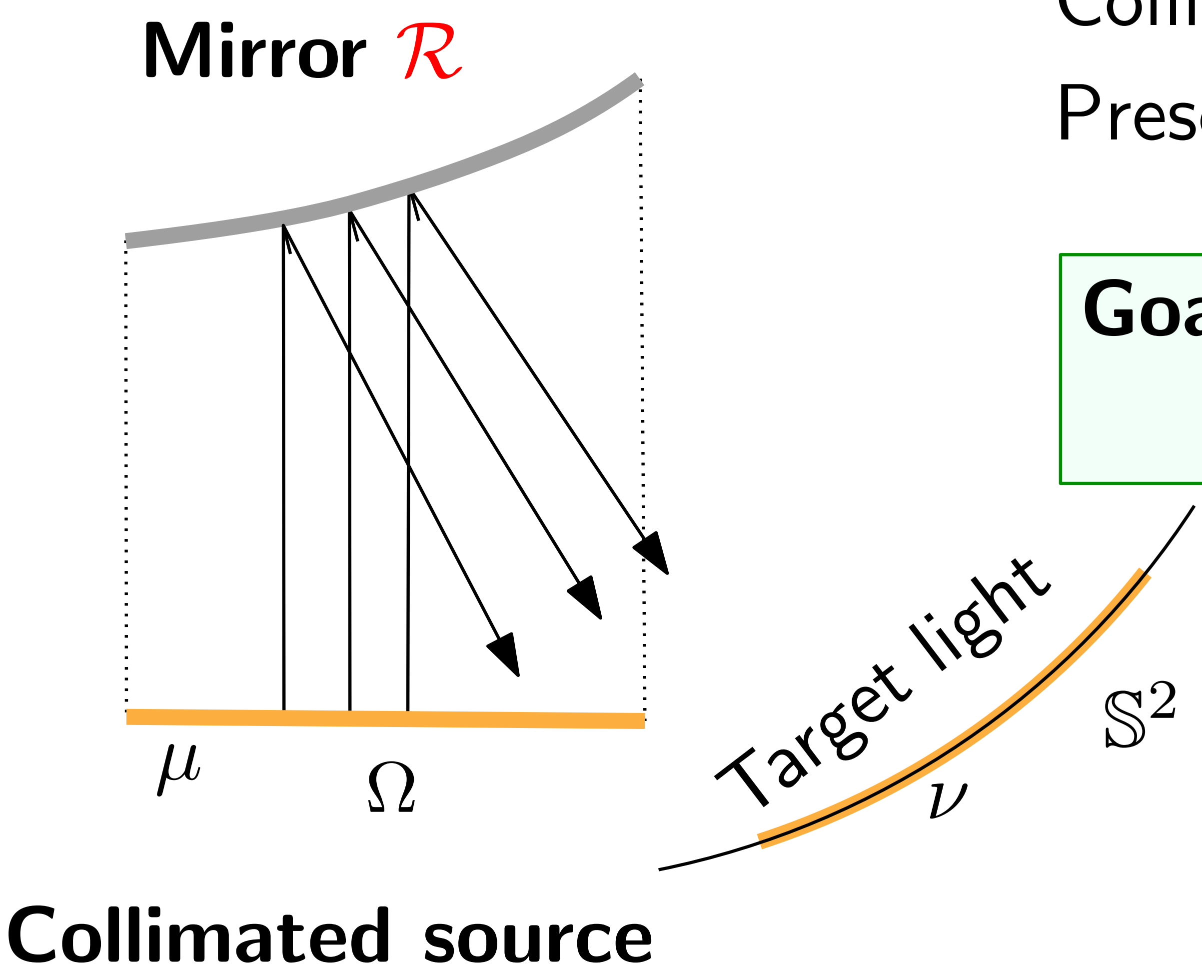
Brenier formulation $(F \circ \nabla u)_\sharp \mu = \nu$

$\Leftrightarrow \forall A \; \mu((F \circ \nabla u)^{-1}(A)) = \nu(A)$

$\Leftrightarrow \forall B \; \mu((\nabla u)^{-1}(B)) = \tilde{\nu}(B)$ with $B = F^{-1}(A) \subset \mathbb{R}^2$

Mirror / Collimated source light



Collimated light μ measure on $\Omega \subset \mathbb{R}^2 \times \{0\}$

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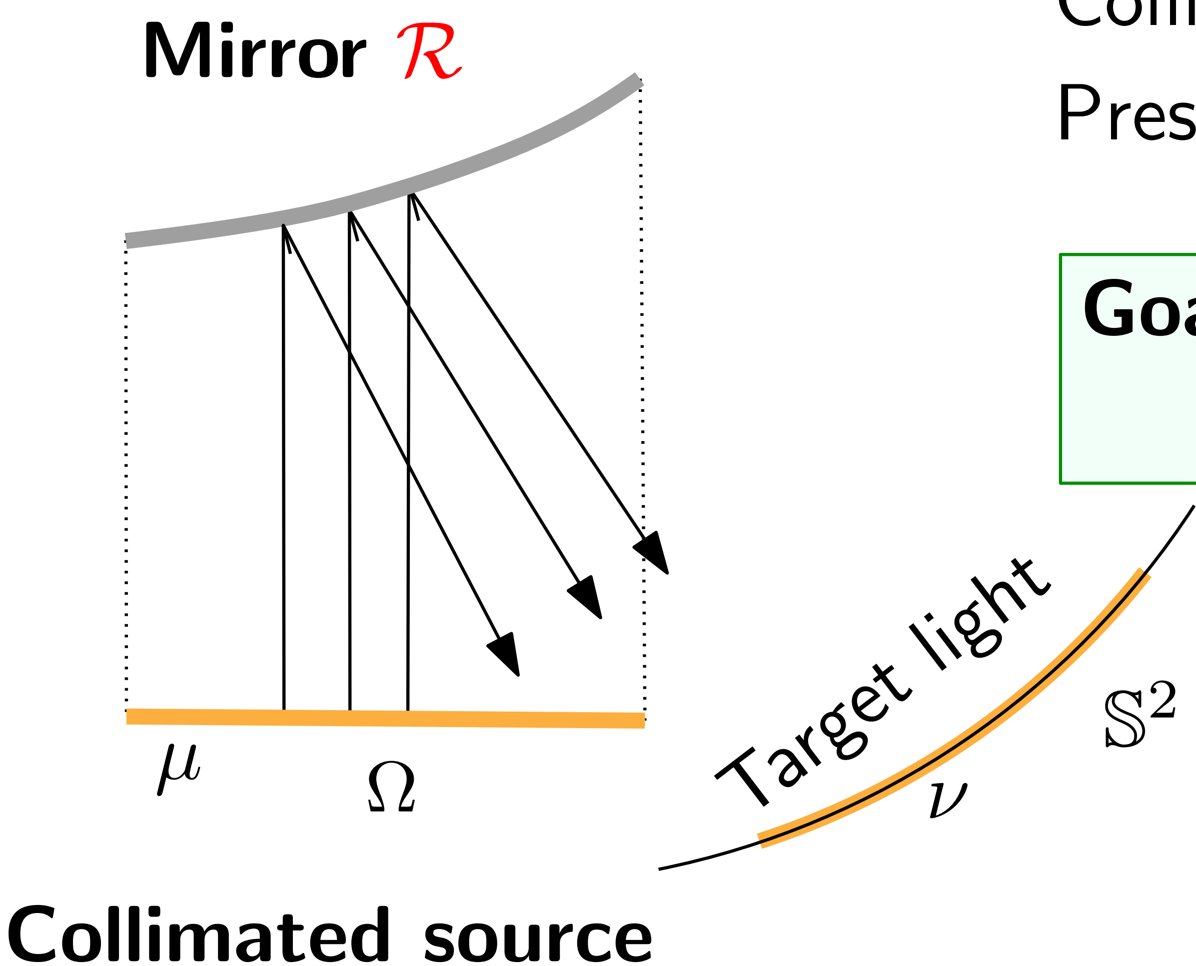
$\Leftrightarrow \det(\nabla^2 u(x)) g(\nabla u(x)) = f(x)$ if $\mu(x) = f(x)dx$ and $\tilde{\nu}(x) = g(x)dx$

Mirror / Collimated source light



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Monge-Ampère equation in $\mathbb{R}^2$

Find $u : \Omega \to \mathbb{R}^2$ such that $\det(\nabla^2 u(x)) g(\nabla u(x)) = f(x)$

Mirror / Collimated source light: semi-discrete

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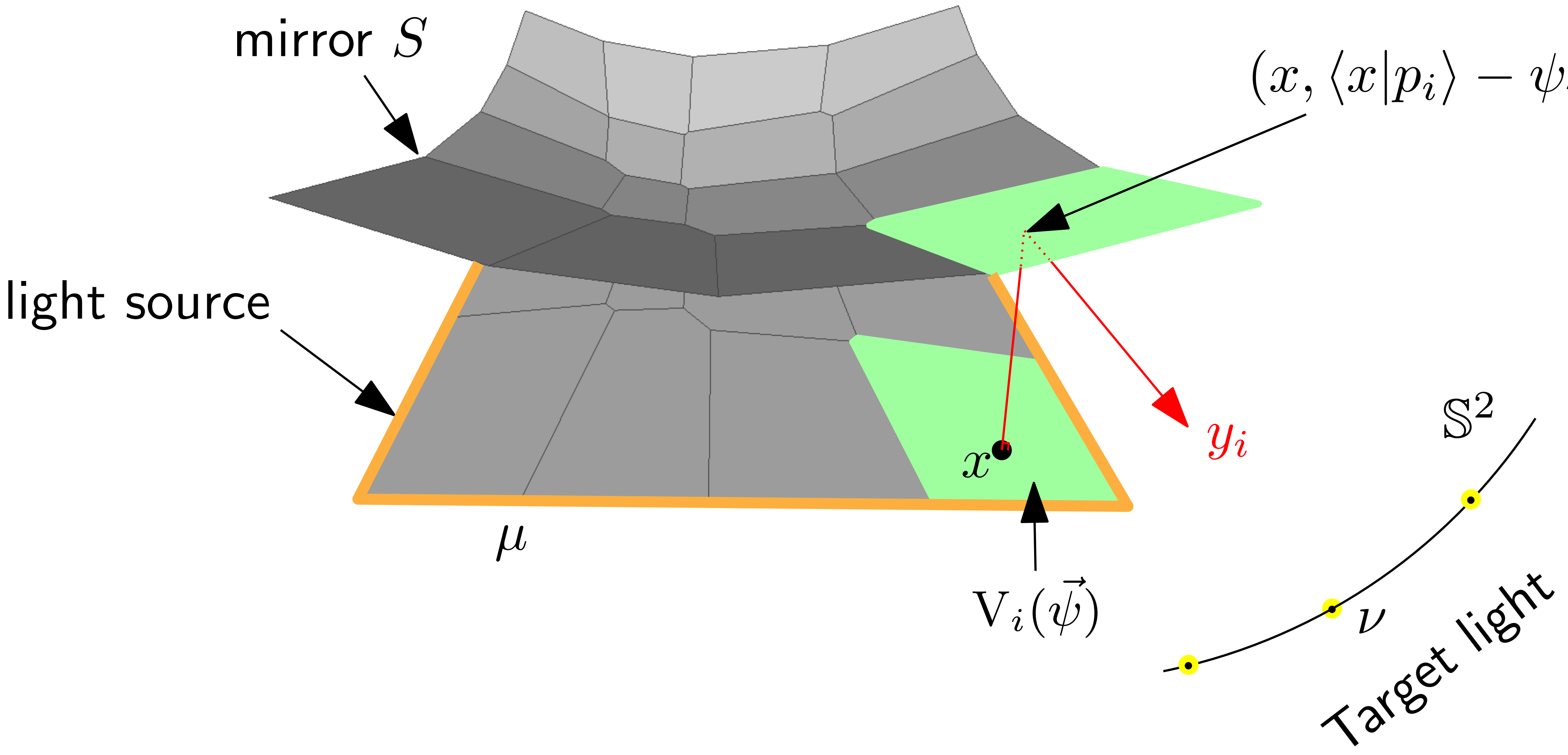
Prescribed far-field: $\nu = \sum_i \nu_i \delta_{y_i}$ on $\mathbb{S}^2$



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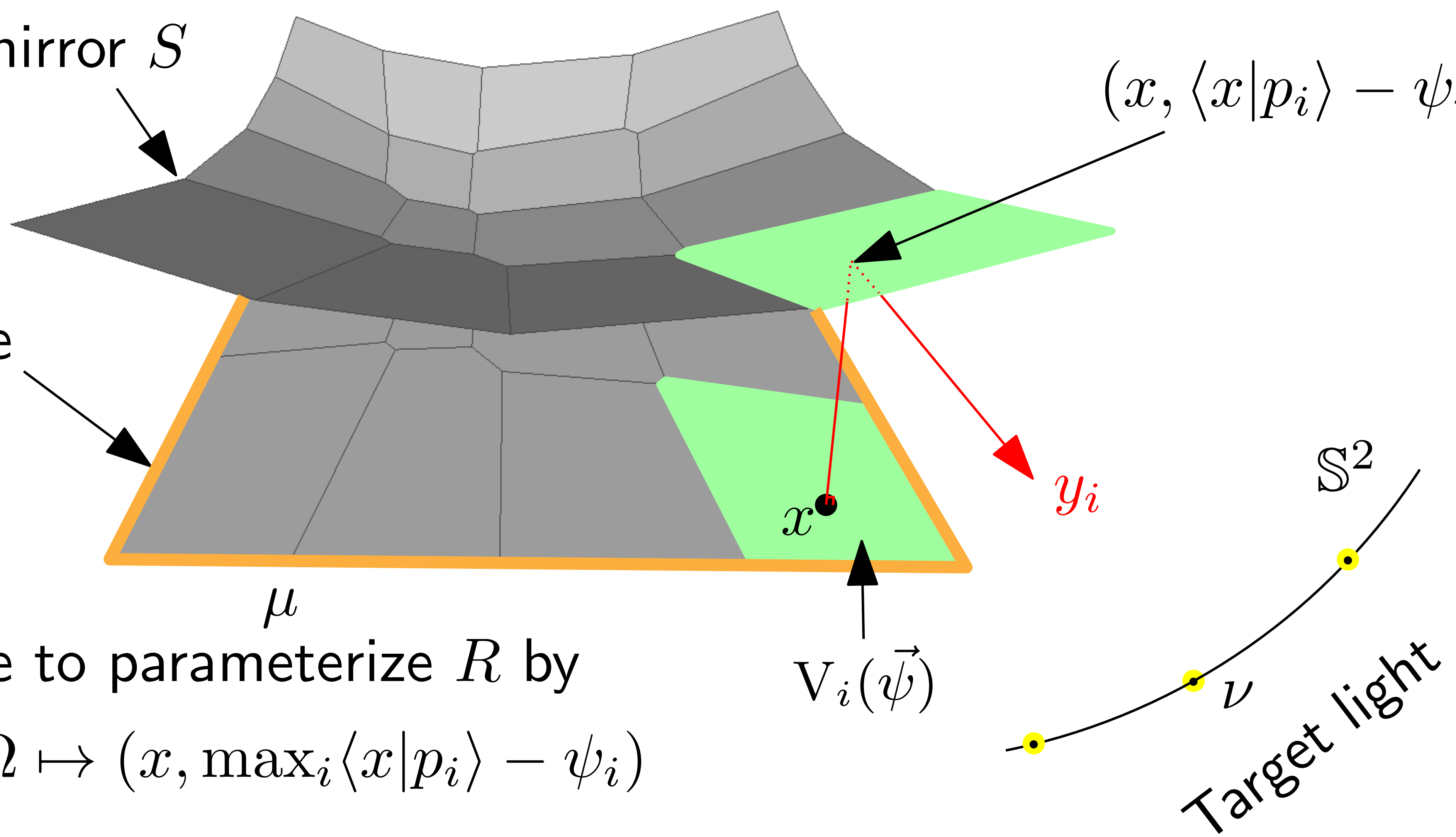

We choose to parameterize R by

$$x \in \Omega \mapsto (x, \max_i \langle x|p_i\rangle - \psi_i)$$

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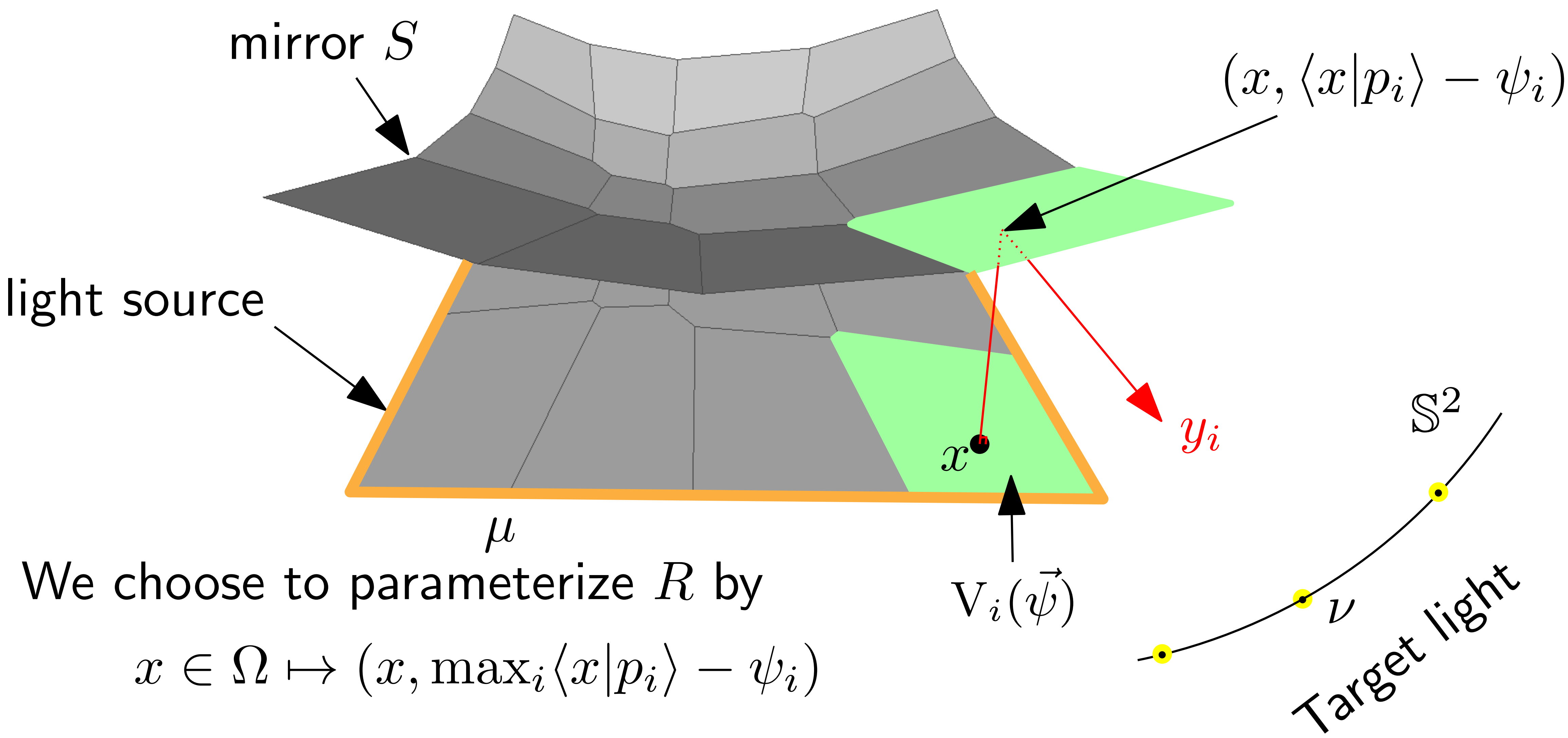

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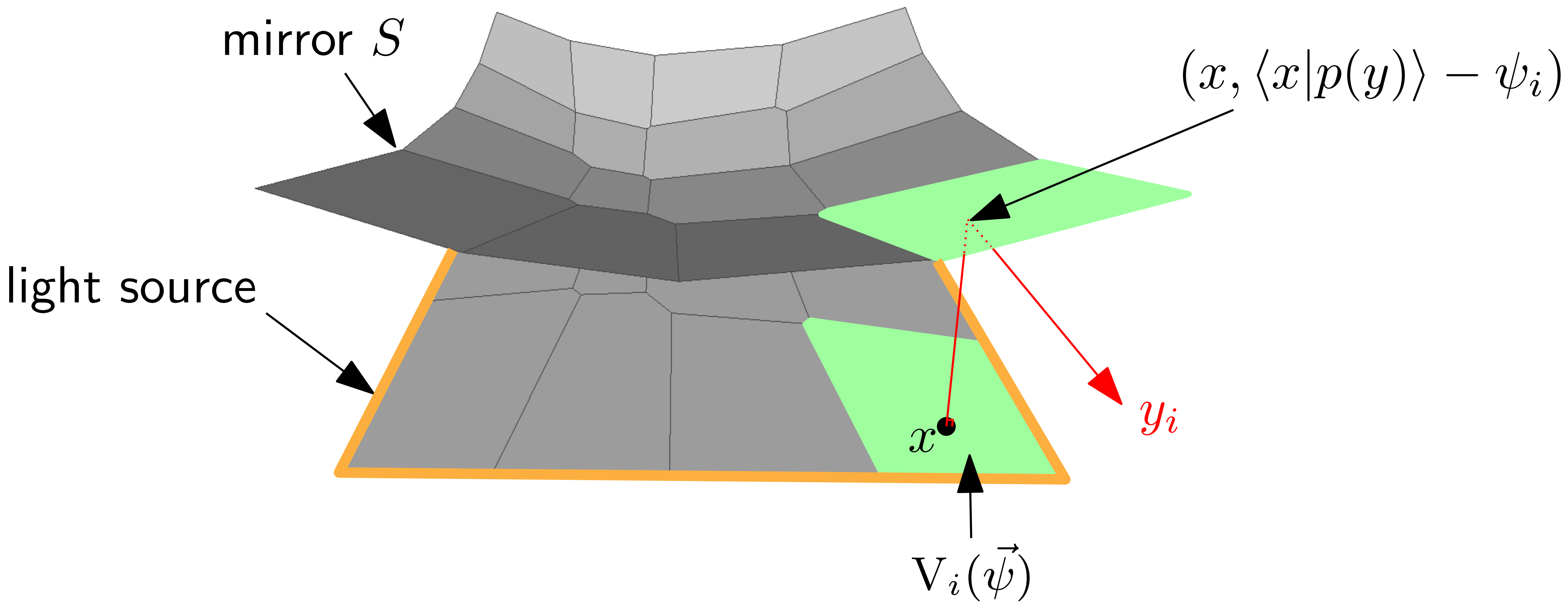
Problem (FF): Find $\psi_1, \ldots, \psi_N$ such that for every i, $\mu(V_i(\vec{\psi})) = \nu_i$.

amount of light reflected in direction y_i.

