The Reduced Basis Method

Part I: Theoretical Introduction

Part II: Applications

The Reduced Basis Method

Part I: Theoretical Introduction

- Motivation
- RB for the Simplest Case
- Extensions

Part II: Applications

The Reduced Basis Method

Part I: Theoretical Introduction

Motivation

RB for the Simplest Case

Extensions

Part II: Applications

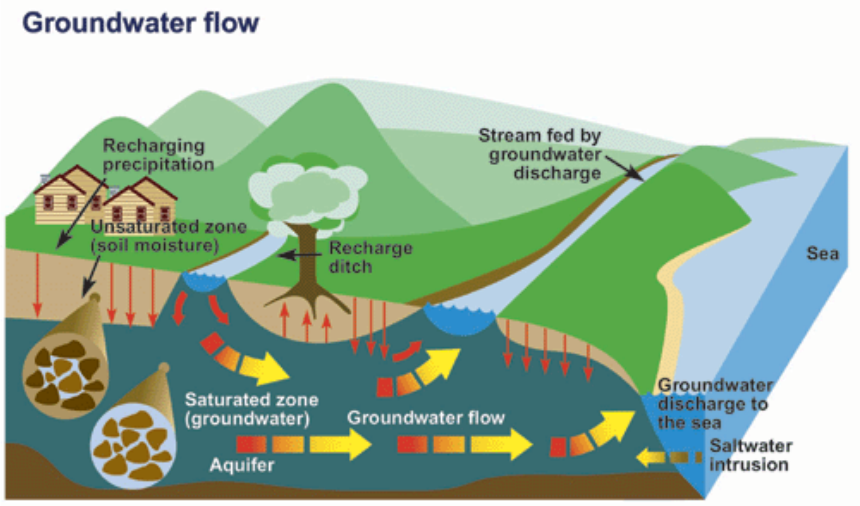
Data Assimilation and Experimental Design

Cancer Treatment Planning

PART I

Theoretical Introduction

Motivation - A Geosciences Example



Source: Environment and Climate Canada
`https://www.ec.gc.ca/eau-water`

Groundwater Flow:

- Groundwater management
- Contaminant transport

Goal:

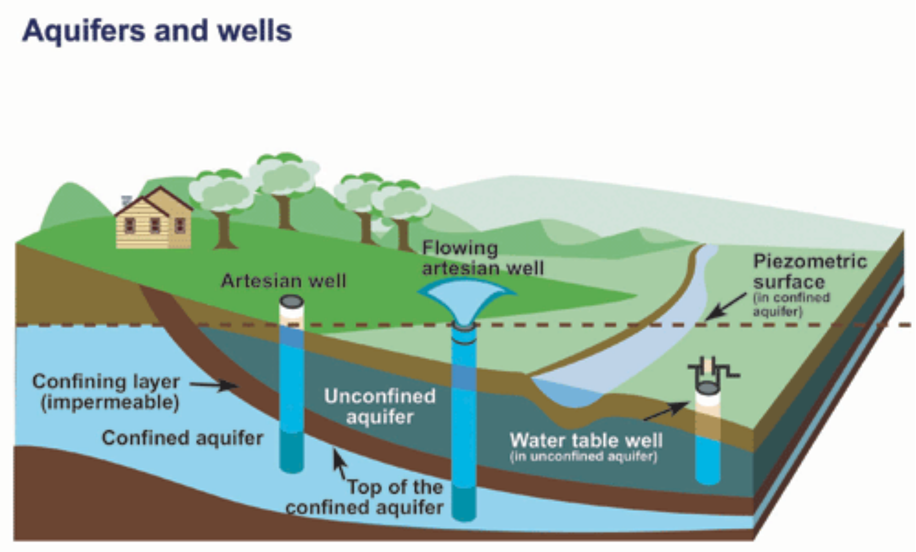
- Predict hydraulic head
- Predict pollutant concentration

Given:

- Parametrized PDE-model

Issues:

- Parameters unknown
- Model, but possibly erroneous
- Boundary or initial conditions uncertain
- Measurements, possibly noisy



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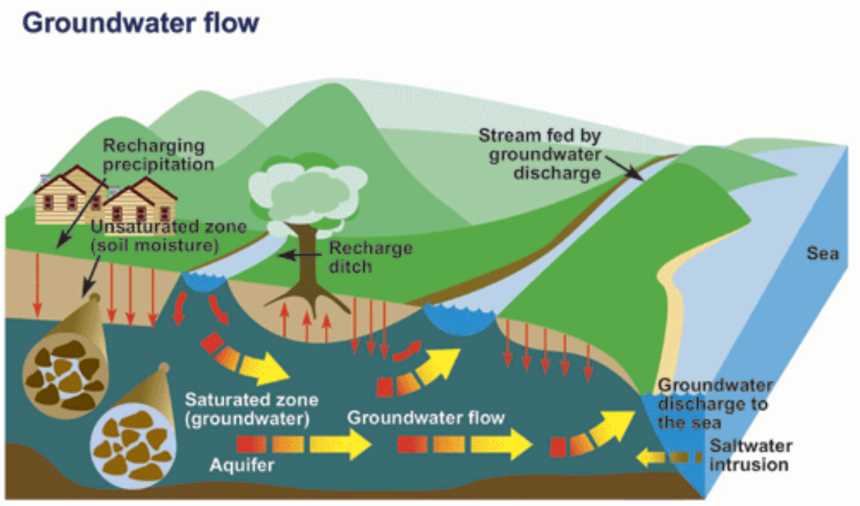
Motivation - A Geosciences Example



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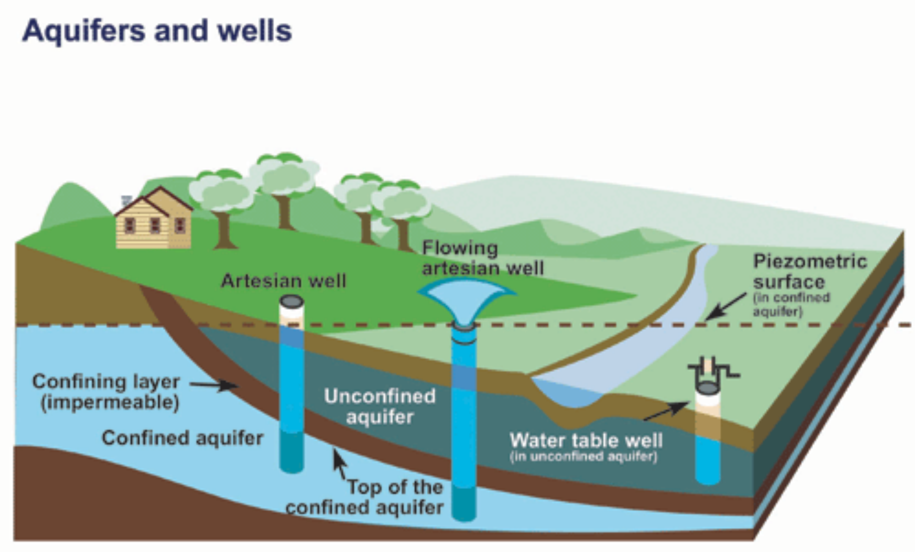

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Objective

Problem: For $\mu \in \mathcal{D}$ compute $s(\mu) = f(y(\mu); \mu)$ where

$$a(y(\mu), v; \mu) = f(v; \mu), \qquad \forall\, v \in \mathcal{Y}$$ PDE

in multi-query, real-time, or slim computing settings.

