Energy-conserving formulation of the two-fluid model for incompressible two-phase flow in channels and pipes

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Energy conservation for multidimensional incompressible flow

The Euler equations can be written in conservative form:

$$\frac{\partial \mathbf{q}}{\partial t} + \boldsymbol{\nabla} \cdot \mathbf{f}(\mathbf{q}) = \mathbf{0}, \tag{1}$$

$$\mathbf{q} = \begin{bmatrix} q_1 \\ \mathbf{q}_2 \end{bmatrix}, \quad q_1 = \rho, \quad \mathbf{q}_2 = \rho \mathbf{u}, \quad \mathbf{f}(\mathbf{q}) = \begin{bmatrix} \mathbf{q}_2 \\ \frac{1}{q_1}\mathbf{q}_2 \otimes \mathbf{q}_2 + p\mathbf{I} \end{bmatrix},$$

with the constraint $\boldsymbol{\nabla} \cdot (\mathbf{q}_2/q_1) = 0.$ (2)

The local kinetic energy is defined by

$$e = \frac{1}{2}\frac{\mathbf{q}_2 \cdot \mathbf{q}_2}{q_1}, \quad \text{with} \quad \mathbf{v} := \left[\frac{\partial e}{\partial \mathbf{q}}\right]^T = \begin{bmatrix} -\frac{1}{2}\frac{\mathbf{q}_2 \cdot \mathbf{q}_2}{q_1^2} \\ \frac{\mathbf{q}_2}{q_1} \end{bmatrix}.$$

We can prove it is conserved by taking the dot product of (1) with $\mathbf{v}$, manipulating derivatives, and substituting (2) to get:

$$\frac{\partial e}{\partial t} + \boldsymbol{\nabla} \cdot \left(\frac{1}{2}\frac{\mathbf{q}_2 \cdot \mathbf{q}_2}{q_1^2}\mathbf{q}_2\right) + \boldsymbol{\nabla} \cdot \left(\frac{\mathbf{q}_2}{q_1}p\right) = 0,$$

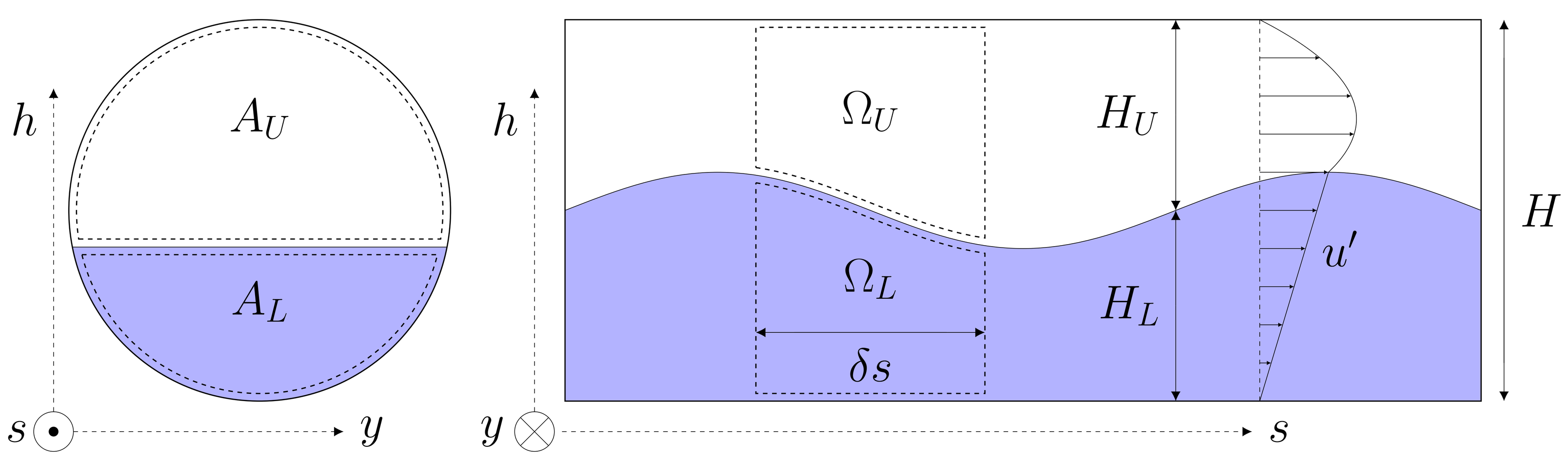
which can be integrated to show that the global energy E is conserved:

$$\frac{\mathrm{d}E}{\mathrm{d}t} = 0, \quad \text{with} \quad E = \int_\Omega e \,\mathrm{d}\Omega.$$

Central interpolation on a staggered grid leads to an energy-conserving discretization which prevents (nonlinear) numerical instability [1].

One-dimensional incompressible two-fluid model (1D TFM)

Can be derived by considering integral mass and momentum balances for control volumes Ω_U and Ω_L for two stratified fluids in a duct separately:



Taking the limit $\delta s \to 0$ yields a system of PDEs describing mass and momentum conservation in terms of cross-sectionally averaged quantities:

$$\frac{\partial \mathbf{q}}{\partial t} + \frac{\partial \mathbf{f}(\mathbf{q})}{\partial s} + \mathbf{d}(\mathbf{q})\frac{\partial p}{\partial s} = \mathbf{0}, \tag{3}$$

$$\mathbf{q} = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix} = \begin{bmatrix} \rho_U A_U \\ \rho_L A_L \\ \rho_U u_U A_U \\ \rho_L u_L A_L \end{bmatrix}, \quad \mathbf{f}(\mathbf{q}) = \begin{bmatrix} q_3 \\ q_4 \\ \frac{q_3^2}{q_1} - \rho_U g_n \widehat{H}_U(A_U) \\ \frac{q_4^2}{q_2} - \rho_L g_n \widehat{H}_L(A_L) \end{bmatrix}, \quad \mathbf{d}(\mathbf{q}) = \begin{bmatrix} 0 \\ 0 \\ \frac{q_1}{\rho_U} \\ \frac{q_2}{\rho_L} \end{bmatrix},$$

geometric terms related to the duct shape and the interface height:

$$\widehat{H}_L = \int_{a_L} (h - H_L)\,\mathrm{d}a, \qquad \widehat{H}_U = \int_{a_U} (h - H_L)\,\mathrm{d}a,$$

and the constraint $\frac{\partial Q}{\partial s} = 0$, with $Q = \frac{q_3}{\rho_U} + \frac{q_4}{\rho_L}.$ (4)

Assumptions made:

- average of a product $\approx$ product of averages
- long-wavelength assumption: $L \gg H$
- hydrostatic balance: vertical velocities negligible

Energy conservation for the 1D TFM

Since (3) includes gravitational terms, the local mechanical energy includes kinetic and potential energy:

$$e = \frac{1}{2}\frac{q_3^2}{q_1} + \frac{1}{2}\frac{q_4^2}{q_2} + \rho_U g_n \tilde{H}_U(A_U) + \rho_L g_n \tilde{H}_L(A_L), \tag{5}$$

$$\text{with} \quad \tilde{H}_L = \int_{a_L} h \,\mathrm{d}a, \qquad \tilde{H}_U = \int_{a_U} h\,\mathrm{d}a,$$

$$\text{and} \quad \mathbf{v}^T = \left[\frac{\partial e}{\partial \mathbf{q}}\right] = \left[-\frac{1}{2}\frac{q_3^2}{q_1^2} + g_n\frac{\mathrm{d}\tilde{H}_U}{\mathrm{d}A_U}, -\frac{1}{2}\frac{q_4^2}{q_2^2} + g_n\frac{\mathrm{d}\tilde{H}_L}{\mathrm{d}A_L}, \frac{q_3}{q_1}, \frac{q_4}{q_2}\right].$$

