Solving Inverse Problems without using Forward Operators

Part II: Minimization Based Formulations

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joint work with

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Outline

- minimization based formulation
  and regularization of inverse problems
- examples
- numerical results

Der Wissenschaftsfonds.
FWF project P30054
Solving Inverse Problems without Forward Operators

examples

Parameter Identification in Differential Equations: Some Examples

- Identify spatially varying coefficients/source a, b, c in linear elliptic boundary value problem on $\Omega \subseteq \mathbb{R}^d$, $d \in \{1,2,3\}$

$$-\nabla(a\nabla u) + cu = b \text{ in } \Omega\,, \qquad \frac{\partial u}{\partial n} = g \text{ on } \partial\Omega\,,$$

from boundary or (restricted) interior observations of u.

- e.g. EIT: identify conductivity σ in

$$-\nabla(\sigma\nabla\phi_i) = 0 \text{ in } \Omega$$

from boundary observations
current $j_i = -\sigma\frac{\partial\phi_i}{\partial n}$ and voltage $\upsilon = \phi_i$ on $\partial\Omega \quad i \in \{1, \ldots I\}$

- Identify parameter ϑ in initial value problem for ODE / PDE

$$\dot{u}(t) = f(t, u(t), \vartheta)\ t \in (0,T)\,, \quad u(0) = u_0$$

from discrete of continuous observations of u.
$y_i = g_i(u(t_i))$, $i \in \{1, \ldots, m\}$ or $y(t) = g(t, y(t))$, $t \in (0,T)$

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Abstract Formulation as Operator Equation

Identify parameter q in (PDE or ODE) model

$$A(q,u) = 0$$

from observations of the state u

$$C(u) = y\,,$$

where $q \in X$, $u \in V$, $y \in Y$, $X, V, Y \ldots$ Hilbert (Banach) spaces
$A : X \times V \to W^* \ldots$ differential operator
$C : V \to Y \ldots$ observation operator

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- reduced approach: operator equation for q

$$F(q) = y,$$

$F = C \circ S$ with $S : X \to V$, $q \mapsto u$ parameter-to-state map

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- minimization based approach . . .

minimization based formulation of inverse problems

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... and beyond, e.g., variational formulation of EIT [Kohn&Vogelius'87]

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...and several other application examples, see below

Formulation and Regularization via Minimization

inverse problem:

$$(q, u) \in \operatorname{argmin}\{\mathcal{J}(q, u; y) \ : (q, u) \in M_{\mathrm{ad}}(y)\}$$

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inverse problem is ill-posed:

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$\rightsquigarrow$ regularized inverse problem:

$$(q_\alpha^\delta, u_\alpha^\delta) \in \text{argmin}\{\mathcal{J}(q,u;y^\delta) + \alpha \cdot \mathcal{R}(q,u) \ : \ (q,u) \in M_{\rm ad}^\delta(y^\delta)\}$$

regularize by

- adding penalties (Tikhonov type) and/or
- imposing constraints (Ivanov type)

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regularize by
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treat data misfit by
- penalty term in cost function (Tikhonov type) or
- constraint (Morozov type)

Regularization with data misfit Penalization

inverse problem (IP):

$$\min_{(q,u)\in X\times V} \mathcal{S}(C(u),y) + \mathcal{Q}(A(q,u))$$

$$\text{s.t. } (q,u) \in M_{\rm ad}(y) = X \times V\,,$$

regularization (RdmP):

$$\min_{(q,u)\in X\times V} \mathcal{S}(C(u),y^\delta) + \mathcal{Q}(A(q,u)) + \alpha\cdot\mathcal{R}(q,u)$$

$$\text{s.t. } (q,u) \in M_{\rm ad}^\delta(y^\delta) = \{(q,u)\in X\times V\ :\ \widetilde{\mathcal{R}}(q,u) \le \rho\}\,.$$

where $\mathcal{S} : Y \times Y \to \overline{\mathbb{R}}$, $\mathcal{Q} : W \to \overline{\mathbb{R}}$ are positive definite functionals

$$\forall y_1, y_2 \in Y\ :\quad \mathcal{S}(y_1,y_2) \ge 0 \quad \text{and} \quad \big(y_1 = y_2 \ \Leftrightarrow\ \mathcal{S}(y_1,y_2) = 0\big)\,,$$

$$\forall w \in W\ :\quad \mathcal{Q}(w) \ge 0 \quad \text{and} \quad \big(w = 0 \ \Leftrightarrow\ \mathcal{Q}(w) = 0\big)\,.$$

e.g., just norms or derived from statistical noise model

Regularization with data misfit Constraint

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$$\min_{(q,u)\in X\times V} \mathcal{Q}(A(q,u))$$

$$\text{s.t. } (q,u) \in M_{\mathrm{ad}}(y) = \{(q,u) \in X \times V \,:\, C(u) = y\}\,,$$

regularization (RdmC):

$$\min_{(q,u)\in X\times V} \mathcal{Q}(A(q,u)) + \alpha \cdot \mathcal{R}(q,u)$$

$$\text{s.t. } (q,u) \in M^{\delta}_{\mathrm{ad}}(y^{\delta}) = \{(q,u) \in X \times V \,:\, \mathcal{S}(C(u), y^{\delta}) \le \tau\delta \text{ and } \widetilde{\mathcal{R}}(q,u) \le \rho\}\,.$$

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e.g., just norms or derived from statistical noise model examples

The variational approach to EIT

see, e.g., [Kohn&Vogelius'87, Kohn&McKenny'90, Knowles'98]

The reduced formulation of EIT

identify conductivity σ in

$$-\nabla(\sigma\nabla\phi_i) = 0 \text{ in } \Omega$$

from boundary observations

current $j_i = -\sigma\frac{\partial\phi_i}{\partial n}$ and voltage $\upsilon = \phi_i$ on $\partial\Omega \quad i \in \{1, \ldots I\}$

The reduced formulation of EIT

identify conductivity σ in

$$-\nabla(\sigma\nabla\phi) = 0 \text{ in } \Omega \qquad (*)$$

from Neumann-Dirichlet operator

$\Lambda_\sigma : j \mapsto \phi|_{\partial\Omega}$ where ϕ solves $(*)$ with $\sigma\frac{\partial\phi}{\partial n} = j$ on $\partial\Omega$, $\int_{\partial\Omega} \phi \, ds = 0$

(Calderón problem)

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Identify spatially distributed conductivity σ in $\Omega \subseteq \mathbb{R}^2$

$$\nabla \cdot J_i = 0\,, \quad \nabla^\perp \cdot E_i = 0\,, \quad J_i = \sigma E_i \quad \text{in } \Omega\,, \quad i = 1, \ldots, I,$$

(with $\nabla^\perp \psi = (-\frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_1})^T$ so that $\nabla^\perp \cdot = \text{curl}$)
from observations of boundary currents j_i and voltages v_i.