Mirror / Collimated source: Optimal Transport

Lemma: With $c(x,y) = -\langle x|y\rangle$

$$\mathrm{V}_i(\vec{\psi}) = \{x \in \mathbb{R}^2,\ c(x,y_i) + \psi_i \leq c(x,y_j) + \psi_j \quad \forall j\}.$$



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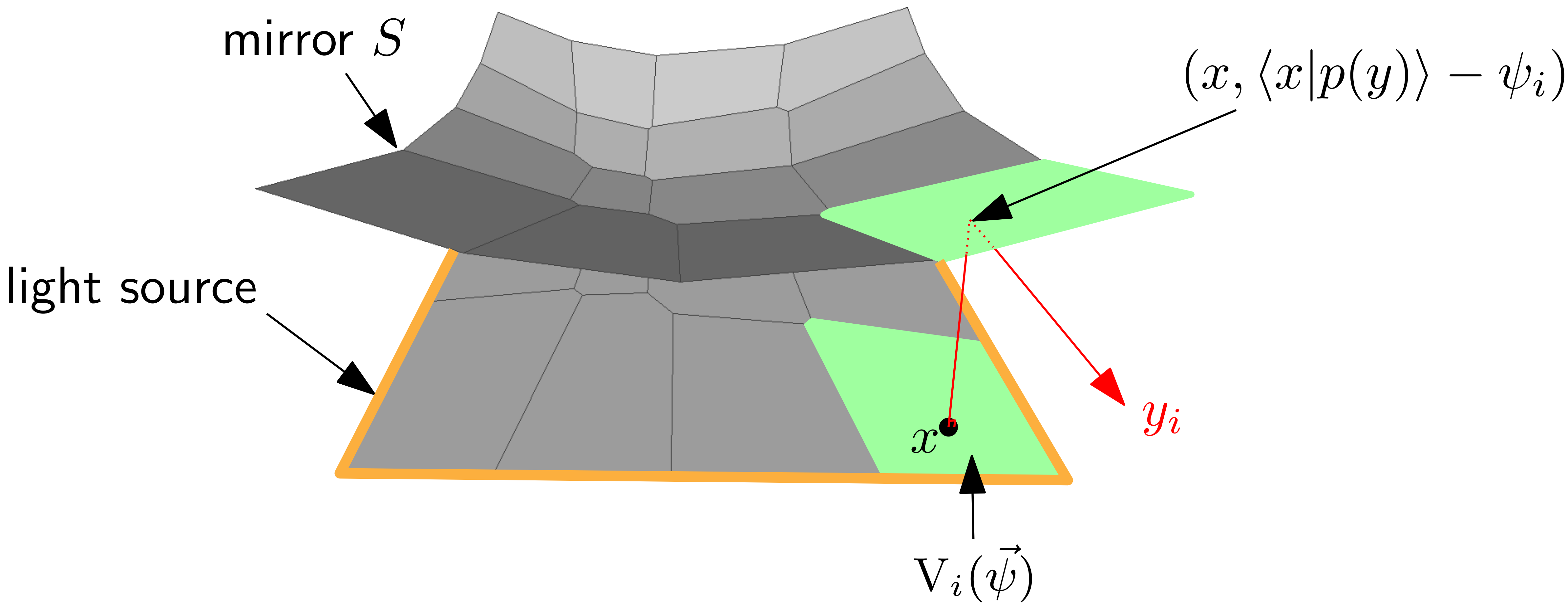

$\rightsquigarrow$ Optimal transport problem in $\mathbb{R}^2$

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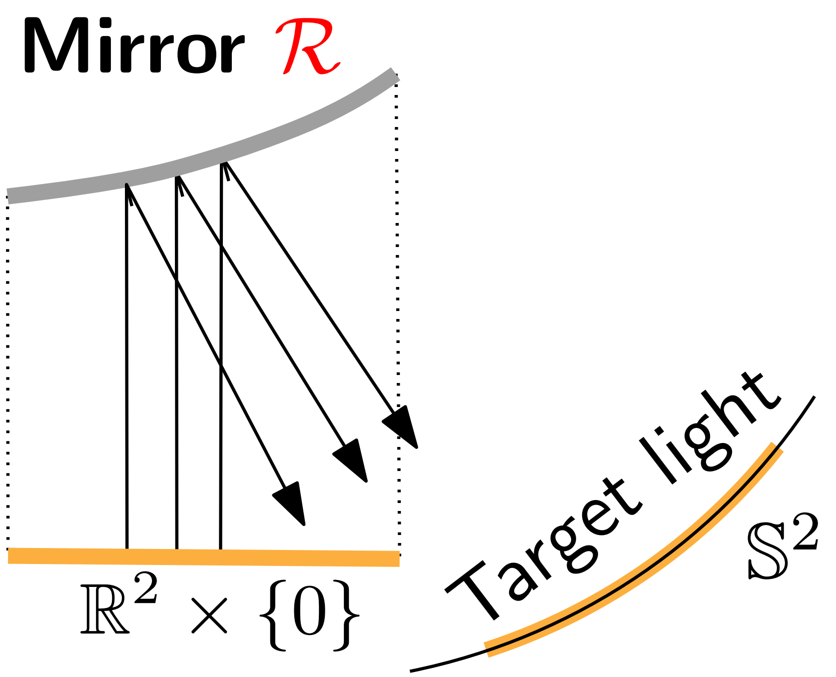
Outline

- Case 1: mirror for point light source
- Case 2: mirror for collimated source light
- Case 3: other cases

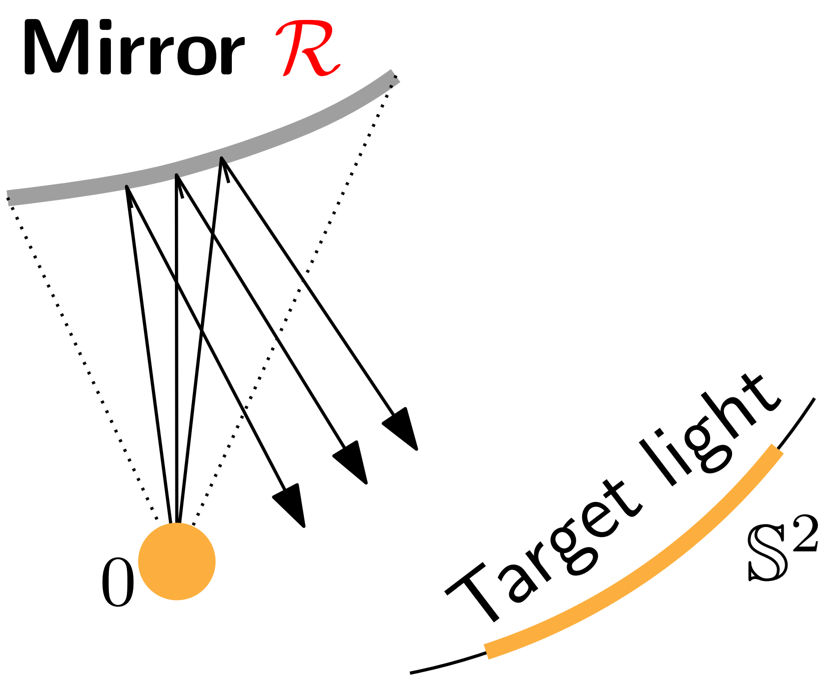
- Semi-discrete optimal transport
- Damped Newton algorithm

- Non-imaging optics: Far-Field target
- Non-imaging optics: Near-Field target

Four inverse problems

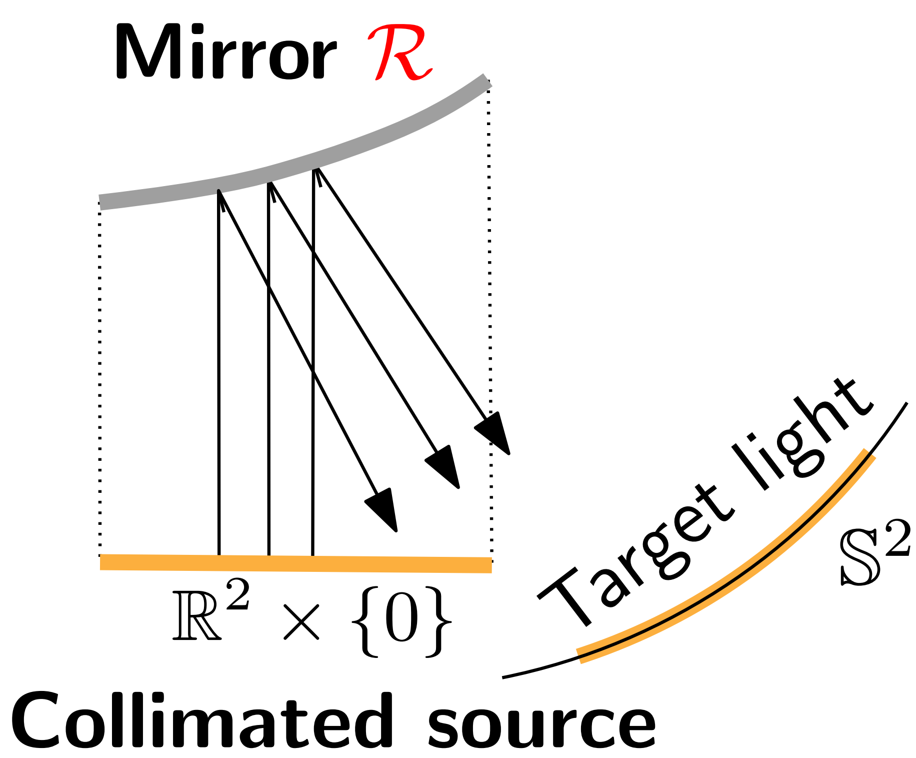


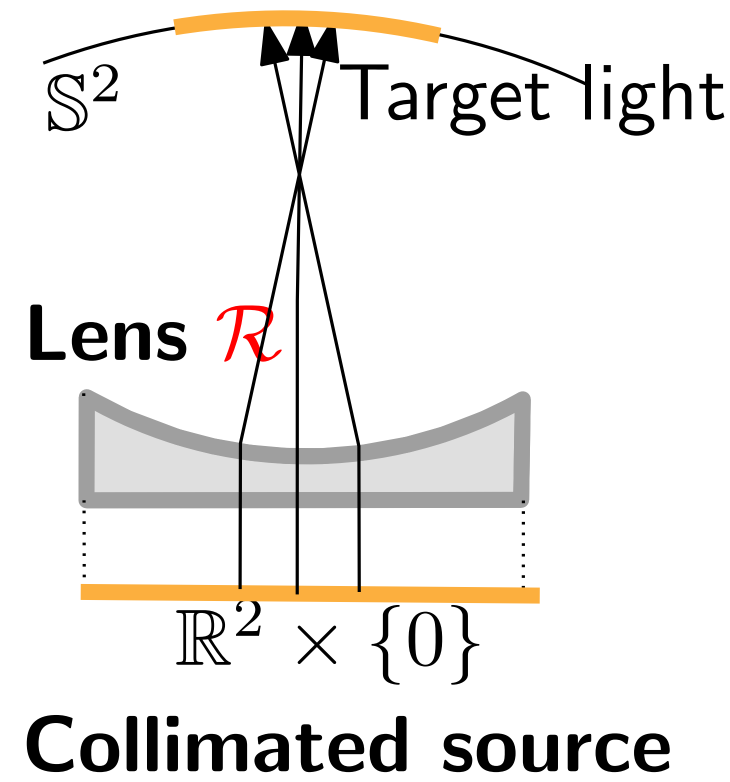
Collimated source

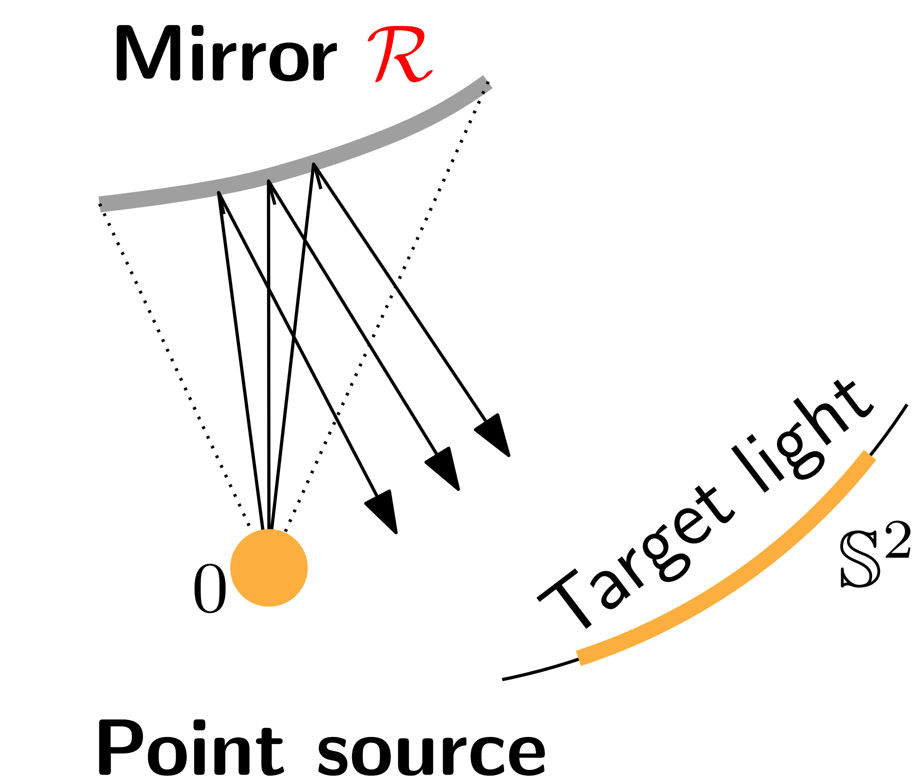


Point source

Four inverse problems







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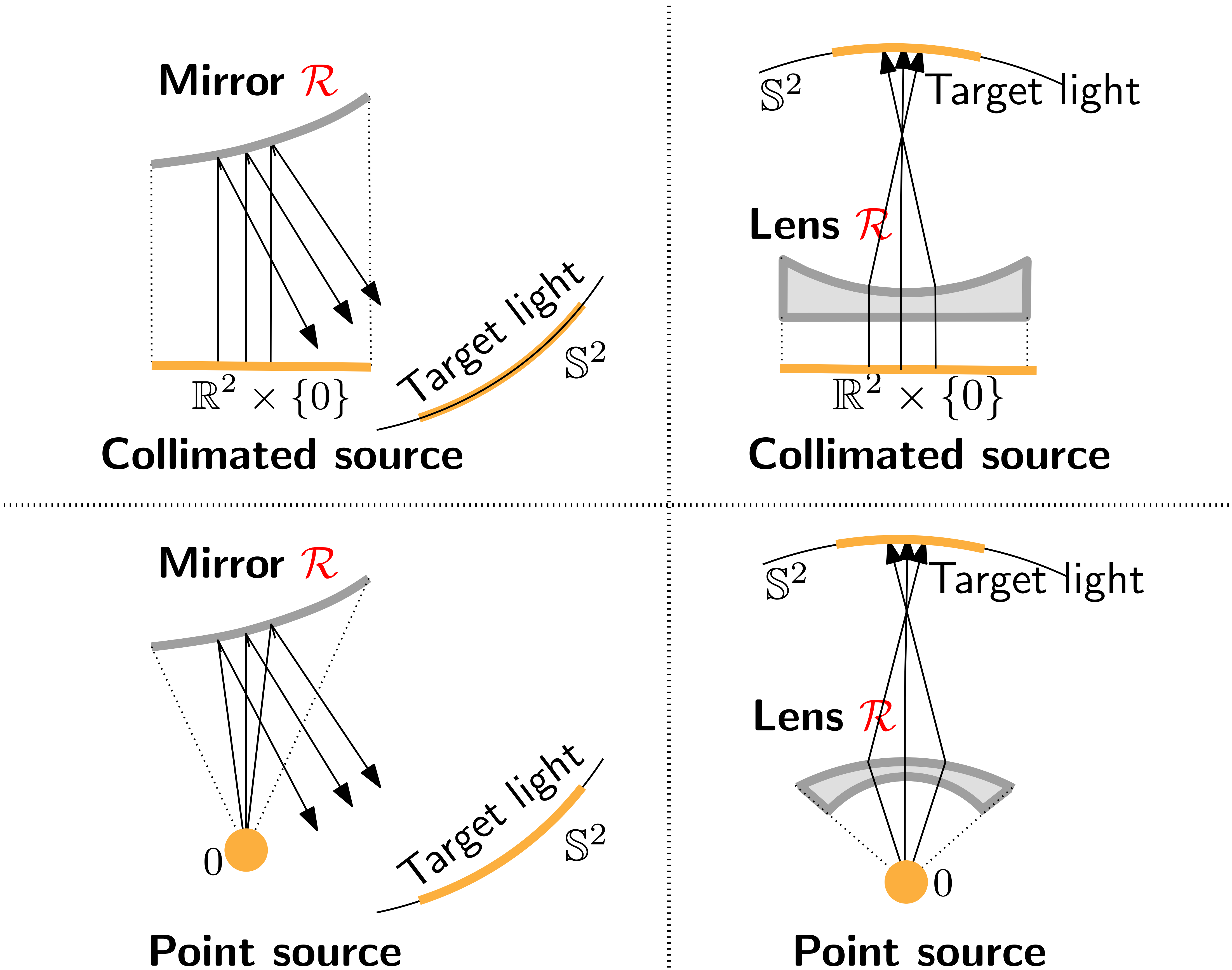

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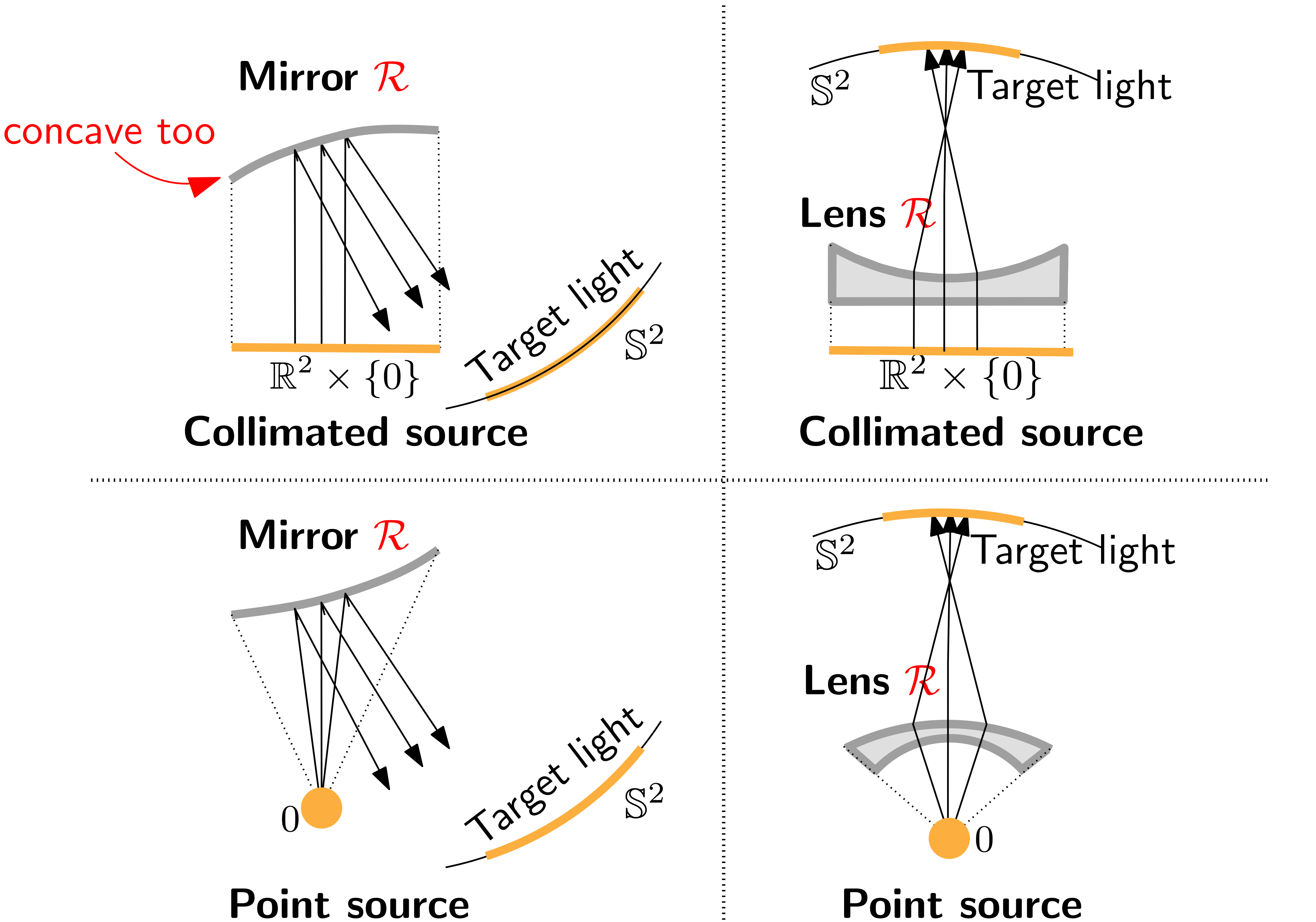
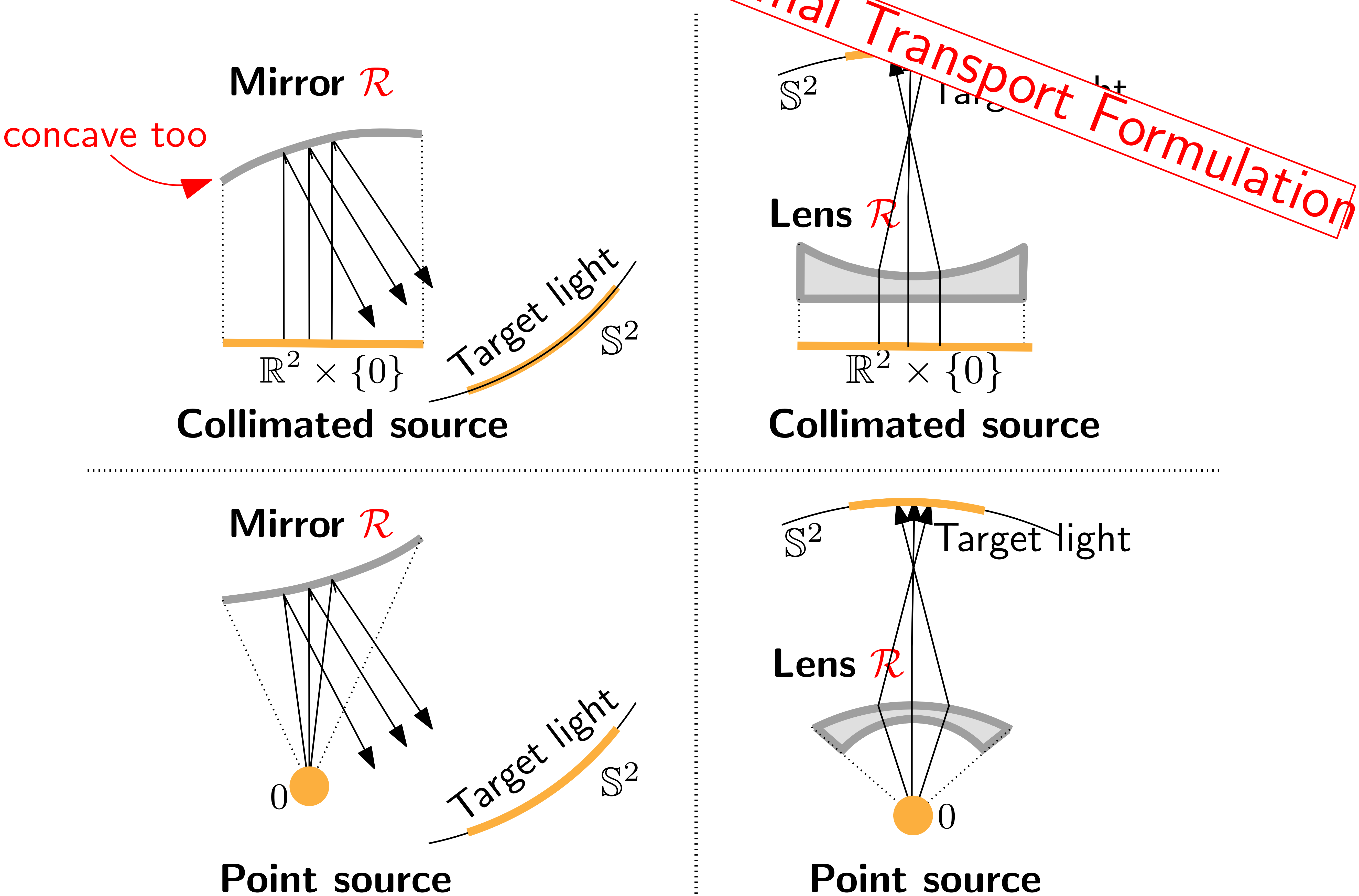


Four inverse problems

Optimal Transport Formulation



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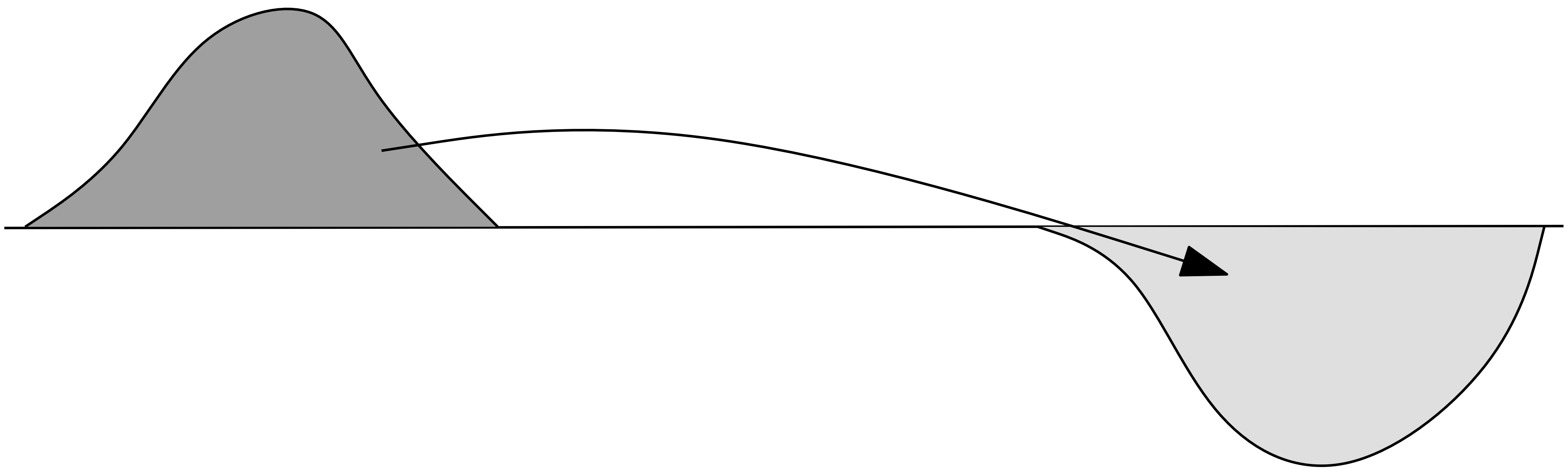
Monge problem (1781)

How to optimally move sand ?