Objective

output of interest **parameter**

Problem: For $\mu \in \mathcal{D}$ compute $s(\mu) = \ell(y(\mu);\mu)$ where

output functional **state variable,** $y(\boldsymbol{x};\mu)$

$$a(y(\mu), v; \mu) = f(v; \mu), \qquad \forall\, v \in \mathcal{Y}$$ PDE

in multi-query, real-time, or slim computing settings.

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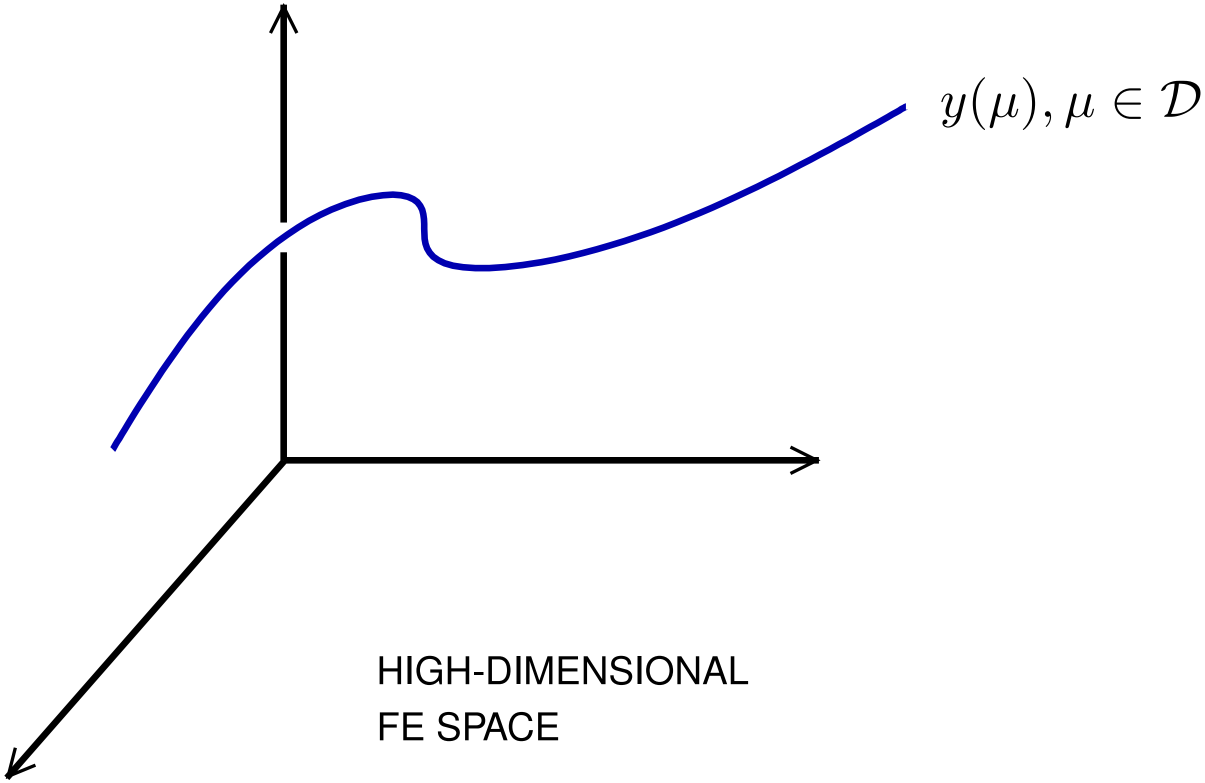
in multi-query, real-time, or slim computing settings.

Goal: Compute approximations

$$y(\mu) \approx y_N(\mu) \qquad s(\mu) \approx s_N(\mu) := f(y_N(\mu); \mu)$$

that are (certifiably) accurate and (online-) inexpensive.