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Using potentials ϕ_i and ψ_i

for current densities J_i and electric fields E_i

$$J_i = -\nabla^\perp \psi_i\,, \quad E_i = -\nabla \phi_i\,, \quad i = 1, \ldots, I\,,$$

we can rewrite the problem as

$$\sqrt{\sigma}\nabla\phi_i = \frac{1}{\sqrt{\sigma}}\nabla^\perp\psi_i \text{ in } \Omega\,;\ \psi_i = \gamma_i\,,\ \phi_i = \upsilon_i \text{ on } \partial\Omega\,,\ i = 1, \ldots, I\,,$$

where $\gamma_i(x(s)) = -\int_0^s j_i(x(r))\,dr$ for $\partial\Omega = \{x(s) : s \in (0, \text{length}(\partial\Omega))\}$.

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equivalent to

$$\min_{\sigma,\underline{\phi},\underline{\psi}} \sum_{i=1}^{I} \frac{1}{2}\int_{\Omega} \left|\sqrt{\sigma}\nabla\phi_i - \frac{1}{\sqrt{\sigma}}\nabla^{\perp}\psi_i\right|^2 dx$$

$$\text{s.t. } \psi_i = \gamma_i\,,\ \phi_i = \upsilon_i \text{ on } \partial\Omega\,, \quad i = 1,\dots,I$$

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$$\text{s.t. } \psi_i = \gamma_i\,,\ \phi_i = \upsilon_i \text{ on } \partial\Omega\,, \quad i = 1,\dots,I$$

equivalent to (since $\int_\Omega \nabla\phi_i\cdot\nabla^{\perp}\psi_i\,dx = \int_{\partial\Omega}\upsilon_i j_i\,dx$)

$$\min_{\sigma,\underline{\phi},\underline{\psi}} \sum_{i=1}^{I} \tfrac{1}{2}\int_{\Omega}\left(\sigma|\nabla\phi_i|^2 + \frac{1}{\sigma}|\nabla^{\perp}\psi_i|^2\right)dx$$

$$\text{s.t. } \psi_i = \gamma_i\,,\ \phi_i = \upsilon_i \text{ on } \partial\Omega\,, \quad i = 1,\dots,I$$

Regularized variational EIT

inverse problem (EIT):

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regularization (RegEIT):

$$\min_{\sigma,\Phi,\Psi} \sum_{i=1}^{I} \left\{ \tfrac{1}{2} \int_\Omega |\sqrt{\sigma}\nabla\phi_i - \frac{1}{\sqrt{\sigma}}\nabla^\perp\psi_i|^2\, dx + \tfrac{\alpha}{2}\left(\|\phi_i\|^2_{H^{1+\epsilon}(\Omega)} + \|\psi_i\|^2_{H^{1+\epsilon}(\Omega)}\right) \right\}$$

$$\text{s.t. } \underline{\sigma} \le \sigma \le \overline{\sigma} \text{ on } \Omega\,,$$

$$\left.\begin{array}{l} \upsilon_i^\delta - \tau\delta \le \phi_i \le \upsilon_i^\delta + \tau\delta\,, \\ \gamma_i^\delta - \tau\delta \le \psi_i \le \gamma_i^\delta + \tau\delta\,, \end{array}\right\} \quad \text{on } \partial\Omega\,, \quad i = 1,\ldots,I\,.$$

$\rightsquigarrow$ special case of regularization with constraint on data misfit (RdmC)

Regularized variational EIT: Function space setting

$$q = \sigma\,, \qquad u = (\phi_1, \dots, \phi_I, \psi_1, \dots, \psi_I)\,, \qquad y = (\upsilon_1, \dots, \upsilon_I, \gamma_1, \dots, \gamma_I)$$

$$\begin{aligned}
X &= L^2(\Omega) \\
Y &= L^\infty(\partial\Omega)^I \times W^{1,1}(\partial\Omega)^I \\
V &= \{(\phi_1, \dots, \phi_I, \psi_1, \dots, \psi_I) \in H^1(\Omega)^{2I} \ : \ \mathrm{tr}^{2I}_{\partial\Omega}(\phi_1, \dots, \phi_I, \psi_1, \dots, \psi_I) \in Y\} \\
W &= L^2(\Omega)^I
\end{aligned}$$

$$A(q,u) = \left(\sqrt{\sigma}\nabla\phi_1 - \tfrac{1}{\sqrt{\sigma}}\nabla^\perp\psi_1, \dots, \sqrt{\sigma}\nabla\phi_I - \tfrac{1}{\sqrt{\sigma}}\nabla^\perp\psi_I\right)\,,$$

$$C = \mathrm{tr}^{2I}_{\partial\Omega}$$

$$\mathcal{Q}(w) = \tfrac{1}{2}\|w\|^2_{L^2(\Omega)^I}$$

$$\mathcal{R}(q,u) = \mathcal{R}(u) = \sum_{i=1}^{I}\left(\|\phi_i\|^2_{H^{1+\epsilon}(\Omega)^{2I}} + \|\psi_i\|^2_{H^{1+\epsilon}(\Omega)^{2I}}\right)$$

$$\widetilde{\mathcal{R}}(q,u) = \widetilde{\mathcal{R}}(q) = \|\sigma - \tfrac{\overline{\sigma}+\underline{\sigma}}{2}|_{L^\infty(\Omega)}\,, \qquad \rho = \tfrac{\overline{\sigma}-\underline{\sigma}}{2}$$

$$\mathcal{S}(y,\tilde{y}) = \max_{i\in\{1,\dots,I\}} \|\upsilon_i - \widetilde{\upsilon}_i\|_{L^\infty(\partial\Omega)} + \|\gamma_i - \widetilde{\gamma}_i\|_{L^\infty(\partial\Omega)}$$

Regularized variational EIT: well-definedness, convergence

$$(\sigma_n, \Phi_n, \Psi_n) \xrightarrow{\mathcal{T}} (\sigma, \Phi, \Psi) \;\Leftrightarrow\; \begin{cases} \sigma_n \overset{*}{\rightharpoonup} \sigma \text{ and } \frac{1}{\sigma_n} \overset{*}{\rightharpoonup} \frac{1}{\sigma} \text{ in } L^\infty(\Omega)\,, \\ (\Phi_n, \Psi_n) \to (\Phi, \Psi) \text{ in } H^1(\Omega)^{2I} \\ (\Phi_n, \Psi_n) \rightharpoonup (\Phi, \Psi) \text{ in } H^{1+\epsilon}(\Omega)^{2I}\,, \\ \mathrm{tr}(\Phi_n, \Psi_n) \to \mathrm{tr}(\Phi, \Psi) \text{ in } L^\infty(\partial\Omega)^{2I} \end{cases}$$

Corollary

For each $y^\delta \in Y$ and $\alpha > 0$ a minimizer of (RegEIT) exists.

