Part 1: Stochastic Galerkin Finite Element Approximation for Forward UQ in PDE Models

Catherine E. Powell

University of Manchester

catherine.powell@manchester.ac.uk

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Forward UQ: PDEs + Uncertain Inputs

Aim: Propagate **uncertainty** from model inputs to outputs

$$-\nabla \cdot (a(\mathbf{x}, \boldsymbol{y}) \nabla u(\mathbf{x}, \boldsymbol{y})) = f(\mathbf{x}), \qquad \mathbf{x} \in D \subset \mathbb{R}^{2,3}.$$

- Represent uncertain inputs as **functions of random parameters** $\boldsymbol{y} \in \Gamma$.
- Approximate QoIs related to solution (e.g., $\mathbb{E}[u]$, $\mathrm{Var}[u]$, $\mathbb{P}(\phi(u) > c)$).

Standard methods:

- Monte Carlo, Stochastic Collocation, Reduced Basis Methods, ...
- **Stochastic Galerkin (SG)** $\rightarrow$ **Not a sampling method!**

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Also known as 'intrusive' **polynomial chaos** methods:

$$u(\boldsymbol{x}, \mathbf{y}) \approx \sum_{\boldsymbol{\alpha} \in J_P} u_{\boldsymbol{\alpha}}(\boldsymbol{x}) \underbrace{\psi_{\boldsymbol{\alpha}}(\mathbf{y})}_{\text{polynomials}} .$$

References: Ghanem & Spanos (1991), Deb, Babuška & Oden (2001), Babuška, Tempone, Zouraris (2004), Frauenfelder, Schwab & Todor (2005), Matthies & Keese (2005), ...

Color Coding

- **Something associated with the uncertain <u>input parameters</u> $\boldsymbol{y} \in \Gamma$, or the chosen discretization on Γ.**

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- **Physical model parameters that are fixed constants.**
- **Statistical parameters in description of random inputs.**

Forward UQ Example: Groundwater Flow

A simple model for steady-state fluid flow in a porous medium is:

$$\begin{aligned} -a(\boldsymbol{x})\nabla p(\boldsymbol{x}) &= \boldsymbol{u}(\boldsymbol{x}) \qquad \boldsymbol{x} \text{ in } D, \\ \nabla \cdot \boldsymbol{u}(\boldsymbol{x}) &= f(\boldsymbol{x}) \qquad \boldsymbol{x} \text{ in } D. \end{aligned}$$

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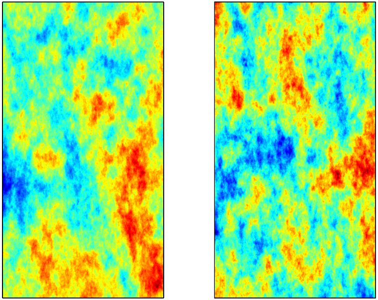
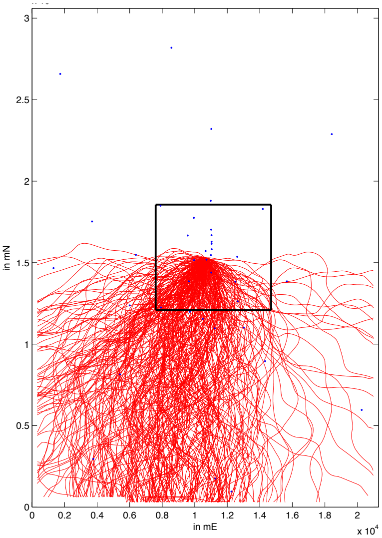
If we are **uncertain** about $a(\boldsymbol{x})$, we model it as a **random field**

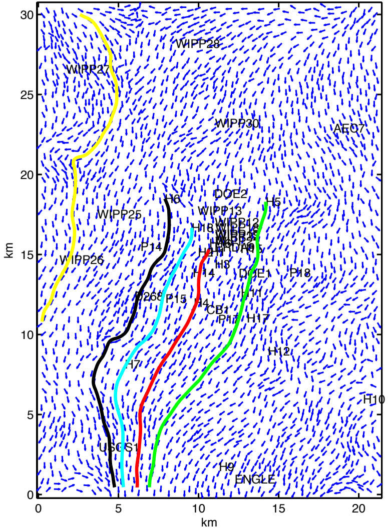
$$a(\boldsymbol{x}, \boldsymbol{\xi}), \qquad \boldsymbol{\xi} = [\xi_1, \ldots, \xi_M]^\top$$

where ξ_i have an 'appropriate' probability distribution.



Aim: Given distribution for **uncertain inputs**, estimate QoIs related to the solution $\mathbf{u}$ (velocity) and p (pressure).





(Stochastic) Galerkin Approximation

1. **Stochastic/Parametric PDE:** Find $u : D \times \Gamma \to \mathbb{R}$ such that

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with

$$0 < a_{\min} \leq a(\boldsymbol{x}, \mathbf{y}) \leq a_{\max} < \infty \quad \text{a.e. in } D \times \Gamma.$$

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2. **Weak Problem:** Find $u \in V := L^2_\pi(\Gamma, H^1(D))$ satisfying

$$\int_\Gamma \left(\int_D a \nabla u \cdot \nabla v \, d\mathbf{x} \right) \, d\pi(\mathbf{y}) = \int_\Gamma \left(\int_D f \, v \, d\mathbf{x} \right) \, d\pi(\mathbf{y}) \qquad \forall v \in V,$$

where π is a **probability measure**.