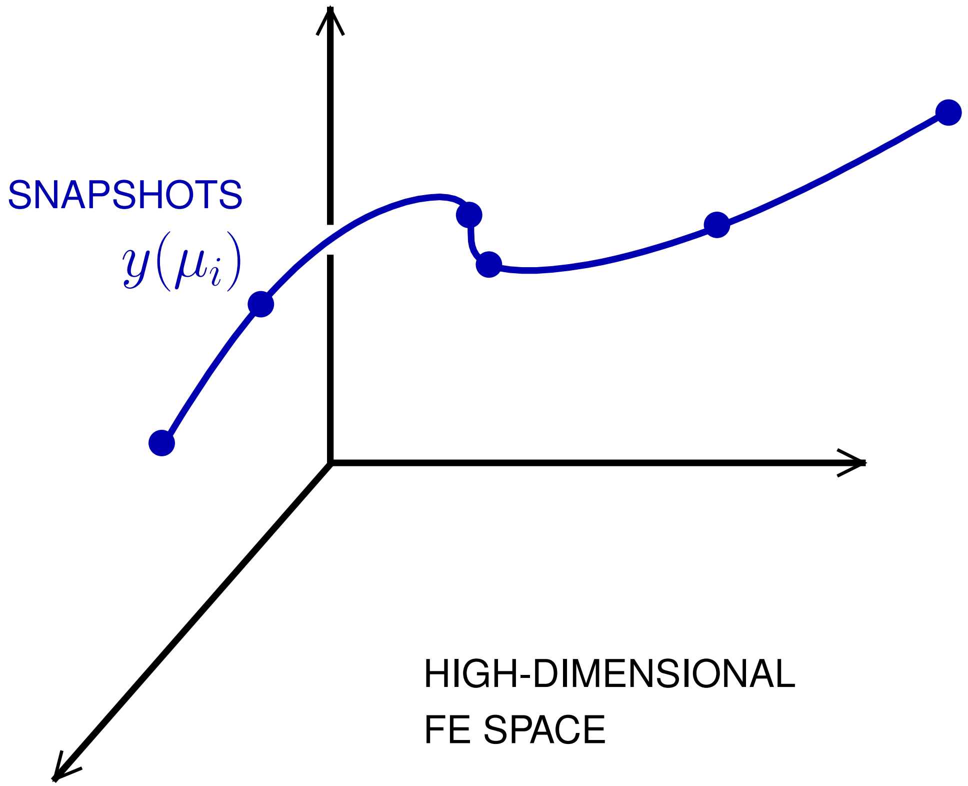
The Reduced Basis Method



HIGH-DIMENSIONAL
FE SPACE

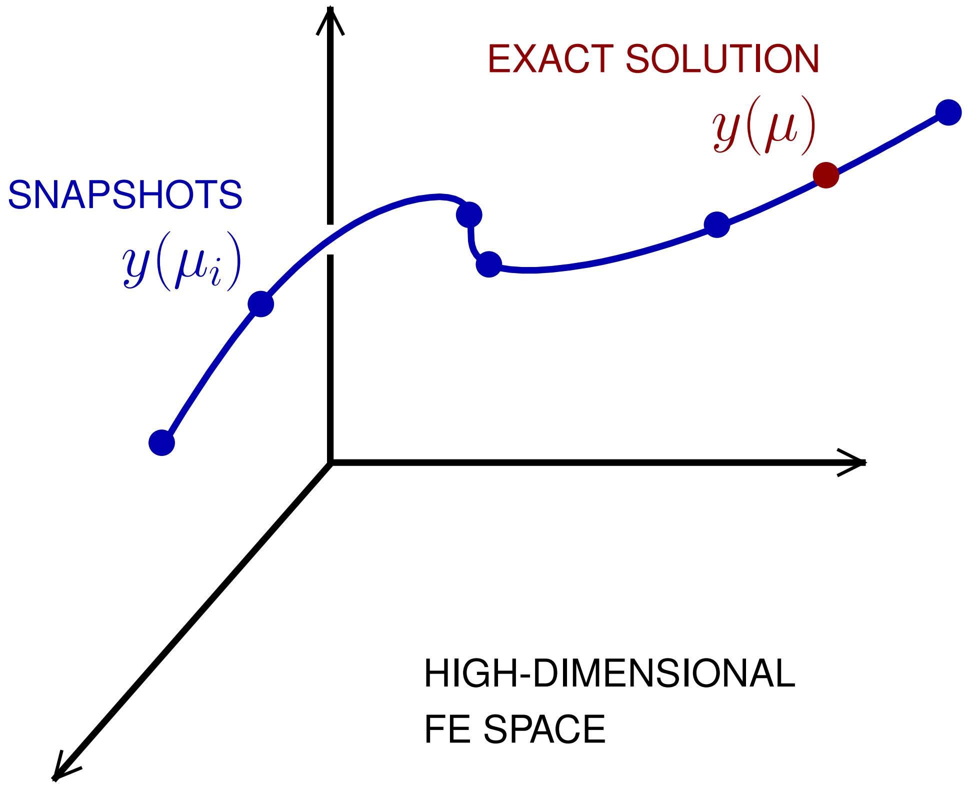
$$a(y(\mu), v; \mu) = f(v; \mu), \quad \text{for all } v \in \mathcal{Y},\ \mu \in \mathcal{D}$$

The Reduced Basis Method



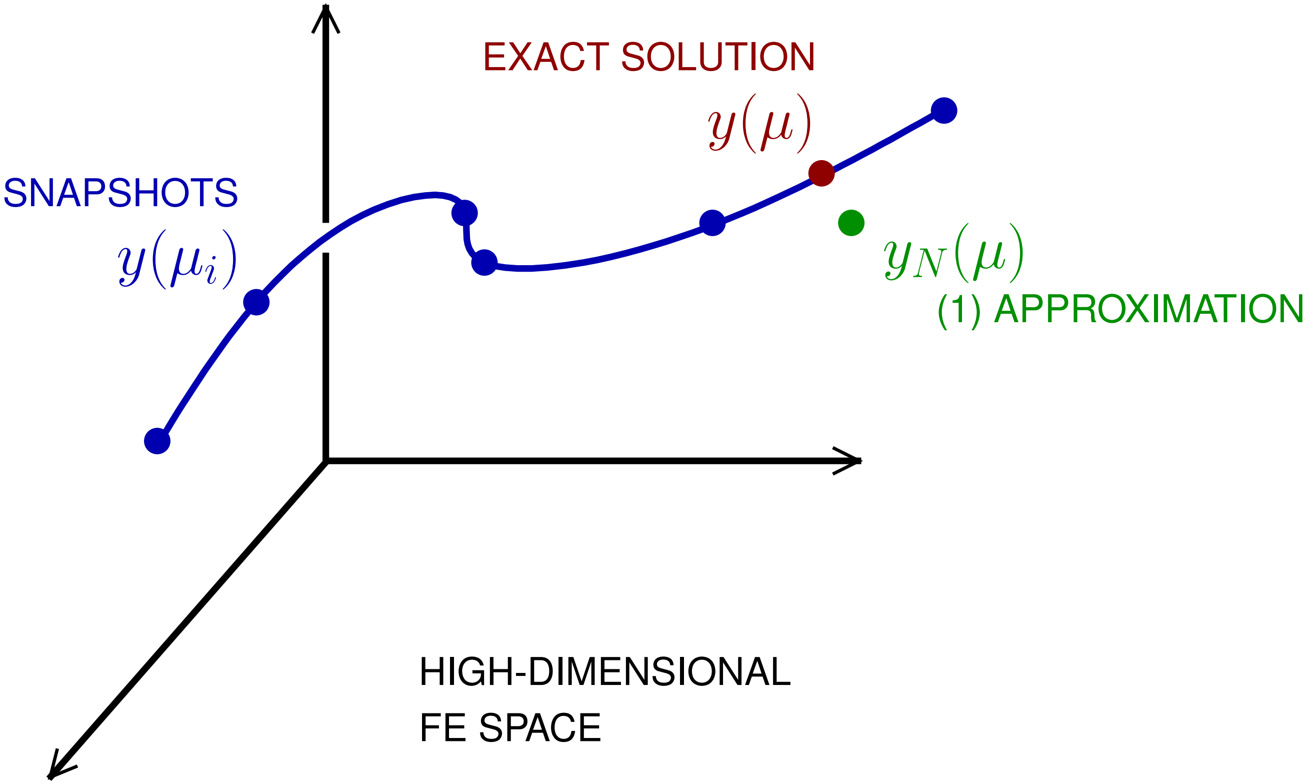
$$\mathcal{Y}_N := \mathrm{span}\{\ y(\mu_i)\ ,\ i = 1, \ldots, N\ \}$$