Let $\mathcal{S}(y, y^\delta) \le \delta$,
$\underline{\sigma} \le \sigma^\dagger \le \overline{\sigma}$ a.e. in Ω
and choose $\alpha = \alpha(\delta, y^\delta)$ such that $\alpha(\delta, y^\delta) \to 0$ as $\delta \to 0$.
Then, as $\delta \to 0$, $(\sigma^\delta_{\alpha(\delta,y^\delta)}, \Phi^\delta_{\alpha(\delta,y^\delta)}, \Psi^\delta_{\alpha(\delta,y^\delta)})) \xrightarrow{\mathcal{T}} (\sigma^\dagger, \Phi^\dagger, \Psi^\dagger)$.

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For each $y^\delta \in Y$ and $\alpha > 0$ a minimizer of (RegEIT) exists.

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and choose $\alpha = \alpha(\delta, y^\delta)$ such that $\alpha(\delta, y^\delta) \to 0$ as $\delta \to 0$.
Then, as $\delta \to 0$, $(\sigma^\delta_{\alpha(\delta,y^\delta)}, \Phi^\delta_{\alpha(\delta,y^\delta)}, \Psi^\delta_{\alpha(\delta,y^\delta)}) \xrightarrow{\mathcal{T}} (\sigma^\dagger, \Phi^\dagger, \Psi^\dagger)$

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- Hilbert spaces X, V for design variables q, u (easier applicability of iterative minimization methods);
- cost function: J^δ differentiable;
- constraints: pointwise bounds can be efficiently implemented [Hungerländer, BK and Rendl 2020] and are practically relevant in view of known a prior bounds on σ;

Remarks on EIT example

- Hilbert spaces X, V for design variables q, u (easier applicability of iterative minimization methods);
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- analgously: permeabilities in magnetostatics, cracks from electrostatic measurements

Identification of sound sources from microphone array measurements

Identify number, locations, amplitudes of sound sources from microphone array measurements.

$$\frac{1}{c_0^2} p_{tt} - \Delta p = \sigma$$

measurements $p(x_\ell)$, $\ell \in \{1, \ldots, L\}$
$x_\ell \in \Omega$ … (known) location of ℓ-th micro
p… acoustic pressure
σ… sound source

Sound source localization: Formulation as 1st order system

linearized conservation of momentum: $\varrho_0 v_t + \nabla p_\sim = f,$

linearized conservation of mass: $\varrho_{\sim t} + \varrho_0 \nabla \cdot v = g,$

linearized equation of state: $\varrho_\sim = \frac{1}{c_0^2} p_\sim,$

$$\varrho_0 v_t + \nabla p_\sim = f, \tag{1}$$

$$\frac{1}{c_0^2} p_{\sim t} + \varrho_0 \nabla \cdot v = g, \tag{2}$$

$_\sim$ . . . fluctuating part $\quad$ $_0$ . . . constant mean value

- $\varrho = \varrho_0 + \varrho_\sim$ . . . mass density
- $v = v_\sim$ . . . acoustic particle velocity,
- $p = p_0 + p_\sim$ . . . pressure,
- c_0 . . . speed of sound.

$\frac{\partial}{\partial t}$ (2) - $\nabla\cdot$ (1) $\Rightarrow$ $\frac{1}{c_0^2} p_{tt} - \Delta p = \sigma = g_t - \nabla \cdot f$ 2nd order wave eq.

Sound source localization

Boundary conditions:

$$\varrho_0 v \cdot \nu + \kappa p = 0 \quad \text{on} \quad \Gamma_a$$
$$v \cdot \nu = 0 \quad \text{on} \quad \Gamma_r$$

$\Gamma_a \ldots$ absorbing boundary part

$\Gamma_r \ldots$ reflecting boundary part

Measurements:

$y_\ell = p(x_\ell)$, $\ell \in \{1, \ldots, L\}$

Sound source localization: Inverse problem

in time domain:

$$\varrho_0 v_t + \nabla p = f \quad \text{in } \Omega \times (0,T)$$
$$\frac{1}{c_0^2} p_t + \varrho_0 \nabla \cdot v = 0 \quad \text{in } \Omega \times (0,T)$$

$$\varrho_0 v \cdot \nu + \kappa p = 0 \quad \text{on } \Gamma_a \times (0,T)$$
$$v \cdot \nu = 0 \quad \text{on } \Gamma_r \times (0,T)$$

$$y_\ell = p(x_\ell)\,, \ \ell \in \{1, \ldots, L\}$$

Sound source localization: Inverse problem

in time domain:

$$\begin{aligned} \varrho_0 v_t + \nabla p &= f && \text{in } \Omega \times (0,T) \\ \frac{1}{c_0^2} p_t + \varrho_0 \nabla \cdot v &= 0 && \text{in } \Omega \times (0,T) \end{aligned} \qquad \begin{aligned} \varrho_0 v \cdot \nu + \kappa p &= 0 && \text{on } \Gamma_a \times (0,T) \\ v \cdot \nu &= 0 && \text{on } \Gamma_r \times (0,T) \end{aligned}$$

$$y_\ell = p(x_\ell)\,, \ \ell \in \{1, \ldots, L\}$$

in frequency domain: (with fixed frequency ω)

$$\begin{aligned} \varrho_0 \imath\omega \hat{v} + \nabla \hat{p} &= f && \text{in } \Omega \\ \frac{1}{c_0^2} \imath\omega \hat{p} + \varrho_0 \nabla \cdot \hat{v} &= 0 && \text{in } \Omega \end{aligned} \qquad \begin{aligned} \varrho_0 \hat{v} \cdot \nu + \kappa \hat{p} &= 0 && \text{on } \Gamma_a \\ \hat{v} \cdot \nu &= 0 && \text{on } \Gamma_r \end{aligned}$$

$$\hat{y}_\ell = \hat{p}(x_\ell)\,, \ \ell \in \{1, \ldots, L\}$$

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$$\hat{p}_\ell = \hat{y}(x_\ell)\,, \ \ell \in \{1, \ldots, L\}$$

