An introduction to quantum algorithms

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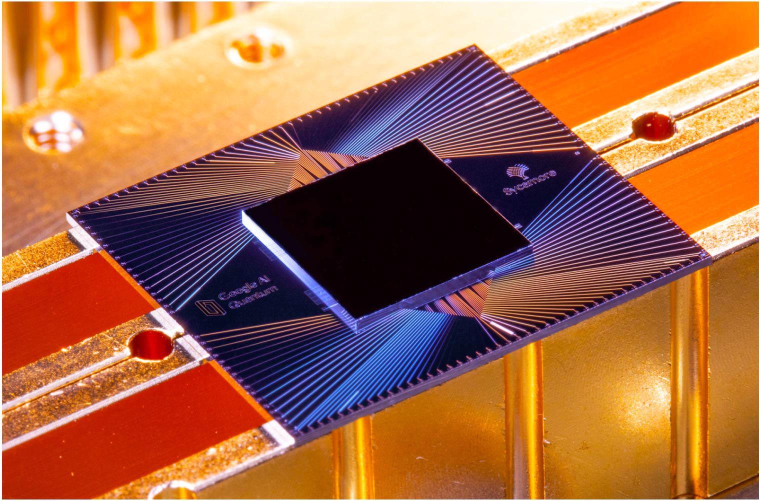

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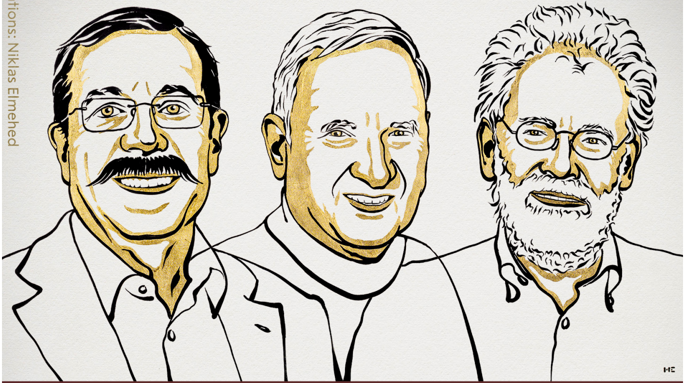

Alain Aspect · John F. Clauser · Anton Zeilinger

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Algorithm	**Speedup**	**Example**
Simulation of quantum systems	Exponential	Lloyd
Breaking cryptographic codes	Exponential	Shor
Optimisation / combinatorial search	Square-root	Grover
High-dimensional linear algebra	Exponential?	HHL
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The Quantum Algorithm Zoo currently lists 430 papers on quantum algorithms.

Quantum simulation

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Quantum systems are represented by Hamiltonians: exponentially big matrices H represented in an efficient way, e.g. the Heisenberg model $H = \sum_{\langle i,j \rangle} X_i X_j + Y_i Y_j + Z_i Z_j$.

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- Even very simple quantum systems can be universal analogue quantum simulators [Cubitt, AM, Piddock, 1701.05182]
- Analogue quantum simulators with > 50 qubits have been implemented experimentally (e.g. [Zhang et al, 1708.01044])

Digital simulation

Dynamics simulation

Given a Hamiltonian H describing a physical system, and an initial state $|\psi_0\rangle$ of that system, produce the state

$$|\psi_t\rangle = e^{-iHt}|\psi_0\rangle.$$

Given such an output state, measurements can be performed to determine quantities of interest about the state.

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- No efficient classical algorithm is known for this task (in full generality), but efficient quantum algorithms exist for many physically reasonable cases.
- A topic of very active research (e.g. [Childs et al 1711.10980])

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- One approach: optimise over quantum circuits using a variational algorithm [McClean et al 1509.04279].

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Theorem [Shor quant-ph/9508027]

There is a quantum algorithm which finds the prime factors of an n-digit integer in time $O(n^3)$.

Shor's algorithm: complexity comparison

Very roughly (ignoring constant factors!):

Number of digits	Timesteps (quantum)	Timesteps (classical)
100	10^6	$\sim 4 \times 10^5$
1,000	10^9	$\sim 5 \times 10^{15}$
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- The fastest computer on the Top500 supercomputer list ($\sim 10^{17}$ operations per second) in $\sim 3 \times 10^{16}$ years.

(see e.g. [Gidney+Ekerå 1905.09749] for a more detailed analysis, showing that a 2048-digit integer can be factorised in 8 hours with 23 million physical qubits)

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- On a classical computer, this task could require 2^n queries to f in the worst case. But on a quantum computer, Grover's algorithm [Grover quant-ph/9605043] can solve the problem with $O(\sqrt{2^n})$ queries to f (and bounded failure probability).

