Part 2: A Posteriori Error Estimation & Adaptive Stochastic Galerkin Finite Element Approximation

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M\cr NA

Stochastic Galerkin Approximation

1. **Parametric PDE:** Find $u : D \times \Gamma \to \mathbb{R}$ such that

$$-\nabla \cdot (a(\boldsymbol{x}, \mathbf{y}) \nabla u(\boldsymbol{x}, \mathbf{y})) = f(\boldsymbol{x}), \qquad \boldsymbol{x} \in D \subset \mathbb{R}^{2,3}, \quad \boldsymbol{y} \in \Gamma.$$

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2. **Weak Problem:** Find $u \in V := L^2_\pi(\Gamma, H^1_0(D))$ satisfying

$$\int_\Gamma \left(\int_D a \nabla u \cdot \nabla v \, d\mathbf{x} \right) d\pi(\mathbf{y}) = \int_\Gamma \left(\int_D f \, v \, d\mathbf{x} \right) d\pi(\mathbf{y}) \qquad \forall v \in V,$$

where π is a **probability measure**.

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2. **Weak Problem:** Find $u \in V := L^2_\pi(\Gamma, H^1_0(D))$ satisfying

$$A(u, v) = \ell(v), \qquad \forall v \in V$$

where $A(\cdot, \cdot)$ is an **inner product** that induces an **energy norm** $\| \cdot \|_A$.

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4. **Error Equations:** The error $e := u - u_X \in V$ satisfies:

$$A(e, v) = \underbrace{\ell(v) - A(u_X, v)}_{\text{residual } R(v)} \qquad \forall\, v \in V.$$

☐ **Ideally:** Choose SGFEM space X so that $\|e\|_A \leq TOL$.

Adaptive SGFEM

- Start with a **low-dimensional** space X and computes $\widehat{u}_0 \in X$.
- Estimate the (energy) error using only **a posterior information**

$$\eta \approx \|u - \widehat{u}_0\|_A = \mathbb{E}\left[\|a^{1/2}\nabla\left(u - \widehat{u}_0\right)\|^2_{L^2(D)}\right]^{1/2}.$$

- Decide how best to **enrich** X if $\eta > TOL$.
- Compute a sequence of approximations $\widehat{u}_0, \ldots, \widehat{u}_L$ until

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We use a strategy called **'Hierarchical Error Estimation'** and work with approximation spaces with the so-called **multilevel** structure:

$$X := \bigoplus_{\boldsymbol{\alpha} \in J_P} H_1^{\boldsymbol{\alpha}} \otimes P^{\boldsymbol{\alpha}}.$$

Disclaimers + Other Work

- **Hierarchical** a posteriori error estimation is an old idea!
  - **Bank + Weiser**, Math. Comp., (1985). **Bank + Smith**, SINUM, 1993.
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- **A priori** results + **convergence rates** for multilevel SGFEM, e.g.
  - **Cohen, DeVore, Schwab**. Analytic regularity and polynomial approx. of parametric and stochastic elliptic PDE's. Anal. Appl., 9(1), 2011.

Background + Collaborators

- Worked presented here based on PhD work of **Adam Crowder**:
  - **PhD thesis**, University of Manchester, 2020.
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- EPSRC-funded project started $\approx$ 10 years ago in Manchester with **David Silvester** and **Alex Bespalov**
  - Energy norm a posteriori error estimation for parametric operator equations, Bespalov, Powell, Silvester, **SISC. 36(2)**, 2014.
  - **S-IFISS MATLAB Toolbox:**

`http://www.manchester.ac.uk/ifiss/sifiss`

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- J_P is a **finite set** of **finitely supported** sequences:

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- **Orthogonal Polynomials:** $P^{\boldsymbol{\alpha}} := \operatorname{span}\{\psi_{\boldsymbol{\alpha}}(\boldsymbol{y})\} \subset L^2_\pi(\Gamma)$ where

$$\psi_{\boldsymbol{\alpha}}(\boldsymbol{y}) = \prod_{m=1}^{\infty} \psi_{\alpha_m}(y_m), \qquad \psi_i \text{ has degree } i \text{ and } \psi_0 = 1.$$

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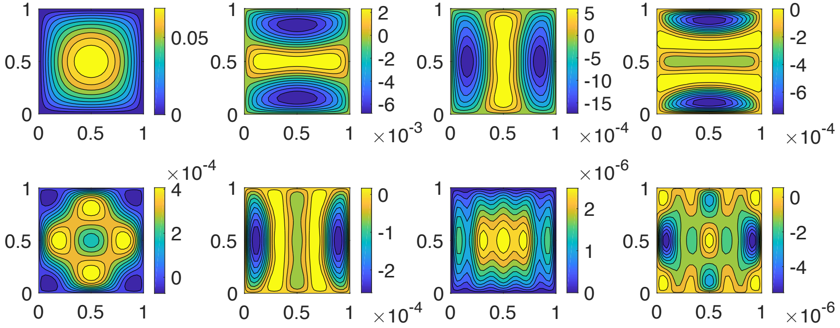
Special case: If $H_1^{\boldsymbol{\alpha}} = H_1$ for all $\boldsymbol{\alpha} \in J_P$ then

$$X := H_1 \otimes P, \qquad \text{where} \quad P := \operatorname{span}\left\{\psi_{\boldsymbol{\alpha}}(\boldsymbol{y}), \boldsymbol{\alpha} \in J_P\right\}.$$