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2. **Galerkin Approximation:** Find $\widehat{u} \in \widehat{V} \subset V$ satisfying:

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3. **Linear System:** $A\mathbf{u} = \mathbf{f}$.

Properties of matrix A depend on **structure** of:

- ☐ PDE, **parameter-dependent input** $a(\boldsymbol{x}, \boldsymbol{y})$, approximation space $\widehat{V}$

Structure is Important!

- **SG methods** most competitive when PDE is **linear** and

$$a(\mathbf{x}, \mathbf{y}) = a_0(\mathbf{x}) + \sum_{m=1}^{\infty} a_m(\mathbf{x}) y_m,$$

$y_m = \xi_m(\omega)$ are images of **independent** & **bounded** random variables.

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- Standard methods use **tensor product** spaces $\widehat{V} = X \otimes \mathcal{P}$ where
  - $X \subset H^1(D)$ is a **finite element** space on D.
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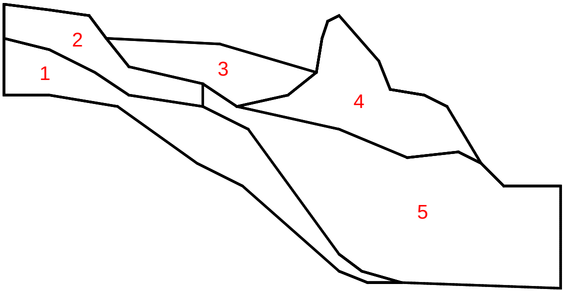
System matrix then has the form

$$A = G_0 \otimes K_0 + \sum_{m=1}^{M} G_m \otimes K_m$$

- K_0, K_m are FEM matrices, G_0, G_m are associated with $\mathcal{P}$.
- M is no. of parameters **activated** via choice of $\mathcal{P}$.



Diffusion coefficient defined **piecewise** (on subdomains):

$$a(\mathbf{x}, \boldsymbol{y}) = \sum_{i=1}^{5} \mathbb{1}_i(\mathbf{x}) y_i, \qquad \boldsymbol{x} \in D, \quad \boldsymbol{y} \in \Gamma,$$

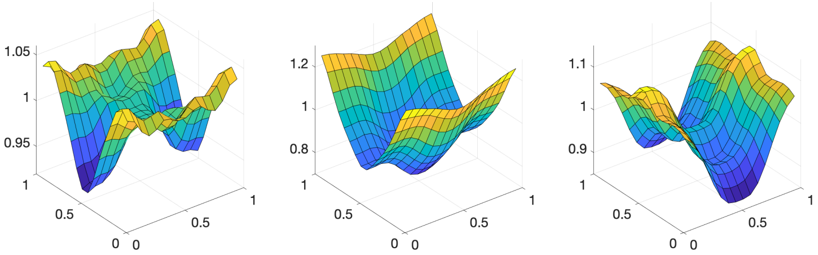
where $y_i \in \Gamma_i$ and $\Gamma := \Gamma_1 \times \Gamma_2 \times \cdots \times \Gamma_5$.

Example: Karhunen-Loève expansion

Second-order random fields $a(\boldsymbol{x}, \omega) \in L^2(\Omega, L^2(D))$ can be decomposed as

$$a\left(\boldsymbol{x}, \boldsymbol{\xi}(\omega)\right) = \underbrace{\mu(\boldsymbol{x})}_{\text{mean}} + \underbrace{\sum_{m=1}^{\infty} \sqrt{\lambda_m} \phi_m^a(\boldsymbol{x}) \xi_m(\omega)}_{\text{random part}}$$

- $(\lambda_m, \phi_m^a(\boldsymbol{x}))$ are **eigenvalues** & **eigenfunctions** associated with an integral operator associated with a **covariance function** $C(\boldsymbol{x}, \boldsymbol{x}')$.
- $\xi_1, \xi_2, \ldots$ are **uncorrelated** with mean zero and unit variance.



Tensor Product Approximation Spaces

$$\widehat{V} = X \otimes \mathcal{P}$$

- $X = \mathrm{span}\{\phi_i(\boldsymbol{x}), i = 1 : n_X\}$ is a **finite element** space on D.
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Example: Let $\boldsymbol{\alpha} = (1, 0, 2, 0, 10, 0, 0, \ldots)$. If $\psi_0(y_m) = 1$, then

$$\psi_{\boldsymbol{\alpha}}(\boldsymbol{y}) = \prod_{m=1}^{\infty} \psi_{\alpha_m}(y_m) = \psi_1(y_1)\psi_2(y_3)\psi_{10}(y_5).$$

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Orthonormal basis: We choose the polynomials $\psi_0, \psi_1, \ldots$ so that

$$\mathbb{E}\left[\psi_{\boldsymbol{\alpha}}(\boldsymbol{y})\psi_{\boldsymbol{\beta}}(\boldsymbol{y})\right] = \delta_{\boldsymbol{\alpha},\boldsymbol{\beta}}.$$

Examples: Legendre (Uniform), Hermite (Gaussian), ...

References: Xiu & Karniadakis (2002).