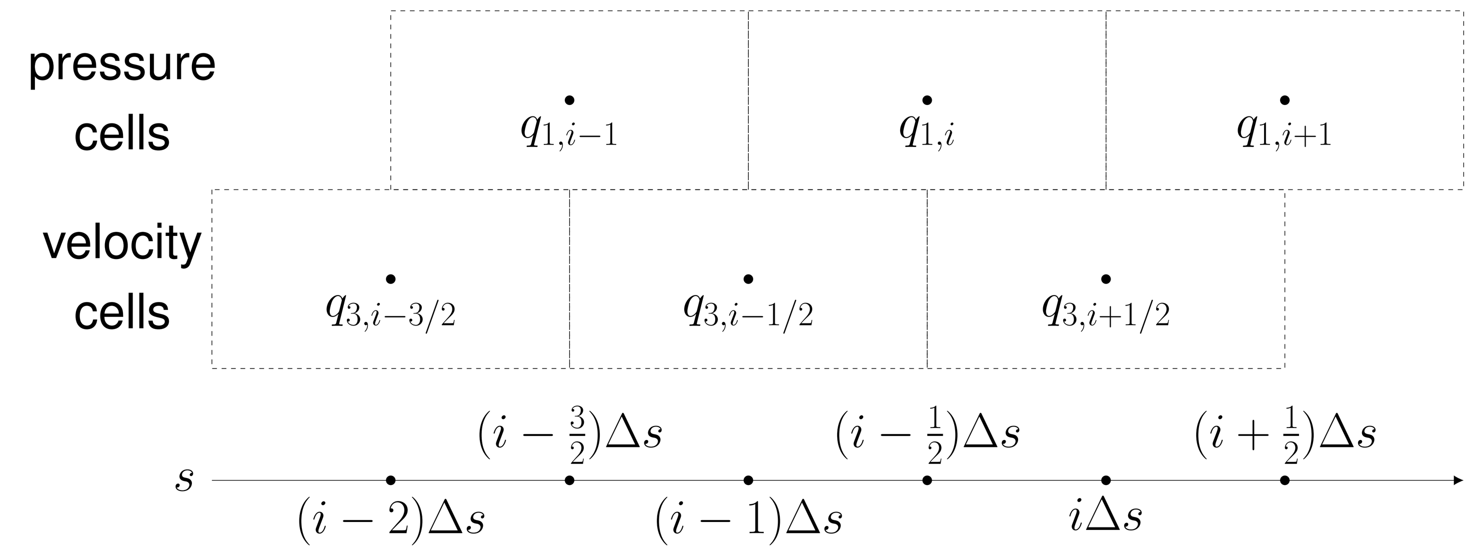
Taking the dot product of $\mathbf{v}$ with (3) and substituting (4) yields

$$\frac{\partial e}{\partial t} + \frac{\partial}{\partial s}(h + j) = 0,$$

$$\text{with} \quad h = \frac{1}{2}\frac{q_3^3}{q_1^2} + \frac{1}{2}\frac{q_4^3}{q_2^2} + g_n q_3 (H - H_U) + g_n q_4 H_L, \quad j = Qp.$$

Semi-discrete 1D TFM

Discretize the model with a finite volume scheme on a staggered grid:



$$\frac{\mathrm{d}\mathbf{q}_i}{\mathrm{d}t} + \left(\mathbf{f}_{i+1/2} - \mathbf{f}_{i-1/2}\right) + \mathbf{d}_i\left(p_i - p_{i-1}\right) = \mathbf{0} \tag{6}$$

$$\mathbf{q}_i = \begin{bmatrix} q_{1,i}(t) \\ q_{2,i}(t) \\ q_{3,i-1/2}(t) \\ q_{4,i-1/2}(t) \end{bmatrix} \quad \mathbf{f}_{i-1/2} = \begin{bmatrix} f_{1,i-1/2}(\mathbf{q}_i) \\ f_{2,i-1/2}(\mathbf{q}_i) \\ f_{3,i-1}(\mathbf{q}_{i-2}, \mathbf{q}_{i-1}, \mathbf{q}_i) \\ f_{4,i-1}(\mathbf{q}_{i-2}, \mathbf{q}_{i-1}, \mathbf{q}_i) \end{bmatrix} \quad \mathbf{d}_i = \begin{bmatrix} 0 \\ 0 \\ d_{3,i-1/2}(\mathbf{q}_{i-1}, \mathbf{q}_i) \\ d_{4,i-1/2}(\mathbf{q}_{i-1}, \mathbf{q}_i) \end{bmatrix}$$