Sound source localization: Inverse problem

in frequency domain: (with fixed frequency ω)

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$$\hat{p}_\ell = \hat{y}(x_\ell)\,,\ \ell \in \{1,\dots,L\}$$

splitting $\hat{v}$, $\hat{p}$ into real and imaginary parts:

$$\left.\begin{aligned} -\varrho_0\omega v_{\Im} + \nabla p_{\Re} - f_{\Re} &= 0 \\ \varrho_0\omega v_{\Re} + \nabla p_{\Im} - f_{\Im} &= 0 \\ -\frac{1}{c_0^2}\omega p_{\Im} + \varrho_0\nabla\cdot v_{\Re} &= 0 \\ \frac{1}{c_0^2}\omega p_{\Re} + \varrho_0\nabla\cdot v_{\Im} &= 0 \end{aligned}\right\} \text{in } \Omega \qquad \begin{aligned} &\left.\begin{aligned} \varrho_0 v_{\Re}\cdot\nu + \kappa p_{\Re} &= 0 \\ \varrho_0 v_{\Im}\cdot\nu + \kappa p_{\Im} &= 0 \end{aligned}\right\} \text{on } \Gamma_a \\ &\left.\begin{aligned} v_{\Re}\cdot\nu &= 0 \\ v_{\Im}\cdot\nu &= 0 \end{aligned}\right\} \text{on } \Gamma_r \end{aligned}$$

$$p_{\Re,\ell} = y_{\Re}(x_\ell)\,,\ p_{\Im,\ell} = y_{\Im}(x_\ell)\,,\ \ell \in \{1,\dots,L\}$$

to avoid the problem of nondifferentiability of $z \mapsto |z|^2$ in $\mathbb{C}$.

Sound source localization: Inverse problem

$$\left.\begin{aligned} res_{mom,\Re} &:= -\varrho_0 \omega v_\Im + \nabla p_\Re - f_\Re = 0 \\ res_{mom,\Im} &:= \quad \varrho_0 \omega v_\Re + \nabla p_\Im - f_\Im = 0 \\ res_{mass,\Re} &:= \; -\frac{1}{c_0^2}\omega p_\Im + \varrho_0 \nabla \cdot v_\Re = 0 \\ res_{mass,\Im} &:= \quad \frac{1}{c_0^2}\omega p_\Re + \varrho_0 \nabla \cdot v_\Im = 0 \end{aligned}\right\} \text{ in } \Omega \qquad \begin{aligned} &\left.\begin{aligned} \varrho_0 v_\Re \cdot \nu + \kappa p_\Re = 0 \\ \varrho_0 v_\Im \cdot \nu + \kappa p_\Im = 0 \end{aligned}\right\} \text{ on } \Gamma_a \\ &\left.\begin{aligned} v_\Re \cdot \nu = 0 \\ v_\Im \cdot \nu = 0 \end{aligned}\right\} \text{ on } \Gamma_r \end{aligned}$$

$$p_{\Re,\ell} = y_\Re(x_\ell)\,, \;\; p_{\Im,\ell} = y_\Im(x_\ell)\,, \;\; \ell \in \{1, \ldots, L\}$$

Sound source localization: Inverse problem

$$\left.\begin{aligned}
\mathrm{res}_{mom,\Re} &:= -\varrho_0\omega v_\Im + \nabla p_\Re - f_\Re = 0\\
\mathrm{res}_{mom,\Im} &:= \quad \varrho_0\omega v_\Re + \nabla p_\Im - f_\Im = 0\\
\mathrm{res}_{mass,\Re} &:= \quad -\frac{1}{c_0^2}\omega p_\Im + \varrho_0 \nabla\cdot v_\Re = 0\\
\mathrm{res}_{mass,\Im} &:= \quad \frac{1}{c_0^2}\omega p_\Re + \varrho_0 \nabla\cdot v_\Im = 0
\end{aligned}\right\} \text{ in } \Omega
\qquad
\begin{aligned}
&\left.\begin{aligned}\varrho_0 v_\Re\cdot\nu + \kappa p_\Re = 0\\ \varrho_0 v_\Im\cdot\nu + \kappa p_\Im = 0\end{aligned}\right\} \text{ on } \Gamma_a\\
&\left.\begin{aligned}v_\Re\cdot\nu = 0\\ v_\Im\cdot\nu = 0\end{aligned}\right\} \text{ on } \Gamma_r
\end{aligned}$$

$$p_{\Re,\ell} = y_\Re(x_\ell)\,,\;\; p_{\Im,\ell} = y_\Im(x_\ell)\,,\;\; \ell\in\{1,\dots,L\}$$

equivalent to

$$\min_{f_\Re,f_\Im,p_\Re,p_\Im,v_\Re,v_\Im} \int_\Omega \left(\mathrm{res}^2_{mom,\Re} + \mathrm{res}^2_{mom,\Im} + \mathrm{res}^2_{mass,\Re} + \mathrm{res}^2_{mass,\Im}\right)\, dx$$

$$\text{s.t.}\quad \varrho_0\hat{v}\cdot\nu + \kappa\hat{p} = 0 \text{ on } \Gamma_a\,,\quad \hat{v}\cdot\nu = 0 \text{ on } \Gamma_r$$

$$\hat{p}(x_\ell) = \hat{y}_\ell\,,\;\; \ell\in\{1,\dots,L\}$$

Sound source localization: Regularized inverse problem

inverse problem (SSL):

$$\min_{f_{\Re}, f_{\Im}, p_{\Re}, p_{\Im}, v_{\Re}, v_{\Im}} \int_{\Omega} \left(\mathrm{res}^2_{mom,\Re} + \mathrm{res}^2_{mom,\Im} + \mathrm{res}^2_{mass,\Re} + \mathrm{res}^2_{mass,\Im} \right) \, dx$$

$$\text{s.t.} \quad \varrho_0 \hat{v} \cdot \nu + \kappa \hat{p} = 0 \text{ on } \Gamma_a \,, \quad \hat{v} \cdot \nu = 0 \text{ on } \Gamma_r$$