Useful as a Surrogate

Solving $A\mathbf{u} = \mathbf{f}$ gives the coefficients $u_{i,\boldsymbol{\alpha}}$ in the approximation

$$\widehat{u}(\boldsymbol{x}, \mathbf{y}) \;=\; \sum_{\boldsymbol{\alpha} \in J_P} \left(\sum_{i=1}^{n_X} u_{i,\boldsymbol{\alpha}} \phi_i(\boldsymbol{x}) \right) \psi_{\boldsymbol{\alpha}}(\mathbf{y}) = \sum_{\boldsymbol{\alpha} \in J_P} u_{\boldsymbol{\alpha}}(\boldsymbol{x}) \psi_{\boldsymbol{\alpha}}(\mathbf{y}).$$

Can be evaluated cheaply for any $\mathbf{y}^*$ of interest. Useful for:

- Forward UQ, Inverse UQ (e.g., in **Bayesian inverse problems**)
- Design/Optimisation, etc.

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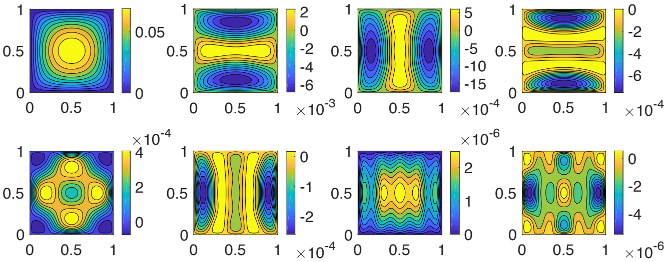
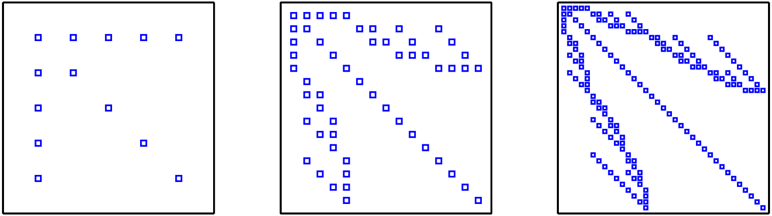
Toy Elliptic Problem: 8 spatial modes $u_{\boldsymbol{\alpha}}(\boldsymbol{x})$



Block Structure (Stochastically Linear)

$$a(\mathbf{x}, \mathbf{y}) = a_0(\mathbf{x}) + \sum_{m=1}^{\infty} a_m(\mathbf{x}) y_m \Rightarrow \textbf{sparse } G_m \text{ matrices}$$

Block sparsity of A when $\mathcal{P}$ is polynomials of **total degree** $\leq k = 1, 2, 3$.



Each $\square$ is a matrix of the same size as the **deterministic FEM problem**.

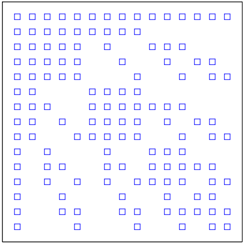
This sparsity can be exploited in the design of efficient iterative solvers.

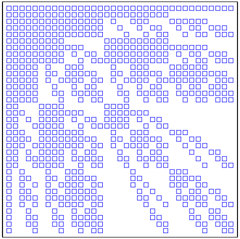
Block Structure (Stochastically Nonlinear)

Contrast with ...

$$a(\mathbf{x}, \mathbf{y}) = a_0(\mathbf{x}) + \sum_{\boldsymbol{\alpha} \in J_P} a_{\boldsymbol{\alpha}}(\mathbf{x}) \underbrace{\psi_{\boldsymbol{\alpha}}(\mathbf{y}),}_{\text{polynomial}}$$

A is **block dense** as polynomial degree k, and no. parameters M increase.





Cost of **matrix-vector products** then becomes more problematic.

Standard Krylov Methods?

Exploiting **Kronecker structure** gives

$$A\mathbf{v} = \mathrm{vec}\left(\sum_{m=0}^{M} K_m \left(G_m V^\top\right)^\top\right), \qquad V = \mathrm{array}(\mathbf{v}) \in \mathbb{R}^{n_X \times n_\mathcal{P}}.$$

Cost of matrix-vector products with A in stochastically linear case is

$$(M+1) \times O(n_X \, n_\mathcal{P}).$$

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Key challenges:

1. Finding **preconditioners** P that are cheap and robust w.r.t
   - Discretization parameters: **FEM mesh size**, (+ possibly k and M)
   - **Statistical parameters**: e.g., σ (**standard deviation** of inputs)
   - Other **physical parameters** in the PDE model
2. Need enough **memory** to store vectors of length $n_X n_{\mathcal{P}}$.

Preconditioning Strategies

$$A = G_0 \otimes K_0 + \sum_{m=1}^{M} G_m \otimes K_m.$$

- **Mean-based/Block-diagonal**

$$P = G_0 \otimes K_0$$

Pellissetti & Ghanem (2000), Le Maitre et al. (2003), P. & Elman (2009)

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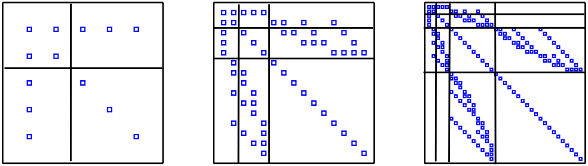
- **Kronecker Product**

$$P = G \otimes K_0, \quad \text{where } G \text{ minimizes } \quad \|A - G \otimes K_0\|_F$$

Ullmann (2010), Van Loan & Pitsianis (1993).

- **Hierarchical**



Sousedik, Ghanem & Phipps (2014), Pultarová (2016).