The Reduced Basis Method



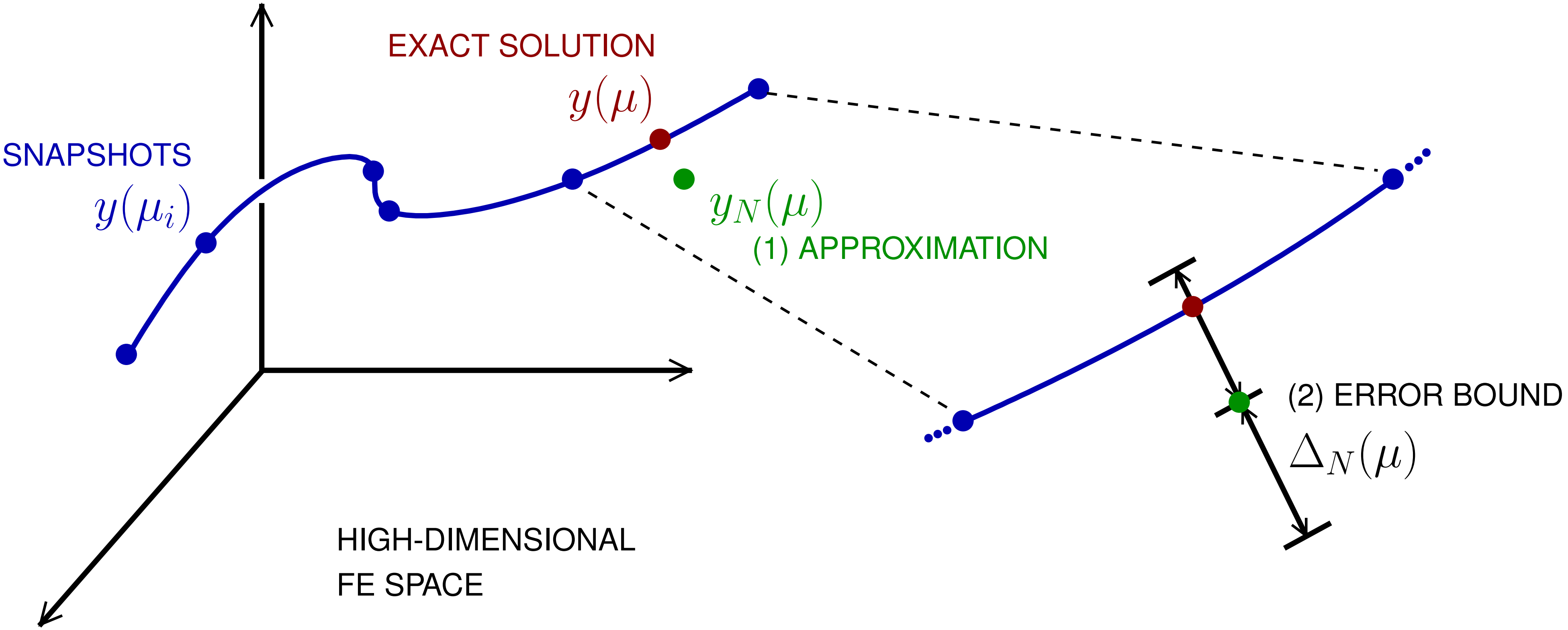
$$a(y(\mu), v; \mu) = f(v; \mu), \quad \text{for all } v \in \mathcal{Y}.$$

The Reduced Basis Method



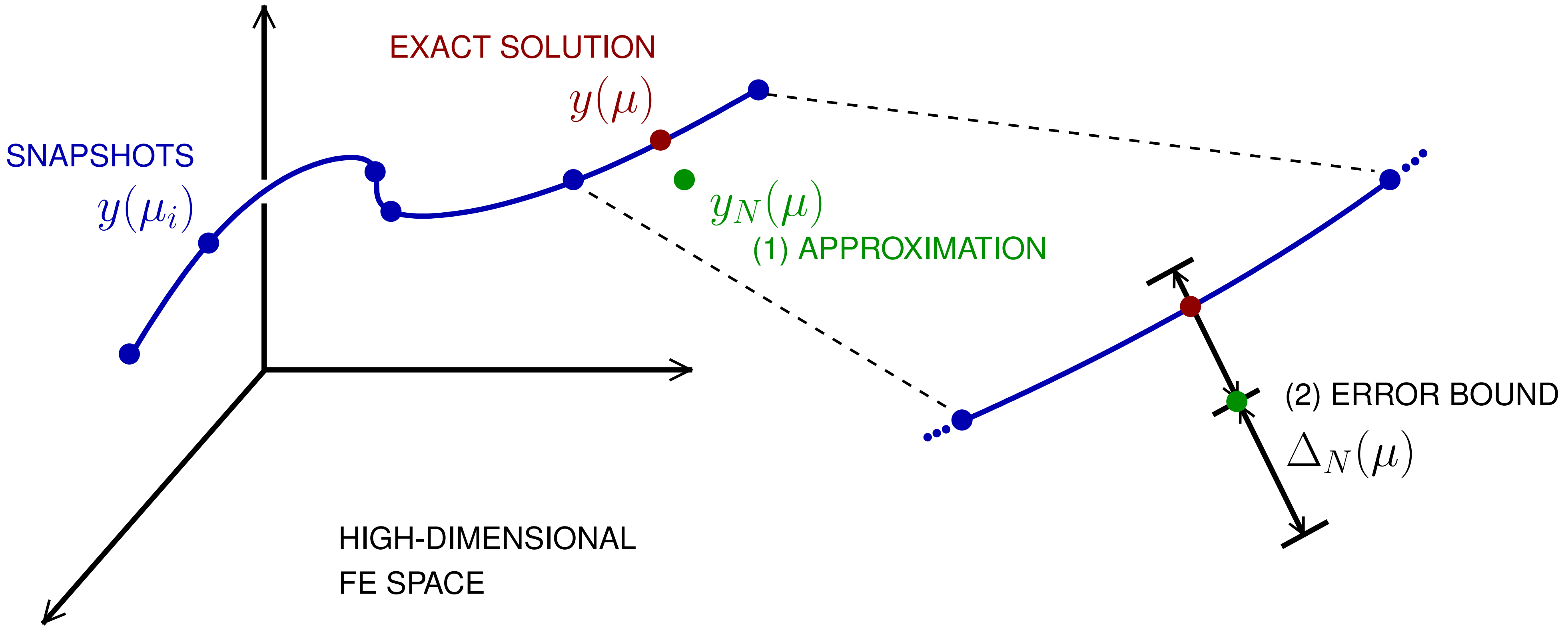
$$a(y_N(\mu), v; \mu) = f(v; \mu), \quad \text{for all } v \in \mathcal{Y}_N.$$