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regularization (RegSSL)
(use measure norm $\| \cdot \|_{\mathcal{M}(\Omega)}$ to enhance sparsity):

$$\min_{\hat{f}, \hat{p}, \hat{v}} \int_{\Omega} \left(\text{res}^2_{mom,\Re} + \text{res}^2_{mom,\Im} + \text{res}^2_{mass,\Re} + \text{res}^2_{mass,\Im} \right) \, dx$$

$$+ \alpha_1 \| (f_{\Re}, f_{\Im}, p_{\Re}, p_{\Im}, v_{\Re}, v_{\Im}) \|^2_{L^2(\Omega)^{14}} + \alpha_2 \| (\nabla \cdot f_{\Re}, \nabla \cdot f_{\Im}) \|_{\mathcal{M}(\Omega)^2}$$

$$\text{s.t.} \quad \varrho_0 \hat{v} \cdot \nu + \kappa \hat{p} = 0 \text{ on } \Gamma_a \,, \quad \hat{v} \cdot \nu = 0 \text{ on } \Gamma_r$$

$$\hat{y}_\ell - \tau\delta \le \hat{p}(x_\ell) \le \hat{y}_\ell + \tau\delta \,, \; \ell \in \{1, \ldots, L\}$$

$\rightsquigarrow$ special case of regularization with constraint on data misfit (RdmC)

Regularized sound source loc.: Function space setting

$q = (f_{\Re}, f_{\Im})\,, \qquad u = (p_{\Re}, p_{\Im}, v_{\Re}, v_{\Im})\,, \qquad y = (y_1, \ldots, y_L)$

$\Omega_{mic} \subseteq \Omega\,,\ \Omega_{mic}$ open

$X = \{(f_{\Re}, f_{\Im}) \in L^2(\Omega)^6\ :\ \text{suppess}(f_{\Re}), \text{suppess}(f_{\Im}) \subseteq \Omega \setminus \Omega_{mic}\}$

$Y = \mathbb{R}^L$

$V = \{(p_{\Re}, p_{\Im}, v_{\Re}, v_{\Im}) \in H^1(\Omega)^2 \times H(\text{div}, \Omega)\ \ (p_{\Re}, p_{\Im})|_{\Omega_{mic}} \in H^2(\Omega_{mic})^2 \subseteq C(\Omega_{mic})^2$

$\varrho_0 \hat{v} \cdot \nu + \kappa \hat{p} = 0 \text{ in } H^{-1/2}(\Gamma_a)^2\,,\ \hat{v} \cdot \nu = 0 \text{ in } H^{-1/2}(\Gamma_r)\}$

$W = L^2(\Omega)^8$

$A(q,u) = (\text{res}_{mom,\Re},\ \text{res}_{mom,\Im},\ \text{res}_{mass,\Re},\ \text{res}_{mass,\Im})\ ,$

$C = (\delta_{x_1} \ldots \delta_{x_L})$ (point evaluation at the microphones)

$\mathcal{Q}(w) = \frac{1}{2}\|w\|^2_{L^2(\Omega)^L}$

$\mathcal{R}_1(q,u) = \|(f_{\Re}, f_{\Im}, p_{\Re}, p_{\Im}, v_{\Re}, v_{\Im})\|^2_{L^2(\Omega)^{14}}$

$\mathcal{R}_2(q,u) = \mathcal{R}_2(q) = \|(\nabla \cdot f_{\Re}, \nabla \cdot f_{\Im})\|_{\mathcal{M}(\Omega)^2}$

$\mathcal{S}(y, \tilde{y}) = \max_{\ell \in \{1,\ldots,L\}} |y_\ell - \tilde{y}_\ell|$

Regularized sound source localization: well-definedness, convergence

$$(f_{\Re,n}, f_{\Im,n}, p_{\Re,n}, p_{\Im,n}, v_{\Re,n}, v_{\Im,n}) \stackrel{\mathcal{T}}{\to} (f_\Re, f_\Im, p_\Re, p_\Im, v_\Re, v_\Im) \ \Leftrightarrow$$

$$\begin{cases} (\nabla\cdot f_{\Re,n}, \nabla\cdot f_{\Im,n}) \stackrel{*}{\rightharpoonup} (\nabla\cdot f_\Re, \nabla\cdot f_\Im) \text{ in } \mathcal{M}(\Omega) \text{ and } (f_{\Re,n}, f_{\Im,n}) \rightharpoonup (f_\Re, f_\Im) \text{ in } L^2(\Omega), \\ (p_{\Re,n}, p_{\Im,n}) \rightharpoonup (p_\Re, p_\Im) \text{ in } H^1(\Omega)^2 \\ (v_{\Re,n}, v_{\Im,n}) \rightharpoonup (v_\Re, v_\Im) \text{ in } H(\text{div}, \Omega)^2 \\ (p_{\Re,n}, p_{\Im,n})|_{\Omega_{mic}} \rightharpoonup (p_\Re, p_\Im)|_{\Omega_{mic}} \text{ in } H^2(\Omega_{mic})^2 \end{cases}$$

Corollary

For each $y^\delta \in Y$ and $\alpha > 0$ a minimizer of (RegSSL) exists.

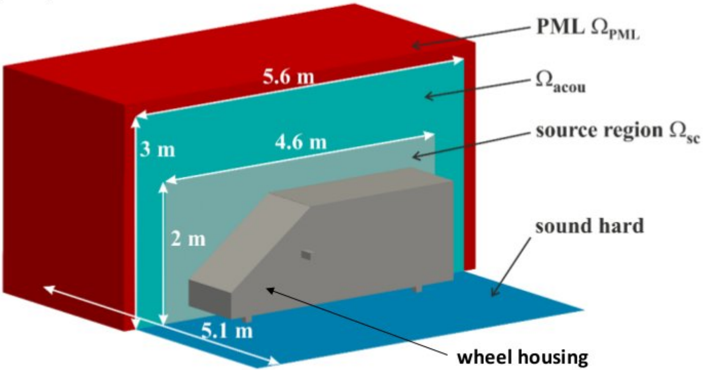
Let $\mathcal{S}(y, y^\delta) \le \delta$ and $\|y^\delta - y\|_Y \to 0$ as $\delta \to 0$, and choose $\alpha = \alpha(\delta, y^\delta) > 0$ such that $\alpha(\delta, y^\delta) \to 0$ as $\delta \to 0$. Then, as $\delta \to 0$, $y^\delta \to y$, the family $(f^\delta_{\alpha(\delta,y^\delta)}, \hat{p}^\delta_{\alpha(\delta,y^\delta)}, \hat{v}^\delta_{\alpha(\delta,y^\delta)})_{\delta\in(0,\bar{\delta}]}$ has a $\mathcal{T}$ convergent subsequence and the limit of every $\mathcal{T}$ convergent subsequence solves (SSL).