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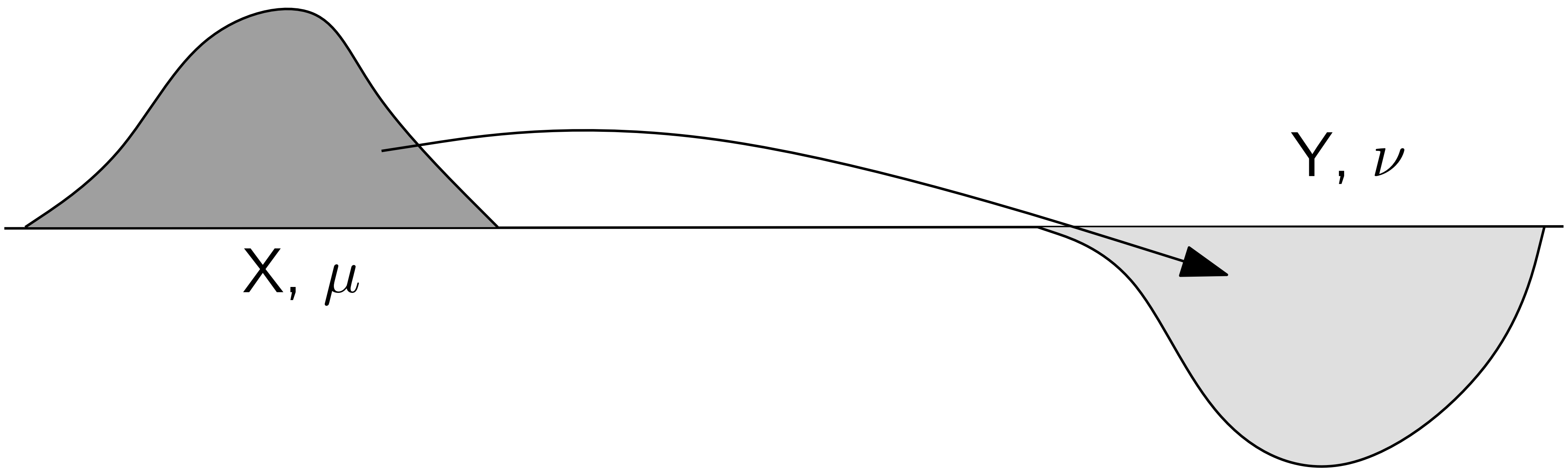

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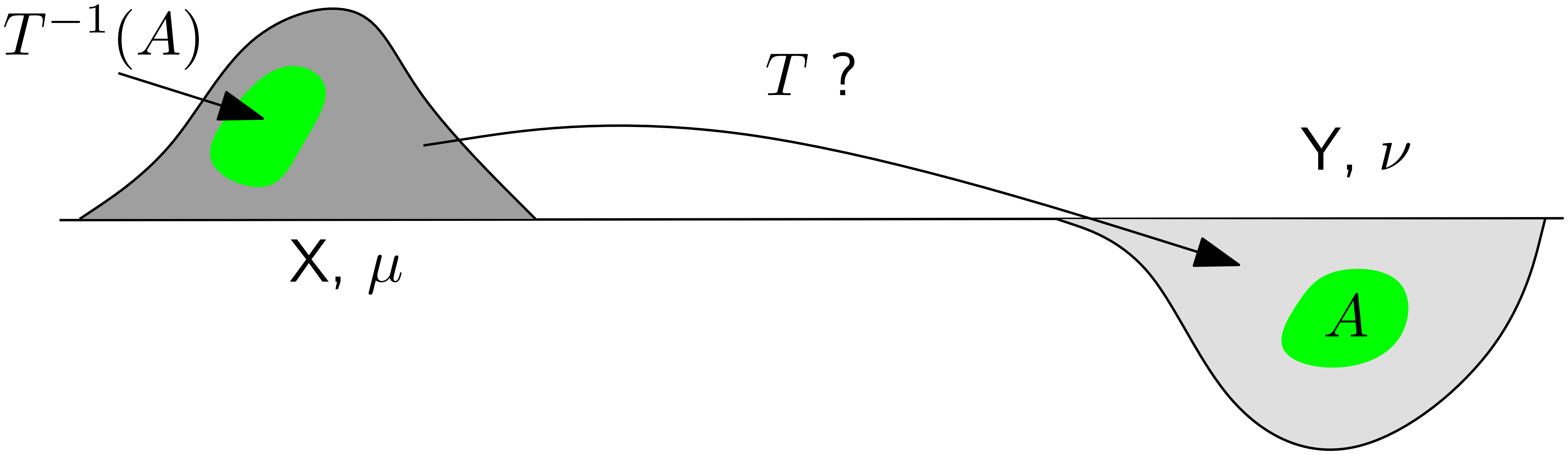

Let $c : X \times Y \to \mathbb{R}$ be a cost function   e.g. $c(x, y) = \|x - y\|^2$

Monge problem. Find a map $T : X \to Y$ such that

- T preserves the mass,   i.e. $\nu(A) = \mu(T^{-1}(A))$
- T minimizes the total cost

$$\min \int_X c(x, T(x)) d\mu(x)$$

The minimizer does not always exist; Constraint not linear

Monge problem (1781)

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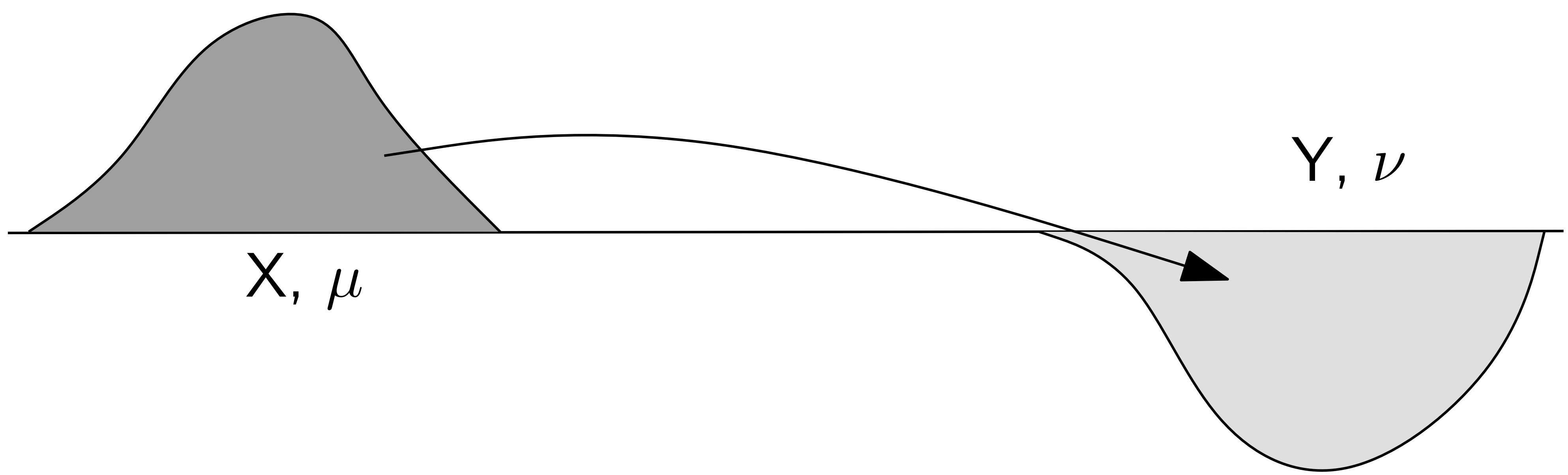

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Kantorovich relaxation – 1940's

Minimise $\int c(x,y)d\pi(x,y)$

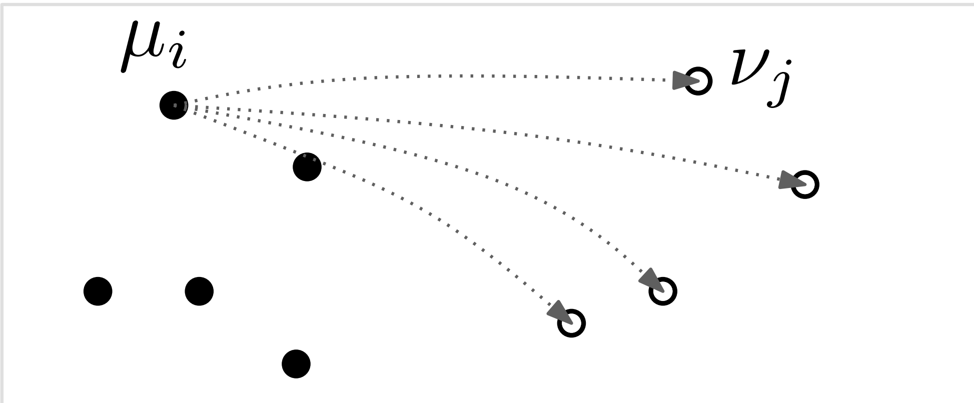
where π is a transport plan, i.e

π is a probability measure on $X \times Y$

$\pi(A \times Y) = \mu(A)$

$\pi(X \times B) = \nu(B)$

Numerical optimal transport



Discrete source and target

linear programming

Bertsekas' auction algorithm

Sinkhorn/IPFP

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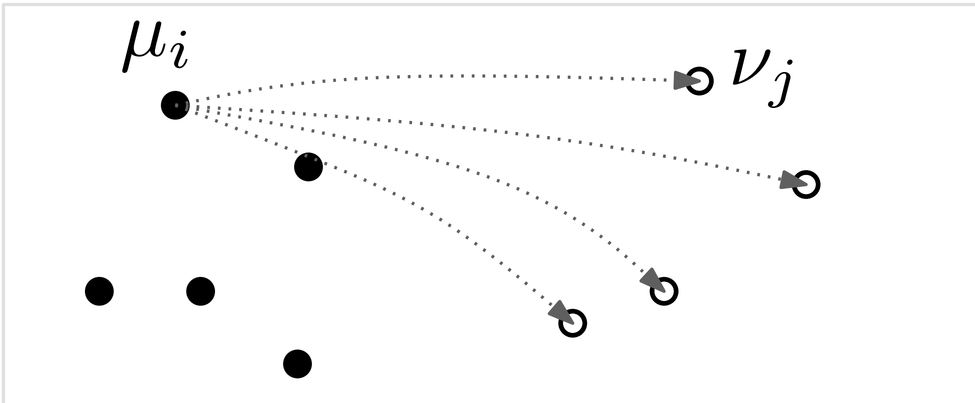

Discrete source and target

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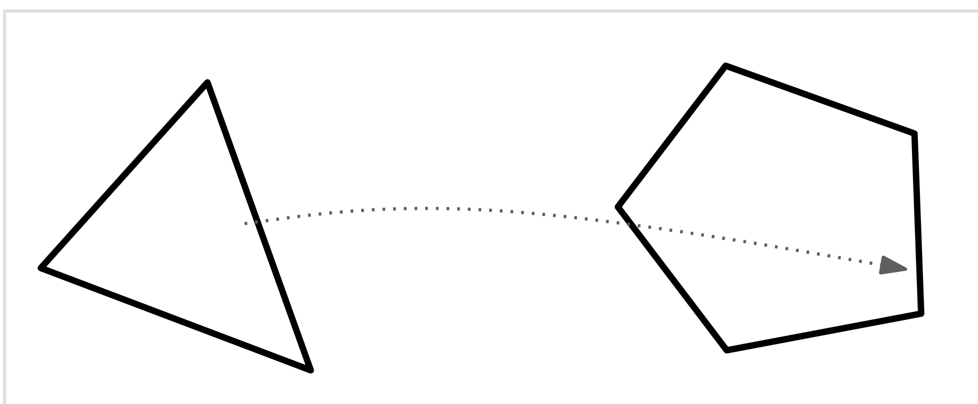

Source and target with density (PDE):

- Benamou-Brenier formulation
- Stencil methods for Monge Ampère equations

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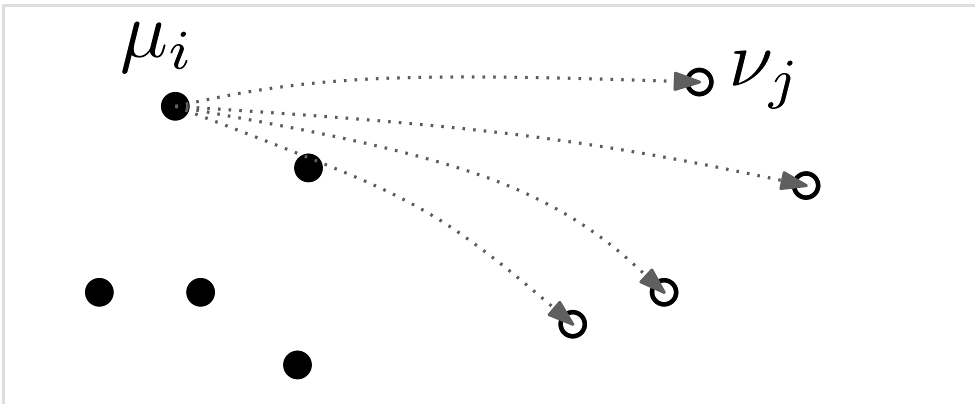

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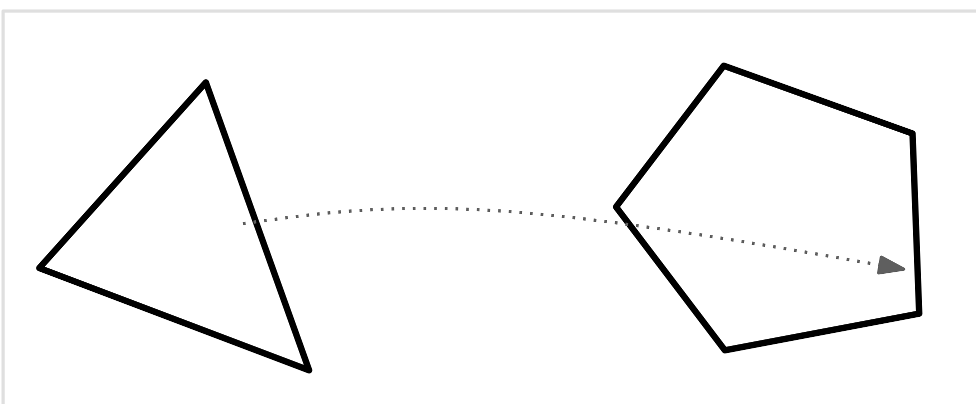

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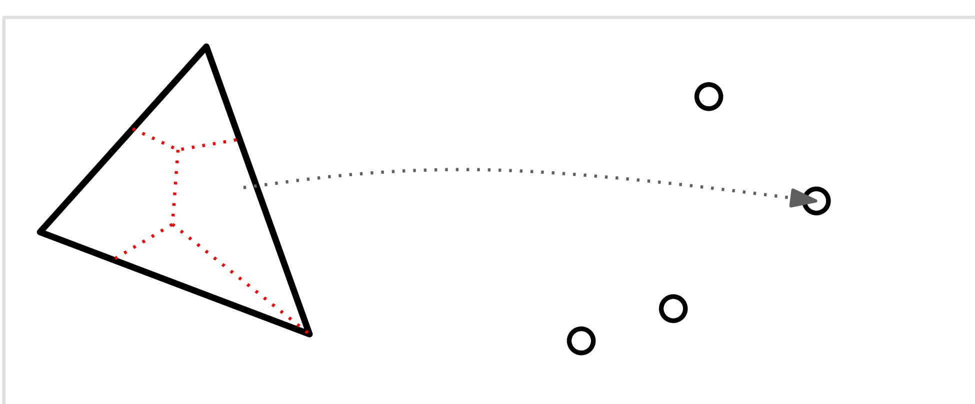


Source with density, discrete target:

Coordinate-wise increment
Oliker-Prussner '89 Caffarelli-Kochengin-Oliker '97
Kitagawa '12

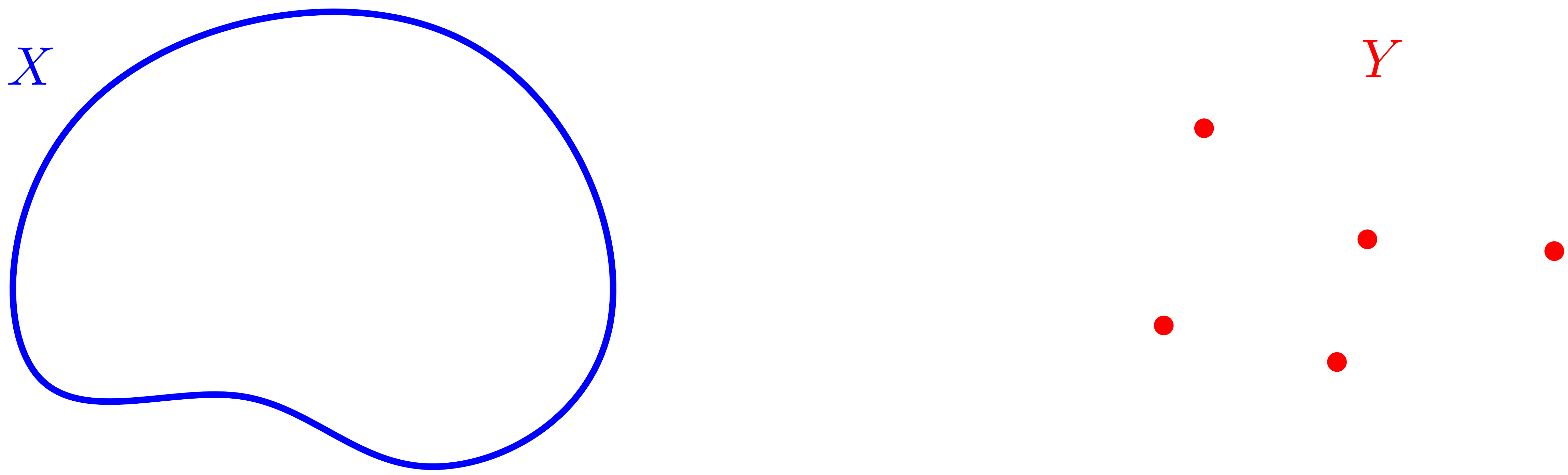
Newton and quasi-Newton methods
Aurenhammer, Hoffmann, Aronov '98
Mérigot '11, Levy'15, Kitagawa-Mérigot-T.'17, etc.

Semi-discrete optimal transport

$\mu(x) = \rho(x)\mathrm{d}x$ probability measure on X

$\nu = \sum_i \nu_i \delta_{y_i}$ prob. measure on finite $Y = \{y_1, \cdots, y_N\}$

$c : X \times Y \to \mathbb{R}$ cost function



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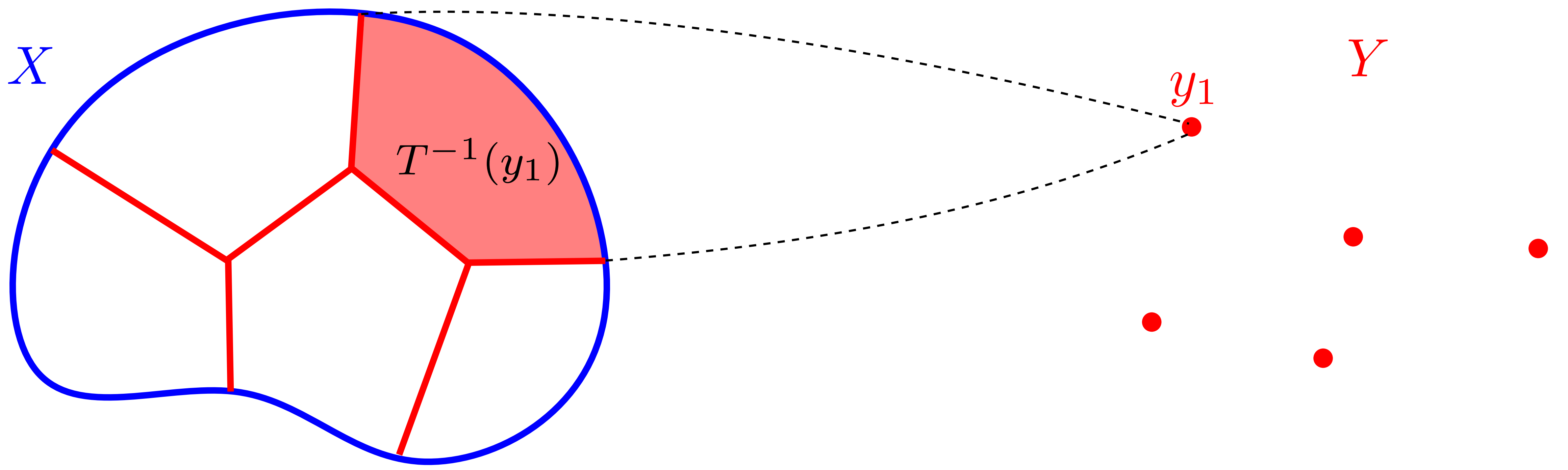

Transport map: $T : X \to Y$ s.t. $\forall i,\ \mu(T^{-1}(\{y_i\})) = \nu_i$ (i.e. $T_\# \mu = \nu$)

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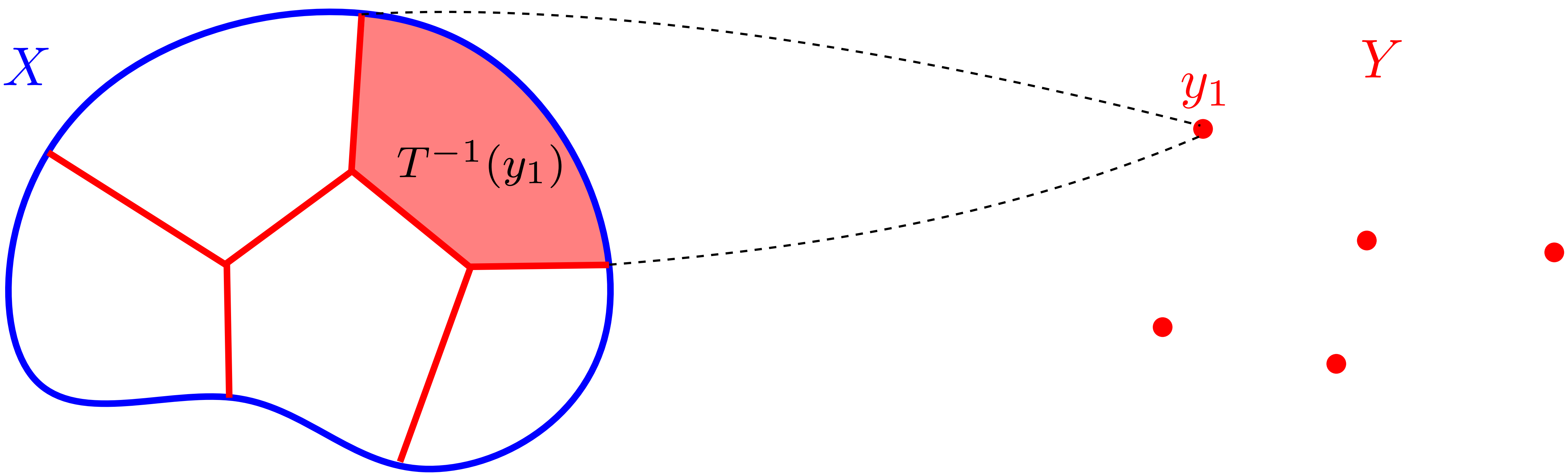

