Matrix-free parallel solution methods for Helmholtz equations

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Introduction

The **Helmholtz equation** applied in many scientific fields

- Both **negative and positive** eigenvalues ⇒ **limits Krylov** based solvers
- Fast **near-origin** moving eigenvalues ⇒ **slows convergence**
  - **CSLP** (Complex Shifted Laplace Preconditioner)
  - Problem remains as wavenumber increases
- **Large wavenumber** & 3D ⇒ **huge** linear systems

Scalable parallel solution method ⇒ solve **larger** problem, solve **faster**

Mathematical Model

The Helmholtz equation reads

$$-\Delta u - k^2 u = g, \text{ on } \Omega \tag{1}$$

The discrete linear system is

$$A_h u_h = g_h \tag{2}$$

The so-called 2D closed-off problem is given by

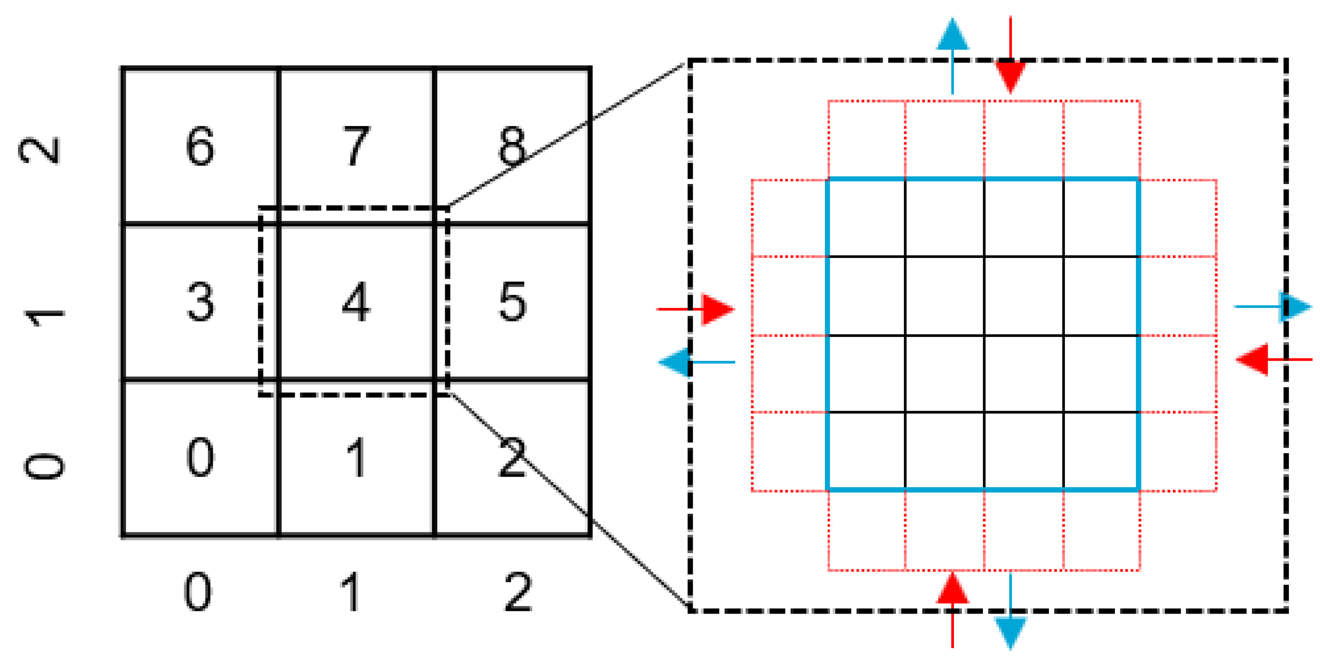
$$g(x,y) = \left(5\pi^2 - k^2\right)\sin(\pi x)\sin(2\pi y), \ \Omega = [0,1]$$
$$u = 0, \text{ on } \partial\Omega$$
$$u = \sin(\pi x)\sin(2\pi y). \tag{3}$$