The Reduced Basis Method



$$\|y(\mu) - y_N(\mu)\|_{\mathcal{Y}} \;\leq\; \Delta_N(\mu).$$

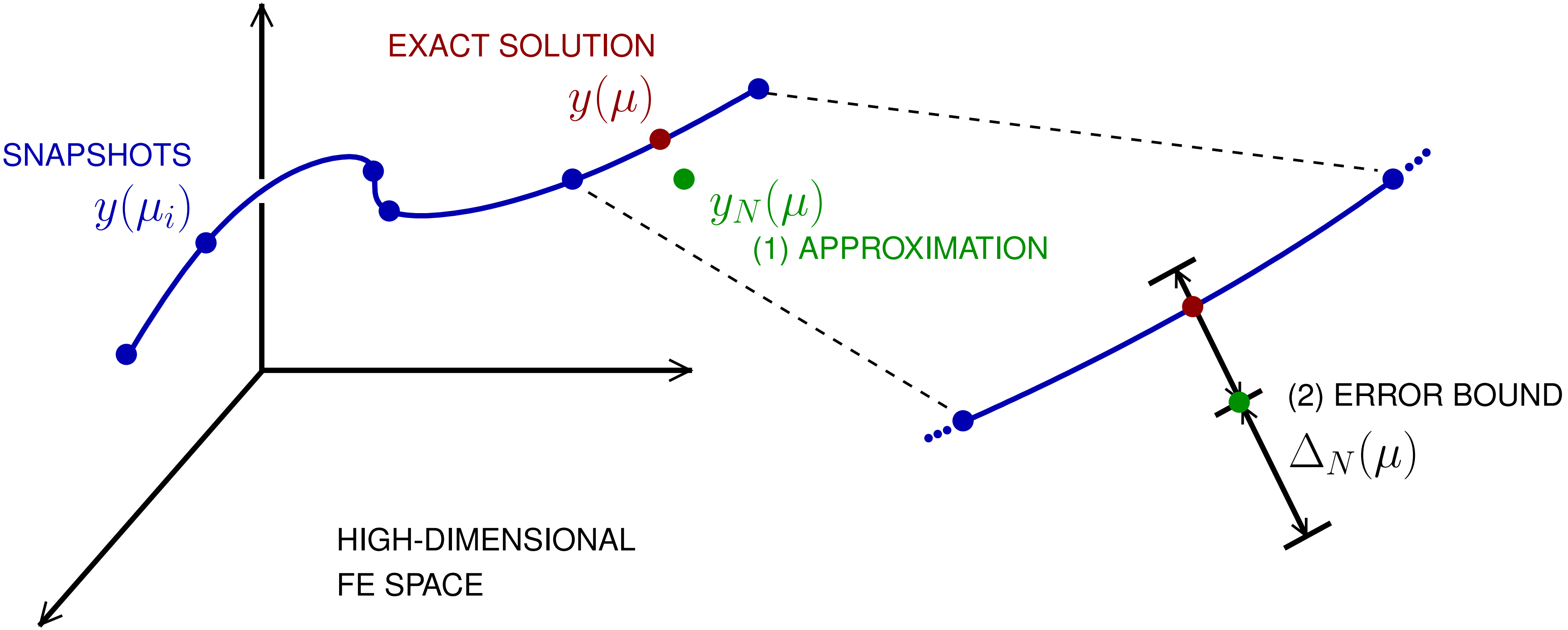
The Reduced Basis Method



(3) OFFLINE-ONLINE COMPUTATIONAL DECOMPOSITION

$$a(w,v;\mu) \;=\; \sum_{q=1}^{Q} \underbrace{\theta^q(\mu)}_{\substack{\mu\text{- DEPENDENT}\\ \text{COEFFICIENTS}}} \quad \underbrace{a^q(w,v)}_{\substack{\mu\text{- INDEPENDENT}\\ \text{BILINEAR FORMS}}}, \quad \forall w,v \in \mathcal{Y}$$

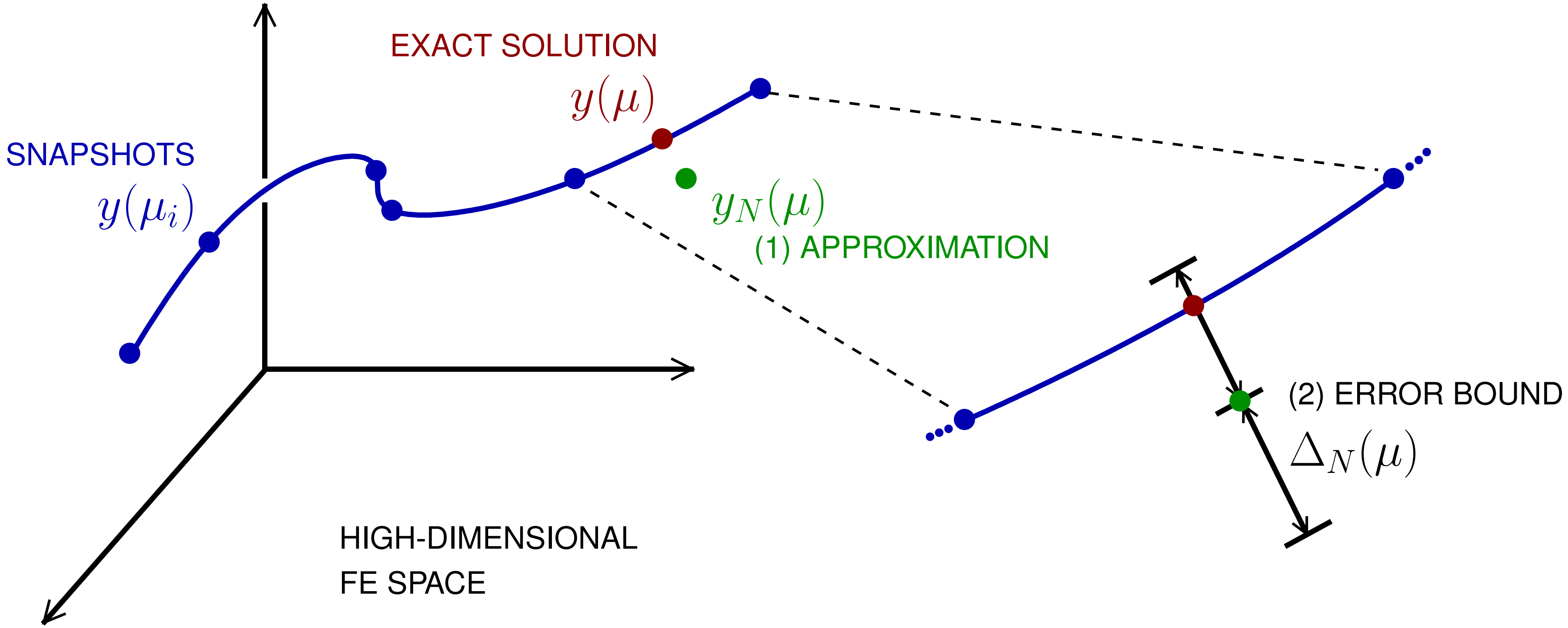
The Reduced Basis Method



(3) OFFLINE-ONLINE COMPUTATIONAL DECOMPOSITION
(4) GREEDY ALGORITHM

$$\mu_{N+1} = \arg\max_{\mu \in \mathcal{D}} \frac{\Delta_N(\mu)}{\|y_N(\mu)\|_{\mathcal{Y}}}$$

The Reduced Basis Method



EXACT SOLUTION

$y(\mu)$

SNAPSHOTS

$y(\mu_i)$

$y_N(\mu)$

(1) APPROXIMATION

(2) ERROR BOUND

$\Delta_N(\mu)$

HIGH-DIMENSIONAL
FE SPACE

(3) OFFLINE-ONLINE COMPUTATIONAL DECOMPOSITION

(4) GREEDY ALGORITHM

The Reduced Basis Method

The **reduced basis method** seeks to provide, for any $\mu \in \mathcal{D}$

accurate	$y_N(\mu) \approx y(\mu)$	**(1) APPROXIMATION**
reliable	$\|y(\mu) - y_N(\mu)\|_{\mathcal{Y}} \geq \Delta_N(\mu)$	**(2) ERR ESTIMATION**
efficient surrogates	cost $(Q^\bullet, N^\bullet)$	**(3) OFFLINE / ONLINE**
	small N	**(4) GREEDY ALGORITHM**

to solutions of **parametrized PDEs**

for the **many-query**, **real-time**,

and **slim-computing** contexts.