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Monge problem: Find a transport map $T : X \to Y$ that minimizes

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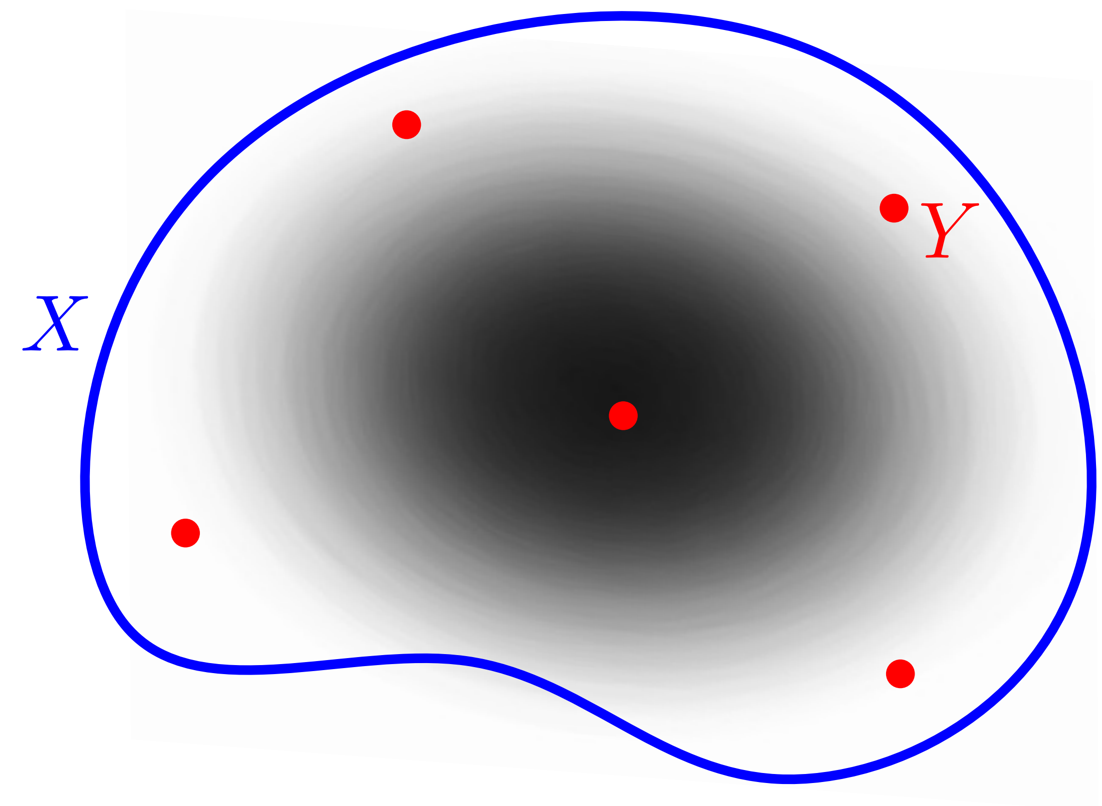
Semi-discrete optimal transport

$\rho : X \to \mathbb{R}$ density of population

Y = location of bakeries

$c(x, y_i) := \|x - y_i\|^2$



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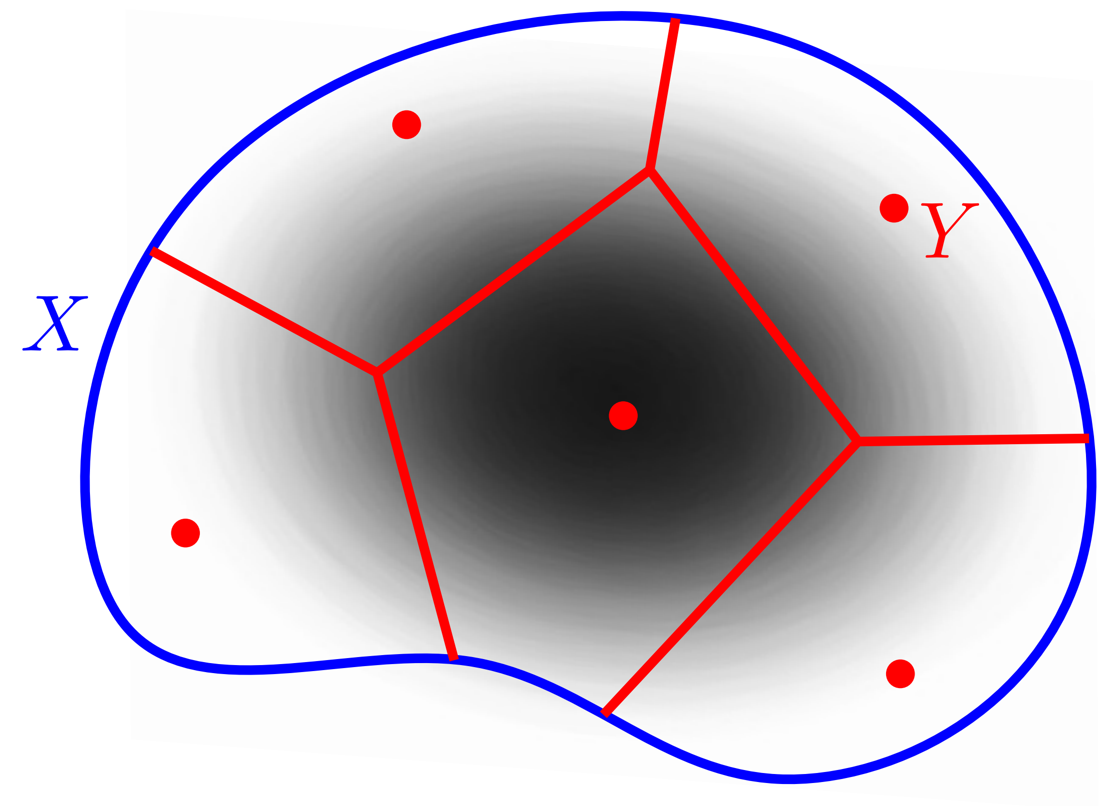

- If the price of bread is uniform, people go the closest bakery:

$$\mathrm{Vor}(y_i) = \{x \in X; \forall j,\ c(x, y_i) \leq c(x, y_j)\}$$

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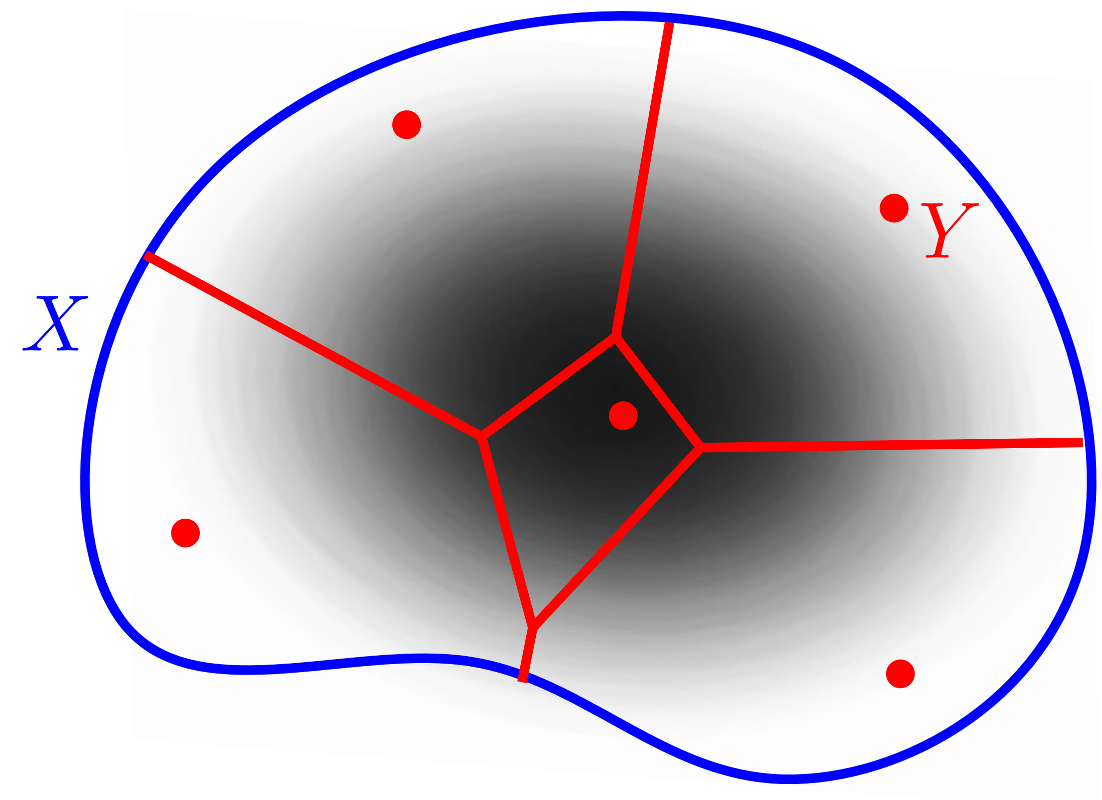

- If prices are given by $\psi_1, \cdots, \psi_N$, people make a compromise:

$$\mathrm{Lag}_i(\psi) = \{x \in X; \forall j,\ c(x, y_i) + \psi_i \leq c(x, y_j) + \psi_j\}$$

Semi-discrete optimal transport

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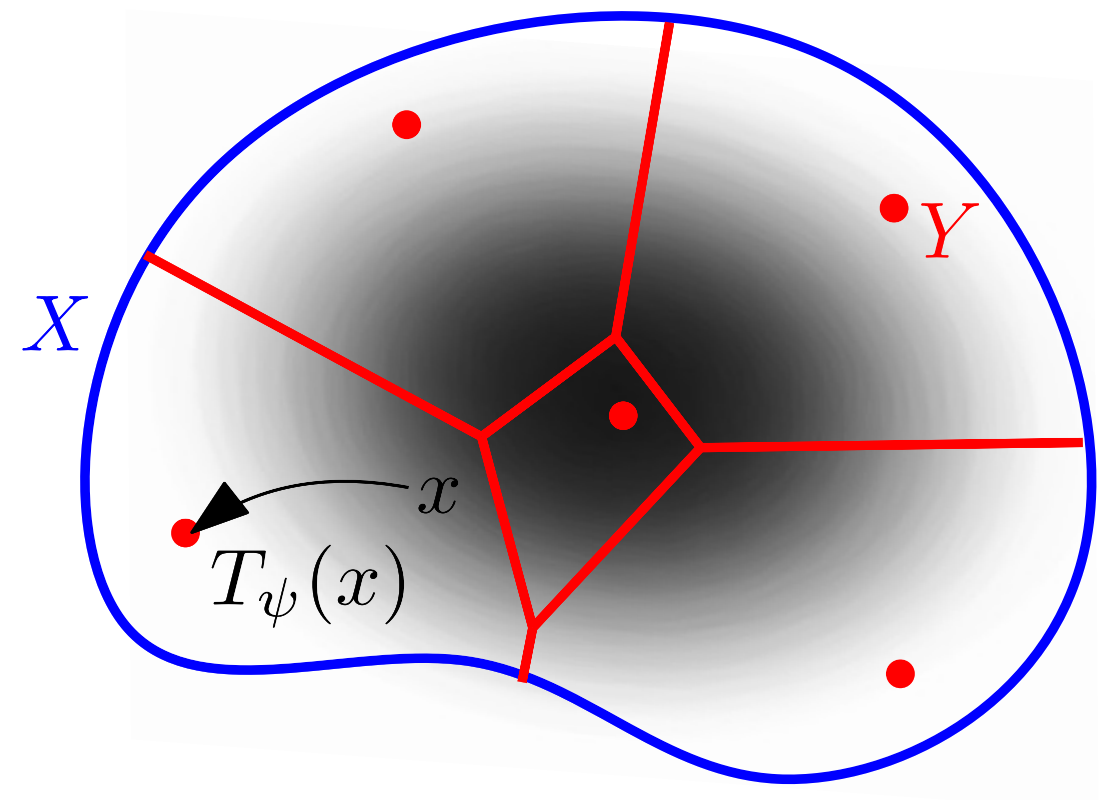

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- We define the function "number of people"

$$\begin{array}{rccc} H: & \mathbb{R}^n & \to & \mathbb{R}^n \\ & (\psi_i)_{1 \leq i \leq n} & \mapsto & (\int_{\mathrm{Lag}_i} \rho(x) dx)_{1 \leq i \leq n} \end{array}$$

Semi-discrete optimal transport

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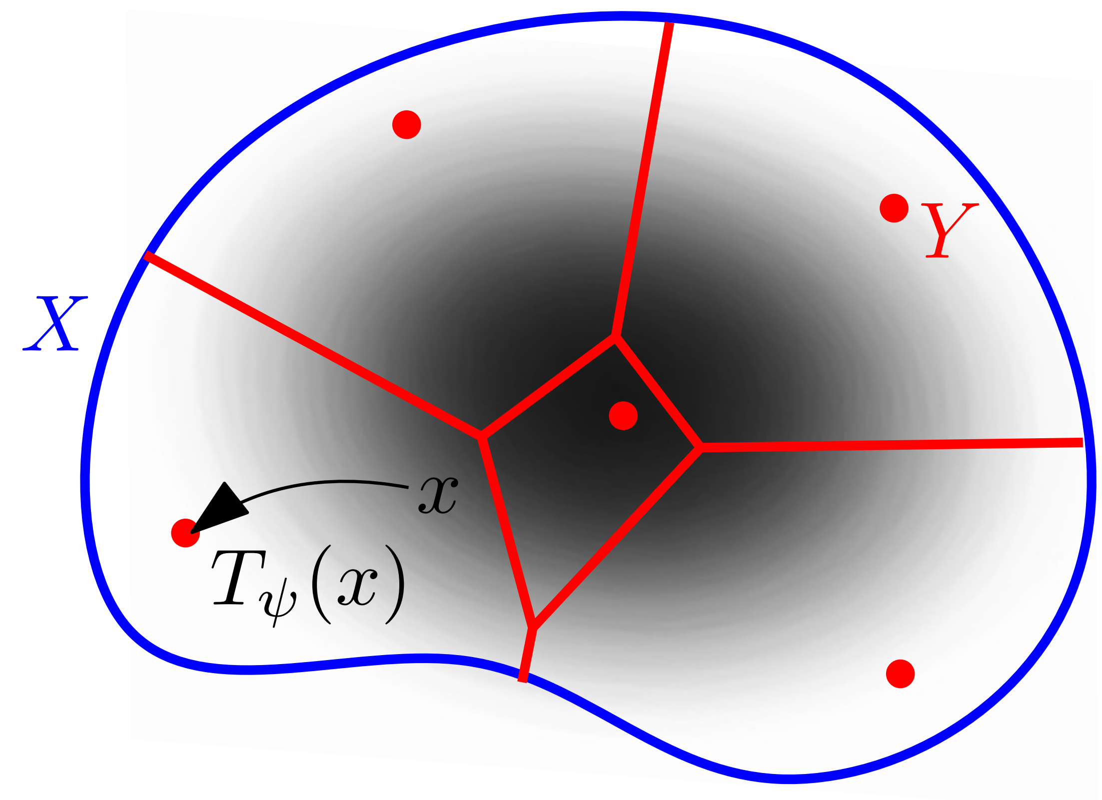

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Trouver $\Psi \in \mathbb{R}^n$ tel que $H(\Psi) = \nu$

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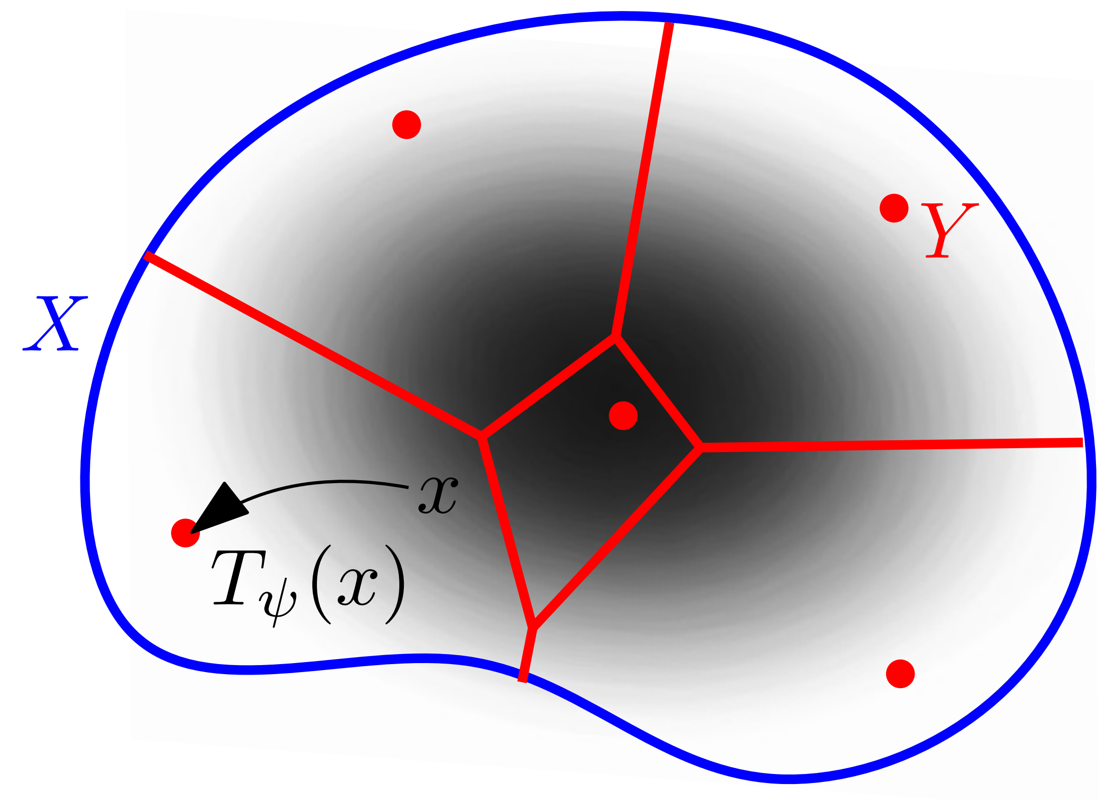

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T is a transport map

Concave optimisation problem

Theorem (variational formulation)

$$H = \nabla \Phi$$

where $\Phi : \mathbb{R}^n \to \mathbb{R}$ is concave

Concave optimisation problem

Theorem (variational formulation)

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Kantorovitch duality

Concave optimisation problem

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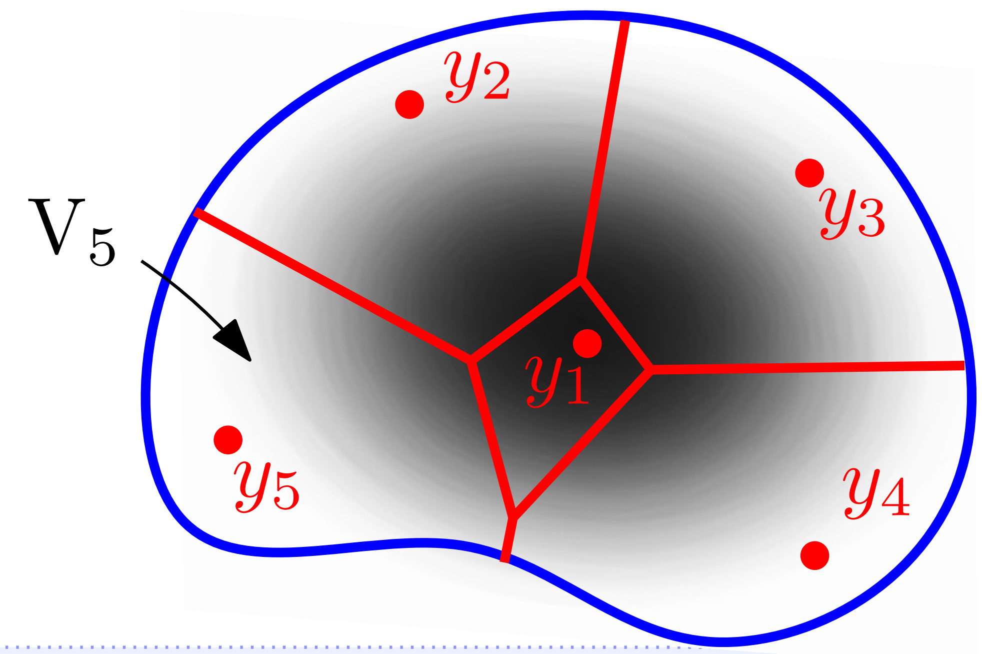
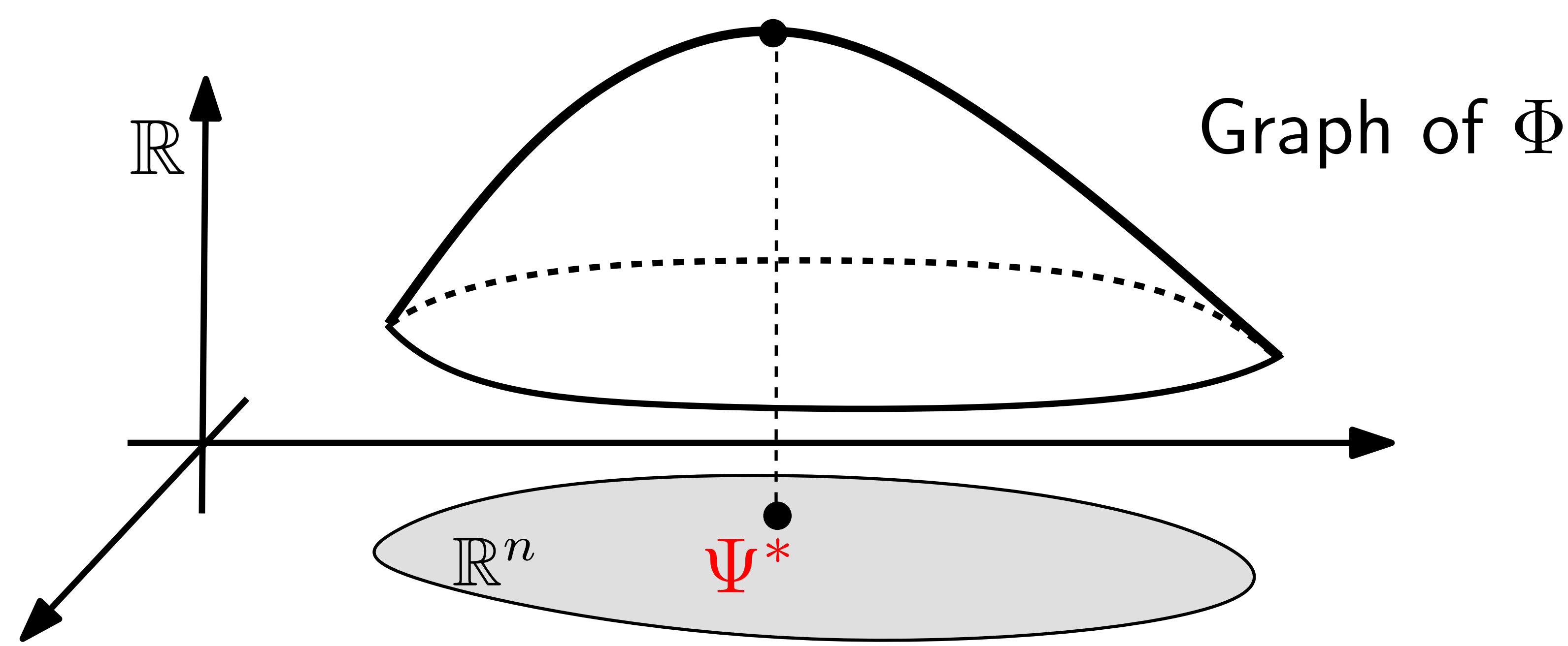


Corollary. Let $\Psi^* = (\psi_1, \cdots, \psi_n)$ prices of breads in $y_1, \cdots, y_n$

Everyone living in V_i has bread in y_i (and all the bread is sold)

$\iff H(\Psi^*) = \nu$ (Monge-Ampère equation)

$\iff \Psi^*$ is a maximum of Φ



Concave optimisation problem

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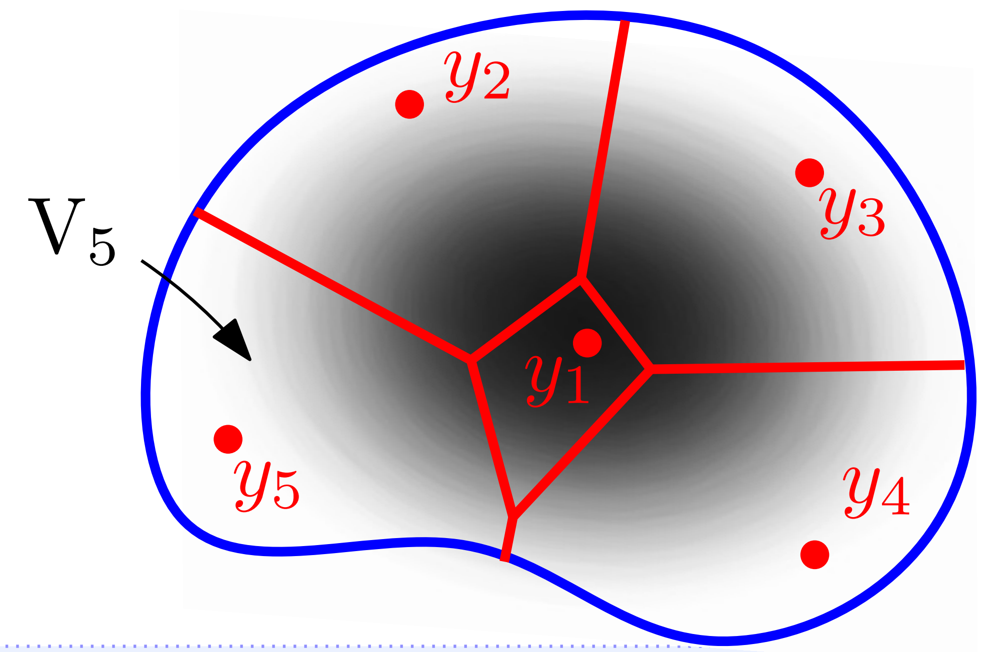

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Algorithms.