Numerical Method

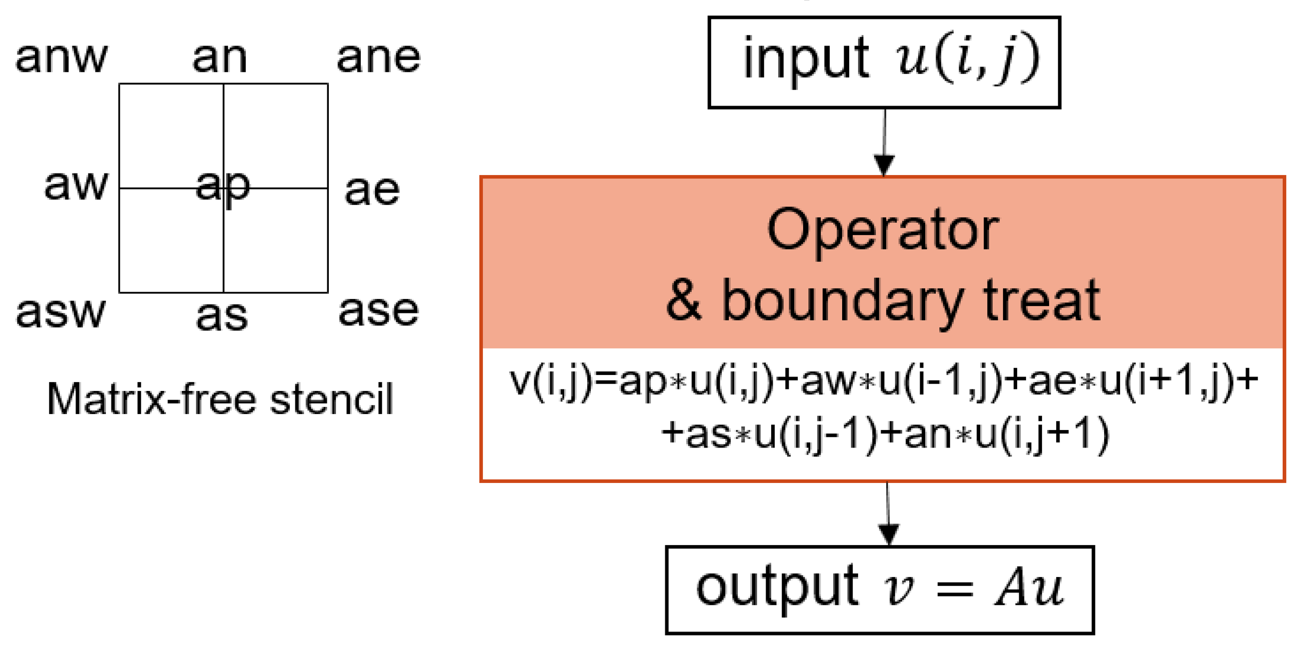
- Krylov subspace method - GMRES for complex system
  - Iterations stop criterion: $res < 10^{-6}$
- Preconditioning - **Multigrid** based CSLP
  - CSLP: $M_h = -\Delta_h - (\beta_1 - \mathrm{i}\beta_2)k^2 I_h$
    *Here $\beta_1 = 1, \ \beta_2 = 0.5$.
- Solve $My = x$ by multigrid method: $y \approx M^{-1}x$
  - ω-Jacobi Smoothers
  - Full weighting Restriction
  - Bilinear Prolongation
  - Coarse grid iteration: GMRES, $res < 10^{-8}$

Parallel implementation

- Partitioning & MPI topology:



- **Matrix-free** matrix vector multiplications:



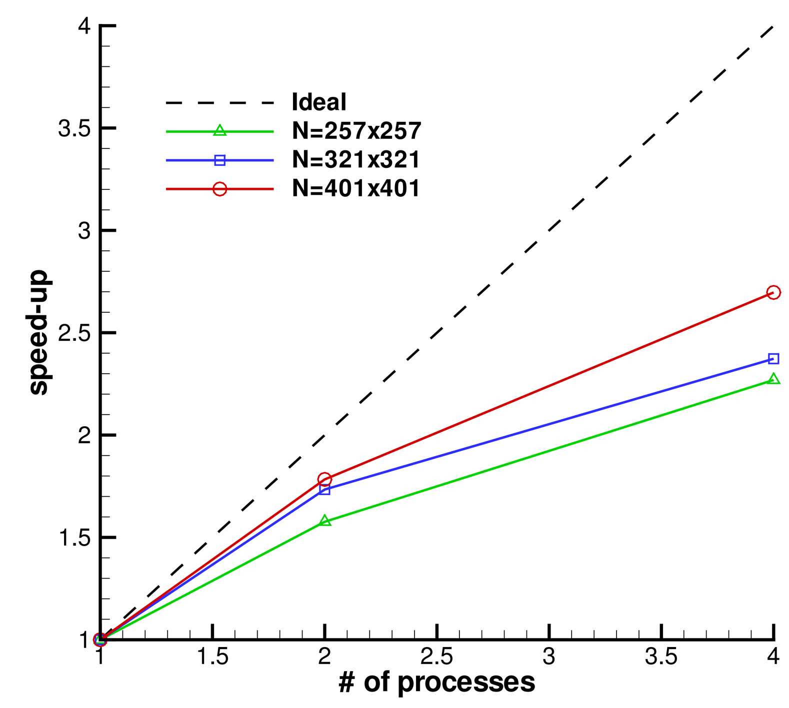
- **Matrix-free** preconditioner & coarse grid operator:
  -**Rediscretized** similarly to A_h
- Dot product:
  -sum(u(i,j)*v(i,j)) + MPI_Allreduce

Results

- Comparison of parallel and serial computing results. $k = 1$, grid size 129×129, "np" denotes the number of processors, "Error" are absulote errors compared to the exact solution.

np	Full GMRES		CSLP(two-cycle)		CSLP(V-cycle)	
	iter	Error	iter	Error	iter	Error
1×1	281	2.669E-04	6	1.739E-04	7	1.744E-04
2×2	281	2.669E-04	6	1.739E-04	7	1.744E-04
3×3	281	2.669E-04	6	1.739E-04	7	1.744E-04

- Parallel efficiency: Here $kh = 0.25$, V-cycle CSLP



Conclusions

- Flexible partitioning
- Equivalent to the serial
- Less memory required
- Good performance trend

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