Adaptive numerical homogenization:
Linearization procedure for multi-scale problems

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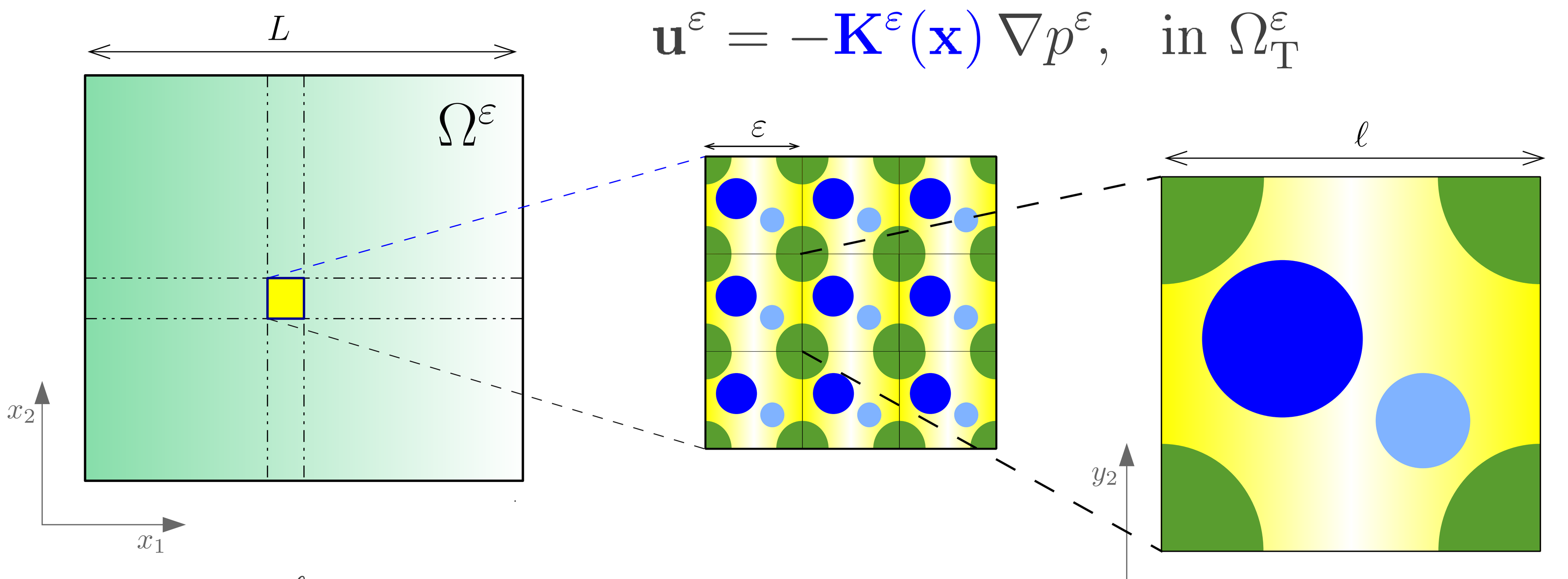
Introduction

- **Non-linear parabolic** problems: partially saturated flow, non-steady filtration and reaction-diffusion systems.
- Main idea: To develop an **efficient** numerical strategy for simulating processes in porous media involving oscillatory characteristics.

Problem definition and upscaling

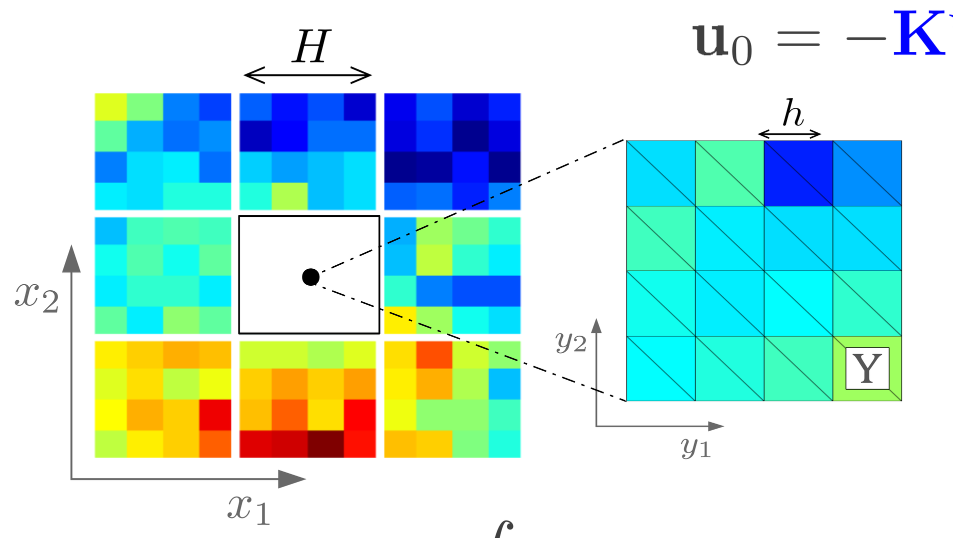
$$\partial_t b^\varepsilon(\mathbf{x}, p^\varepsilon) + \nabla\cdot\mathbf{u}^\varepsilon = f^\varepsilon, \quad \text{in } \Omega^\varepsilon_T$$
$$\mathbf{u}^\varepsilon = -\mathbf{K}^\varepsilon(\mathbf{x})\,\nabla p^\varepsilon, \quad \text{in } \Omega^\varepsilon_T$$