- Oliker Prussner: coordinate-wise increment with minimum step, with complexity $O(\frac{N^3}{\varepsilon}\log(N))$, $\varepsilon =$ precision.
- Quasi Newton methods for $c(x, y) = \|x - y\|^2$ on $\mathbb{R}^2/\mathbb{R}^3$ $\mathbb{S}^2$ No analysis

  [Mérigot. '11] [Lévy '14] [de Goes et al '12] [Machado, Mérigot, Thibert '16]
- Newton method in $\mathbb{R}^2$, $\mathbb{R}^3$, when μ supported on a triangulation.

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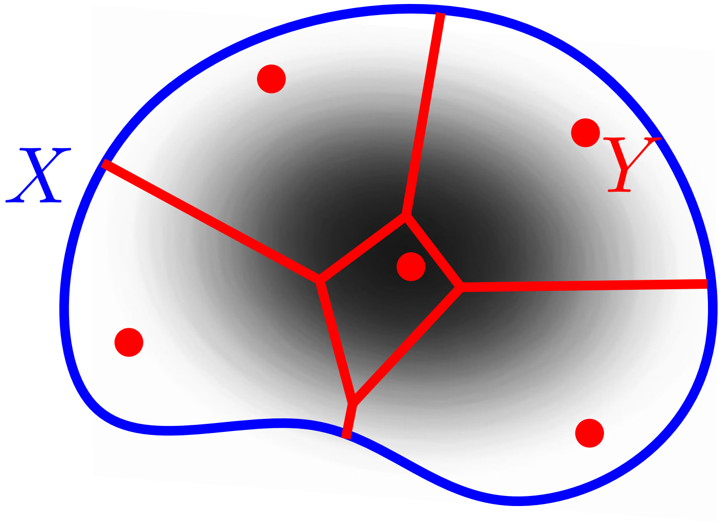
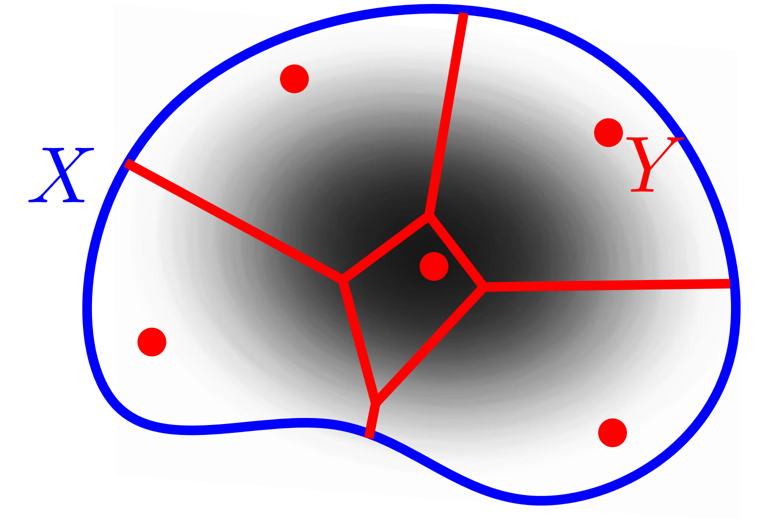
- Non-imaging optics: Near-Field target

Newton Algorithm



Newton Algorithm

Equation $H(\psi) = \nu$



where $H : \mathbb{R}^N \to \mathbb{R}^N$ by $H(\psi) = (\rho(\mathrm{Lag}_i(\psi)))_{1 \leq i \leq N}$

Remark: H is invariant by addition of a vector $\lambda(1, \cdots, 1)$.

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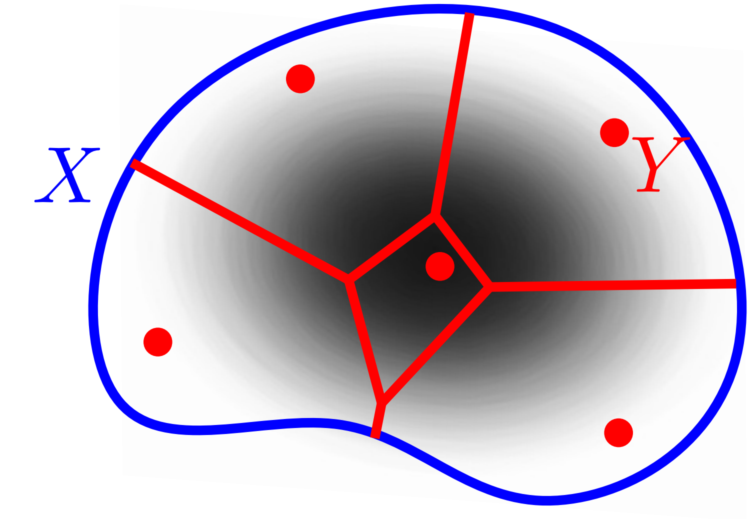

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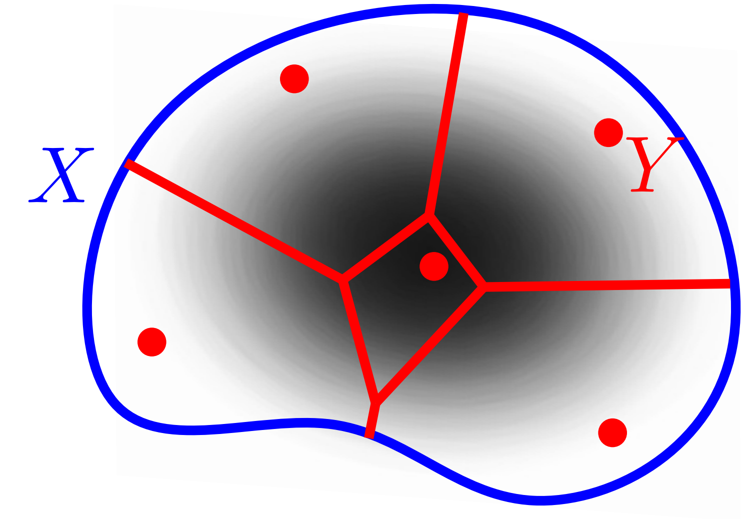

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Local convergence : if ψ^0 is close to a solution ψ^*, then it converges.

Damped Newton Algorithm [Kitagawa, Mérigot, Thibert]

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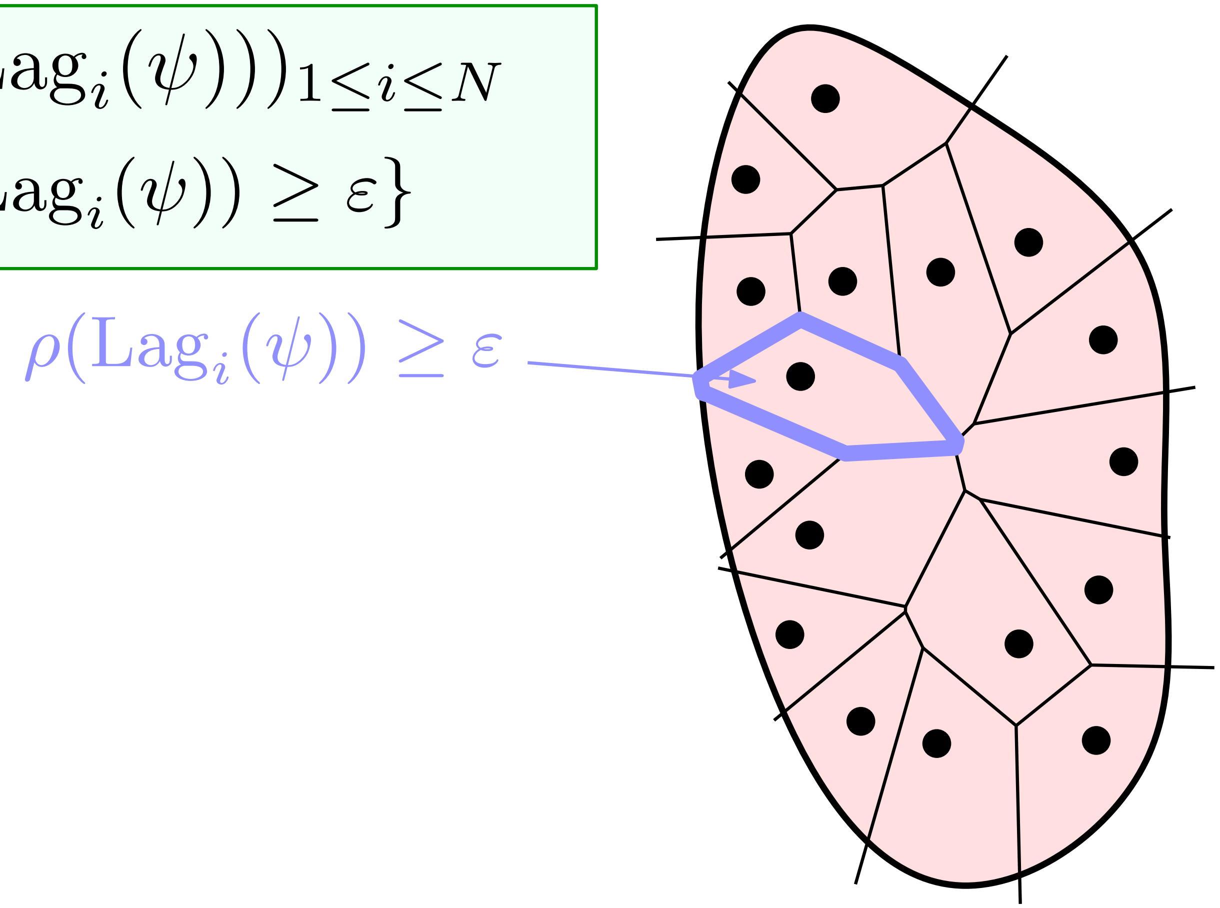

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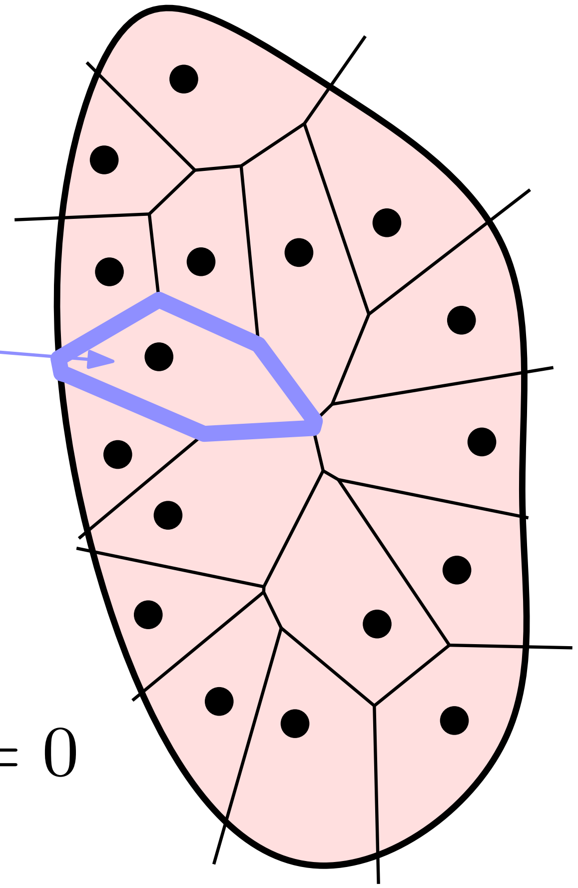

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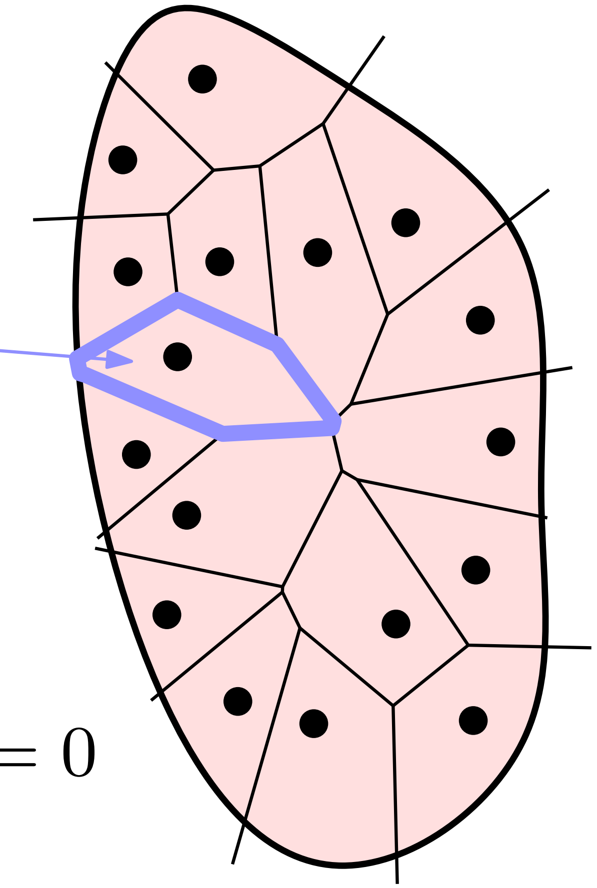

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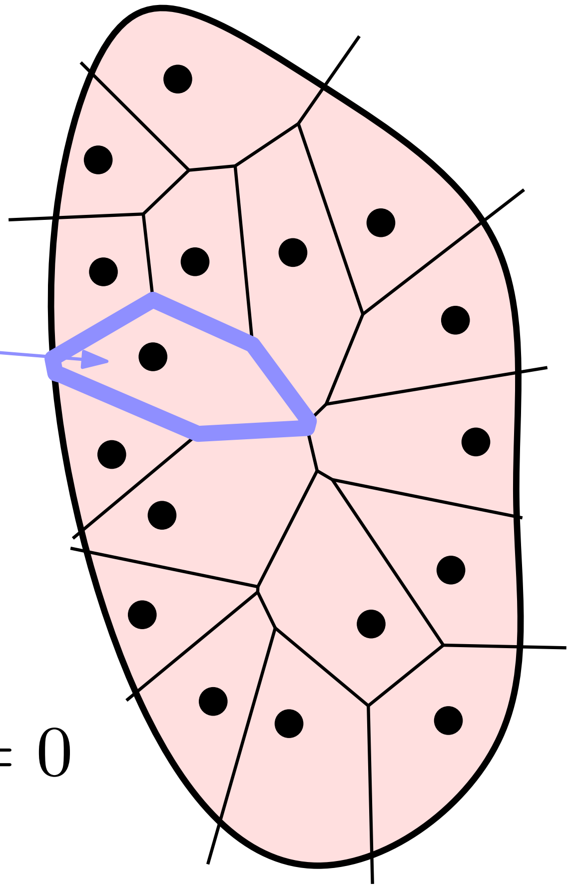

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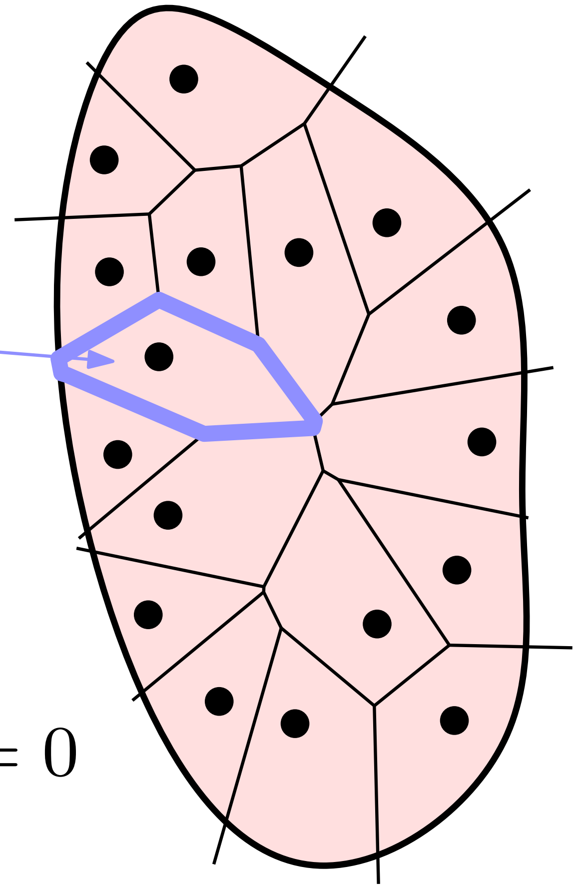

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$\Rightarrow$ We have to show smoothness and strict monotonicity

Quadratic cost : smoothness of G

we have $H_i(\psi) = \rho(\mathrm{Lag}_i(\psi)) \quad c(x,y) := \|x-y\|^2$

Proposition: For $\psi \in E_\varepsilon$, and assuming that $\rho \in \mathcal{C}^0_c(\mathbb{R}^d)$ one has

(A) $\frac{\partial H_i}{\partial \psi_j}(\psi) = \frac{1}{2\|y_i - y_j\|} \int_{\mathrm{Lag}_{ij}(\psi)} \rho(x) \,\mathrm{d}\, x$ (B) $\frac{\partial H_i}{\partial \psi_i}(\psi) = -\sum_{j\neq i} \frac{\partial G_i}{\partial \psi_j}(\psi)$

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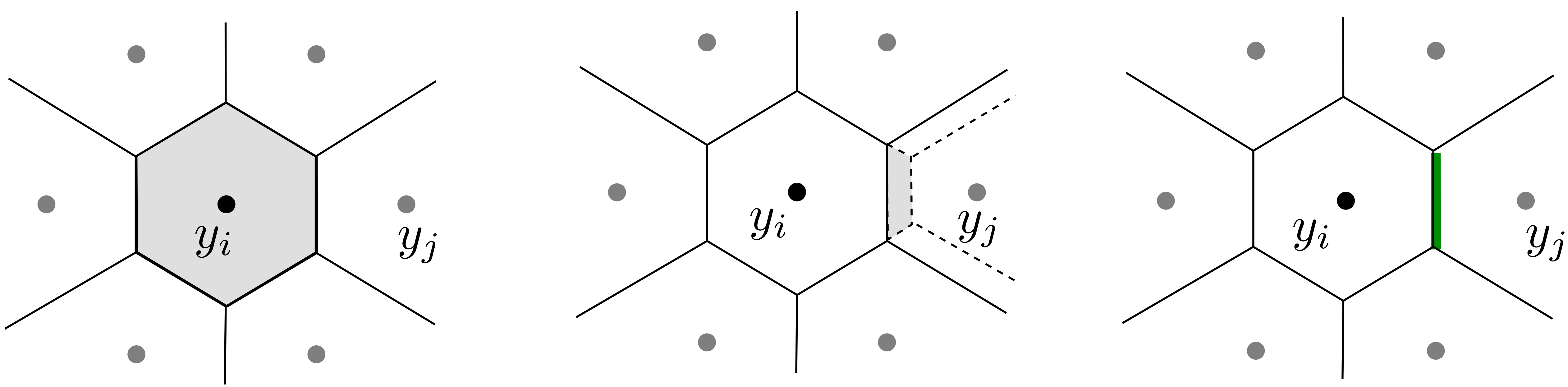
sketch of proof:



$H_i(\psi)$ $\quad \frac{H_i(\psi+\epsilon e_j) - H_i(\psi)}{\varepsilon} \xrightarrow{\varepsilon \to 0} \frac{\partial H_i}{\partial \psi_j}(\psi)$

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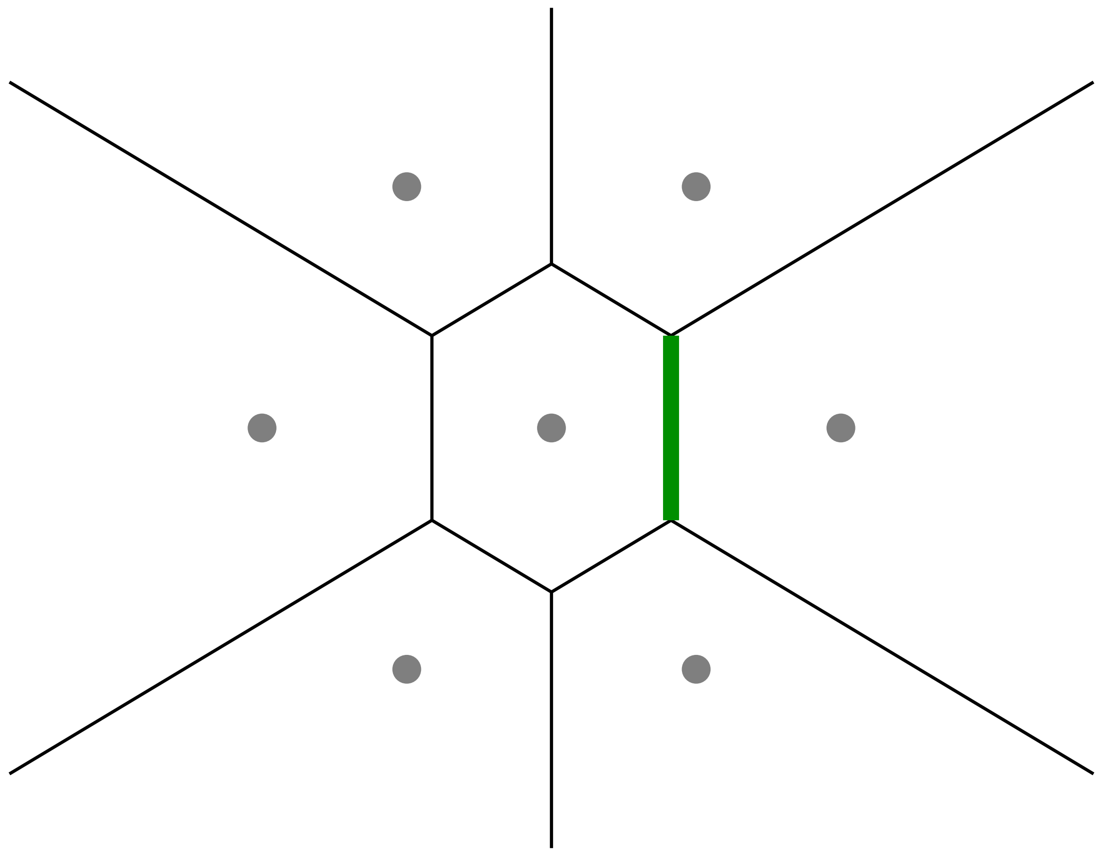

Continuity of $\frac{\partial H_i}{\partial \psi_j}(\psi)$

When t varies, $\frac{\partial H_i}{\partial \psi_j}(\psi_t)$ increases ...