Polynomial-based Surrogate

$$u_X(\boldsymbol{x},\mathbf{y}) \quad = \quad \sum_{\boldsymbol{\alpha}\in J_P}\left(\sum_{i=1}^{n_{\boldsymbol{\alpha}}} u_{i,\boldsymbol{\alpha}}\phi_i^{\boldsymbol{\alpha}}(\boldsymbol{x})\right)\psi_{\boldsymbol{\alpha}}(\mathbf{y}) = \sum_{\boldsymbol{\alpha}\in J_P}\underbrace{u_{\boldsymbol{\alpha}}(\boldsymbol{x})}_{\in H_1^{\boldsymbol{\alpha}}}\psi_{\boldsymbol{\alpha}}(\mathbf{y}).$$

Test Problem: 8 spatial modes $u_{\boldsymbol{\alpha}}(\boldsymbol{x})$



Hierarchical Error Estimation

For $u_X \in X \subset V$, we know $e := u - u_X \in V$ satisfies:

$$A(e, v) = R(v) \qquad \forall\, v \in V.$$

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1. Consider $e_W \in W \supset X$ such that:

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$$W = X \oplus \underbrace{Y}_{\textbf{`detail'}}, \qquad X \cap Y = \{0\}$$

and define **error estimate** $\eta := \|e_Y\|_A$ where

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If we can choose the **detail space** Y so that

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1. **Saturation Assumption:** $\exists\, \beta \in [0,1)$ such that

$$\|u - u_W\|_A \leq \beta\, \|u - u_X\|_A$$

2. **CBS Inequality:** $\exists\, \gamma \in [0,1)$ such that

$$|A_0(u,v)| \leq \gamma\, \|u\|_{A_0} \|v\|_{A_0}, \qquad \forall u \in X,\, \forall v \in Y$$

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- Constants depend on the **choice of detail space** Y and $A_0(\cdot,\cdot)$.
- Here, choose $A_0(\cdot,\cdot)$ to be inner product associated with $a_0(\mathbf{x})$.

Detail Space (1)

$$X = \bigoplus_{\boldsymbol{\alpha} \in J_P} H_1^{\boldsymbol{\alpha}} \otimes P^{\boldsymbol{\alpha}}$$

- For each $\boldsymbol{\alpha} \in J_P$ choose $H_2^{\boldsymbol{\alpha}}$ **(new FEM space)** such that

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- Choose J_Q **(new set of multi-indices)** with

$$J_P \cap J_Q = \emptyset$$

and for each $\boldsymbol{\beta} \in J_Q$ define $Q^{\boldsymbol{\beta}} := \mathrm{span}\,\{\psi_{\boldsymbol{\beta}}(\boldsymbol{y})\}$.

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Define **detail space**:

$$Y = \left(\bigoplus_{\boldsymbol{\alpha} \in J_P} H_2^{\boldsymbol{\alpha}} \otimes P^{\boldsymbol{\alpha}} \right) \oplus \left(\bigoplus_{\boldsymbol{\beta} \in J_Q} H \otimes Q^{\boldsymbol{\beta}} \right) := Y_1 \oplus Y_2$$

where H is <u>**one**</u> of the **FEM spaces** $H_1^{\boldsymbol{\alpha}}$.

Detail Space (2)

Re-write detail space as

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$$\eta := \|e_Y\|_{A_0} = \left(\sum_{\boldsymbol{\alpha} \in J_P} \|e_{Y_{1,\boldsymbol{\alpha}}}\|_{A_0}^2 + \sum_{\boldsymbol{\beta} \in J_Q} \|e_{Y_{2,\boldsymbol{\beta}}}\|_{A_0}^2 \right)^{1/2},$$

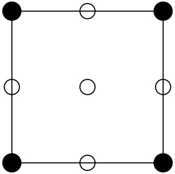
where the components satisfy

$$\begin{aligned} e_{Y_{1,\boldsymbol{\alpha}}} \in Y_{1,\boldsymbol{\alpha}} : \quad A_0(e_{Y_{1,\boldsymbol{\alpha}}}, v) &= R(v), \quad \forall\, v \in Y_{1,\boldsymbol{\alpha}}, \\ e_{Y_{2,\boldsymbol{\beta}}} \in Y_{2,\boldsymbol{\beta}} : \quad A_0(e_{Y_{2,\boldsymbol{\beta}}}, v) &= R(v), \quad \forall\, v \in Y_{2,\boldsymbol{\beta}}. \end{aligned}$$

How to Choose H_2^{α}?

CBS constant $\gamma := \max_{\boldsymbol{\alpha} \in J_P}\{\gamma_{\boldsymbol{\alpha}}\}$ where $\gamma_{\boldsymbol{\alpha}}$ depends only on $H_2^{\boldsymbol{\alpha}}$ and $H_1^{\boldsymbol{\alpha}}$.

$H_1^{\boldsymbol{\alpha}} = \mathbb{Q}_1(h_{\boldsymbol{\alpha}})$



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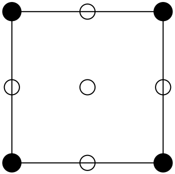

Example: Construct $H_2^{\boldsymbol{\alpha}}$ locally using 'bubble' functions associated with $(\circ)$.