'Mean-Based' Preconditioning: Toy Problem

$$-\nabla \cdot (a(\mathbf{x},\mathbf{y})\nabla u(\mathbf{x},\mathbf{y})) = f(\mathbf{x}), \qquad a(\boldsymbol{x},\boldsymbol{y}) = \mu + \sigma \sum_{m=1}^{10} \sqrt{\lambda_m}\varphi_m(\boldsymbol{x}) y_m,$$

on $D = [-1,1]^2$ where $y_m \sim U(-\sqrt{3},\sqrt{3})$ and (λ_m, φ_m) are eigenpairs associated with covariance function

$$C(\boldsymbol{x},\boldsymbol{x}') = \sigma^2 \exp\left(-\frac{1}{2}\|\boldsymbol{x}-\boldsymbol{x}'\|_1\right).$$

PCG Iterations

n_X	k	n_P	$\frac{\sigma}{\mu} = 0.1$	$\frac{\sigma}{\mu} = 0.2$	$\frac{\sigma}{\mu} = 0.3$
3,963	4	1,001	**8**	**11**	**17**
	5	3,003	**8**	**12**	**20**
16,129	4	1,001	**8**	**11**	**17**
	5	3,003	**8**	**12**	**20**
65,025	4	1,001	**8**	**11**	**17**
	5	3,003*	**8**	**12**	**20**

***195 million equations**. Requires 24,024 **decoupled** PDE solves with K_0.

Linear Elasticity + Uncertain Young Modulus

Herrmann model for linear elasticity for **nearly incompressible** materials.

$$-\nabla \cdot \boldsymbol{\sigma}(\mathbf{u}) = \mathbf{f}, \qquad \nabla \cdot \mathbf{u} + \lambda^{-1} p = \mathbf{0}$$

where the stress + strain tensors are

$$\boldsymbol{\sigma}(\mathbf{u}) := 2\mu\, \epsilon(\mathbf{u}) - p\boldsymbol{I}, \quad \epsilon(\mathbf{u}) := (\nabla \mathbf{u} + (\nabla \mathbf{u})^\top)/2$$

+ the Lamé coefficients λ, μ are:

$$\mu(\boldsymbol{x}, \boldsymbol{y}) = \frac{E(\boldsymbol{x}, \boldsymbol{y})}{2(1+\nu)}, \quad \lambda(\boldsymbol{x}, \boldsymbol{y}) = \frac{E(\boldsymbol{x}, \boldsymbol{y})\,\nu}{(1+\nu)(1-2\nu)}.$$

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- **Solution fields**: $\mathbf{u}$, p (displacement, 'pressure')
- **Uncertain input:** Young modulus $E(\mathbf{x}, \mathbf{y}) := e_0(\mathbf{x}) + \sum_{m=1}^{\infty} e_m(\mathbf{x})\, y_m$
- **Physical parameters:** $\nu \in (0, 1/2)$ **(Poisson ratio)**

Stochastically Linear Three-field Formulation

$$E(\mathbf{x}, \mathbf{y}) := e_0(\mathbf{x}) + \sum_{m=1}^{\infty} e_m(\mathbf{x})\, y_m$$

To avoid a stochastically **nonlinear** problem + maintain **SPARSITY** in the SGFEM matrix we can introduce an additional auxiliary variable

$$\tilde{p} = \frac{1}{E} p := -\underbrace{\frac{\lambda}{E}}_{=:\tilde{\lambda}} \nabla \cdot \mathbf{u}$$

and solve a **three-field formulation**:

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$$-\nabla \cdot \boldsymbol{\sigma}(\mathbf{u}) = \mathbf{f}$$

$$\nabla \cdot \mathbf{u} + \frac{1}{\tilde{\lambda}} \tilde{p} = 0$$

$$\frac{1}{\tilde{\lambda}} p - \frac{E}{\tilde{\lambda}} \tilde{p} = 0$$

in which E but **not** E^{-1} appears.

Well-Posed Weak Formulation

Find $(\mathbf{u}, p, \tilde{p}) \in \mathbf{V} \times W \times W$ such that

$$
\begin{aligned}
a(\mathbf{u}, \mathbf{v}) + b(\mathbf{v}, p) &= f(\mathbf{v}) && \forall \mathbf{v} \in \mathbf{V}, \\
b(\mathbf{u}, q) - c(\tilde{p}, q) &= 0 && \forall q \in W, \\
-c(p, \tilde{q}) + d(\tilde{p}, \tilde{q}) &= 0 && \forall \tilde{q} \in W,
\end{aligned}
$$

where $\mathbf{V} := L^2_\pi(\Gamma, \mathbf{H}^1_0(D))$ and $W := L^2_\pi(\Gamma, L^2(D))$.

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where $\mathbf{V} := L^2_\pi(\Gamma, \mathbf{H}^1_0(D))$ and $W := L^2_\pi(\Gamma, L^2(D))$.

