Symplectic Model Order Reduction for the Seismic Wave Problem

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DeepNL: Comprehensive monitoring and predicting seismicity within Groningen gas field using large-scale field observations

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Summary

- Modeling the physics of seismic wave propagation including source mechanism is computationally expensive and challenging.
- Model Order Reduction (MOR) can speed up the calculations, but standard MOR methods experience bad convergence rates and instabilities for wave-type problems.
- In the following, we present a symplectic-MOR strategy to construct stable reduced models (with optimal error decay) for the forward problem.

Model Problem

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain representing isotopic, homogeneous medium with Lipschitz boundary $\Gamma = \partial\Omega$, such that $\bar{\Gamma} = \bar{\Gamma}_N \cup \bar{\Gamma}_D$, $\Gamma_N \cap \Gamma_D = \Phi$, $|\Gamma_D| > 0$. We are interested in approximating $\boldsymbol{u} = (u_1, u_2)^{\mathrm{T}}$ that solves

$$\rho\, \ddot{\boldsymbol{u}} - \nabla \cdot \boldsymbol{\sigma}(\boldsymbol{u}) = \boldsymbol{f} \quad \text{in } \Omega \times [0,T],$$
$$\boldsymbol{\sigma}(\boldsymbol{u}).\mathbf{n} = 0 \quad \text{on } \Gamma_N \times [0,T],$$
$$\boldsymbol{u} = 0 \quad \text{on } \Gamma_D \times [0,T],$$
$$\boldsymbol{u}(x,0) = 0 \text{ and } \dot{\boldsymbol{u}}(x,0) = 0 \quad \text{in } \Omega,$$

where ρ denotes the density, $\boldsymbol{f}$ the source (spatial Gaussian point source and temporal sinc signal), $\mathbf{n}$ the outer normal of Ω, and $\dot{\boldsymbol{u}} = \partial u / \partial t$.

For a linear elastic material with plane strain conditions, we have

- Cauchy stress tensor: $\boldsymbol{\sigma}(\boldsymbol{u}) = \mathbf{C} : \boldsymbol{\varepsilon}(\boldsymbol{u}) = \sum_{k,l}^{2} \mathbf{C}_{ijkl}\, \boldsymbol{\varepsilon}_{kl}$,
- infinitesimal strain tensor: $\boldsymbol{\varepsilon}(\boldsymbol{u}) = 0.5\,(\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^T)$,
- fourth-order stiffness tensor: $\mathbf{C}_{ijkl} = \lambda \delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$, where
- λ and μ are the Lame parameters, and δ_{ij} is the Kronecker delta.

Laplace transformation:

$\mathcal{L}(\boldsymbol{u}(x,t)) = \hat{\mathbf{u}}(x,s) = \int_0^\infty \boldsymbol{u}(x,t) \exp(-st)dt$, where, $s = s_R + i s_I \in \mathbb{C}$, and $s_R, s_I \in \mathbb{R}$.

Weak formulation using finite element method (FEM):

Given a frequency $s \in \mathbb{C}$, *find* $\hat{\mathbf{u}} \in V = \{\hat{\boldsymbol{v}} \in [H^1(\Omega; \mathbb{C}^2)]^2 : \hat{\boldsymbol{v}}|_{\Gamma_D} = 0\}$ *where,*

$$s^2 \rho \int_\Omega \hat{\mathbf{u}} \cdot \hat{\boldsymbol{v}}^c\, dx + \int_\Omega \varepsilon(\hat{\mathbf{u}}) : \varepsilon(\hat{\boldsymbol{v}}^c)\, dx = \int_\Omega \hat{\boldsymbol{f}}(s) \cdot \hat{\boldsymbol{v}}^c dx, \qquad \forall\, \hat{\boldsymbol{v}} \in V.$$

Discretized problem using FEM:

$$s^2 \mathbf{M}\, \hat{\mathbf{u}} + \mathbf{K}\, \hat{\mathbf{u}} = \hat{\mathbf{F}}, \text{ where } \mathbf{M} \in \mathbb{R}^{n\times n},\ \boldsymbol{K} \in \mathbb{R}^{n\times n},\ \hat{\mathbf{F}} \in \mathbb{C}^n, \text{ and } \hat{\mathbf{u}} \in \mathbb{C}^n.$$