Applications of Grover's algorithm

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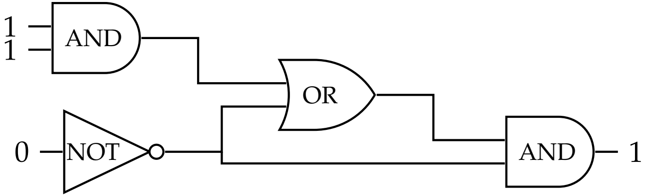
- For example, in the Circuit SAT problem we would like to find an input to a circuit on n bits such that the output is 1:



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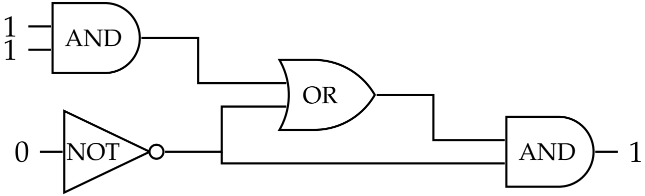

- Grover's algorithm improves the runtime from $O(2^n)$ to $O(2^{n/2})$: applications to design automation, circuit equivalence, model checking, ...

Applications of Grover's algorithm

An important generalisation of Grover's algorithm is known as amplitude amplification.

Amplitude amplification [Brassard et al quant-ph/0005055]

Assume we are given access to a "checking" function f, and a probabilistic algorithm $\mathcal{A}$ such that

$$\Pr[\mathcal{A} \text{ outputs } w \text{ such that } f(w) = 1] = \epsilon.$$

Then we can find w such that $f(w) = 1$ with $O(1/\sqrt{\epsilon})$ uses of f.

Gives a quadratic speed-up over classical algorithms which are based on heuristics.

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These primitives can be used to obtain many speedups over classical algorithms, e.g.:

- Finding the minimum of n numbers in $O(\sqrt{n})$ time
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They can also speed up Monte Carlo methods [AM 1504.06987, Hamoudi+Magniez 1807.06456]:

- The mean of a random variable with variance σ^2 can be approximated up to ϵ in time roughly $O(\sigma/\epsilon)$, as opposed to the classical $O(\sigma^2/\epsilon^2)$.

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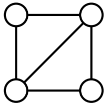

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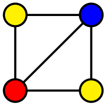

Backtracking algorithms solve CSPs by "trial and error": exploring a tree of partial solutions.

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Theorem [AM 1509.02374] (informal)

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Some applications:

- Quantum speedup of the Travelling Salesman Problem on bounded-degree graphs [Moylett, Linden and AM 1612.06203]
- Finding shortest vectors in lattices for cryptographic applications [Alkim et al. '15, del Pino et al. '16]
- Accelerating classical branch-and-bound algorithms for optimisation problems [AM 1906.10375]

“Solving” linear equations

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Theorem: If A has condition number κ $(= \|A^{-1}\| \|A\|)$, $|x\rangle$ can be approximately produced in time $\mathrm{poly}(\log N, d, \kappa)$ [Harrow et al 0811.3171]

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- "Solving" differential equations [Leyton+Osborne 0812.4423] [Berry 1010.2745]
- Recommendation systems and other problems in machine learning (e.g. [Kerenidis+Prakash 1603.08675]) – but note "quantum-inspired" competition [Tang 1807.04271]!

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Examples:

- The adiabatic algorithm / quantum annealing [Farhi et al quant-ph/0001106]
- The Quantum Approximate Optimisation Algorithm (QAOA) [Hogg+Portnov quant-ph/0006090, Farhi et al 1411.4028]

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Evidence that they outperform classical algorithms is mixed, but we at least know they are probably hard to simulate classically [Farhi+Harrow 1602.07674].

Analysing real quantum algorithm complexity

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Some fully worked-out applications with large speedups (for quantum runtime ~ 1 day) include:

- Nitrogen fixation [Reiher et al 1605.03590]
- Many-body localisation [Childs et al 1711.10980]
- Other problems in quantum chemistry and condensed-matter physics, e.g. [Babbush et al 1805.03662]
- Integer factorisation [Kutin quant-ph/0609001] [Gidney and Ekerå 1905.09749]

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In constraint satisfaction the speedups are smaller and quantum hardware requirements larger…

- Graph colouring / boolean satisfiability: speedup factor of $\sim 10^5$ (ignoring cost of fault-tolerance processing) but $\sim 10^{12}$ physical qubits required [Campbell et al 1810.05582]

Conclusions

There are many quantum algorithms, solving many different problems, some of which achieve substantial speedups over their classical counterparts.

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Important future research directions include:

- Finding more practical applications for these algorithms;
- Analysing their complexity in detail;
- New ideas for quantum algorithm design;
- Getting the most out of near-term quantum computers.

Further reading:

- Quantum algorithms: an overview [AM, 1511.04206]
- Quantum algorithm design: techniques and applications [Shao et al, Journal of Systems Science and Complexity, 2019]
- Noisy intermediate-scale quantum (NISQ) algorithms [Bharti et al, 2101.08448]
- Tutorial talk by Andrew Childs at QIP 2021 `https://www.cs.umd.edu/~amchilds/talks/qip21.pdf`

Thanks!