$H_1^{\boldsymbol{\alpha}}$	$H_2^{\boldsymbol{\alpha}}$	$\gamma_{\boldsymbol{\alpha}}^2$	$\sqrt{1-\gamma_{\boldsymbol{\alpha}}^2}$
$\mathbb{Q}_1(h_{\boldsymbol{\alpha}})$	$\mathbb{Q}_2(h_{\boldsymbol{\alpha}})$	**0.4545**	0.7385
$\mathbb{Q}_1(h_{\boldsymbol{\alpha}})$	$\mathbb{Q}_1(h_{\boldsymbol{\alpha}}/2)$	**0.3750**	0.7905
$\mathbb{Q}_1(h_{\boldsymbol{\alpha}})$	$\mathbb{Q}_2(h_{\boldsymbol{\alpha}}/2)$	**0.0446**	0.9774
$\mathbb{Q}_1(h_{\boldsymbol{\alpha}})$	$\mathbb{Q}_4(h_{\boldsymbol{\alpha}})$	**0.0121**	0.9939

How to Choose J_Q?

$$X = \bigoplus_{\boldsymbol{\alpha} \in J_P} H_1^{\boldsymbol{\alpha}} \otimes P^{\boldsymbol{\alpha}}, \qquad Y = \left(\bigoplus_{\boldsymbol{\alpha} \in J_P} H_2^{\boldsymbol{\alpha}} \otimes P^{\boldsymbol{\alpha}} \right) \oplus \left(\bigoplus_{\boldsymbol{\beta} \in J_Q} H \otimes Q^{\boldsymbol{\beta}} \right)$$

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To get an **accurate** error estimate, need to choose Y so that

$$\sqrt{1-\beta^2}\,\sqrt{1-\gamma^2} \approx 1$$

where

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- We choose

$$J_Q = \{\boldsymbol{\beta} \in J \setminus J_P,\ \boldsymbol{\beta} = \boldsymbol{\alpha} \pm \mathbf{e}^m, \boldsymbol{\beta} \in J_P, m = 1, \ldots, M_P + \Delta_M\}$$

where M_P is $\#$ highest parameter activated in J_P.

How to Choose J_Q?

$$X = \bigoplus_{\boldsymbol{\alpha}\in J_P} H_1^{\boldsymbol{\alpha}} \otimes P^{\boldsymbol{\alpha}}, \qquad Y = \left(\bigoplus_{\boldsymbol{\alpha}\in J_P} H_2^{\boldsymbol{\alpha}} \otimes P^{\boldsymbol{\alpha}}\right) \oplus \left(\bigoplus_{\boldsymbol{\beta}\in J_Q} H \otimes Q^{\boldsymbol{\beta}}\right)$$

To get an **accurate** error estimate, need to choose Y so that

$$\sqrt{1-\beta^2}\,\sqrt{1-\gamma^2} \approx 1$$

where

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  where M_P is # highest parameter activated in J_P.

- Δ_M needs to be **larger** for problems where $\|a_m\|_\infty$ decays **slowly**.

Decoupled Error Reduction Indicators

Define the spaces

$$W_1 := X \oplus \left(\bigoplus_{\boldsymbol{\alpha} \in \overline{J_P}} Y_{1,\boldsymbol{\alpha}} \right), \qquad W_2 := X \oplus \left(\bigoplus_{\boldsymbol{\beta} \in \overline{J_Q}} Y_{2,\boldsymbol{\beta}} \right)$$

where $\overline{J_P} \subset J_P$ and $\overline{J_Q} \subset J_Q$ and consider the Galerkin approximations:

- $u_{W_1} \in W_1$ **(spatial refinement)**, $u_{W_2} \in W_2$ **(parametric enrichment)**.

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Then one can **prove** that

$$\lambda\, \eta_1 \leq \quad \|u_{W_1} - u_X\|_E^2 \quad \leq \frac{\Lambda}{1-\gamma^2}\, \eta_1$$

$$\lambda\, \eta_2 \leq \quad \|u_{W_2} - u_X\|_E^2 \quad \leq \Lambda\, \eta_2.$$

This allows us to do **adaptivity!**

Basic Adaptive Algorithm (1)

- **INITIALIZE:** J_P and $\{H_1^{\boldsymbol{\alpha}}, \forall \boldsymbol{\alpha} \in J_P\}$
- **SOLVE:** Find $u_X \in X$

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$$\{\|e_{Y_{1,\boldsymbol{\alpha}}}\|_{A_0},\ \boldsymbol{\alpha} \in J_P\}, \qquad \{\|e_{Y_{2,\boldsymbol{\beta}}}\|_{A_0},\ \boldsymbol{\beta} \in J_Q\}$$

- **ENERGY ERROR ESTIMATE:**

$$\eta = \left(\sum_{\boldsymbol{\alpha} \in J_P} \|e_{Y_{1,\boldsymbol{\alpha}}}\|_{A_0}^2 + \sum_{\boldsymbol{\beta} \in J_Q} \|e_{Y_{2,\boldsymbol{\beta}}}\|_{A_0}^2 \right)^{1/2}$$

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- **IF** $\eta \leq TOL$ **STOP**;
- **ELSE**

Compute **ESTIMATED ERROR REDUCTION RATIOS:**

$$\left\{ \frac{\|e_{Y_{1,\boldsymbol{\alpha}}}\|_{A_0}^2}{\dim(Y_{1,\boldsymbol{\alpha}})}, \ \boldsymbol{\alpha} \in J_P \right\}, \qquad \left\{ \frac{\|e_{Y_{2,\boldsymbol{\beta}}}\|_{A_0}^2}{\dim(Y_{2,\boldsymbol{\beta}})}, \ \boldsymbol{\beta} \in J_Q \right\}$$

Basic Adaptive Algorithm (2)

- **IDENTIFY** 'important' subsets

$$\overline{J_P} \subset J_P, \qquad \overline{J_Q} \subset J_Q$$

- **ERROR REDUCTION INDICATORS**

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IF SPATIAL

- Freeze J_P
- **improve** $H_1^{\boldsymbol{\alpha}}$ for $\boldsymbol{\alpha} \in \overline{J_P}$