where $\varepsilon := \frac{\ell}{L} \rightarrow$ **Scale separation**.

- **Upscaled model**: effective parameters homogeneous within sub-domains.

$$\partial_t b^\star(\mathbf{x}, p_0) + \nabla\cdot\mathbf{u}_0 = f^\star, \quad \text{in } \Omega_T$$
$$\mathbf{u}_0 = -\mathbf{K}^\star(\mathbf{x})\,\nabla p_0, \quad \text{in } \Omega_T$$



Micro-scale problems:

$$-\nabla_y\cdot\left(\mathbf{K}\left(\nabla_y\omega^j + \mathbf{e}_j\right)\right) = 0$$

ω^j is Y-periodic

$$\mathbf{K}^\star_{i,j}(\mathbf{x}) = \int_Y \left(\mathbf{K}(\mathbf{x},\mathbf{y})\left(\mathbf{e}_j + \nabla_y\omega^j(\mathbf{x},\mathbf{y})\right)\right)\cdot\mathbf{e}_i\, d\mathbf{y}$$

$$b^\star(\mathbf{x},p) := \int_Y b(\mathbf{x},\mathbf{y},p)\,d\mathbf{y} \quad \text{and} \quad f^\star(\mathbf{x},t) := \int_Y f(\mathbf{x},\mathbf{y},t)\,d\mathbf{y}.$$

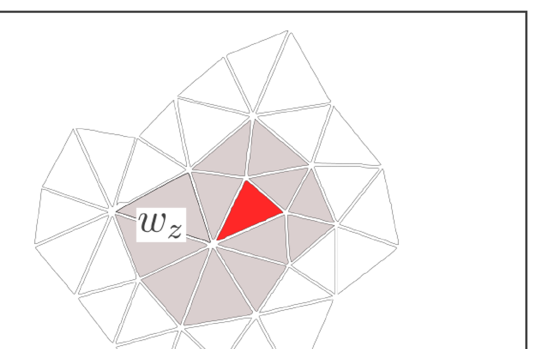
Adaptive strategy

Error control by the flux averaging

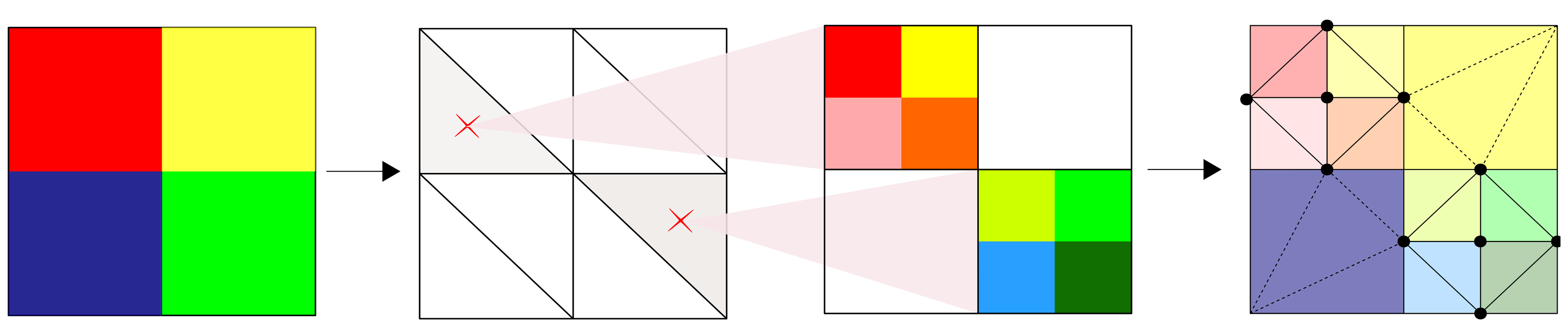
$$\eta_{\mathcal{T}} := \|\mathbf{u}_H - \mathfrak{A}\mathbf{u}_H\|_{L^2(\mathcal{T})}$$