Assume that:

- $0 < E_{\min} \leq E(\boldsymbol{x}, \mathbf{y}) \leq E_{\max} < \infty \quad \text{a.e. in } D \times \Gamma$
- $0 < e_0^{\min} \leq e_0(\boldsymbol{x}) \leq e_0^{\max} < \infty \quad \text{a.e. in } D.$

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$$\begin{aligned} a(\mathbf{u},\mathbf{v}) + b(\mathbf{v},p) &= f(\mathbf{v}) && \forall \mathbf{v} \in \mathbf{V}, \\ b(\mathbf{u},q) - c(\tilde{p},q) &= 0 && \forall q \in W, \\ -c(p,\tilde{q}) + d(\tilde{p},\tilde{q}) &= 0 && \forall \tilde{q} \in W, \end{aligned}$$

where $\mathbf{V} := L^2_\pi(\Gamma, \mathbf{H}^1_0(D))$ and $W := L^2_\pi(\Gamma, L^2(D))$.

Assume that:

- $0 < E_{\min} \le E(\boldsymbol{x},\mathbf{y}) \le E_{\max} < \infty \quad \text{a.e. in } D \times \Gamma$
- $0 < e_0^{\min} \le e_0(\boldsymbol{x}) \le e_0^{\max} < \infty \quad \text{a.e. in } D.$

There **exists a unique** solution $(\mathbf{u}, p, \tilde{p}) \in \mathbf{V} \times W \times W$ satisfying

$$|||(\mathbf{u},p,\tilde{p})||| \le \underbrace{(C/E_{\min})\,\alpha^{-1/2}}_{\text{bounded as } \nu \to 1/2} ||\mathbf{f}||_{L^2(D)}$$

where **weighted** norm $|||\cdot|||$ is defined via

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$$|||(\mathbf{u},p,\tilde{p})|||^2 := \alpha\,||\nabla\mathbf{u}||^2_{\mathbf{W}} + (\alpha^{-1} + \tilde{\lambda}^{-1})||p||^2_W + \tilde{\lambda}^{-1}||\tilde{p}||^2_W.$$

Stochastic Galerkin Mixed FEM

We need to choose **compatible** approximation spaces.

- $\boldsymbol{V}_h \subset \boldsymbol{H}_0^1(D)$, $W_h \subset L^2(D)$ (**inf-sup stable** FEM pair).

$$\sup_{0 \neq \mathbf{v} \in \mathbf{V}_h} \frac{\int_D q \, \nabla \cdot \mathbf{v}}{||\nabla \mathbf{v}||_{\mathbf{L}^2(D)}} \geq \beta \, ||q||_{L^2(D)} \quad \forall q \in W_h$$

with $\beta > 0$ independent of FEM mesh parameter h.

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The associated weak problem also has a **unique solution** that is bounded w.r.t $||| \cdot |||$ as the **Poisson ratio** $\nu \to 1/2$.

Reference: Khan, Powell., Silvester, **SIAM J. Comp. Sci., 41(1)**, (2019).

Stochastic Galerkin Matrix

We obtain a **symmetric** and **indefinite** coefficient matrix

$$\left(\begin{array}{ccc|c}
\alpha \sum_{m=0}^{M} G_m \otimes A_{11}^m & \alpha \sum_{m=0}^{M} G_m \otimes A_{12}^m & \mathbf{0} & G_0 \otimes B_1^\top \\
\alpha \sum_{m=0}^{M} G_m \otimes A_{21}^m & \alpha \sum_{m=0}^{M} G_m \otimes A_{22}^m & \mathbf{0} & G_0 \otimes B_2^\top \\
\mathbf{0} & \mathbf{0} & \tilde{\lambda}^{-1} \sum_{m=0}^{M} G_m \otimes D_m & -\tilde{\lambda}^{-1} G_0 \otimes C \\
\hline
G_0 \otimes B_1 & G_0 \otimes B_2 & -\tilde{\lambda}^{-1} G_0 \otimes C & \mathbf{0}
\end{array}\right)$$

where α and $\tilde{\lambda}$ depend on the **Poisson ratio** ν.

Saddle Point Structure

$$\begin{pmatrix} \mathcal{A} & \mathcal{B}^\top \\ \mathcal{B} & \mathbf{0} \end{pmatrix}$$

Schur-complement Preconditioning

Ideal preconditioner

$$P = \begin{pmatrix} \mathcal{A} & \mathbf{0} \\ \mathbf{0} & \mathcal{S} \end{pmatrix}, \quad \text{where} \quad \mathcal{S} := \mathcal{B}\mathcal{A}^{-1}\mathcal{B}^\top.$$

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- We can approximate $\mathcal{A}$ with the **block-diagonal** matrix

$$\mathcal{A}_{approx} := \begin{pmatrix} \alpha G_0 \otimes \mathbb{A}_0 & 0 & 0 \\ 0 & \alpha G_0 \otimes \mathbb{A}_0 & 0 \\ 0 & 0 & \tilde{\lambda}^{-1} G_0 \otimes D_0 \end{pmatrix}$$

where $\mathbb{A}_0$ is a standard FEM stiffness matrix associated with $-\nabla \cdot e_0 \nabla$.