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- Freeze J_P
- <u>**improve**</u> H_1^{α} for $\alpha \in \overline{J_P}$

IF PARAMETRIC

- Freeze H_1^{α} for $\alpha \in J_P$
- initialize H for new $\alpha \in \overline{J_Q}$
- $J_P \leftarrow J_P \cup \overline{J_Q}$;

END

Example 1: Synthetic KL Expansion

$D = [0,1]^2$, $f = 1$, $a(\mathbf{x},\mathbf{y}) = 1 + \sum_{m=1}^{\infty} a_m(\mathbf{x})\, y_m$ with $y_m \sim U(-1,1)$ and

$$a_m(\mathbf{x}) := 0.547 m^{-2} \cos(2\pi\beta_m^1 x_1)\cos(2\pi\beta_m^2 x_2)$$

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- **INITIALIZE:** $J_P = \{\mathbf{0}, (1,0,\ldots,0)\}$ and $H_1^{\boldsymbol{\alpha}} = \mathbb{Q}_1(h)$ on uniform mesh, with $h = 2^{-4}$ **('level' 4)**.

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- ▷ **DETAIL SPACE:** Choose $H_2^{\boldsymbol{\alpha}} = \mathbb{Q}_2(h)$ and $\Delta_M = 5$.
- ▷ **ERROR ESTIMATION:** Compute $\eta \approx \|u - u_X\|_A$.
  If $\eta \leq TOL$, then **STOP**. Otherwise,
  - Improve $H_1^{\boldsymbol{\alpha}}$ (e.g. refine the mesh) for one or more $\boldsymbol{\alpha} \in J_P$, **OR**
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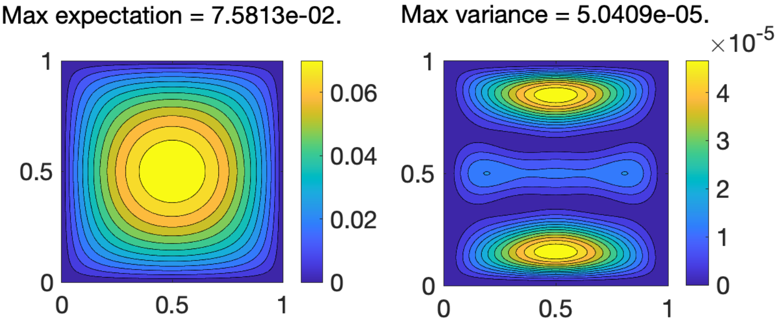
- ▷ Choose $TOL = 1.5\text{e-}3$.
- ▷ **Target convergence rate:** $N_{\mathrm{dof}}^{-1/2}$.

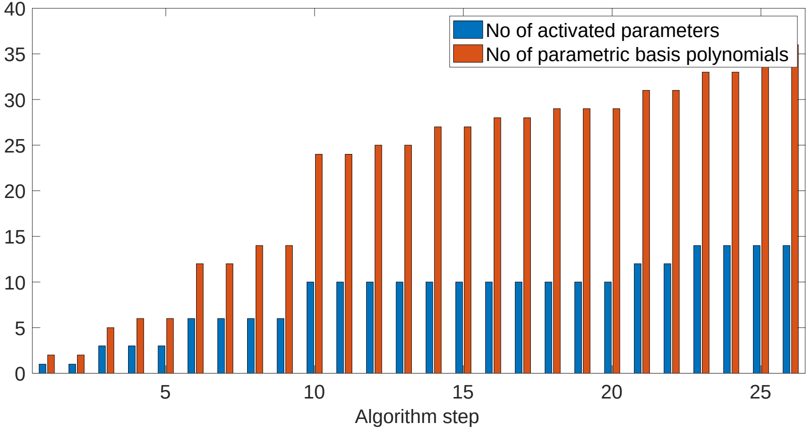
Eigel, Gittelson, Schwab, Zander. Adaptive stochastic Galerkin FEM. **CMAME** (2014).

Example 1: Final Mean & Variance




```
Time = 20.90 seconds.
Total iterations = 26.
No. parametric polynomials = 36
No. activated variables = 14.
Total DOF = 104,452.
```

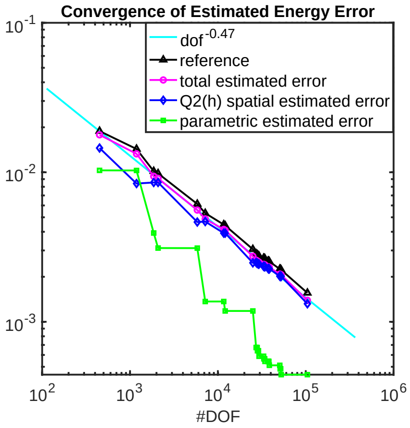
Example 1: Final Approximation Space

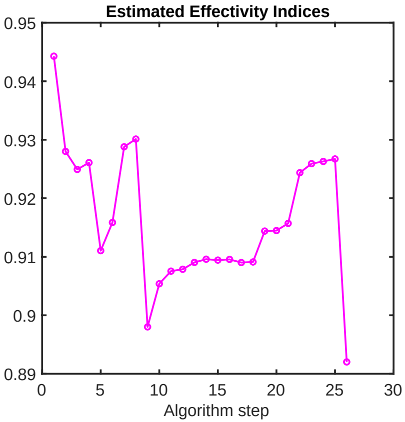


At the final step $X := \bigoplus_{\boldsymbol{\alpha} \in J_P} H_1^{\boldsymbol{\alpha}} \otimes P^{\boldsymbol{\alpha}}$

