Low-Rank Tensor-Product Approximations for the Radiative Transfer Equation in Plane-Parallel Geometry

R. Bardin[1], **M. Schlottbom**[1] and **M. Bachmayr**[2]

[1]University of Twente, [2] Johannes Gutenberg-Universität Mainz

Introduction

The radiative transfer equation (RTE) describes transport, absorption, and scattering of energy through a medium, and has important applications in several fields, from medical imaging [1] and tumour treatment, to climate sciences [2], astrophysics, geosciences, and efficient generation of white light.

Challenge and proposal

Goal: break the *curse of dimensionality*, i.e. the fact that the computational complexity scales exponentially with the dimension.

Proposal: application of low-rank structures to the RTE hyperbolic problem through a special variational formulation, introducing also preconditioning techniques.

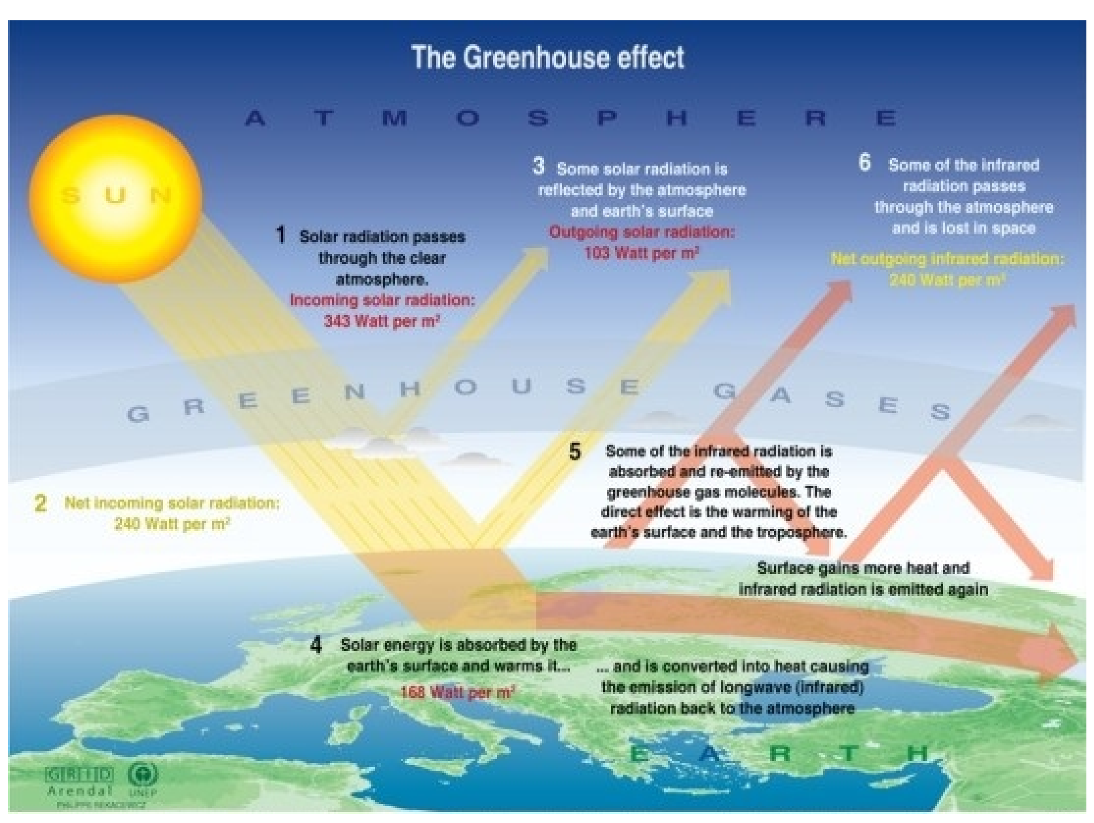

Figure: Sources: Okanagan university college in Canada, Department of geography. University of Oxford, school of geography. United States Environmental Protection Agency (EPA), Washington; Climate change 1995. The science climate change, contribution of working group 1 to the second assessment report of the intergovernmental panel on climate change, UNEP and WMO, Cambridge university press, 1996.