Original papers: [Prud'homme, et al., 2002], [Maday, et al., 2002], …

Books: [Hesthaven, Rozza & Stamm, 2015], [Quarteroni, Manzoni & Negri, 2015]

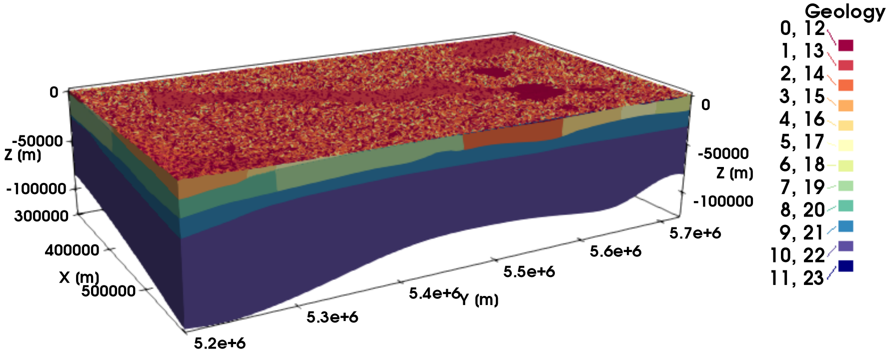
Other MOR Methods: [Schilders, Van der Vorst, Rommes, 2008] [Benner, Ohlberger, Cohen & Willcox, 2017]

…

Simplest Case

Strong Form

Find y such that

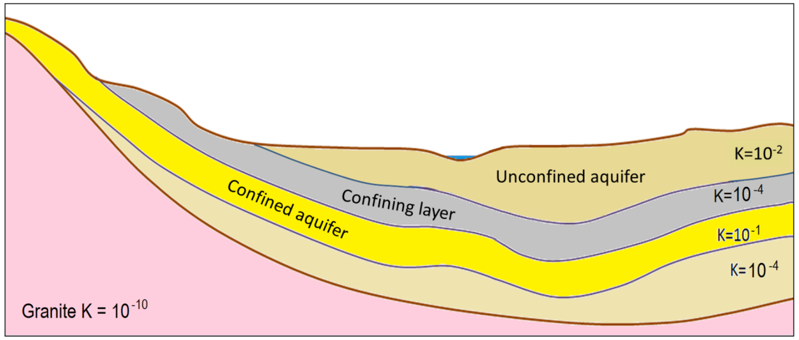


Upper Rhine Graben (Germany)
Courtesy of Prof. Scheck-Wenderoth, GFZ Postdam.

PDE $$-\kappa \nabla^2 y = f \quad \text{in} \quad \Omega \subset \mathbb{R}^3$$

BC $$y = 0 \quad \text{on} \quad \Gamma_N$$

where $\kappa = \begin{cases} \kappa_0 := 1 & \text{in} \quad \Omega_o \\ \kappa_i & \text{in} \quad \Omega_i, \; i = 1, \ldots, P \end{cases}$



Source: Physical Geology (Ch. 14) by S. Earle
(Licence: CC-BY-4.0)

Let the parameter $\mu := \{\kappa_1, \ldots, \kappa_P\} \in \mathcal{D} \subset \mathbb{R}^P$

Simplest Case

Weak Form

For $\mu \in \mathcal{D}$, find $y \equiv y(\mu) \in \mathcal{Y}$ such that

$$\sum_{i=0}^{P} \int_{\Omega_i} \kappa_i \nabla y \cdot \nabla v \, d\Omega \quad = \quad \int_{\Omega} v \, d\Omega, \qquad \forall \, v \in \mathcal{Y},$$

and compute $s(\mu) = \displaystyle\int_{\Omega} y(\mu) \, d\Omega$.

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Abstract Form

For $\mu \in \mathcal{D}$, find $y \equiv y(\mu) \in \mathcal{Y}$ such that

$$a(y(\mu), v; \mu) \quad = \quad f(v; \mu), \qquad \forall \, v \in \mathcal{Y},$$

and compute $s(\mu) \;=\; \ell(y(\mu); \mu)$ where $\ell = f$.

Simplest Case

Weak Form

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Simplest Case:
- a is coercive
- compliant, $\ell = f$
- linear
- time-independent.

Simplest Case

General Form

Find $y(\mu) \in \mathcal{Y}$ s.t.

$$a(y(\mu), v; \mu) \;=\; f(v; \mu) \qquad \forall\, v \in \mathcal{Y}.$$

Matrix Form

Find $\boldsymbol{y}(\mu) \in \mathbb{R}^{\mathcal{N}}$ s.t.

$$\boldsymbol{\mathcal{A}}(\mu)\boldsymbol{y}(\mu) \;=\; \boldsymbol{f}(\mu),$$

where $\boldsymbol{\mathcal{A}} \in \mathbb{R}^{\mathcal{N}\times\mathcal{N}}$, $\boldsymbol{f} \in \mathbb{R}^{\mathcal{N}}$.

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Questions:

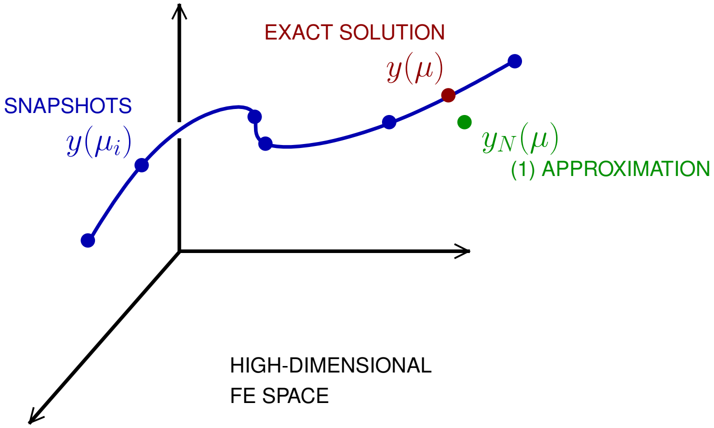
How can we compute the approximation?

How do we know the error is small?

How do we know what value of N to take?

How do we compute the approximations efficiently?

How do we choose the sample points optimally?



