Reliable eigenvalue approximations in solid mechanics

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The eigenvalues

- determines how a physical object behaves under external influence.
- depend on the structure and material of the object.

State of the Art

- Standard methods are not accurate enough: the minimization of an energy depending on the displacement variable leads to locking for nearly incompressible materials
- Usual remedies cannot lead to guaranteed accuracy: usually involves stabilization parameter that needs to be tuned and obstructs robust, guaranteed error estimates
- Current stress-based approach is robust and guaranteed but too costly
  - main challenge: the symmetric gradient is difficult to impose at the discrete level.
  - The conforming discretization of those stresses (Arnold-Winther approach) sophisticated and increasing computational cost enormously.

- ✓ given a tolerance, the true error is less and greater than the tolerance
- ✓ no dependence on model parameter in the error bounds.
- ✓ moderate computational cost, discretisation elements present in commercial software.
- ✓ constants in error bounds only depend on local quantities allowing for adaptive strategies.

Mathematical Framework

Stress-based approaches seek for a stress-tensor $\boldsymbol{\sigma}$, displacements $\boldsymbol{u}$ and an eigenvalue $\omega \in \mathbb{R}$ satisfying a generalized eigenvalue problem of the form

$$A(\boldsymbol{\sigma},\boldsymbol{u}) = \omega M(\boldsymbol{\sigma},\boldsymbol{u}) \qquad \text{(EVP)}$$

involving linear operators A and M defined on some vector space V.

The analysis of such an eigenvalue problem is closely related to the corresponding source problem, a boundary value problem

$$A(\boldsymbol{\sigma},\boldsymbol{u}) = \boldsymbol{f}\ . \qquad \text{(BVP)}$$

References

[1] D. N. Arnold and R. Winther. *Mixed finite elements for elasticity*, Numer. Math. 92, (2002), pp. 401–419.

[2] J. Gedicke and A. Khan. *Arnold-Winther mixed finite elements for Stokes eigenvalue problems*, SIAM J. Sci. Comput. 40.5 (2018)

[3] F. Bertrand, Z. Cai, and E. Y. Park. *Least-squares methods for elasticity and Stokes equations with weakly imposed symmetry*, Comput. Methods Appl. Math. 19.3 (2019)

[4] D. Boffi. *Finite element approximation of eigenvalue problems*, Acta Numer. 19 (2010)

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Scientific Challenge

the solution operator T is defined such that T(**f**) is the solution ($\boldsymbol{\sigma}$,**u**) of (BVP) with right-hand side **f.**

convergence of eigenvalues ⟺ convergence of T

Babuska-Osborn theorem for T compact

Three major challenges occurs:

- **Lack of compactness:**
  Stress-based methods lead to high accuracy of the stress $\boldsymbol{\sigma}$,
  - allows for the momentum conservation div $\boldsymbol{\sigma}$
  - T cannot be compact.

  Remedy: restrict solution operator to a part of the solution excluding $\boldsymbol{\sigma}$.
- **Lack of conformity:** due to the weakly imposed symmetry condition.
  - Remedy: treat the saddle-point formulation as a non-conforming method.
  - Use Least-Squares and dPG methods to estimate the non-conforming part.
- **Lack of accuracy:**
  - Finite element spaces have to obey the compatibility condition
  - The trial space for the displacements does not usually lead to sufficient accuracy.

Remedy: recent supercloseness results

F. Bertrand, H. Schneider, Least-Squares Methods for Linear Elasticity: Refined Error Estimates, 14th WCCM-ECCOMAS Congress 2020