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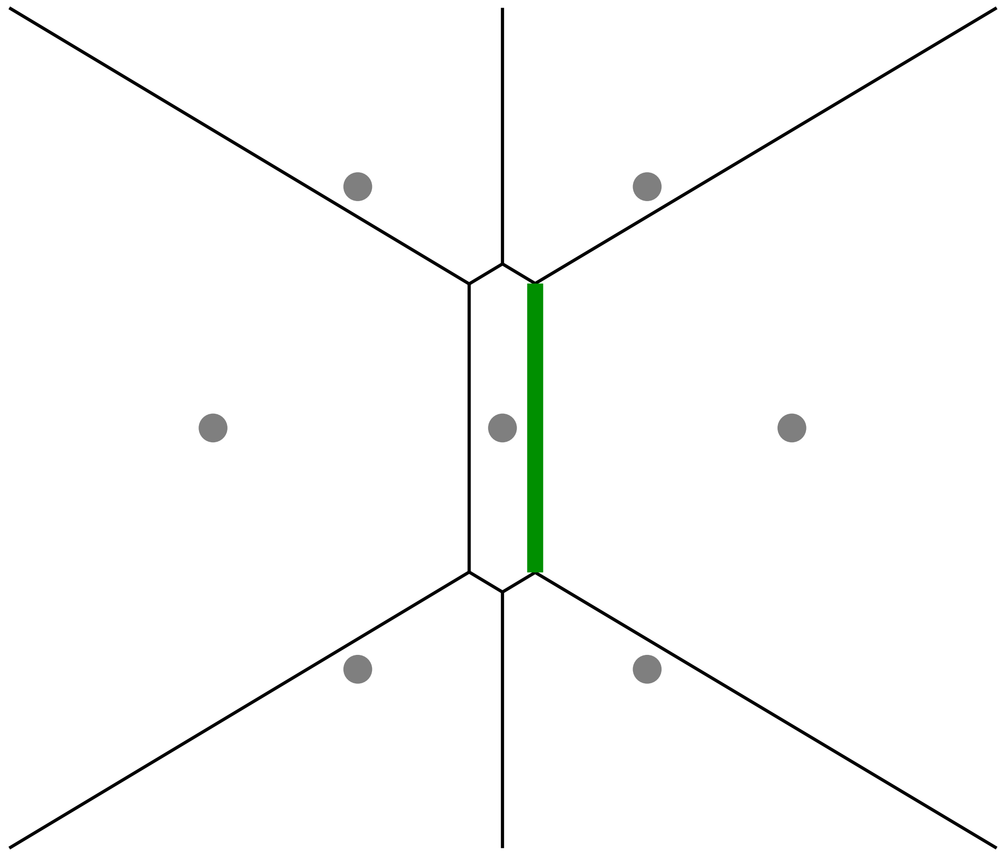

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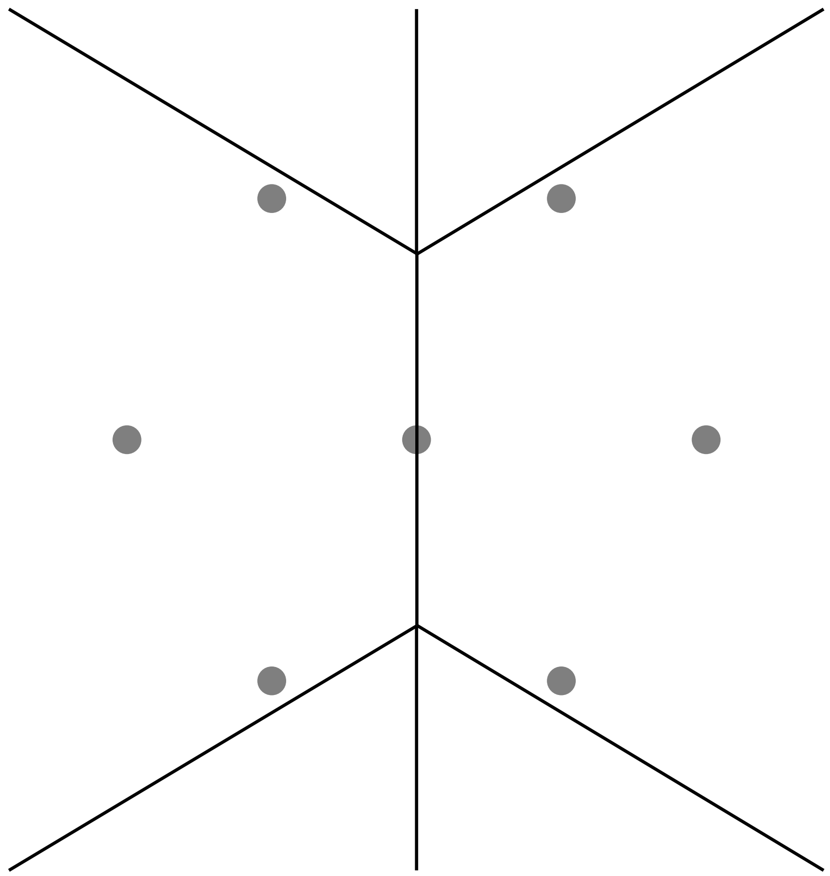

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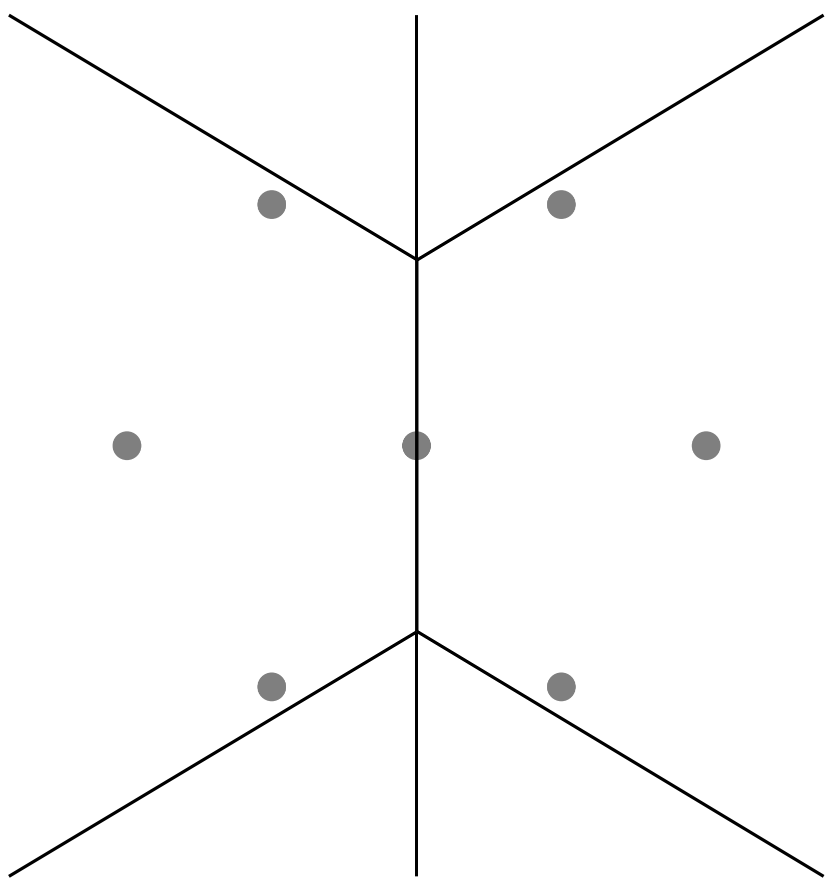

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$\rightsquigarrow$ we require $-$ $\rho(\mathrm{Lag}_i(\psi)) > 0$ at all times

$-$ or a genericity condition (three points not aligned)

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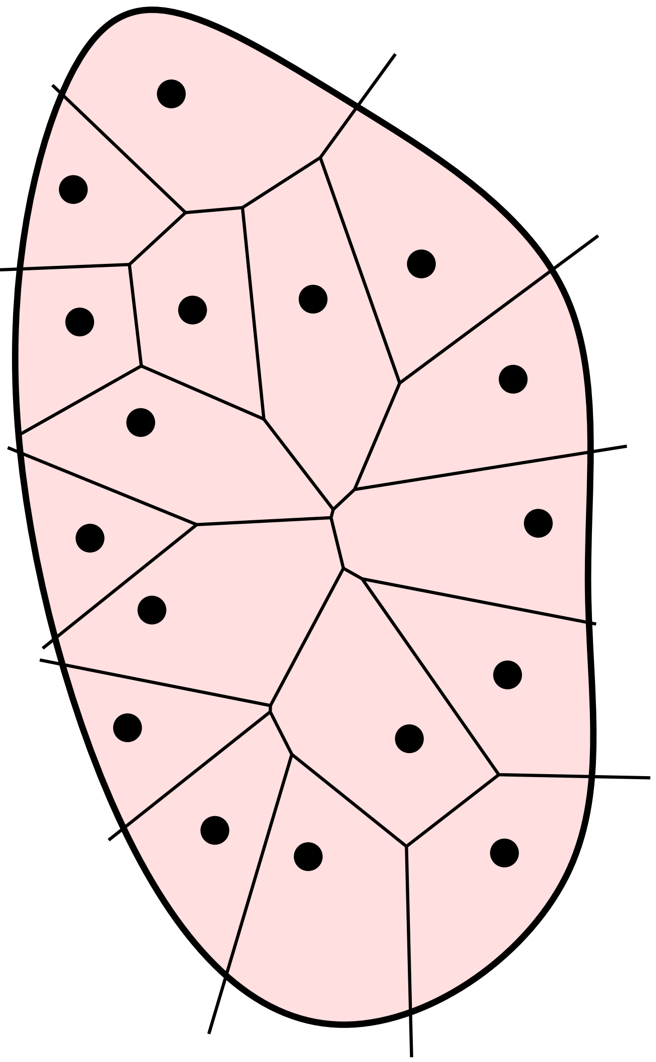

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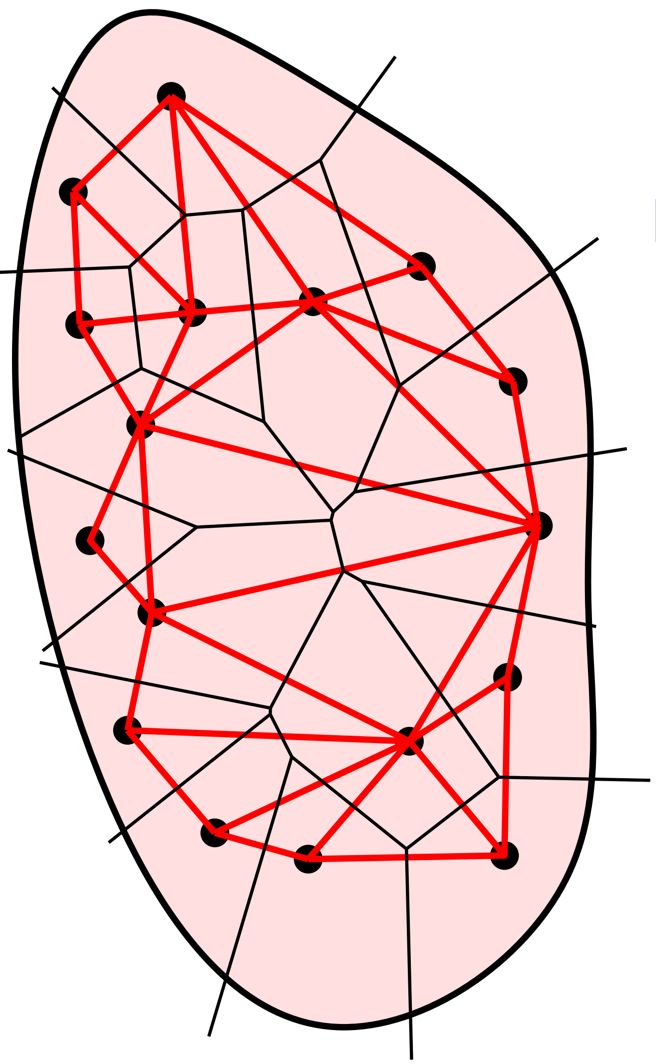

- Consider the matrix of $DH := (\frac{\partial H_i}{\partial \psi_j}(\psi))$ and the graph $\mathcal{G}$:

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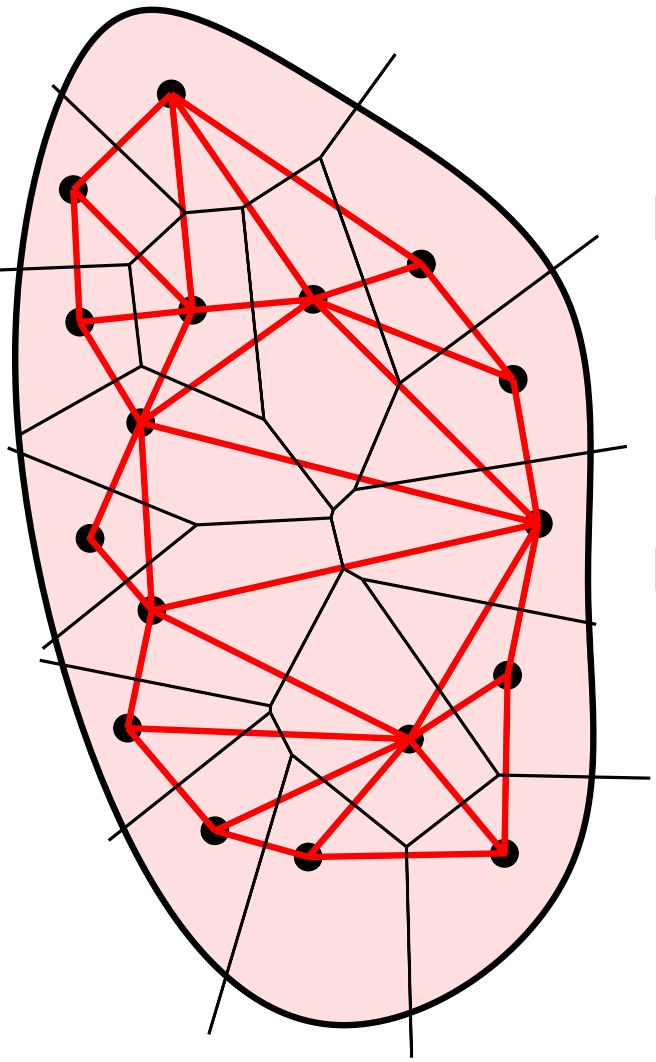

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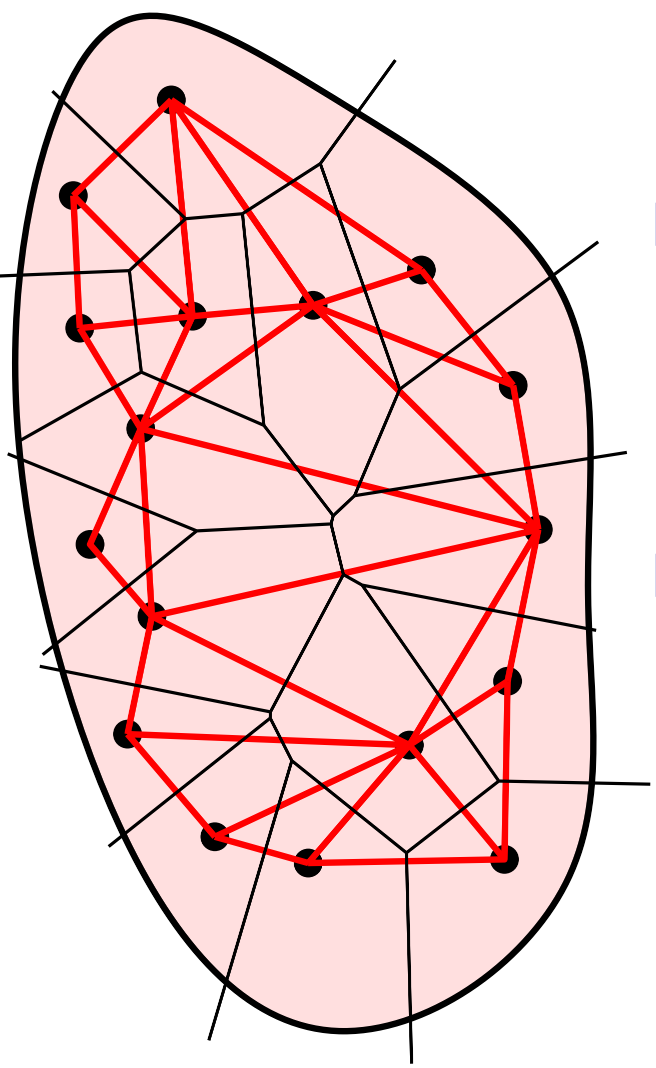

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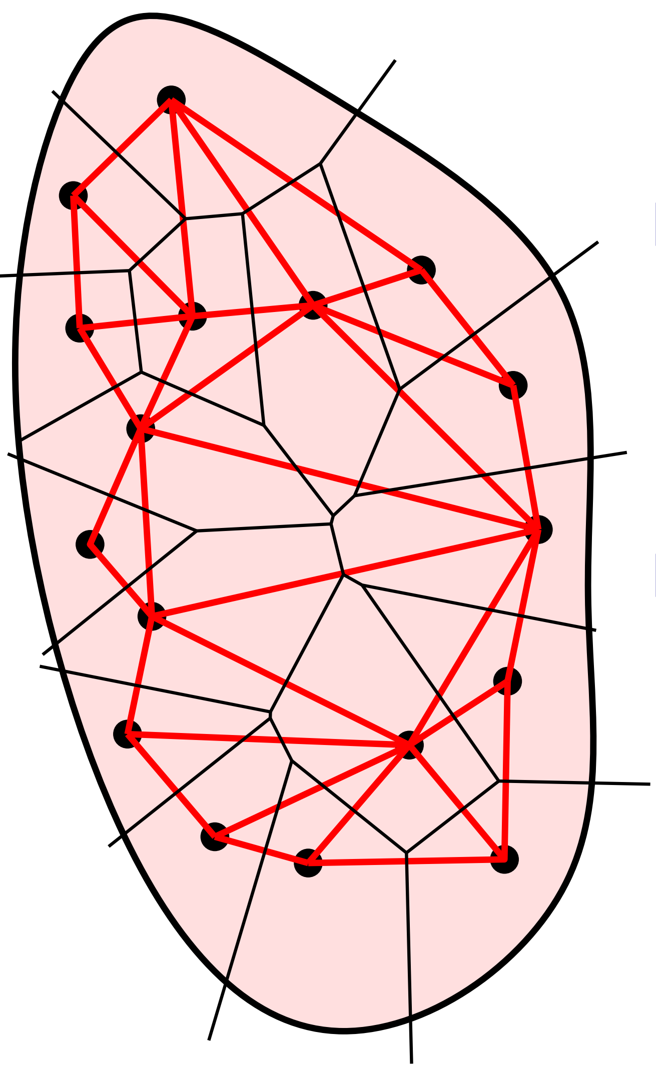

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$\Rightarrow$ monotonicity

Convergence in the quadratic case

Theorem: Let X be a (closed) convex bounded domain of $\mathbb{R}^d$ with $Y \subset \mathbb{R}^d$ be a finite set, ρ of class C^1 and $\{\rho > 0\}$ connected.

Then, the damped Newton algorithm for SD-OT converges **globally** with linear rate and locally with quadratic rate.

$$\|H(\psi^{k+1}) - \nu\| \leq \left(1 - \frac{\tau^*}{2}\right)^2 \|H(\psi^k) - \nu\|$$

[Kitagawa, Mérigot, T., JEMS 2019]

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- Holds when $X \subset M$ Riemannian manifold, $c \in C^2$ satistifes Twist, MTW.
- Holds when $X \subset \mathbb{R}^d$, c satistifes Twist.
  No convexity assumption but genericity conditions [Mérigot, T., 2020]

Quadratic cost: numerics

Exemple: ρ uniform on $X = [0,1]^2$; $\nu = \frac{1}{N}\sum_i \delta_{y_i}$

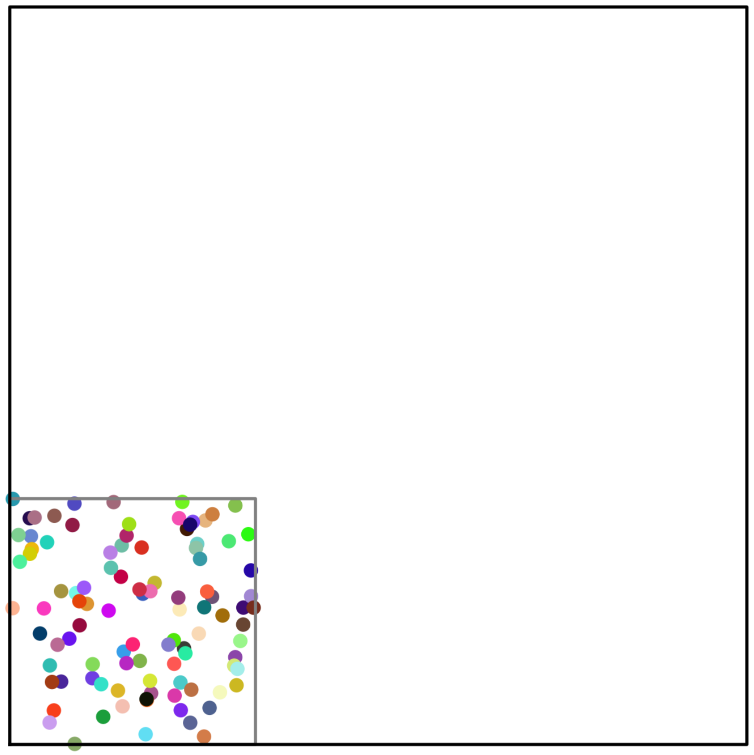

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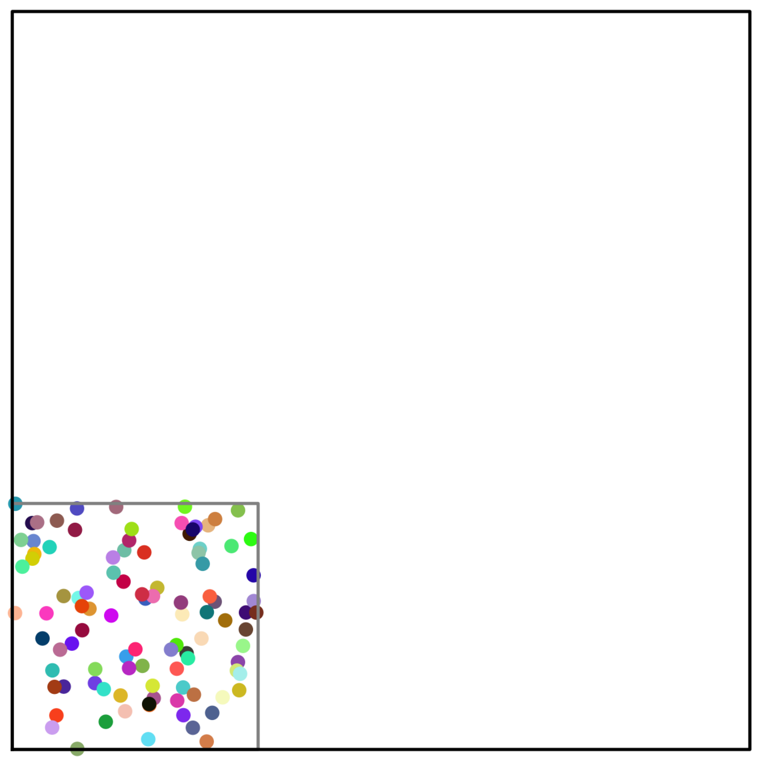
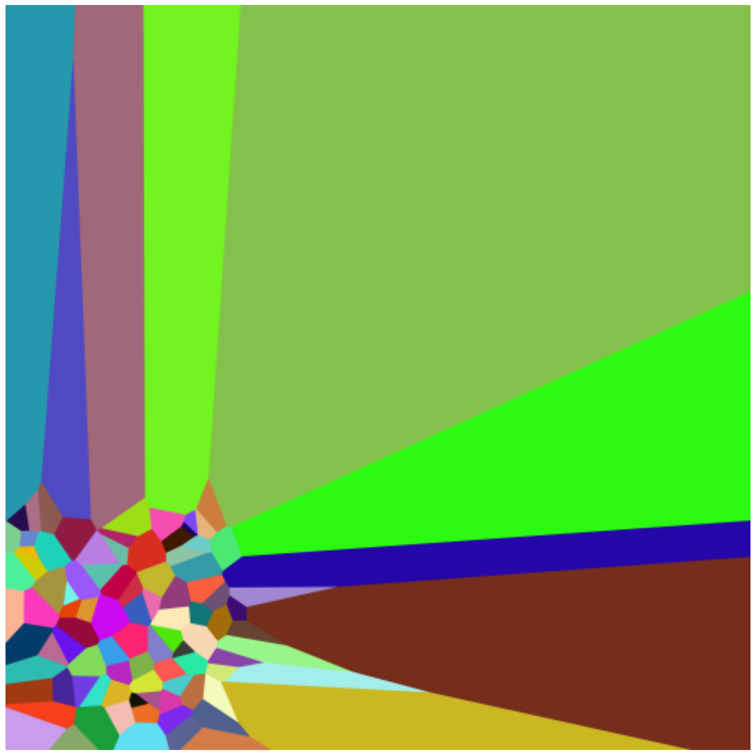
diagramme de Laguerre