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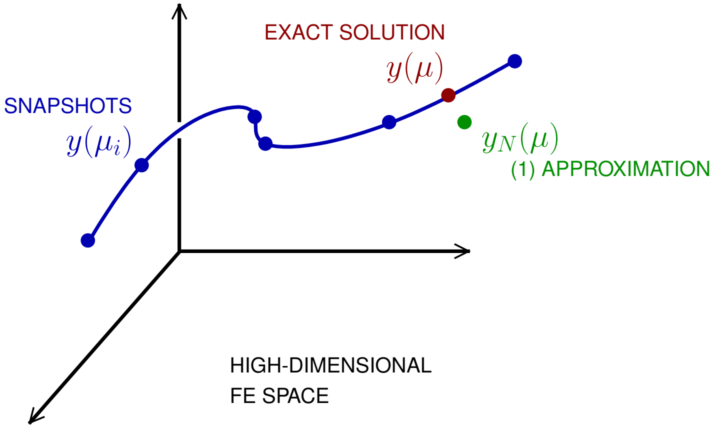

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Simplest Case

General Form

Find $y_N(\mu) \in \mathcal{Y}_N$ s.t.

$$a(y_N(\mu), v; \mu) \;=\; f(v; \mu) \qquad \forall\, v \in \mathcal{Y}_N.$$

Matrix Form

Find $\mathbf{y}_N(\mu) \in \mathbb{R}^N$ s.t.

$$\mathbf{A}_N(\mu)\mathbf{y}_N(\mu) \;=\; \mathbf{f}_N(\mu),$$

where $\mathbf{A}_N \in \mathbb{R}^{N\times N},\ \mathbf{f}_N \in \mathbb{R}^N.$

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where $\mathbf{A}_N \in \mathbb{R}^{N\times N}$, $\mathbf{f}_N \in \mathbb{R}^N$.

Let $y_N(\mu) \in \mathcal{Y}_N$, $\quad \mathcal{Y}_N \;:=\; \text{span}\{\underbrace{y(\mu_1), \ldots, y(\mu_N)}_{\text{snapshots}}\} \;=\; \text{span}\{\underbrace{\varphi_1, \ldots, \varphi_N}_{\text{orthogonal basis}}\}$

$$y_N(\mu) \;=\; \sum_{i=1}^{N} (\mathbf{y}_N)_i\, \varphi_i$$

Simplest Case

General Form

Find $y(\mu) \in \mathcal{Y}$ s.t.

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$$\forall\, v \in \mathcal{Y}.$$

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Find $\boldsymbol{y}(\mu) \in \mathbb{R}^{\mathcal{N}}$ s.t.

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Problem: Find $\boldsymbol{y}(\mu) \in \mathbb{R}^{\mathcal{N}}$ s.t. $\boldsymbol{v}^T \boldsymbol{\mathcal{A}}(\mu)\boldsymbol{y}(\mu) \; = \; \boldsymbol{v}^T \boldsymbol{f}(\mu), \;\; \forall\, \boldsymbol{v} \in \mathbb{R}^{\mathcal{N}}.$

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Let $\boldsymbol{y}_N(\mu) \in \mathrm{span}\{\, \boldsymbol{\varphi}_i, \ldots, \boldsymbol{\varphi}_N \,\}$, and $\boldsymbol{\mathcal{W}}_N = [\, \boldsymbol{\varphi}_i, \ldots, \boldsymbol{\varphi}_N \,]$

$$\underbrace{\boldsymbol{y}_N(\mu)}_{\text{approximation to } \boldsymbol{y}(\mu)} = \sum_{i=1}^{N} \underbrace{(\mathbf{y}_N)_i(\mu)}_{\text{coefficients}} \; \underbrace{\boldsymbol{\varphi}_i}_{\text{snapshots/ basis functions}} \qquad = \boldsymbol{\mathcal{W}}_N \mathbf{y}_N$$

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Similarly, let $\boldsymbol{v}_N = \boldsymbol{\mathcal{W}}_N \, \mathrm{v}_N .$

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$$\Rightarrow \ (\boldsymbol{\mathcal{W}}_N \mathbf{v}_N)^T \boldsymbol{\mathcal{A}}(\mu)\, \boldsymbol{\mathcal{W}}_N\, \mathbf{y}_N(\mu) = (\boldsymbol{\mathcal{W}}_N \mathbf{v}_N)^T \boldsymbol{f}(\mu), \quad \forall\, \mathbf{v}_N \in \mathbb{R}^{\mathrm{N}}$$

Simplest Case

$$\mathbf{v}_N^T \underbrace{\boldsymbol{\mathcal{W}}_N^T \boldsymbol{\mathcal{A}}(\mu) \ \boldsymbol{\mathcal{W}}_N}_{=:\ \mathbf{A}_N(\mu)} \mathbf{y}_N(\mu) = \mathbf{v}_N^T \underbrace{\boldsymbol{\mathcal{W}}_N^T \boldsymbol{f}(\mu)}_{=:\ \mathbf{f}_N(\mu)} \quad \forall\, \mathbf{v}_N \in \mathbb{R}^{\mathrm{N}}$$

Simplest Case

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$$\Leftrightarrow \quad \mathbf{A}_N(\mu) \, \underset{\uparrow}{\mathbf{y}_N(\mu)} \; = \; \mathbf{f}_N(\mu)$$

coefficients in expansion
in terms of the basis

Simplest Case

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How do we know the error is small?

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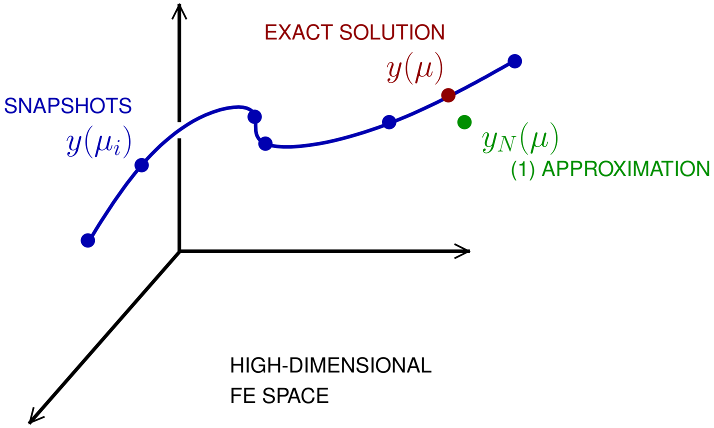

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Problem: Find $y(\mu) \in \mathcal{Y}$ s.t. $a(y(\mu), v; \mu) \;=\; f(v;\mu), \quad \forall\, v \in \mathcal{Y}$.

(Lax-Milgram) Let $\mathcal{Y}$ be a Hilbert space, and for all $\mu \in \mathcal{D}$, assume

- $a(\cdot,\cdot;\mu)$ is continuous and coercive,

$$\gamma_a(\mu) \;:=\; \sup_{v\in\mathcal{Y}} \sup_{w\in\mathcal{Y}} \frac{a(v,w;\mu)}{\|v\|_{\mathcal{Y}}\|w\|_{\mathcal{Y}}} < \infty \;,$$

$$\alpha_a(\mu) := \inf_{v\in\mathcal{Y}} \frac{a(v,v;\mu)}{\|v\|^2_{\mathcal{Y}}} > 0 \;,$$

- f is bounded,

$$\|f\|_{\mathcal{Y}'} := \sup_{v\in\mathcal{Y}} \frac{f(v)}{\|v\|_{\mathcal{Y}}} < \infty.$$

Simplest Case

Problem: Find $y(\mu) \in \mathcal{Y}$ s.t. $a(y(\mu), v; \mu) \; = \; f(v; \mu), \quad \forall\, v \in \mathcal{Y}.$