The elements to be refined are $\mathcal{T}\in\mathfrak{T}_H$:

$$\eta_{\mathcal{T}} \geq \Theta\left(\max_{\mathcal{K}\in\mathfrak{T}_H}\eta_{\mathcal{K}}\right)$$



$$\mathfrak{A}\mathbf{u}_H(z) = \frac{1}{|w_z|}\int_{w_z}\mathbf{u}_H$$



References

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Linearization (Upscaled model)

Take $\mathfrak{L} \geq \max_{p\in\mathbb{R}}\{\partial_p b^\star\}$ and $p^0_n := p(t_{n-1})$ given.
While $\|p^i_n - p^{i-1}_n\| \leq \delta$ solve:

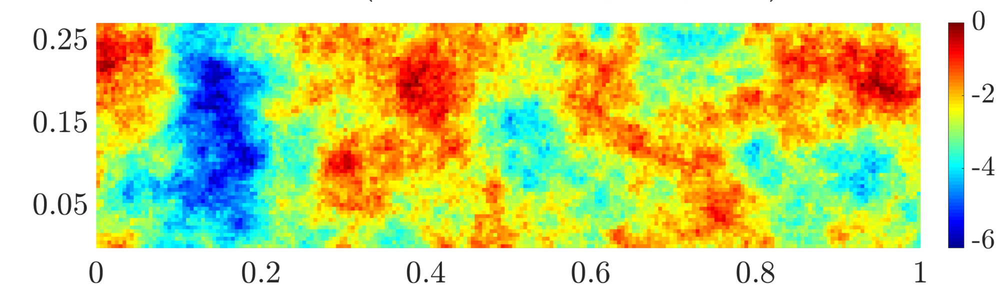
$$\mathfrak{L}\left(p^i_n - p^{i-1}_n\right) + b^\star\left(p^{i-1}_n\right) + \Delta t\,\nabla\cdot\mathbf{u}^i_n = \Delta t\, f^\star + b^\star(p_{n-1})$$
$$\mathbf{u}^i_n - \mathbf{K}^\star\,\nabla p^i_n = 0$$

- Fixed point iteration scheme.
- Unconditional convergence.
- Convergence rate: $\alpha = \frac{\mathfrak{L}-m}{\mathfrak{L}+C\Delta t}$ for some $C>0$ and $0<m<\mathfrak{L}$.

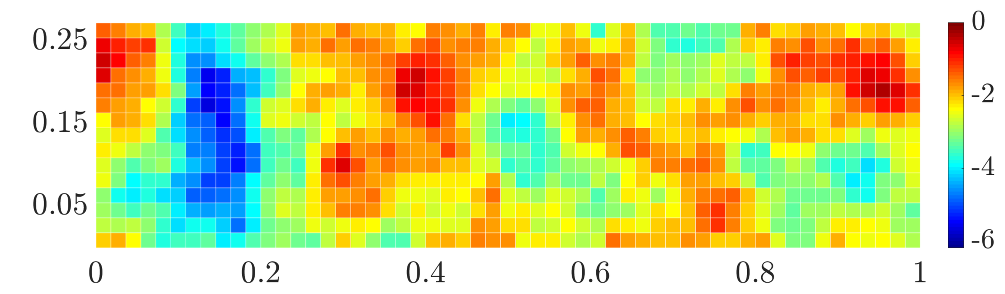
Numerical Results

Highly heterogeneous, non-periodic medium (SPE10th project)

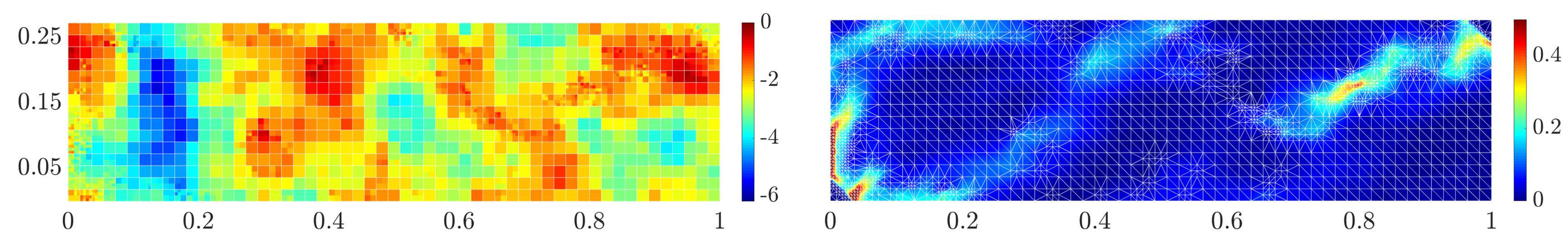
- $b^\varepsilon(p^\varepsilon) = \mathcal{R}\cdot(p^\varepsilon)^3$
- $\mathrm{T} = 1$
- $\Delta t = 0.02$
- $\mathfrak{L} \geq \frac{\max\left(3\cdot\mathcal{R}\cdot(p^\varepsilon)^2\right)}{2}$
- $\Theta = 0.2$