$\|H(\psi^0) - \nu\|_1$
$\simeq 1.8$

Quadratic cost: numerics

Exemple: ρ uniform on $X = [0,1]^2$; $\nu = \frac{1}{N}\sum_i \delta_{y_i}$

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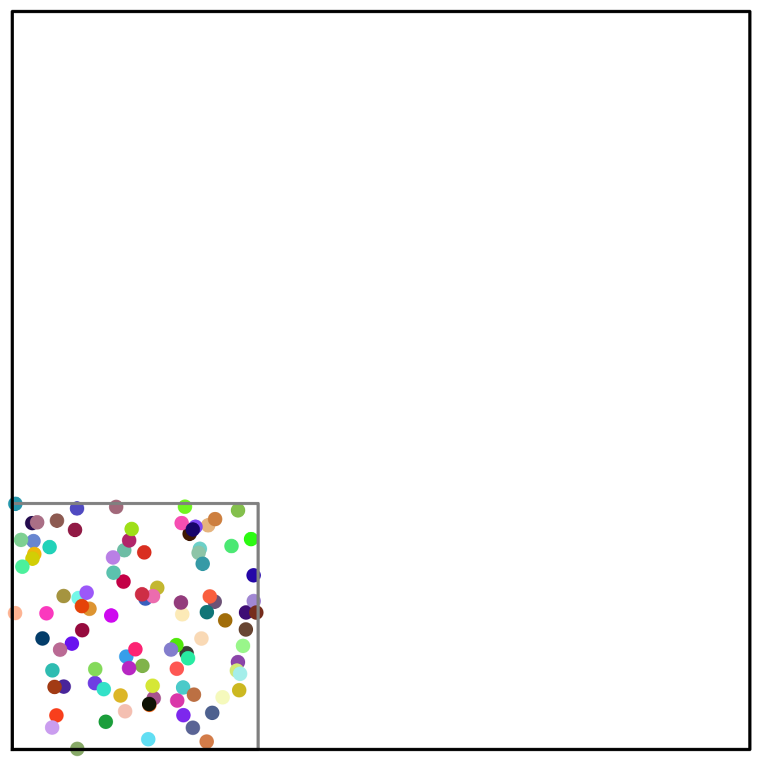

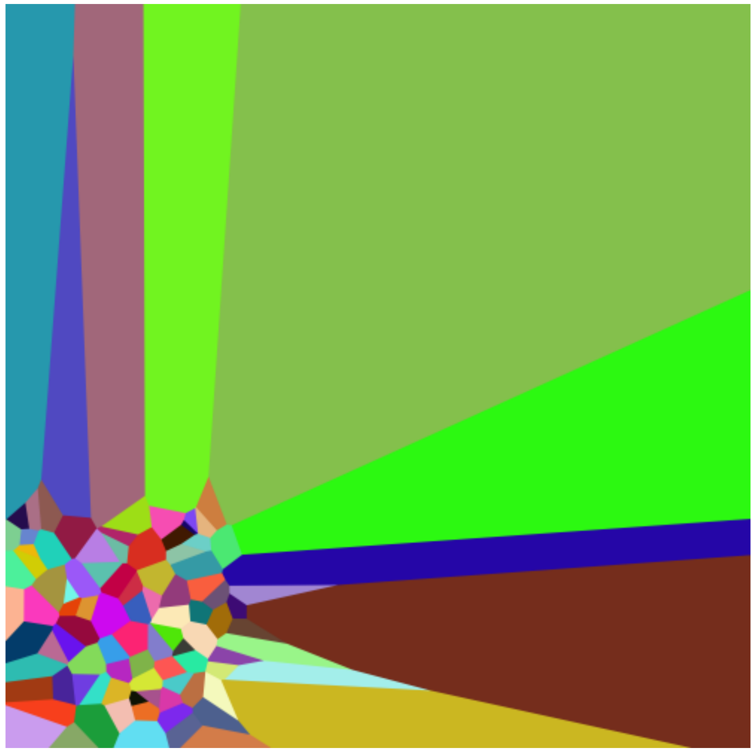

$\|H(\psi^0) - \nu\|_1$
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$\|H(\psi^1) - \nu\|_1$
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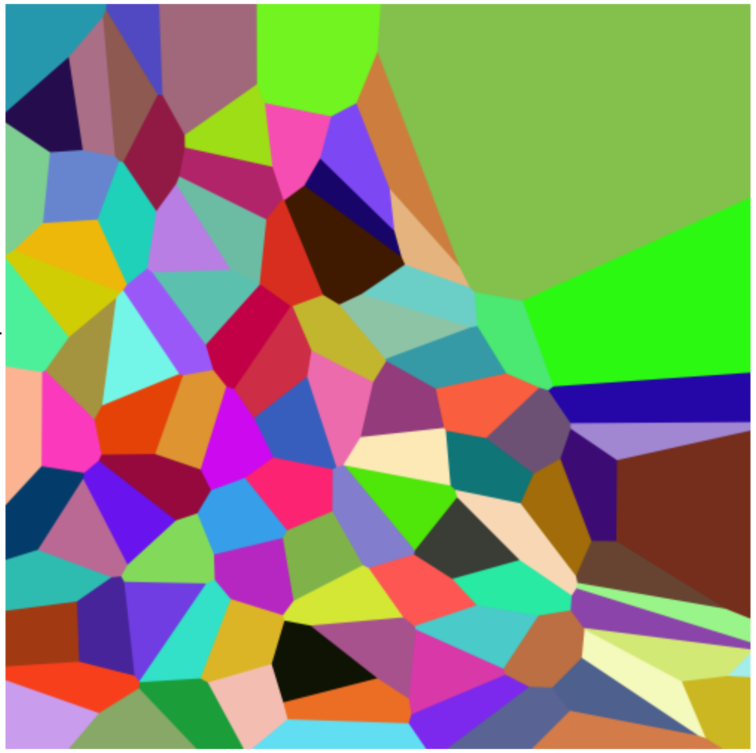

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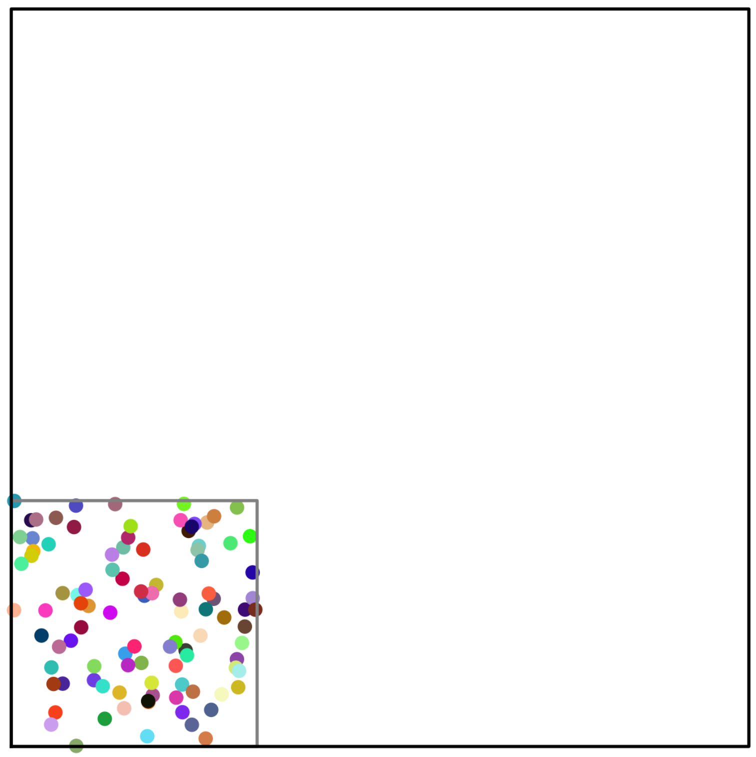

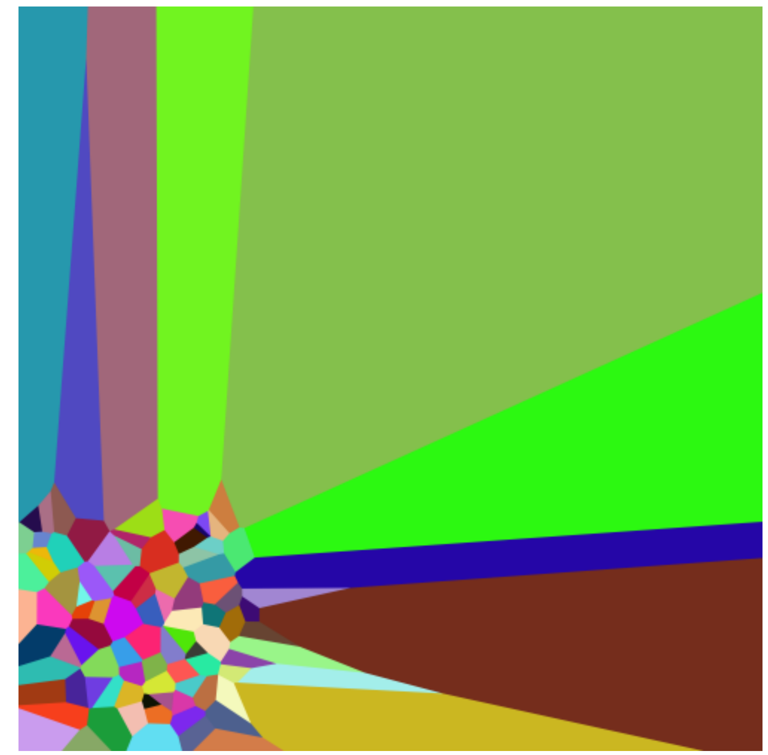

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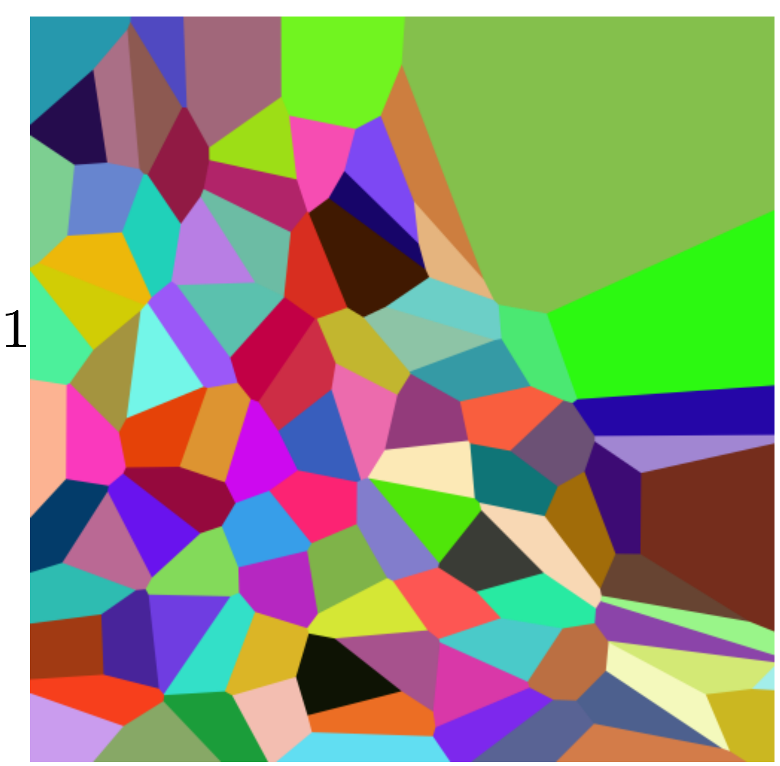
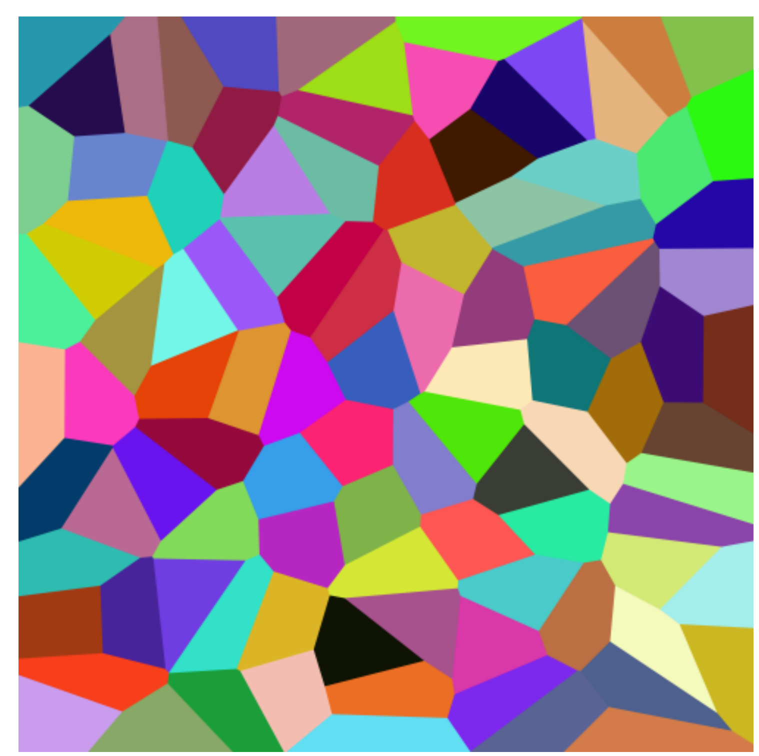




$\|H(\psi^3) - \nu\|_1 \simeq 10^{-9}$

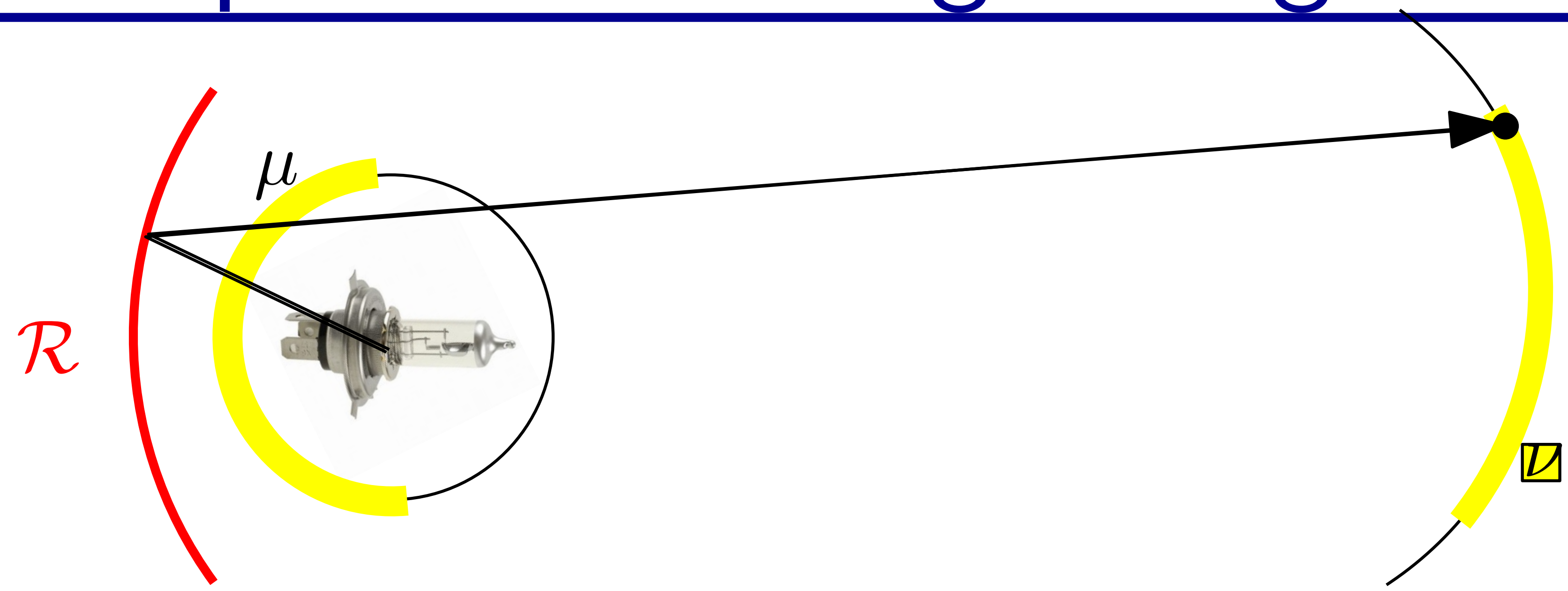
Outline

- Case 1: mirror for point light source
- Case 2: mirror for collimated source light
- Case 3: other cases

- Semi-discrete optimal transport
- Damped Newton algorithm

- Non-imaging optics: Far-Field target
- Non-imaging optics: Near-Field target

Mirror for point source light: algorithm



Mirror for point source light: algorithm

- **Damped Newton algorithm:**

⇝ Computation of Newton direction at each time step

⇝ Evaluation of H and DH:

$$\int_{\mathrm{Lag}_i} \mathrm{d}\,\mu(x) \qquad \text{and} \qquad \int_{\mathrm{Lag}_{i,j}} \mathrm{d}\,\mu(x)$$

Computation of Laguerre cells

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Computation of Laguerre cells

- **Power diagrams:**

CGAL

Definition: Given $P = \{p_i\}_{1\leq i\leq N} \subseteq \mathbb{R}^d$ and $(\omega_i)_{1\leq i\leq N} \in \mathbb{R}^N$

$$\mathrm{Pow}_P^\omega(p_i) := \{x \in \mathbb{R}^d; i = \arg\min_j \|x - p_j\|^2 + \omega_j\}$$

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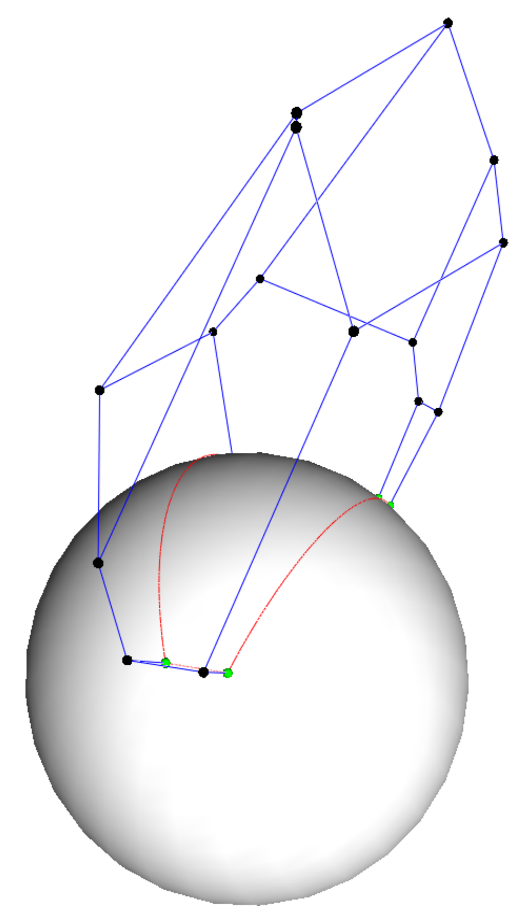


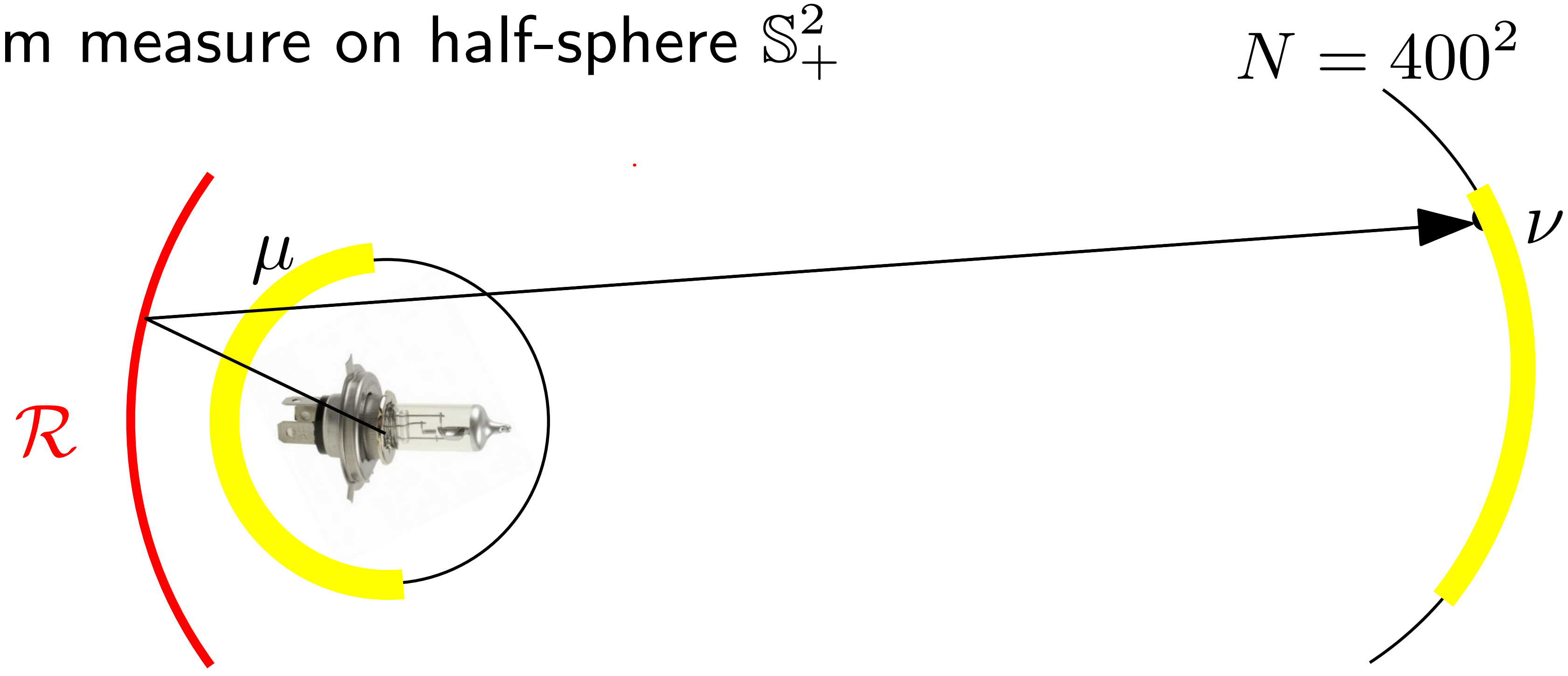
Lemma: $\mathrm{Lag}_i(\kappa) = \mathrm{Pow}_P^\omega(p_i) \cap \mathbb{S}^2$

with $p_i := -\frac{e_i y_j}{2d_i\|y_i\|}$ and $\omega_i := -\frac{e_i^2}{4+d_i^2} - \frac{1}{d_i}$,

Mirror for point source light: algorithm

$\nu = \sum_{i=1}^{N} \nu_i \delta_{x_i}$ obtained by discretizing a picture of Cameraman.