Energy conservation for the semi-discrete TFM

The energy (5), defined on the velocity grid, requires interpolation:

$$e_{i-1/2} = \frac{1}{2}\frac{q_{3,i-1/2}^2}{\overline{q}_{1,i-1/2}} + \frac{1}{2}\frac{q_{4,i-1/2}^2}{\overline{q}_{2,i-1/2}} + \rho_U g_n \overline{\tilde{H}}_{U,i-1/2}\Delta s + \rho_L g_n \overline{\tilde{H}}_{L,i-1/2}\Delta s,$$

$$\text{with} \quad \overline{x}_{i-1/2} = \frac{1}{2}\left(x_{i-1} + x_i\right), \quad \overline{x}_i = \frac{1}{2}\left(x_{i-1/2} + x_{i+1/2}\right),$$

$$\text{and} \quad \mathbf{v}_{i-1/2,i-1} = \left[\frac{\partial e_{i-1/2}}{\partial \mathbf{q}_{i-1}}\right]^T, \quad \mathbf{v}_{i-1/2,i} = \left[\frac{\partial e_{i-1/2}}{\partial \mathbf{q}_i}\right]^T.$$

If the discretization is chosen as

$$\mathbf{f}_{i-1/2} = \begin{bmatrix} \frac{q_{3,i-1/2}}{\Delta s} \\ \frac{q_{4,i-1/2}}{\Delta s} \\ \frac{1}{\Delta s}\overline{\left(\frac{q_{3,i-1}}{\overline{q}_{1,i-1}}\right)}\overline{q}_{3,i-1} - \rho_U g_n \widehat{H}_{U,i-1} \\ \frac{1}{\Delta s}\overline{\left(\frac{q_{4,i-1}}{\overline{q}_{2,i-1}}\right)}\overline{q}_{4,i-1} - \rho_L g_n \widehat{H}_{L,i-1} \end{bmatrix}, \quad \mathbf{d}_i = \begin{bmatrix} 0 \\ 0 \\ \frac{\overline{q}_{1,i-1/2}}{\rho_U \Delta s} \\ \frac{\overline{q}_{2,i-1/2}}{\rho_L \Delta s} \end{bmatrix},$$

then taking $\mathbf{v}_{i-1/2,i-1} \cdot (6)_{i-1} + \mathbf{v}_{i-1/2,i} \cdot (6)_i$,