Problem statement

Strong even-parity formulation of radiative transfer [3]:

$$\begin{cases} \partial_z \left[(\mu^2/\sigma_t)\partial_z u\right] + \sigma_t u = \sigma_s \int_0^1 u(z,\mu')\,d\mu' + q & \text{in } \mathcal{D} \\ u + (\mu/\sigma_t)\partial_n u = g & \text{on } \partial\mathcal{D}_- \end{cases}$$

where $\mathcal{D} := (0,Z)\times(0,1)$ and $\partial\mathcal{D}_- := 0\times(0,1)\cup Z\times(0,1)$. The variational formulation reads [3]: *find* $u \in V := H^1(0,Z)\otimes_{\|\cdot\|} L^2(0,1)$ *such that*

$$a(u,v) = f(v) \qquad \forall v \in V$$

The bilinear and linear forms are defined as

$$a(u,v) = \left(\frac{1}{\sigma_t}\mu\partial_z u, \mu\partial_z v\right)_{L^2(\mathcal{D})} + (\sigma_t u, v)_{L^2(\mathcal{D})} - (\sigma_s \mathcal{P}u, v)_{L^2(\mathcal{D})} + (u,v)_{L^2_-}$$
$$f(v) = (q,v)_{L^2(\mathcal{D})} + (g,v)_{L^2_-}$$

where, $L^2_- := L^2(\partial\mathcal{D}_-\times(0,1))$, $\mathcal{P}$ is the scattering operator and the energy norm $\|\cdot\|$ is the one associated with the variational formulation, i.e. $\|v\|^2 = a(v,v)$.

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Contact Information

- Web: https://people.utwente.nl/r.bardin
- Email: r.bardin@utwente.nl
- Phone: +31534893630

Approach

- Discretization: Standard FEM in space and Legendre polynomials in angle allow to recast the variational formulation as a linear system $\mathbf{Au}=\mathbf{f}$, where $\mathbf{A}$ is symmetric positive definite and can be represented by a short sum of Kronecker products of matrices.
- Application of system matrix: the Kronecker products are never assembled thanks to the identity
$$(\mathbf{B}\otimes\mathbf{C})\text{vec}(\mathbf{X}) = \text{vec}(\mathbf{CXB}^T), \qquad \mathbf{C}\in\mathbf{R}^{m\times n},\ \mathbf{X}\in\mathbf{R}^{n\times p},\ \mathbf{B}\in\mathbf{R}^{q\times p}$$
- Solution of the linear system: two issues are addressed in the solution of the system with iterative solvers:
  - Rank growth: Let $\mathbf{U}_k = \text{mat}(u_k) = \mathbf{U\Sigma V}^T$ be the SVD of the matricization of the iterate u_k, with $\mathbf{U}$, $\mathbf{V}$ having r columns. Then $\text{rank}(u_k) = \text{rank}(\mathbf{U}_k) = r$. Application of the Kronecker identity leads to rank increase at every iteration.
  - Preconditioning: computational complexity is further reduced constructing a proper preconditioner through change of basis.

Iterative methods with recompression

Richardson method to solve the linear system:

$$\mathbf{u}^{k+1} = \mathbf{u}^k - \mathcal{B}(\mathbf{Au}^k - \mathbf{f}) = \mathcal{F}(\mathbf{u}^k)$$

for $k \geq 0$.

- Contraction $\mathcal{F}$, with suitable choice of the preconditioner, like $\mathcal{B} = \mathcal{I}\omega$, $\omega >> 1$, provides error reduction.
- Rank truncation is modelled by a non-expansive thresholding operator $\mathcal{S}_\alpha$ [4], based on the SVD of the iterates. The full iteration then reads
$$\mathbf{u}^{k+1} = \mathcal{S}_\alpha(\mathcal{F}(\mathbf{u}^k)) \qquad k \geq 0$$
- Storage of the solution is reduced from $\mathcal{O}(h^{-2d+1})$ to $\mathcal{O}(h^{-d}k)$ for these operations.

Preconditioning

Construction of the preconditioner:

- Trace lemma [5]: the differential operator associated with a is spectrally equivalent to
$$\mathcal{B} = \mu^2 \otimes \partial_z'\partial_z + \mathcal{I}\otimes\mathcal{I}$$
- After discretization, the eigendecomposition of the operators $\mu^2 = \mathbf{VDV}^T$ and $\partial_z'\partial_z = \mathbf{WFW}^T$ yields the factorization
$$\mathbf{B} = (\mathbf{V}\otimes\mathbf{W})(\mathbf{D}\otimes\mathbf{F}+\mathbf{I})^{-1}(\mathbf{V}\otimes\mathbf{W})^T$$
- Exponential sums to approximate the entries of the diagonal matrix $\mathbf{G} = (\mathbf{D}\otimes\mathbf{F}+\mathbf{I})^{-1}$, which is not in Kronecker form:
$$g_{ij} = \frac{d_i^{-1}}{d_i^{-1}+f} \approx d_i^{-1}\sum_{n=1}^{N}\omega_n e^{\alpha_n(d_i^{-1}+f_j)}$$

Conclusions and developments

- Kronecker product structure from the even-parity formulation is amenable to low-rank approximations.
- Error reduction and rank control are ensured by the combination of a contractive iterative mapping $\mathcal{F}$ with a non-expansive mapping $\mathcal{S}_\alpha$, which performs complexity reduction.

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