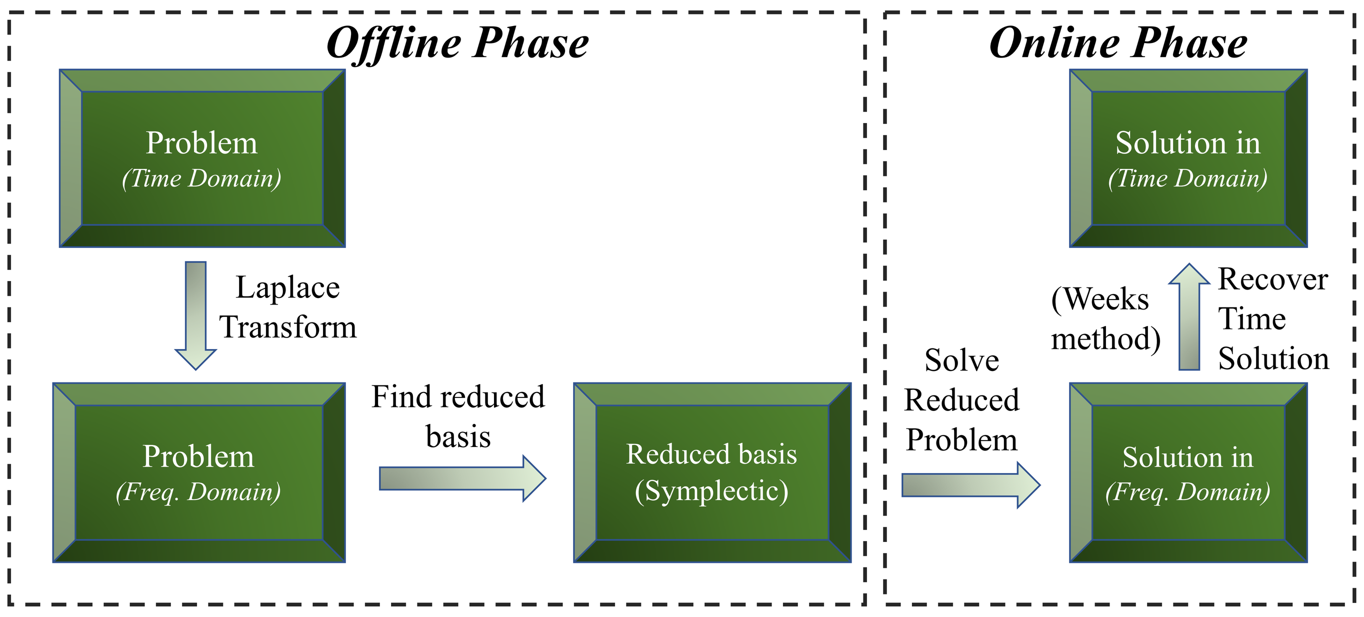
Weeks method

Numerical Laplace inversion performed using Weeks method:

$$\boldsymbol{u}(x,t) = e^{(s_R - s_I)t} \sum_{k=0}^{N_z - 1} \left(\frac{s_I}{N_z} \sum_{j=-N_z}^{N_z - 1} \frac{e^{-ik\theta_{j+1/2}}}{1 - e^{i\theta_{j+1/2}}} \hat{\mathbf{u}}(x, s^j) \right) L_k(2 s_I t),$$

where, $s^j = s_R + i s_I \cot(\frac{\theta_{j+1/2}}{2})$, $\theta_j = \frac{j\pi}{N_z}$, $L_k(\cdot)$ are the Laguerre polynomials of degree k, and N_z the number of frequencies points.

Methodology



Symplectic-MOR

- We seek a low-dimensional subspace $B \subset V$, such that

  $r = \dim(B) \ll \dim(V) = n$, and $s \mapsto \hat{\mathbf{u}}(x,s)$
- Estimation of the symplectic basis matrix (B_{2r}) is performed using Algorithm 1, such that, $\hat{\mathbf{u}}(s) \approx \mathrm{B}_{2r} \hat{\mathbf{u}}_{red}(s)$

Algorithm 1: Symplectic–Greedy: Laplace Domain

Result: B_{2r}: Symplectic (reduced basis)
Input: Full-order model (FOM), Training set ($s \in \Xi_{n_s}$), Tolerance (δ), r_{max}