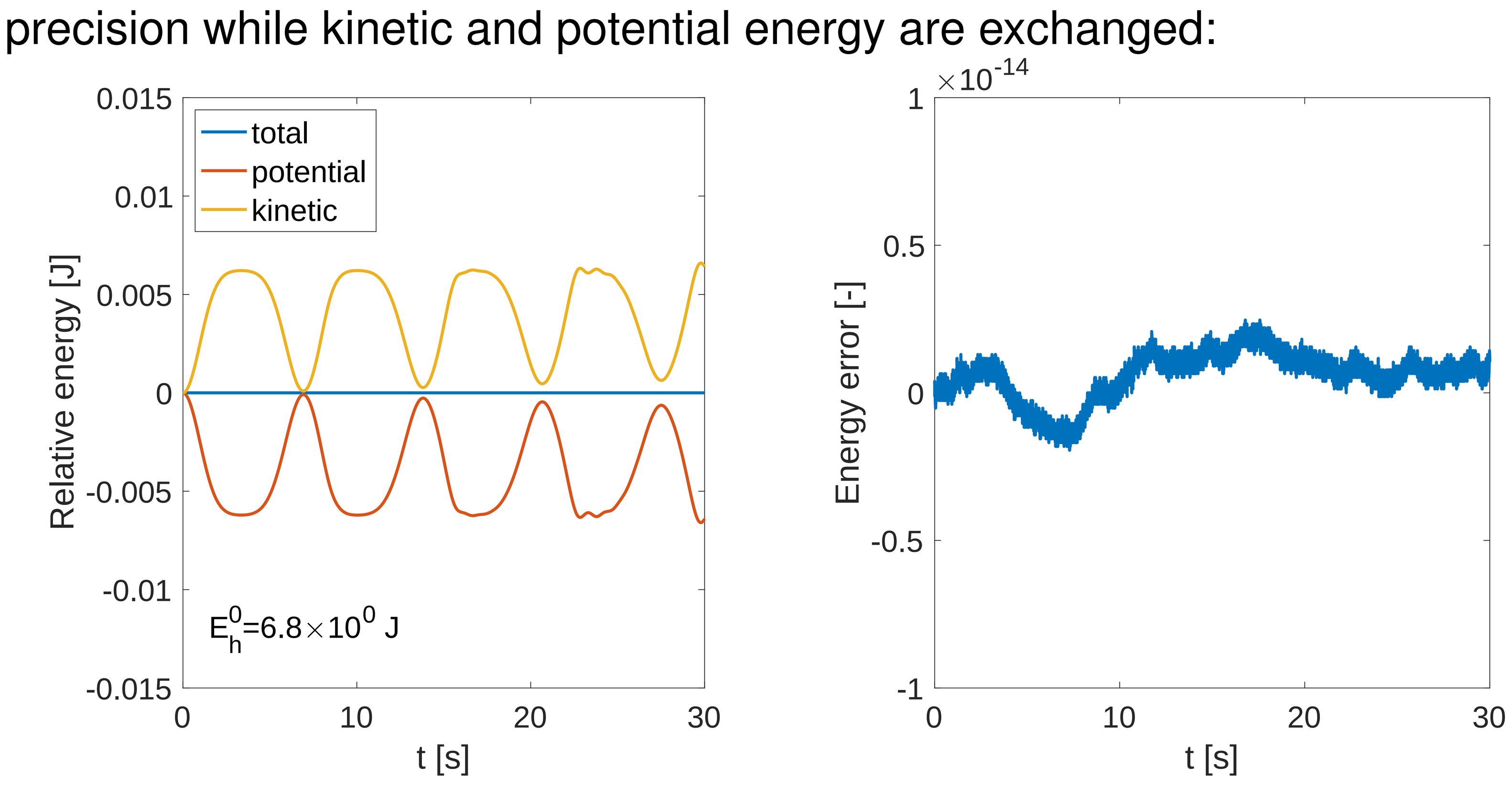
and substituting a discrete version of (4), yields

$$\frac{\mathrm{d}e_{i-1/2}}{\mathrm{d}t} + (h_i - h_{i-1}) + (j_i - j_{i-1}) = 0.$$

Therefore, this spatial discretization is energy-conserving.

Simulation results

If the time step is small, the total energy is conserved up to machine precision while kinetic and potential energy are exchanged:



Conclusions

- Quantities that are conserved *implicitly* by the continuous model can be conserved in the discrete setting using a carefully chosen discretization.
- The discretization is chosen such that the continuous derivation of the conservation equation can be replicated in the discrete setting.

[1] F. H. Harlow and J. E. Welch. Numerical calculation of time-dependent viscous incompressible flow of fluid with free surface. *The Physics of Fluids*, 8:2182–2189, 1965.

[2] J. F. H. Buist, B. Sanderse, S. Dubinkina, R. A. W. M. Henkes, and C. W. Oosterlee. Energy-conserving formulation of the two-fluid model for incompressible two-phase flow in channels and pipes. 2021. URL: http://arxiv.org/abs/2104.07728.