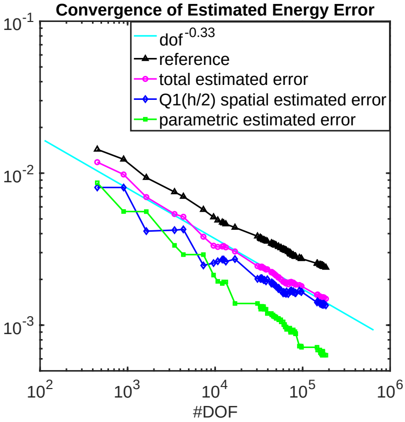
- J_P contains 36 multi-indices, $(M = 14$ **activated parameters**$)$
- $H_1^{\boldsymbol{\alpha}} = \mathbb{Q}_1(h)$ with $h = 2^{-8}, 2^{-7}, 2^{-6}, 2^{-5}, 2^{-4}$ **(1,1,3,6,25 terms)**

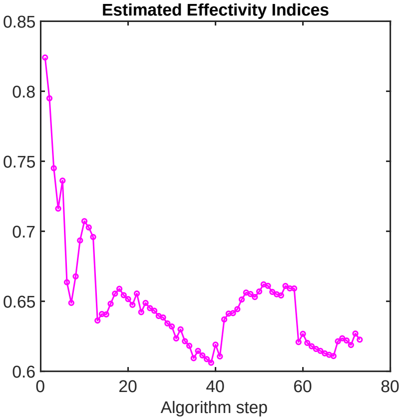
Example 1: Convergence & Accuracy





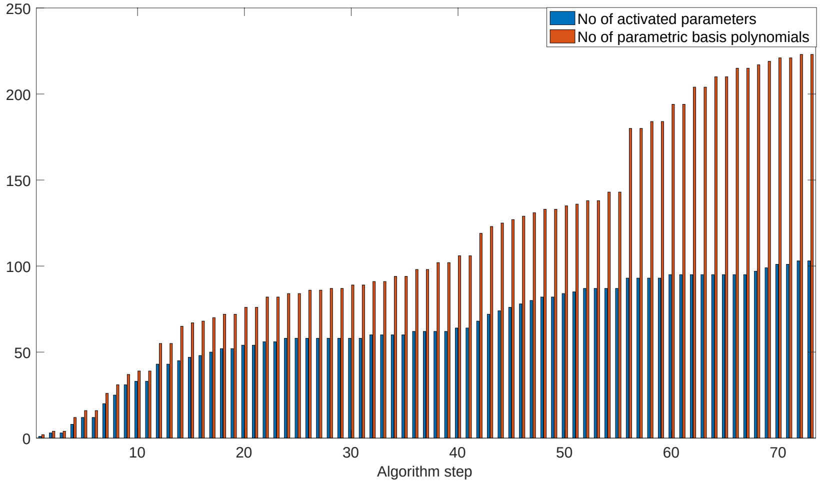
Example 2: Separable Exponential Covariance





Poorer convergence rate and error estimator is less accurate.

Example 2: Separable Exponential Covariance



At the final step $X := \bigoplus_{\boldsymbol{\alpha}\in J_P} H_1^{\boldsymbol{\alpha}} \otimes P^{\boldsymbol{\alpha}}$

- J_P contains 223 multi-indices, ($M = 103$ **activated parameters**)
- $H_1^{\boldsymbol{\alpha}} = \mathbb{Q}_1(h)$ with $h = 2^{-8}, 2^{-6}, 2^{-5}, 2^{-4}$ **(1,5,69,148 terms)**

Example 3: Synthetic KL Expansion

Test Problem: exponential decay case

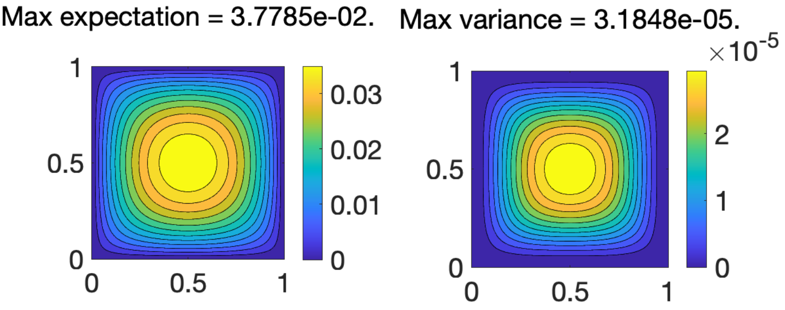
$D = [0,1]^2$, $f(\boldsymbol{x}) = 1$, $y_m \sim U(-1,1)$ and

$$a(\boldsymbol{x},\boldsymbol{y}) = 1 + \sum_{m=1}^{\infty} a_m(\boldsymbol{x}) y_m \qquad \|a_m(\boldsymbol{x})\|_\infty \sim e^{-m}.$$

- **INITIALIZE:** $J_P = \{\boldsymbol{0}, (1,0,\ldots,0)\}$ and $H_1^{\alpha} = \mathbb{Q}_2(h)$ on uniform mesh with $h = 2^{-4}$ **(level 4)**.
- **DETAIL SPACE:** Choose $H_2^{\alpha} = \mathbb{Q}_4(h)$ and $\Delta_M = 2$.
- Choose $TOL = 1\text{e}{-4}$.
- **Target convergence rate:** N_{dof}^{-1}.