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B_2r = [ ], r = 1, δ_r = δ + 1
Starting point: s^r = Ξ_{n_s}(1)
while δ_r > δ & r < r_max do
    û(s^r) ← Solve the FOM for s^r
    [ζ^r, J^T_{2n} ζ^r] ← Symplectic orthonormalization of û(s^r)
    B_2r ← Symplectic extension [B_2r] ∪ [ζ^r, J^T_{2n} ζ^r]
    ROM ← Symplectic projections using B_2r
    for s ∈ Ξ_{n_s} do
        û_red(s) ← Solve the ROM for s
        Δ_r(s) ← Error measure
    end
    [δ_{r+1}, s^{r+1}] ← max Δ_r(s)
    r ← r + 1
end
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Results

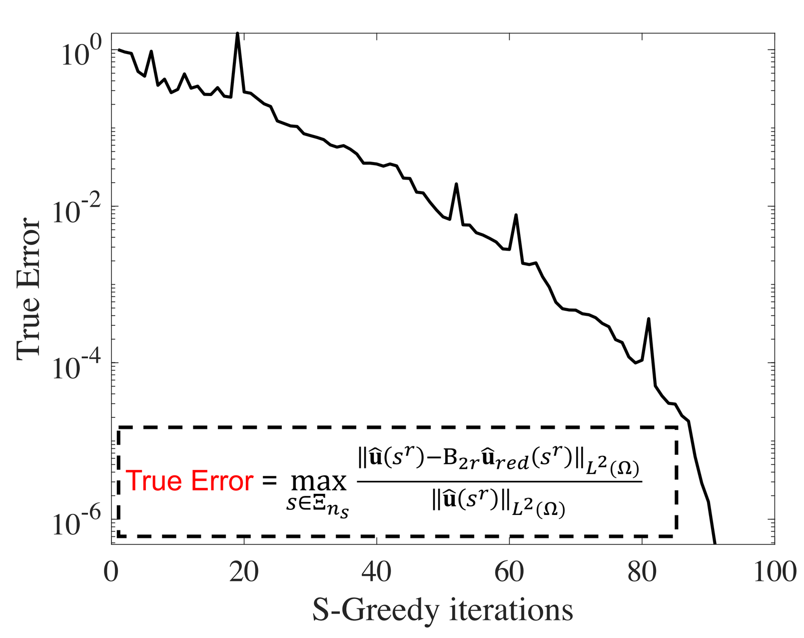


Figure: Error decay illustrating reduction from n=40000 to r=90 (Laplace domain).