Remarks on sound source localization example

- Hilbert spaces X, V for design variables q, u (easier applicability of iterative minimization methods).
- cost function: J^δ differentiable;
- constraints: pointwise bounds can be efficiently implemented [Hungerländer, BK and Rendl 2020];
- first order least squares formulation of the PDE model;
- Euler-Lagrange equation for unregularized problem yields second order PDE model $-\frac{\omega^2}{c_0^2}\hat{p} - \Delta p = 0$;
- due to finite dimensional data space partial data inversion can be employed, see [Huynh and BK, 2020].

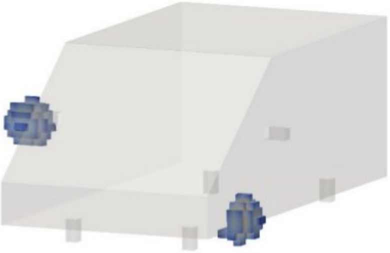
numerical results for sound source localization

- Computational setup
  - Simplified SAE Type 4 Body[5]
  - Two acoustic sources with equal intensity
    - Near the side mirror and near the wheel housing
    - Frequency of 500 Hz

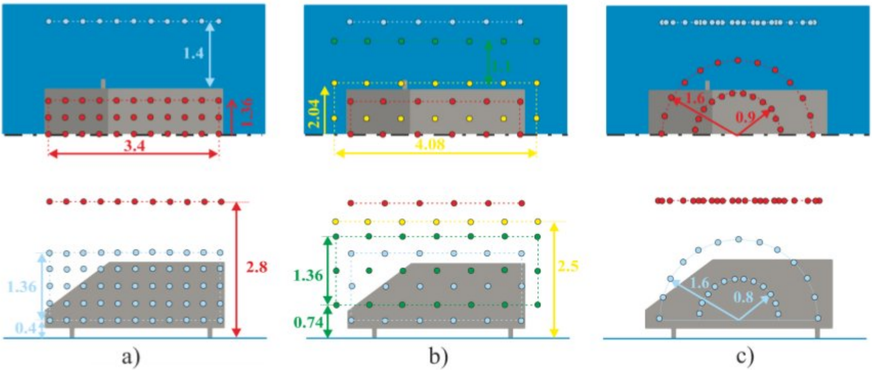


[1] Society of Automotive Engineers: Aerodynamic Testing of Road Vehicles in Open Jet Wind Tunnels. SAE Special Publication 1465 (1999).

- Realistic pressure values at the microphone positions
  - Forward simulation on a much finer computational grid as then used in the identification process
    - 4.6 million degrees of freedoms in contrast to 0.5 million
  - PML was twice as thick than on the coarse grid
  - Random noise was added (SNR of 26 dB)
  - Microphone positions on the fine and coarse differ slightly
  - Original source distribution



- Three different microphone configurations have been considered

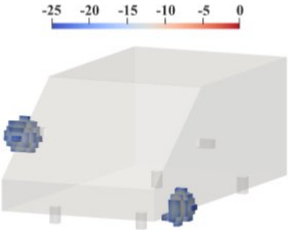


165 microphones
equally spaced (0.34 m)

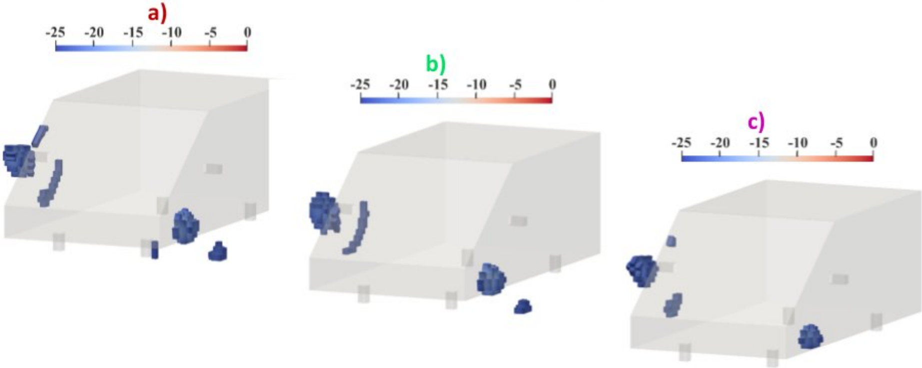
124 microphones
different planes

98 microphones

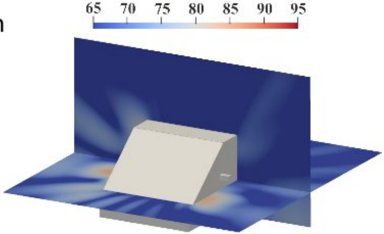
- Original source distribution



- Microphone configurations

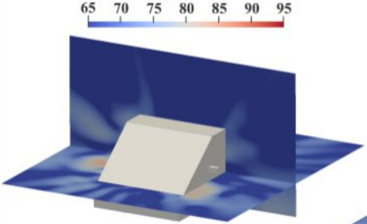


- Original sound pressure distribution

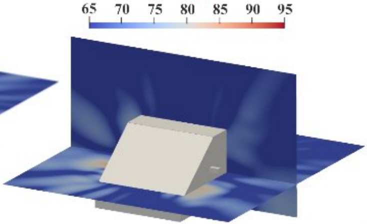


- Sound pressure based on identified sources

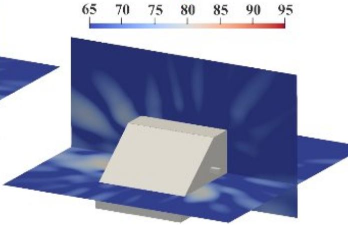
a)



b)



c)



numerical experiments for a model problem

Numerical Experiments

Identify spatially varying coefficient c in

$$-\Delta + cu = b \text{ in } (-1,1)^2$$

with homogeneous $\begin{cases} \text{Dirichlet} \\ \text{Neumann} \end{cases}$ boundary data on $\begin{cases} \{-1,1\} \times (-1,1) \\ (-1,1) \times \{-1,1\} \end{cases}$

from interior observations of u.