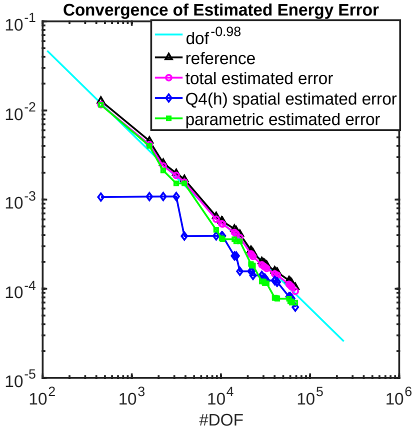
Lord, Powell, Shardlow. Introduction to Computational Stochastic PDEs, **CUP**, 2014.

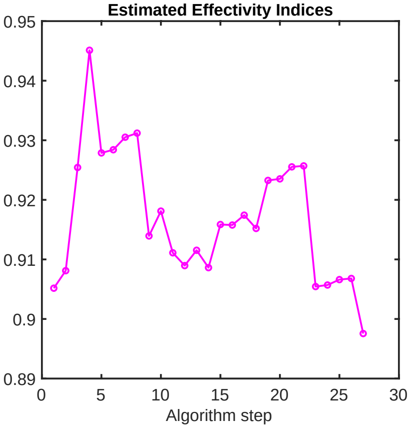
Example 3: Final Mean & Variance




```
Time = 26.80 seconds.
Total iterations = 27
No. parametric polynomials = 150
No. activated variables = 13
Total DOF = 68,246
```

Example 3: Convergence & Accuracy





Summary: Adaptive ML-SGFEM for Scalar Elliptic PDEs

- If you are prepared to exploit **structure**, SG methods can be used to build **surrogates** with automated and **rigorous error control**.

- For ‘nice enough’ problems, **multilevel** SG methods can achieve **convergence rates** associated with the chosen spatial discretization for the analogous **deterministic** problem.

- Accurate a posteriori **error estimation** is key for driving adaptive algorithms and designing X in a smart way.

- Classical **hierarchical**[1] error estimation can be modified for use in the parametric PDE setting.

- **MATLAB code:**

`https://github.com/ceapowell/ML-SGFEM`

[1] Bank & Weiser, Bank & Smith, Ainsworth & Oden

Linear Elasticity + Uncertain Young Modulus

Herrmann model for linear elasticity for **nearly incompressible** materials.

$$-\nabla \cdot \boldsymbol{\sigma}(\mathbf{u}) = \mathbf{f}, \qquad \nabla \cdot \mathbf{u} + \lambda^{-1} p = \mathbf{0}$$

where the stress + strain tensors are

$$\boldsymbol{\sigma}(\mathbf{u}) := 2\mu\, \epsilon(\mathbf{u}) - p\boldsymbol{I}, \quad \epsilon(\mathbf{u}) := (\nabla \mathbf{u} + (\nabla \mathbf{u})^\top)/2$$

+ the Lamé coefficients λ, μ are:

$$\mu(\boldsymbol{x}, \boldsymbol{y}) = \frac{E(\boldsymbol{x}, \boldsymbol{y})}{2(1+\nu)}, \quad \lambda(\boldsymbol{x}, \boldsymbol{y}) = \frac{E(\boldsymbol{x}, \boldsymbol{y})\,\nu}{(1+\nu)(1-2\nu)}.$$

- **Solution fields**: $\mathbf{u}$, p (displacement, pressure)
- **Uncertain input:** Young modulus $E(\mathbf{x}, \mathbf{y}) := e_0(\mathbf{x}) + \sum_{m=1}^{\infty} e_m(\mathbf{x})\, y_m$
- **Physical parameters:** $\nu \in (0, 1/2)$ **(Poisson ratio)**

Error Estimation: Single-Level Setting

Denote the stochastic Galerkin approximation

$$\mathbf{u}_{\mathrm{gal}} \in \widehat{\mathbf{V}} := \mathbf{V}_h \otimes \mathcal{P}, \qquad p_{\mathrm{gal}},\ \tilde{p}_{\mathrm{gal}} \in \widehat{W} := W_h \otimes \mathcal{P}.$$

Error equations

Substituting $\mathbf{u} = \mathbf{u}_{\mathrm{gal}} + e^{\mathbf{u}}$, $p = p_{\mathrm{gal}} + e^{p}$ and $\tilde{p} = \tilde{p}_{\mathrm{gal}} + e^{\tilde{p}}$ gives:

$$\begin{aligned} a(e^{\mathbf{u}}, \mathbf{v}) + b(\mathbf{v}, e^{p}) &= \mathcal{R}^{\mathbf{u}}(\mathbf{v}) \quad \forall \mathbf{v} \in \mathbf{V}, \\ b(e^{\mathbf{u}}, q) - c(e^{\tilde{p}}, q) &= \mathcal{R}^{p}(q) \quad \forall q \in W, \\ -c(e^{p}, \tilde{q}) + d(e^{\tilde{p}}, \tilde{q}) &= \mathcal{R}^{\tilde{p}}(\tilde{q}) \quad \forall \tilde{q} \in W, \end{aligned}$$

where $\mathcal{R}^{\mathbf{u}}(\mathbf{v})$, $\mathcal{R}^{p}(q)$ and $\mathcal{R}^{\tilde{p}}(\tilde{q})$ are the residuals.

▷ **Approximate** $e^{\mathbf{u}} \in \mathbf{V}$, $e^{p}, e^{\tilde{p}} \in W$ in **richer** spaces than $\widehat{\mathbf{V}}$ and $\widehat{W}$.