$\mu =$ uniform measure on half-sphere $\mathbb{S}^2_+$



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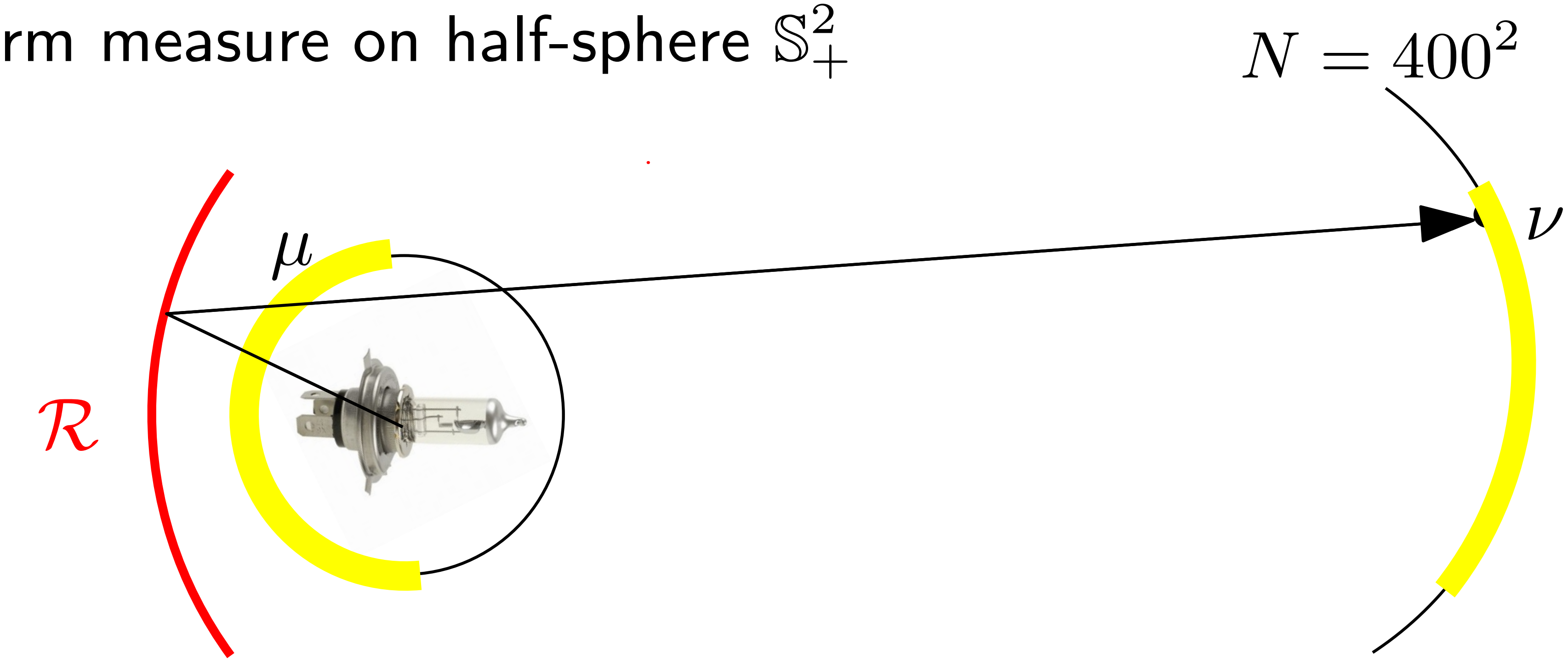
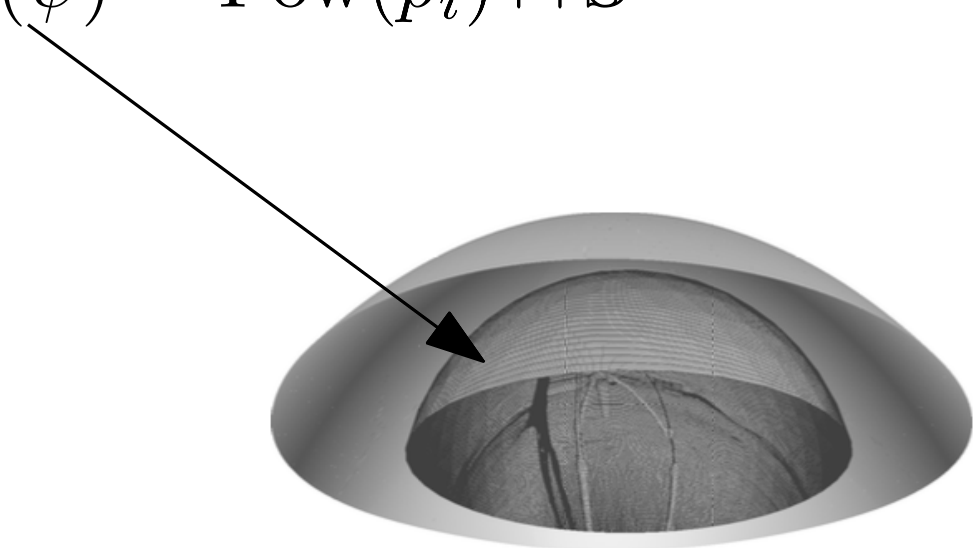
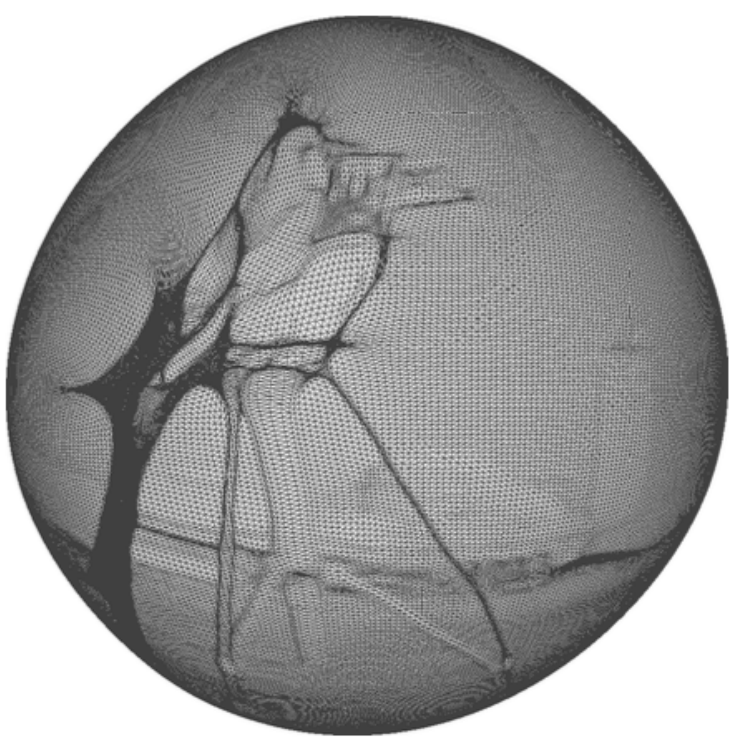
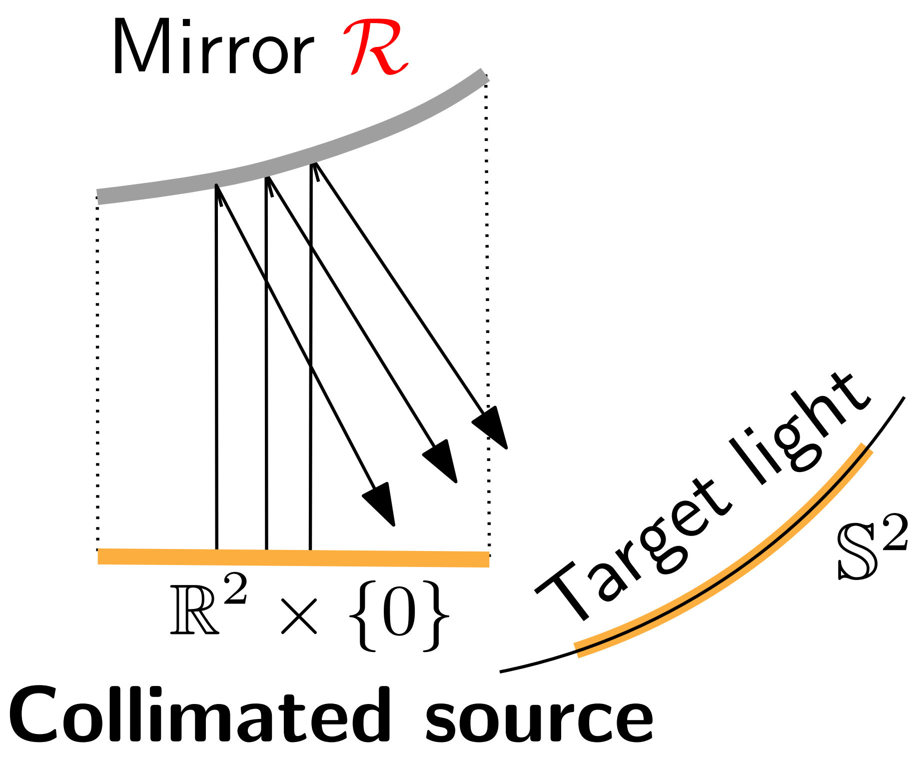


$V_i(\psi) = \mathrm{Pow}(p_i) \cap \mathbb{S}^2$



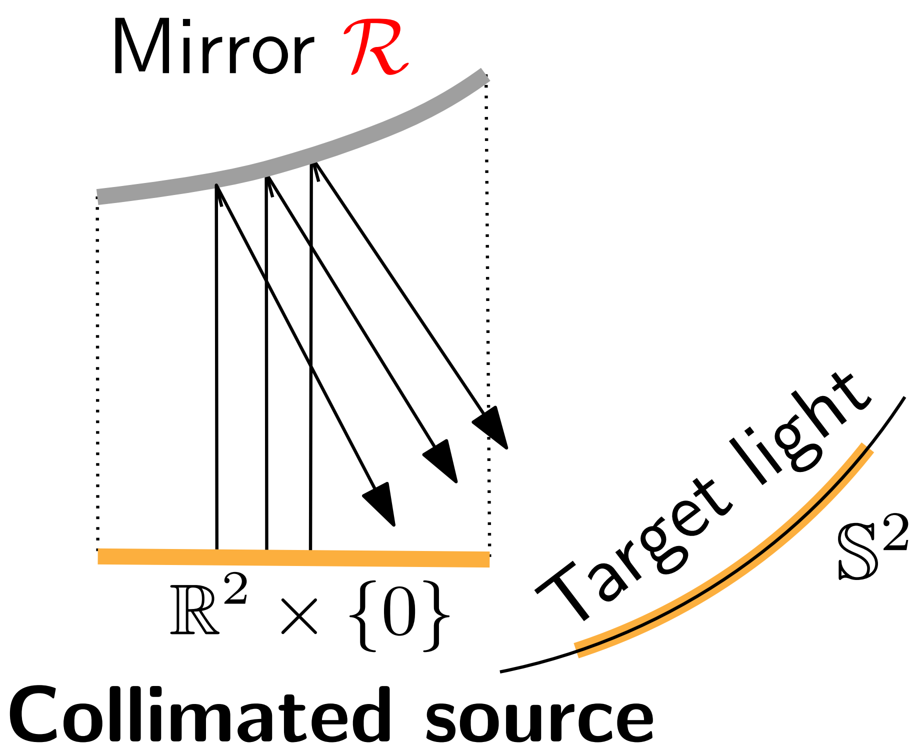


Collimated source / Far Field Target



targeted image $N = 400 \times 480$

Collimated source / Far Field Target



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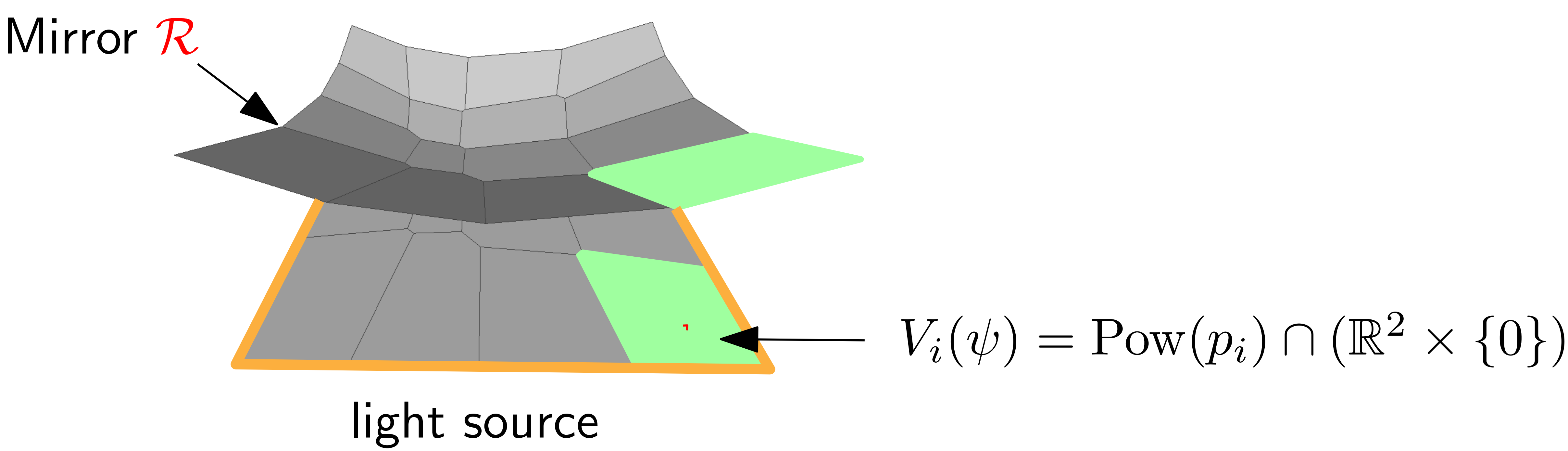

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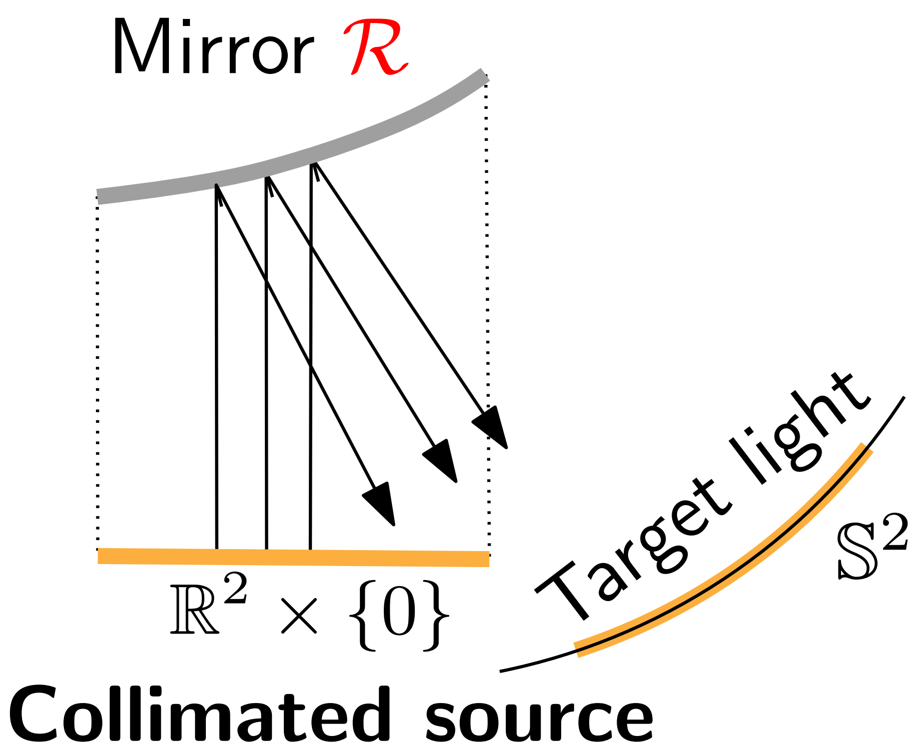

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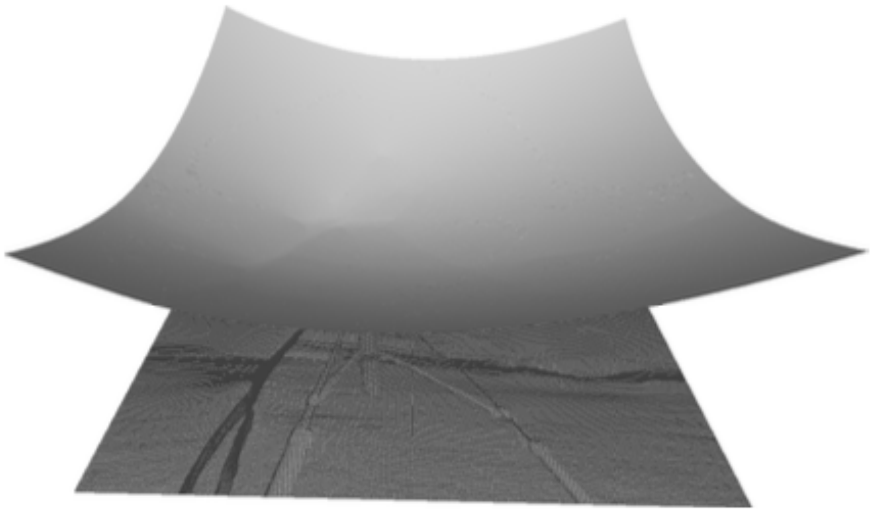
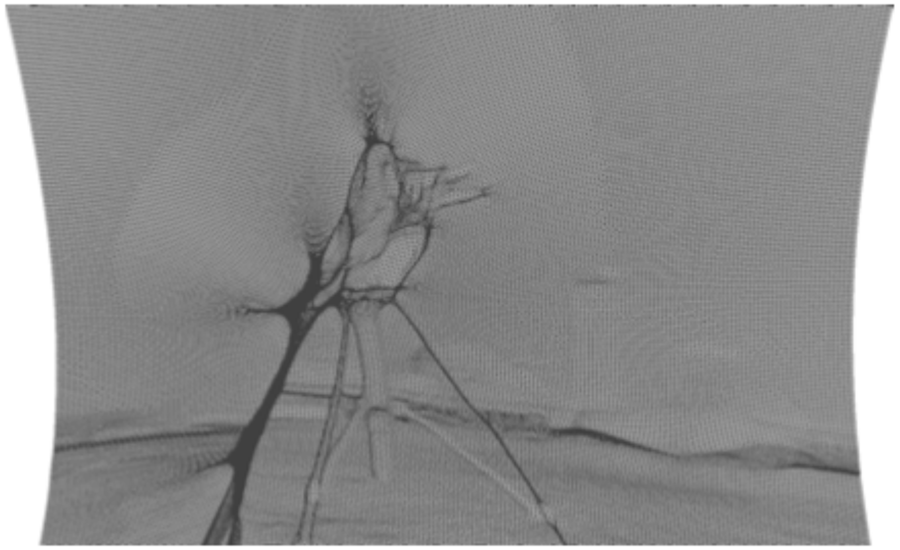
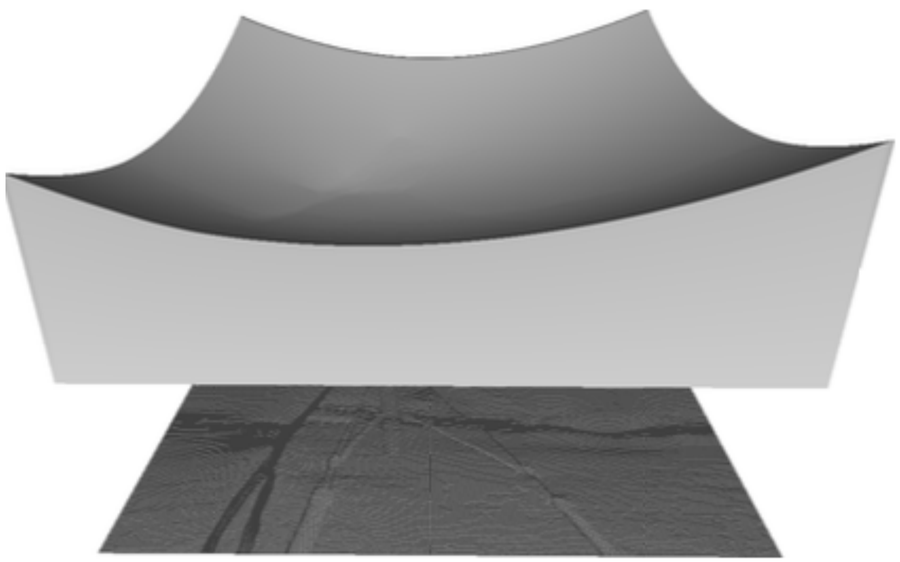
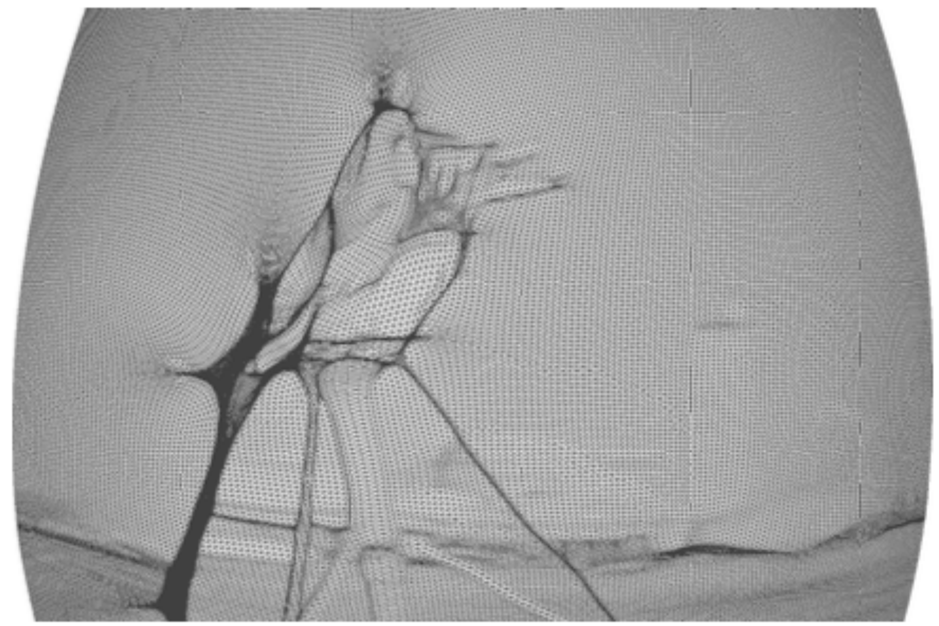
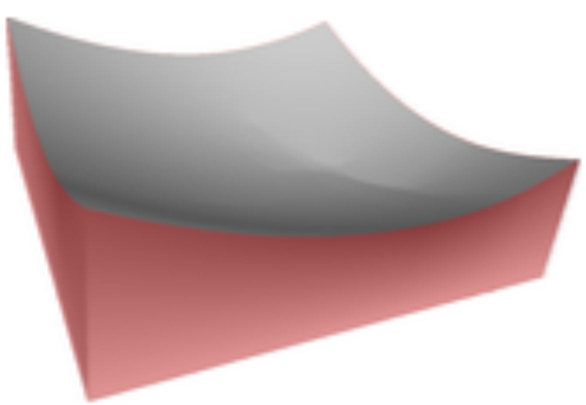
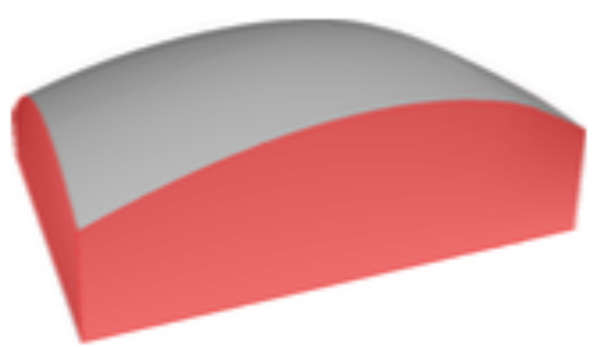




Lenses

We solve 8 optical problems with one program

$\rightsquigarrow V_i(\psi) = \mathrm{Pow}(p_i) \cap X$ where $X = \mathbb{S}^2, \mathbb{R}^2 \times \{0\}$

$\rightsquigarrow$ Automatic differentiation

Conclusion

We solved 4 inverse problems arising in nonimaging optics
using semi-discrete approach and optimal transport

- Each problem is a Monge-Ampère equation
- For far-field target, OT problem on $\mathbb{R}^2$ or $\mathbb{S}^2$ $\rightsquigarrow$ Newton algorithm
- Iterative procedure for real-life light target

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Tomorrow

$\rightsquigarrow$ Generalization to generated jacobian equations (application to optics, near field target)

$\rightsquigarrow$ Stability results

Thank you!