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from interior observations of u.

$$\min_{c,u} \|-\Delta+cu-b\|^2_{H^{-1}} \text{ s.t. } -\tau\delta \le u(x)-y^\delta \le \tau\delta\,,\ \underline{c} \le c(x) \le \overline{c} \text{ a.e.}$$

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$$\min_{c,u} \|-\Delta + cu - b\|^2_{H^{-1}} \text{ s.t. } -\tau\delta \leq u(x) - y^\delta \leq \tau\delta\,,\ \underline{c} \leq c(x) \leq \overline{c} \text{ a.e.}$$

test 1: $c_{ex}(x,y) = 1 + 10 \cdot \mathbb{1}_{B_1}$ $\quad \underline{c} = 1, \quad \overline{c} = 11\,,$

test 2: $c_{ex}(x,y) = 1 - 10 \cdot \mathbb{1}_{B_1} + 5 \cdot \mathbb{1}_{B_2}$ $\quad \underline{c} = -9, \quad \overline{c} = 6\,,$

test 3: $c_{ex}(x,y) = -10 \cdot \mathbb{1}_{B_1} - 5 \cdot \mathbb{1}_{B_2}$ $\quad \underline{c} = -10, \quad \overline{c} = 0\,,$

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$$\begin{aligned}
&\text{test 1: } c_{ex}(x,y) = 1 + 10 \cdot \mathbb{1}_{B_1} && \underline{c} = 1, && \overline{c} = 11\,, \\
&\text{test 2: } c_{ex}(x,y) = 1 - 10 \cdot \mathbb{1}_{B_1} + 5 \cdot \mathbb{1}_{B_2} && \underline{c} = -9, && \overline{c} = 6\,, \\
&\text{test 3: } c_{ex}(x,y) = -10 \cdot \mathbb{1}_{B_1} - 5 \cdot \mathbb{1}_{B_2} && \underline{c} = -10, && \overline{c} = 0\,,
\end{aligned}$$

- $B_1 = B_{0.2}(-0.4,-0.3)$, $B_2 = B_{0.1}(0.5,0.5)$
- piecewise linear/constant FE discretization of u/c
- Gauss-Newton method starting at $c_0 \equiv \frac{1}{2}(\underline{c} + \overline{c})$
- stopping criterion $\frac{J(x_k^\delta, u_k^\delta)}{J(x_0, u_0)} < 1.e-5$
- $\tau = 1.1$

Test 1

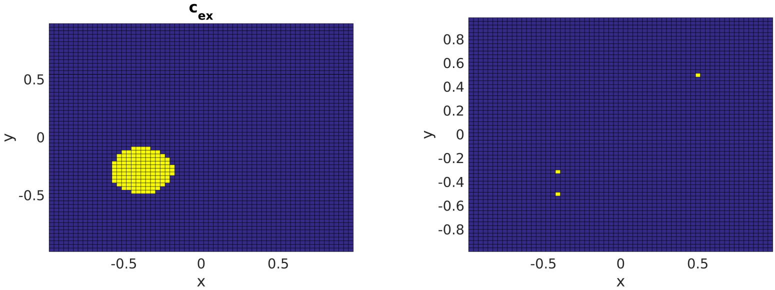


Figure: left: exact coefficient c_{ex}; $\underline{c} = 1$, $\overline{c} = 11$
right: locations of spots for testing weak * L^∞ convergence

Comparison

- mkr_box ... recursive globalization of semismooth Newton
- mSN2_box ... combinatorial globalization of semismooth Newton
- quadprog (Matlab) with trust-region-reflective
  (subspace trust-region method based on interior-reflective Newton [Coleman&Li'96]

	quadprog	mSN2_box	mkr_box
k	5	4	4
$\frac{J(x_k^\delta, u_k^\delta)}{J(x_0, u_0)}$	4.6671e-06	9.8449e-06	9.8449e-06
err_{spot_1}	3.7548e-13	0	0
err_{spot_2}	5.1669e-06	0	0
err_{spot_3}	0.5280	1.3360	1.3360
$\text{err}_{L^1(\Omega)}$	0.0882	0.0972	0.0972
CPU	30.77	35.22	6.55

k ... number of Gauss-Newton steps

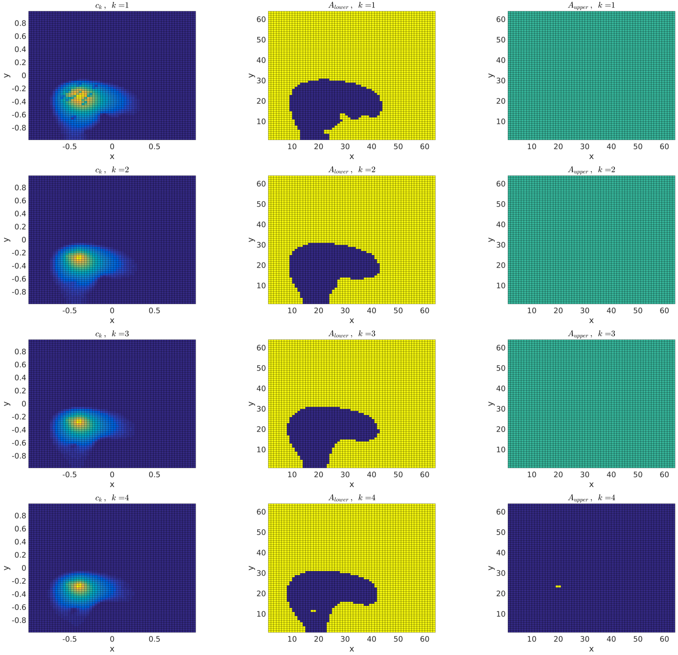
Convergence as $\delta \to 0$

δ	0.001	0.01	0.1
err_{spot_1}	0	0	0
err_{spot_2}	0	0.7960	4.8689
err_{spot_3}	1.0840	2.1512	2.5862
$err_{L^1(\Omega)}$	0.1472	0.2136	0.3671

Table: Averaged errors of five test runs on each noise level, with random uniform noise

(using `mkr_box`)

reconstruction c_k; active set lower bound; active set upper bound; $\delta = 10\%$