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- We can approximate $\mathcal{S}$ with the **block-diagonal** matrix

$$\mathcal{S}_{approx} := (\alpha^{-1} + \tilde{\lambda}^{-1})\, G_0 \otimes C,$$

and C is a pressure FEM mass matrix since it can be shown that

$$\frac{\beta^2}{e_0^{\max}} \leq \frac{\mathbf{w}^{\top}\mathcal{B}\mathcal{A}_{approx}^{-1}\mathcal{B}^{\top}\mathbf{w}}{\mathbf{w}^{\top}\mathcal{S}_{approx}\mathbf{w}} \leq \frac{2}{e_0^{\min}}.$$

Preconditioner

Block-diagonal Preconditioner

$$P = \left(\begin{array}{ccc|c} \alpha G_0 \otimes \mathbb{A}_0 & 0 & 0 & 0 \\ 0 & \alpha G_0 \otimes \mathbb{A}_0 & 0 & 0 \\ 0 & 0 & \tilde{\lambda}^{-1} G_0 \otimes D_0 & 0 \\ \hline 0 & 0 & 0 & (\alpha^{-1} + \tilde{\lambda}^{-1})\, G_0 \otimes C \end{array}\right)$$

▷ **Eigenvalue bounds** for preconditioned system are independent of

- **Poisson ratio** ν
- finite element **mesh size** h
- polynomial degree k and M (assuming bounded random variables)

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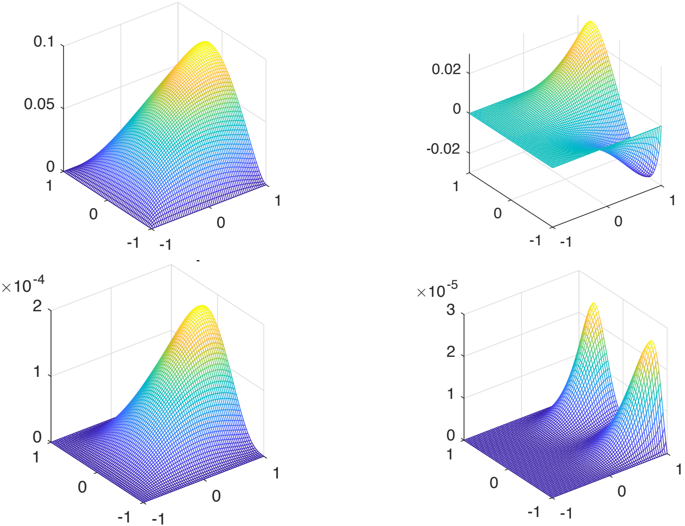
  - Korn constant C_K, inf-sup constant β.

- **Cost of applying** P^{-1}
  - $2n_{\mathcal{P}}$ **decoupled** solves with $\mathbb{A}_0$ (Poisson solves)
  - $n_{\mathcal{P}}$ **decoupled** solves with the FEM mass matrices C and D_0.

Test Problem: Clamped Plate Problem

$\mathbb{E}[u_x]$ (top) and $Var[u_x]$ (bottom) for $\nu = .4$ (left), $\nu = 0.49999$ (right).



Preconditioned MINRES ($\sigma/\mu = 0.1$)

$$E(\mathbf{x},\mathbf{y}) := \mu + \sigma \sum_{i=1}^{\infty} \sqrt{\lambda_i}\varphi_i(\mathbf{x})\, y_i, \qquad y_i \sim U(-1,1).$$

▷ $\widehat{\mathbf{V}} \times \widehat{W} \times \widehat{W} = \mathbf{Q}_2 \times P_{-1} \times P_{-1}$ (Mixed FEM),

▷ $\mathcal{P}$: polynomials of total degree $k \leq 3$ in M variables

---

M	$\nu = .4$	$\nu = .49$	$\nu = .499$	$\nu = .4999$	$\nu = .49999$
			$h = 1/32$		
5	**58**	**76**	**79**	**79**	**79**
10	**58**	**77**	**79**	**80**	**81**
			$h = 1/64$		
5	**58**	**77**	**79**	**79**	**79**
10	**58**	**77**	**81**	**81**	**81**

▷ Timings scale **linearly** with the problem size $n_X n_{\mathcal{P}}$.

Preconditioned MINRES ($\sigma/\mu = 0.2$)

$$E(\mathbf{x},\mathbf{y}) := \mu + \sigma \sum_{i=1}^{\infty} \sqrt{\lambda_i}\varphi_i(\mathbf{x})\, y_i, \qquad y_i \sim U(-1,1).$$

▷ $\widehat{\mathbf{V}} \times \widehat{W} \times \widehat{W} = \mathbf{Q}_2 \times P_{-1} \times P_{-1}$ (Mixed FEM),

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M	$\nu = .4$	$\nu = .49$	$\nu = .499$	$\nu = .4999$	$\nu = .49999$
$h = 1/32$					
5	**70**	**91**	**95**	**95**	**95**
10	**72**	**93**	**98**	**98**	**98**
$h = 1/64$					
5	**70**	**91**	**96**	**96**	**96**
10	**72**	**94**	**98**	**98**	**99**

▷ Timings scale **linearly** with the problem size $n_X n_{\mathcal{P}}$.

Biot Consolidation Model (Poroelasticity)

▷ **Applications:** geophysical flows, fluid flow in central nervous system.

$$
\begin{aligned}
-\nabla \cdot \boldsymbol{\sigma}(\mathbf{u}) &= \mathbf{f}, \\
-\nabla \cdot \mathbf{u} - \lambda^{-1}(p_T - \alpha p_F) &= 0 \\
\lambda^{-1}(\alpha p_T - \alpha^2 p_F) - s_0 p_F + \nabla \cdot (\kappa(\boldsymbol{x}, \boldsymbol{y}) \nabla p_F) &= g,
\end{aligned}
$$

where, again, the stress tensor is $\boldsymbol{\sigma} := 2\mu\epsilon(\mathbf{u}) - p_T \mathbf{I}$ and

$$
\mu = \frac{E(\boldsymbol{x}, \boldsymbol{y})}{2(1+\nu)}, \quad \lambda = \frac{E(\boldsymbol{x}, \boldsymbol{y})\,\nu}{(1+\nu)(1-2\nu)}.
$$