- Choose FEM detail spaces $\widetilde{\mathbf{V}}_h \subset \mathbf{H}_0^1(D)$, $\widetilde{W}_h \subset L^2(D)$ with

$$\mathbf{V}_h \cap \widetilde{\mathbf{V}}_h = \{\mathbf{0}\}, \qquad W_h \cap \widetilde{W}_h = \{0\},$$

such that the enriched spaces are an **inf-sup stable** pair:

$$\mathbf{V}_h^* := \mathbf{V}_h \oplus \widetilde{\mathbf{V}}_h \quad \text{and} \quad W_h^* := W_h \oplus \widetilde{W}_h.$$

Hierarchical Approach: Single-Level Setting

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- Choose a new set of multi-indices to obtain a polynomial space $\mathcal{Q}$ with

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$$\mathcal{P} \cap \mathcal{Q} = \{0\}.$$

- 'Enriched' error approximation spaces:

$$\mathbf{V}^* := \widehat{\mathbf{V}} \oplus \left(\widetilde{\mathbf{V}}_h \otimes \mathcal{P}\right) \oplus (\mathbf{V}_h \otimes \mathcal{Q}),$$
$$W^* := \widehat{W} \oplus \left(\widetilde{W}_h \otimes \mathcal{P}\right) \oplus (W_h \otimes \mathcal{Q}).$$

Reference: Khan, Bespalov, Powell., Silvester, **Math. Comp., 90(328)**, (2020).

Error Estimation: Single-Level Setting

Define detail spaces:

$$\mathbf{V}_{\rm new} = \left(\widetilde{\mathbf{V}}_h \otimes \mathcal{P}\right) \oplus (\mathbf{V}_h \otimes \mathcal{Q}), \quad W_{\rm new} = \left(\widetilde{W}_h \otimes \mathcal{P}\right) \oplus (W_h \otimes \mathcal{Q}).$$

Solve Simplified Error Equations

Find $\mathbf{e}^{\mathbf{u}}_{\rm approx} \in \mathbf{V}_{\rm new}$, $e^{p}_{\rm approx} \in W_{\rm new}$ and $e^{\tilde{p}}_{\rm approx} \in W_{\rm new}$ such that:

$$\begin{aligned} a_0(\mathbf{e}^{\mathbf{u}}_{\rm approx}, \mathbf{v}) &:= \mathcal{R}^{\mathbf{u}}(\mathbf{v}), \qquad && \forall \mathbf{v} \in \mathbf{V}_{\rm new} \\ c_0(e^{p}_{\rm approx}, q) &:= \mathcal{R}^{p}(q), && \forall q \in W_{\rm new}, \\ d_0(e^{\tilde{p}}_{\rm approx}, \tilde{q}) &:= \mathcal{R}^{\tilde{p}}(\tilde{q}), && \forall \tilde{q} \in W_{\rm new}. \end{aligned}$$

Error Estimation: Single-Level Setting

Define detail spaces:

$$\mathbf{V}_{\mathrm{new}} = \left(\widetilde{\mathbf{V}}_h \otimes \mathcal{P}\right) \oplus (\mathbf{V}_h \otimes \mathcal{Q}), \quad W_{\mathrm{new}} = \left(\widetilde{W}_h \otimes \mathcal{P}\right) \oplus (W_h \otimes \mathcal{Q}).$$

Solve Simplified Error Equations

Find $\mathbf{e}^{\mathbf{u}}_{\mathrm{approx}} \in \mathbf{V}_{\mathrm{new}}$, $e^{p}_{\mathrm{approx}} \in W_{\mathrm{new}}$ and $e^{\tilde{p}}_{\mathrm{approx}} \in W_{\mathrm{new}}$ such that:

$$\begin{aligned} a_0(\mathbf{e}^{\mathbf{u}}_{\mathrm{approx}}, \mathbf{v}) &:= \mathcal{R}^{\mathbf{u}}(\mathbf{v}), && \forall \mathbf{v} \in \mathbf{V}_{\mathrm{new}} \\ c_0(e^{p}_{\mathrm{approx}}, q) &:= \mathcal{R}^{p}(q), && \forall q \in W_{\mathrm{new}}, \\ d_0(e^{\tilde{p}}_{\mathrm{approx}}, \tilde{q}) &:= \mathcal{R}^{\tilde{p}}(\tilde{q}), && \forall \tilde{q} \in W_{\mathrm{new}}. \end{aligned}$$

A posteriori error estimate is defined as

$$\eta := (\eta_{\mathbf{u}}^2 + \eta_p^2 + \eta_{\tilde{p}}^2)^{1/2},$$

where $\eta_{\mathbf{u}} := \|\mathbf{e}^{\mathbf{u}}_{\mathrm{approx}}\|_{a_0}$, $\quad \eta_p := \|e^{p}_{\mathrm{approx}}\|_{c_0}$, $\quad \eta_{\tilde{p}} := \|e^{\tilde{p}}_{\mathrm{approx}}\|_{d_0}$.

Error Estimation: Single-Level Setting

$$\eta := (\eta_{\mathbf{u}}^2 + \eta_p^2 + \eta_{\tilde{p}}^2)^{1/2}.$$

Two-sided error bounds

$$C_1\,\eta \le |||(e^{\mathbf{u}}, e^p, e^{\tilde{p}})||| \le C_2\,\eta,$$

where C_1, C_2 are **independent** of the **discretization** parameters and ν.

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where C_1, C_2 are **independent** of the **discretization** parameters and ν.