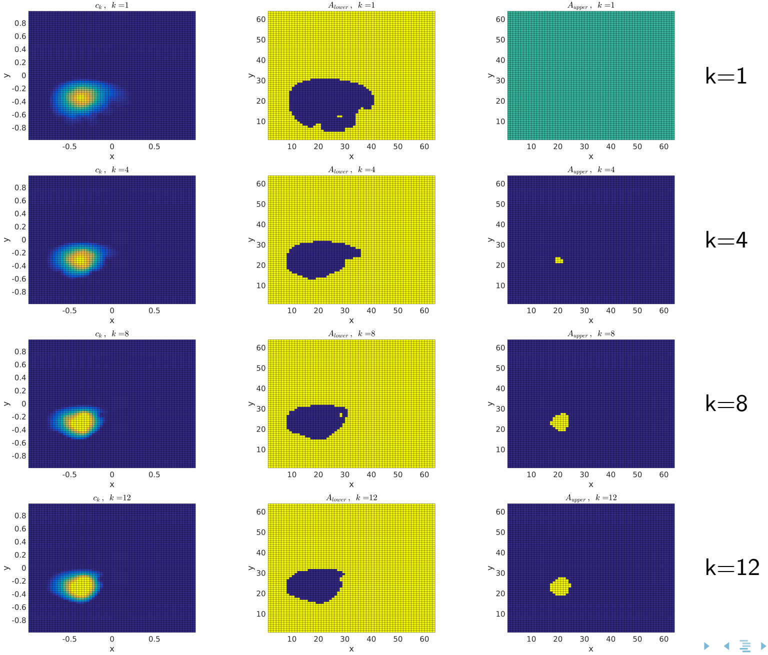
k=1

k=2

k=3

k=4

reconstruction c_k; active set lower bound; active set upper bound; $\delta = 1\%$



Test 2

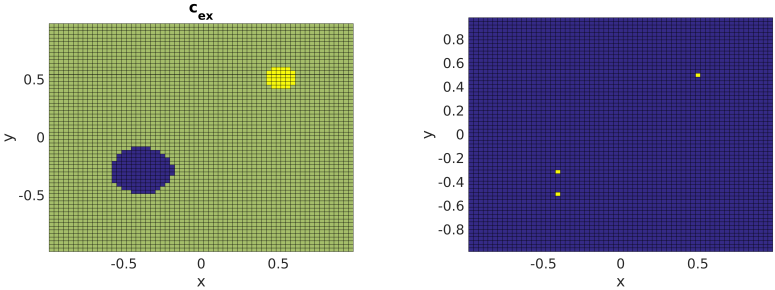
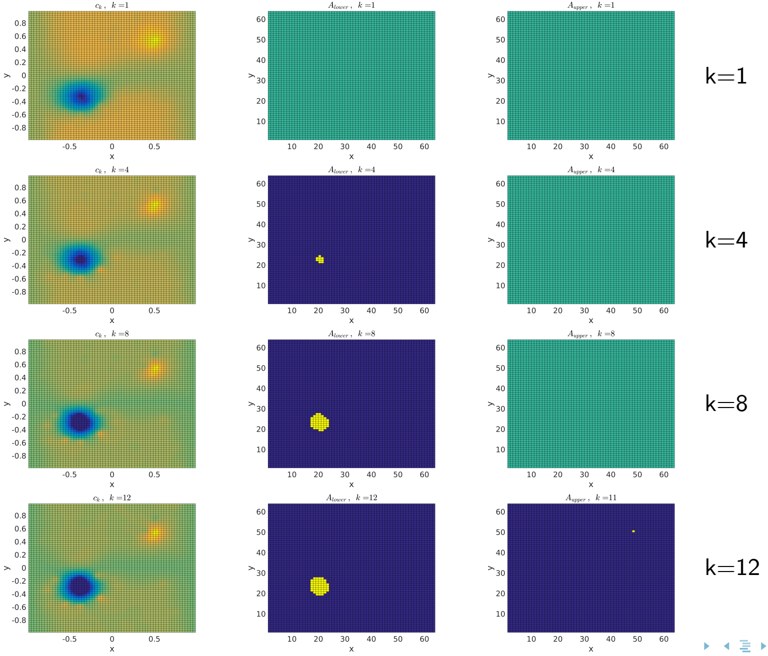
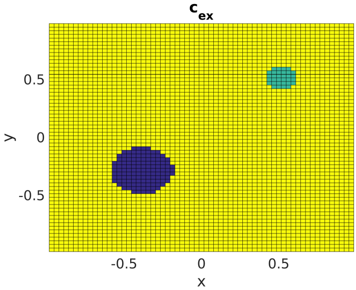


Figure: left: exact coefficient c_{ex}; $\underline{c} = -9$, $\overline{c} = 6$
right: locations of spots for testing weak * L^∞ convergence

reconstruction c_k; active set lower bound; active set upper bound; $\delta = 1\%$



Test 3



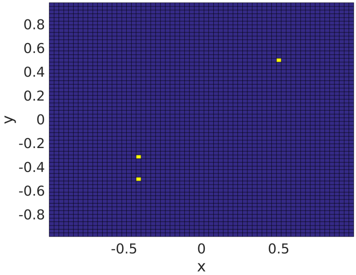
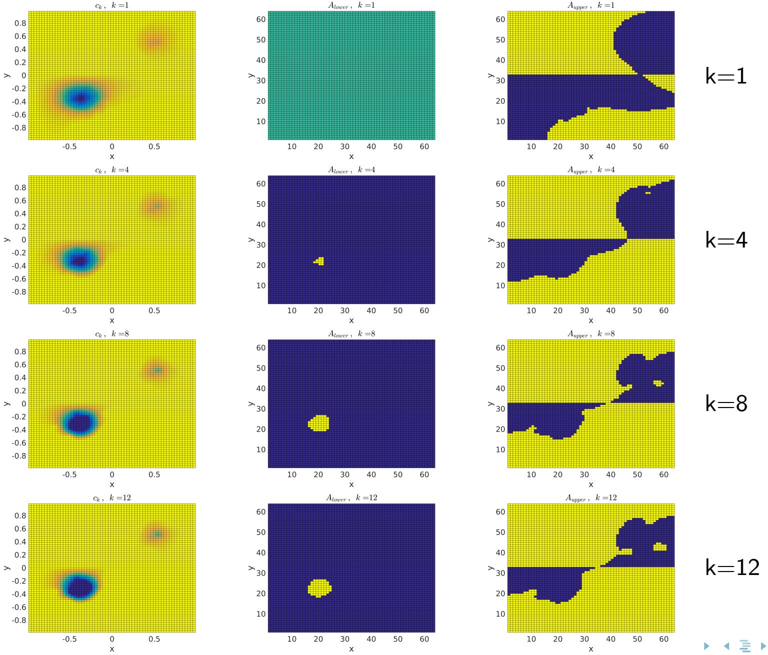


Figure: left: exact coefficient c_{ex}; $\underline{c} = -10$, $\overline{c} = 0$
right: locations of spots for testing weak * L^∞ convergence

reconstruction c_k; active set lower bound; active set upper bound; $\delta = 1\%$



Conclusions

- Convergence analysis for a nonstandard variational regularization of a variational formulation

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- for EIT

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→ iterative methods

→ other applications (e.g., distributed or nonlinear permeabilities in magnetostatics, Lamé parameters in elastostatics, cracks)

Thank you for your attention!

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