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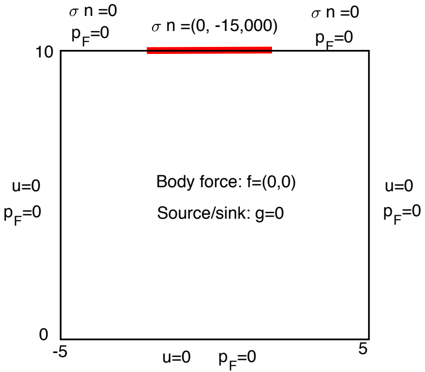
$$

- ☐ **Solution fields**: $\mathbf{u}$ p_F, p_T (displacement, fluid pressure, total pressure)
- ☐ **Multiple physical parameters:** ν, α, s_0
- ☐ **Uncertain inputs:** Young modulus $E(\boldsymbol{x}, \boldsymbol{y})$, conductivity field $\kappa(\boldsymbol{x}, \boldsymbol{y})$.

References: Lee, Mardal & Winther (2017), Oyarzúa & Ruiz–Baier (2016), **Khan** & P., (2021).

Example: Footing Problem



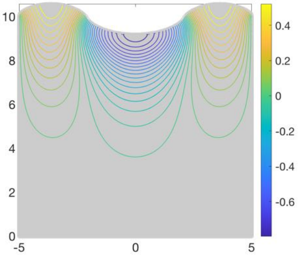
▷ **Uncertain Young's modulus & hydraulic conductivity:**

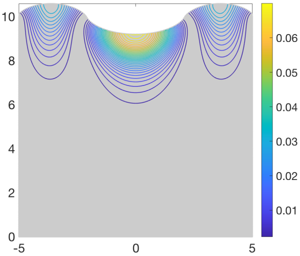
$$E = e_0 + e_1 y_1, \qquad \kappa = \kappa_0 + \kappa_1 y_2, \qquad y_1, y_2 \sim U(-1, 1)$$

where $e_0 = 3 \times 10^4$, $e_1 = 0.5 \times e_0$, and $\kappa_0 = 10^{-4}$, $\kappa_1 = 0.5 \times \kappa_0$.

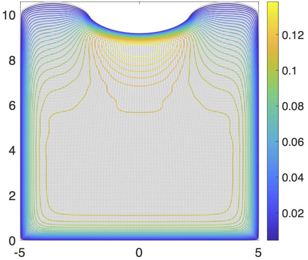
Mean (left) & Variance (right), $\nu = 0.4995$, $\tilde{s}_0 = 30$

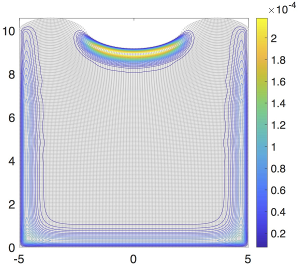
Vertical displacement





Fluid Pressure





Mean-Based Schur-Complement Preconditioner

Saddle Point Structure

$$\begin{pmatrix} \mathcal{A} & \mathcal{B}^\top \\ \mathcal{B} & -\mathcal{C} \end{pmatrix}, \qquad P = \begin{pmatrix} \mathcal{A} & \mathbf{0} \\ \mathbf{0} & \mathcal{S} \end{pmatrix}.$$

Using standard **operator** arguments, can find mean-based/block-diagonal approximations that respect the **'right norm'**.

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$$\mathcal{A}_{\mathrm{approx}} = \begin{pmatrix} \tilde{\mu} G_0 \otimes \mathbb{A}_0 & 0 & 0 & 0 \\ 0 & \tilde{\mu} G_0 \otimes \mathbb{A}_0 & 0 & 0 \\ 0 & 0 & \tilde{\lambda}^{-1} G_0 \otimes \tilde{C}_0 & 0 \\ 0 & 0 & 0 & \tilde{s}_0\, G_0 \otimes \tilde{C}_0 \end{pmatrix},$$

$$\mathcal{S}_{\mathrm{approx}} = \begin{pmatrix} (\alpha^2 \tilde{\lambda}^{-1} + \tilde{s}_0) G_0 \otimes (\overline{C}_b + D_0) & 0 \\ 0 & (\tilde{\mu} + \tilde{\lambda}^{-1}) G_0 \otimes C \end{pmatrix}.$$

For bounded random inputs, eigenvalue bounds independent of **physical parameters** and **SGFEM discretization** parameters.

Footing Problem: Preconditioned MINRES

- ▷ $\mathbf{Q}_2$–Q_1–Q_1 (Mixed FEM), $\mathcal{P}$: polynomials of total degree $k \leq 3$.
- ▷ $E = 10^5 + 10^4 y_1$ and $\kappa = \kappa_0 + \kappa_1 y_2$, where $\kappa_1/\kappa_0 = 0.1$.