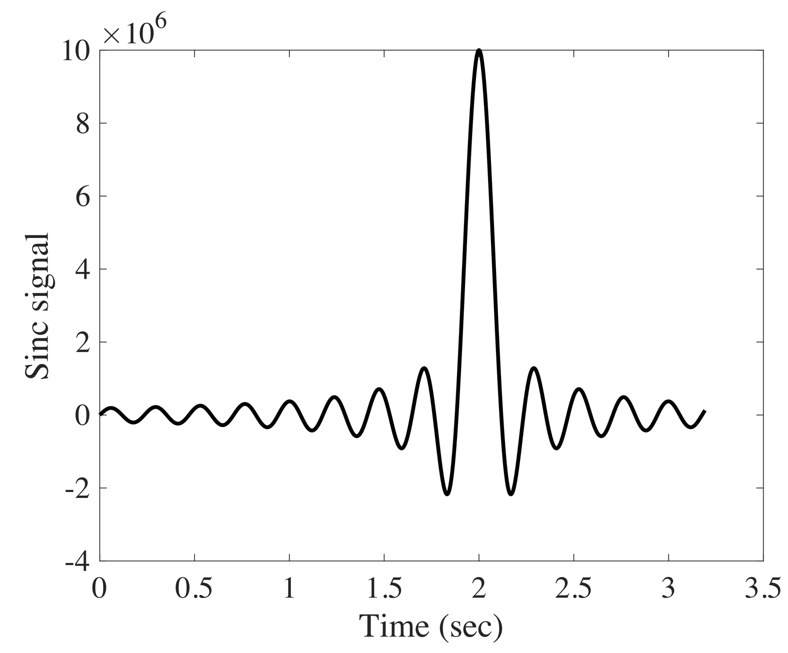
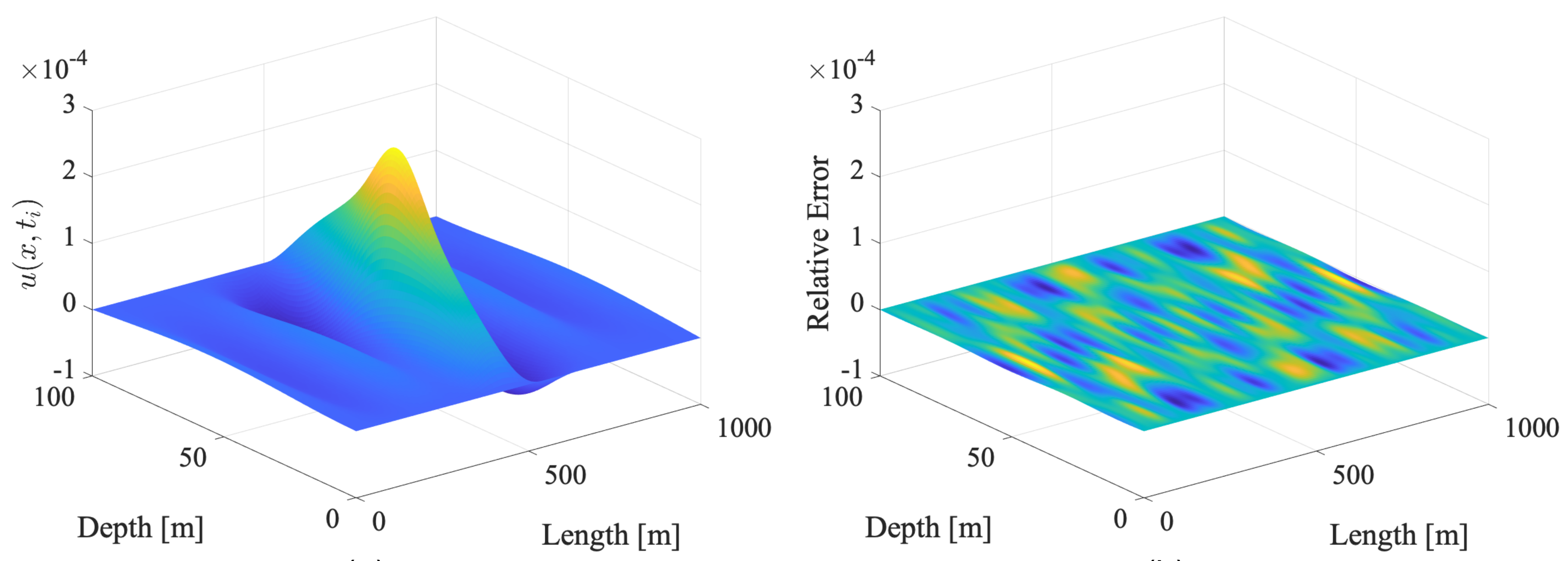


Figure: Time domain sinc signal with a width of 8.5.



(a) (b)

Figure: Comparison of S-MOR with the FOM approximations at the peak of sinc signal, i.e., t_i = 2 sec (a) S-MOR (b) Relative error FOM vs S-MOR.

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References

1. Weeks, W. T. (1966). Numerical inversion of Laplace transforms using Laguerre functions. *Journal of the ACM (JACM)*, *13*(3), 419-429.
2. Maday, Y., Patera, A. T., & Turinici, G. (2002). A priori convergence theory for reduced-basis approximations of single-parameter elliptic partial differential equations. *Journal of Scientific Computing*, *17*(1), 437-446.
3. Kano, P. O., Brio, M., & Moloney, J. V. (2005). Application of Weeks method for the numerical inversion of the Laplace transform to the matrix exponential. *Communications in Mathematical Sciences*, *3*(3), 335-372.
4. Buffa, A., Maday, Y., Patera, A. T., Prud'homme, C., & Turinici, G. (2012). A priori convergence of the greedy algorithm for the parametrized reduced basis method. *ESAIM: Mathematical modelling and numerical analysis*, *46*(3), 595-603.
5. Peng, L., & Mohseni, K. (2016). Symplectic model reduction of Hamiltonian systems. *SIAM Journal on Scientific Computing*, *38*(1), A1-A27.
6. Buchfink, P., Bhatt, A., & Haasdonk, B. (2019). Symplectic model order reduction with non-orthonormal bases. *Mathematical and Computational Applications*, *24*(2), 43.
7. Bigoni, C., & Hesthaven, J. S. (2020). Simulation-based anomaly detection and damage localization: an application to structural health monitoring. *Computer Methods in Applied Mechanics and Engineering*, *363*, 112896.