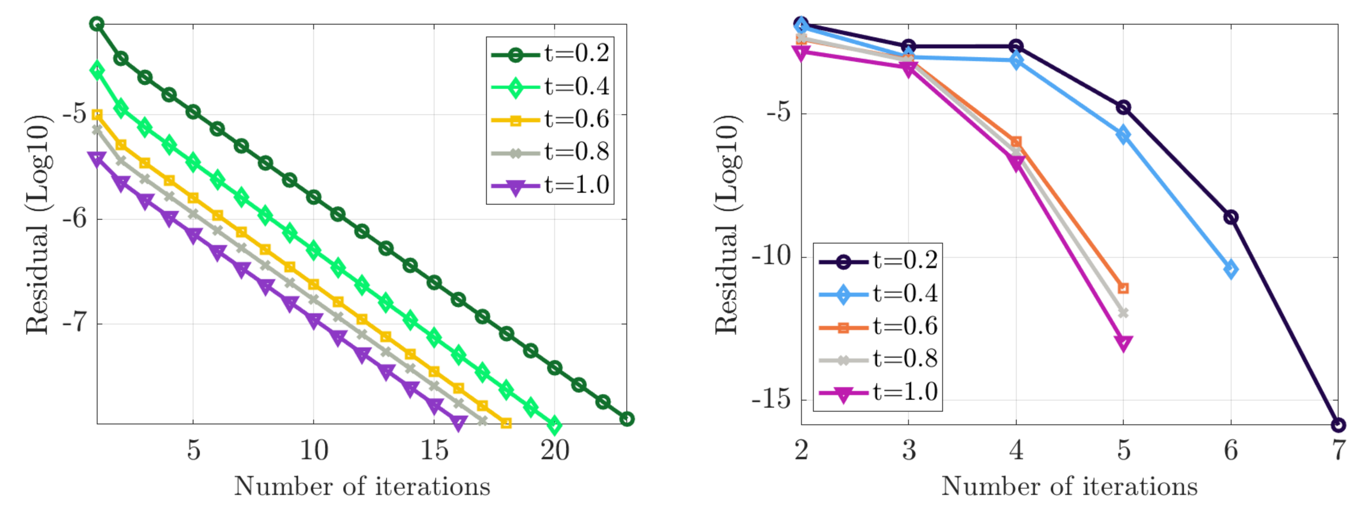
Fine-scale permeability (Log10)



Coarse-scale permeability (55 × 15)



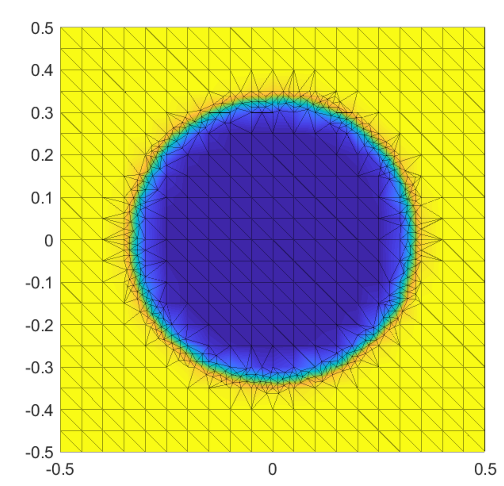
Refined permeability (L); Magnitude of the velocity field (R)



Convergence L-scheme (L); Improved convergence L-scheme & Newton (R)

Conclusions and future work

- Backward Euler and lowest order Raviart-Thomas MFEM.
- Improved accuracy without compromising the efficiency.
- Update macro-scale parameters only when it is necessary.
- Unconditional convergence of the linear solver.



- Complex micro-scale models.
- Reactive transport.
- Moving interfaces.
- **Phase-field modelling.**

Acknowledgements and contact

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