---

		h	$\nu = .4$	$\nu = .499$	$\nu = .49999$
$\kappa_0 = 1$	$\alpha = 1$	2^{-5}	**56**	**71**	**71**
		2^{-6}	**56**	**71**	**71**
	$\alpha = 10^{-4}$	2^{-5}	**55**	**70**	**70**
		2^{-6}	**55**	**70**	**70**
$\kappa_0 = 10^{-5}$	$\alpha = 1$	2^{-5}	**60**	**72**	**71**
		2^{-6}	**60**	**73**	**72**
	$\alpha = 10^{-4}$	2^{-5}	**56**	**70**	**70**
		2^{-6}	**56**	**71**	**71**
$\kappa_0 = 10^{-10}$	$\alpha = 1$	2^{-5}	**71**	**70**	**72**
		2^{-6}	**72**	**73**	**73**
	$\alpha = 10^{-4}$	2^{-5}	**58**	**71**	**71**
		2^{-6}	**58**	**71**	**71**

Convergence **independent** of **discretization** and **physical parameters**.

Matrix Equation Formulation

- What if I can't store vectors of length $n_X n_{\mathcal{P}}$?

Matrix Equation Formulation

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Reshape vector $\mathbf{u}$ into a matrix $U = [\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_{n_P}]$ and rewrite $A\mathbf{u} = \mathbf{f}$ as

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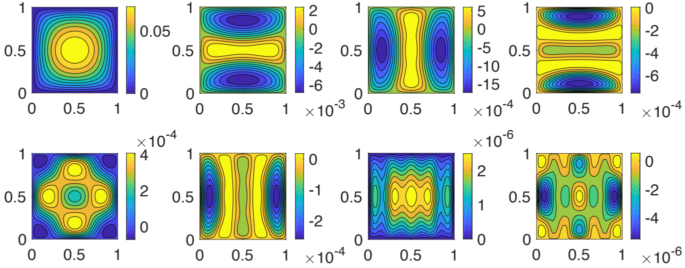
Each **column** of U represents a different **spatial mode** in

$$\widehat{u}(\boldsymbol{x}, \mathbf{y}) = \sum_{\boldsymbol{\alpha} \in J_P} u_{\boldsymbol{\alpha}}(\boldsymbol{x}) \psi_{\boldsymbol{\alpha}}(\mathbf{y}) \in X \otimes \mathcal{P}.$$

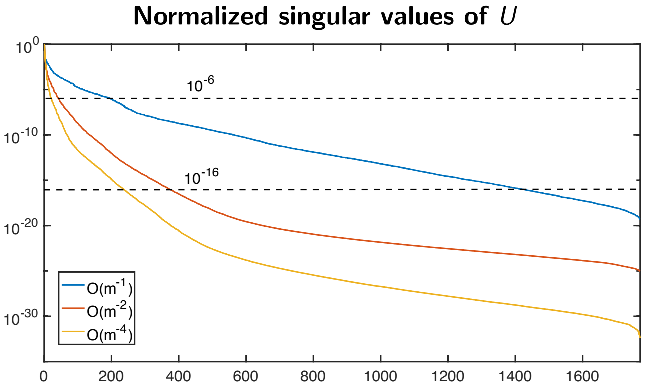


Example: Singular Values of Solution Matrix

$$-\nabla \cdot (a(\mathbf{x},\mathbf{y})\nabla u(\mathbf{x},\mathbf{y})) = f(\mathbf{x}), \quad a(\boldsymbol{x},\boldsymbol{y}) = a_0(\boldsymbol{x}) + \sum_{m=1}^{\infty} a_m(\boldsymbol{x}) y_m$$

Three Cases: $\|a_m\|_\infty \sim O(m^{-1}), O(m^{-2}), O(m^{-4})$.

SGFEM discretisation: $n_X = 3,969$, $n_{\mathcal{P}} = 1,771$.



Galerkin Projection

If U is **low rank**, approximate $U \in \mathbb{R}^{n_X \times n_{\mathcal{P}}}$ by

$$U_k = V_k Y_k, \qquad V_k \in \mathbb{R}^{n_X \times n_k}, \; Y_k \in \mathbb{R}^{n_k \times n_{\mathcal{P}}}.$$

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If A is **symmetric & positive definite** then we minimize

$$\frac{1}{2}\|U - U_k\|_A^2$$

if we can find V_k and Y_k such that $V_k^\top R_k = 0$ and $R_k Y_k^\top = 0$.

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1. $(V_k^\top K_0 V_k) Y_k G_0 + \sum_{m=1}^{M} (V_k^\top K_m V_k) Y_k G_m = V_k^\top F$
2. $K_0 V_k (Y_k G_0 Y_k^\top) + \sum_{m=1}^{M} K_m V_k (Y_k G_m Y_k^\top) = F\, Y_k^\top$

Various strategies for **alternating** between these two equations and **adaptively** constructing factors V_k and Y_k.

References: Anthony Nouy (2007, 2010), [. . . . . .], Kookjin Lee (2022).

Summary

- SG methods can be used to build **surrogates**.
- For **stochastically linear** problems, highly efficient solvers can be designed if one is prepared to exploit structure.
- **Mean-based preconditioners** allow re-use of existing FEM solvers.
- More sophisticated solvers needed for problems with stochastically **nonlinear** inputs, **high variance** and/or **unbounded** random variables.
- For **saddle-point problems**, usual recipes to construct parameter-robust block-diagonal preconditioners apply.
- Galerkin-based projection methods for **matrix equation formulation** promising, but several open questions.
- **Low rank** structure often stems from naive choice of $\widehat{V} = X \otimes \mathcal{P}$.

References: UQ, SGFEM & Approximation Theory

Books

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