Each component of the estimator can be **decomposed**, e.g.,

$$\eta_{\mathbf{u}}^2 = \|\mathbf{e}^{\mathbf{u}}_{\text{approx}}\|_{a_0}^2 = \|\mathbf{e}^{\mathbf{u}}_{\text{spatial}}\|_{a_0}^2 + \|\mathbf{e}^{\mathbf{u}}_{\text{param}}\|_{a_0}^2$$

where

$$\mathbf{e}^{\mathbf{u}}_{\text{spatial}} \in \widetilde{\mathbf{V}}_h \otimes \mathcal{P}, \qquad \mathbf{e}^{\mathbf{u}}_{\text{param}} \in \mathbf{V}_h \otimes \mathcal{Q}.$$

Error Estimation: Single-Level Setting

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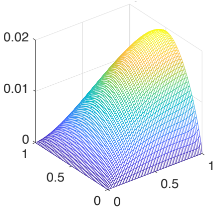
Can use separate contributions as indicators for the **error reduction** that would be achieved by enriching either (i) $\mathbf{V}_h$-W_h or (ii) $\mathcal{P}$ at the next step.

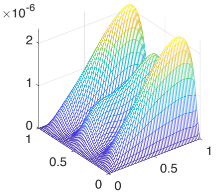
Reference: Khan, Bespalov, Powell., Silvester, **Math. Comp., 90(328)**, (2020).

Clamped Plate Problem - Revisited

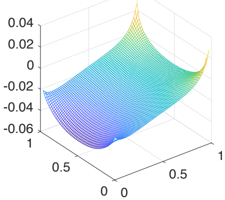
$$E(\mathbf{x},\mathbf{y}) = 1 + \sum_{m=1}^{\infty} a_m(\mathbf{x})\, y_m, \qquad \|a_m(\mathbf{x})\|_\infty \sim m^{-2}, \quad y_m \sim U(-1,1).$$

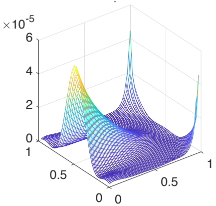
Horizontal Displacement (Mean, Variance)





Pressure (Mean, Variance)





Adaptive: Poisson ratio $\nu = 0.4$

Initial discretization:

- $\mathbf{V}_h$–$W_h = \mathbf{P}_2$–P_1 on coarse uniform mesh.
- $\mathcal{P} = \mathrm{span}\{\psi_{\boldsymbol{\alpha}}(\boldsymbol{y}), \boldsymbol{\alpha} \in J_P\}$, with $J_P = \{(0, 0, \dots,), (1, 0, \dots)\}$.

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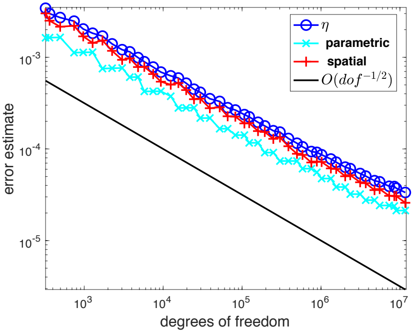
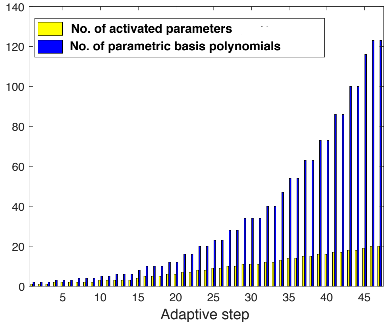
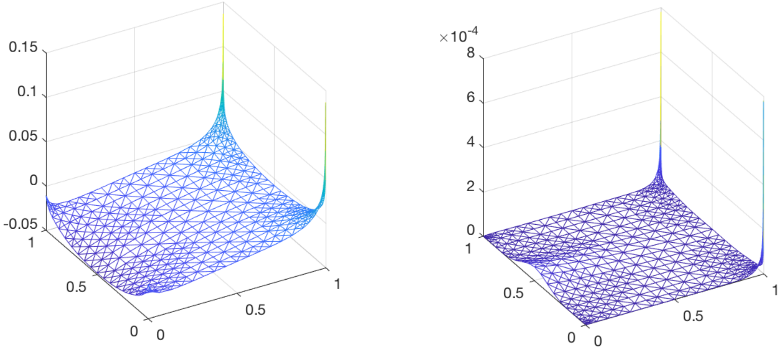




Improved Pressure Approximation ($\nu = 0.4$)



Mean (left) and **variance** (right).

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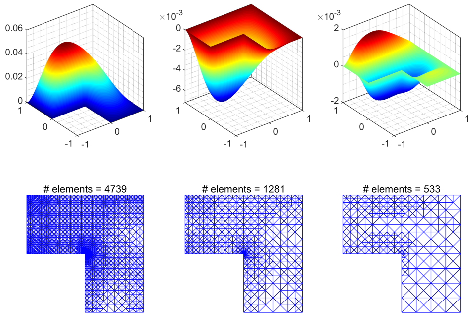
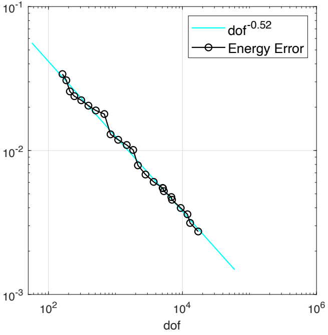
Locally Adapted Meshes

IF **SPATIAL**

- Freeze J_P
- **improve** H_1^{α} for $\alpha \in \overline{J_P}$





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