(Lax-Milgram) Let $\mathcal{Y}$ be a Hilbert space, and for all $\mu \in \mathcal{D}$, assume

- $a(\cdot, \cdot; \mu)$ is continuous and coercive,

$$\gamma_a(\mu) \; := \; \sup_{v \in \mathcal{Y}} \; \sup_{w \in \mathcal{Y}} \frac{a(v, w; \mu)}{\|v\|_{\mathcal{Y}} \|w\|_{\mathcal{Y}}} < \infty \; ,$$

$$\alpha_a(\mu) := \inf_{v \in \mathcal{Y}} \frac{a(v, v; \mu)}{\|v\|_{\mathcal{Y}}^2} > 0 \; ,$$

- f is bounded,

$$\|f\|_{\mathcal{Y}'} := \sup_{v \in \mathcal{Y}} \frac{f(v)}{\|v\|_{\mathcal{Y}}} < \infty.$$

Then there exists a unique solution $y(\mu)$ satisfying

$$\|y(\mu)\|_{\mathcal{Y}} \leq \frac{\|f\|_{\mathcal{Y}'}}{\alpha_a(\mu)}.$$

Simplest Case

Consider the following

$$a(y(\mu), v; \mu) = f(v; \mu), \quad \forall\, v \in \mathcal{Y}, \qquad \text{FE problem}$$

$$a(y_N(\mu), v; \mu) = f(v; \mu), \quad \forall\, v \in \mathcal{Y}_N. \qquad \text{RB approximation}$$

Simplest Case

Consider the following

$$a(y(\mu), v; \mu) = f(v; \mu), \quad \forall\, v \in \mathcal{Y}, \qquad \text{FE problem}$$

$$a(y_N(\mu), v; \mu) = f(v; \mu), \quad \forall\, v \in \mathcal{Y}_N. \qquad \text{RB approximation}$$

Define the error $e_N(\mu) := y(\mu) - y_N(\mu)$ and the residual

$$\begin{aligned} r(v; y_N(\mu); \mu) &:= f(v; \mu) - a(y_N(\mu), v; \mu) \\ &= a(y(\mu), v; \mu) - a(y_N(\mu), v; \mu) \\ &= a(y(\mu) - y_N(\mu), v; \mu) \\ &= a(e_N(\mu), v; \mu), \qquad \forall\, v \in \mathcal{Y}. \end{aligned}$$

Simplest Case

From the error-residual equation

$$a(e_N(\mu), v; \mu) \;=\; r(v; y_N(\mu), \mu)$$

Simplest Case

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$$a(e_N(\mu), v; \mu) \; = \; r(v; y_N(\mu), \mu)$$

and the Lax-Milgram Theorem, we have that

$$\|e_N(\mu)\|_{\mathcal{Y}} \; \leq \; \frac{\|r(\cdot; y_N(\mu), \mu)\|_{\mathcal{Y}'}}{\alpha_a(\mu)}.$$

Simplest Case

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and the Lax-Milgram Theorem, we have that

$$\|e_N(\mu)\|_{\mathcal{Y}} \;\leq\; \frac{\|r(\cdot; y_N(\mu), \mu)\|_{\mathcal{Y}'}}{\alpha_a(\mu)}.$$

Assume we have a computable lower bound $\alpha_a^{\mathrm{LB}}(\mu) \leq \alpha_a(\mu),\ \forall\, \mu \in \mathcal{D}$.

Then $$\|e_N(\mu)\|_{\mathcal{Y}} \;\leq\; \Delta_N(\mu) \;:=\; \frac{\|r(\cdot; y_N(\mu))\|_{\mathcal{Y}'}}{\alpha_a^{\mathrm{LB}}(\mu)}$$ ERR BOUND

Simplest Case

General Form

Find $y_N(\mu) \in \mathcal{Y}_N$ s.t.

$$a(y_N(\mu), v; \mu) = f(v; \mu) \qquad \forall\, v \in \mathcal{Y}_N.$$

Matrix Form

Find $\mathbf{y}_N(\mu) \in \mathbb{R}^N$ s.t.

$$\mathbf{A}_N(\mu)\mathbf{y}_N(\mu) = \mathbf{f}_N(\mu),$$

where $\mathbf{A}_N \in \mathbb{R}^{N \times N}$, $\mathbf{f}_N \in \mathbb{R}^N$.

Questions:

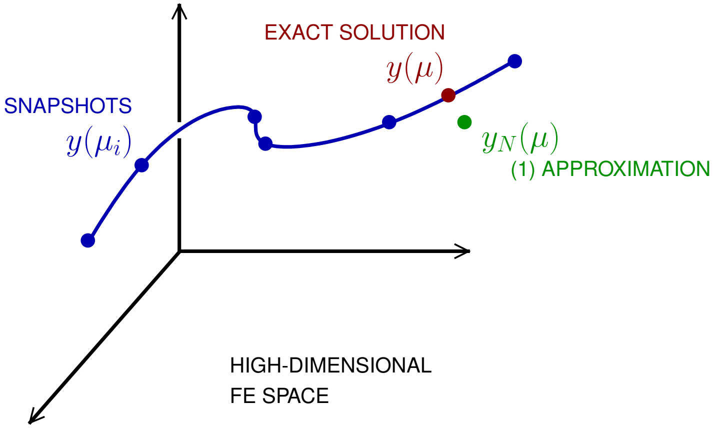
How can we compute the approximation?

How do we know the error is small?

How do we know what value of N to take?

How do we compute the approximations efficiently?

How do we choose the sample points optimally?



Simplest Case

We have $\mathbf{A}_N(\mu)\,\mathbf{y}_N(\mu) = \mathbf{f}_N(\mu)$ where

$$\mathbf{A}_N(\mu) = \boldsymbol{\mathcal{W}}_N^T \boldsymbol{\mathcal{A}}(\mu)\, \boldsymbol{\mathcal{W}}_N \quad \text{and} \quad \mathbf{f}_N(\mu) = \boldsymbol{\mathcal{W}}_N^T \boldsymbol{f}(\mu)$$

Simplest Case

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Assume that $\boldsymbol{\mathcal{A}}_N(\mu)$ and $\boldsymbol{f}_N(\mu)$ are affine in the parameter, i.e.

$$\boldsymbol{\mathcal{A}}_N(\mu) = \sum_{q=1}^{Q_a} \underbrace{\theta_a^q(\mu)}_{\mu\text{-dependent coefficients}} \quad \underbrace{\boldsymbol{\mathcal{A}}^q}_{\mu\text{-independent matrices}} \quad \text{and} \quad \boldsymbol{f}_N(\mu) = \sum_{q=1}^{Q_f} \theta_f^q(\mu) \quad \boldsymbol{f}^q.$$

Simplest Case

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For our Poisson example

$$a(w,v;\mu) \;=\; \sum_{q=0}^{P} \kappa_q \int_{\Omega_q} \nabla w \cdot \nabla v \; d\Omega$$

Simplest Case

We have $\mathbf{A}_N(\mu)\,\mathbf{y}_N(\mu) \;=\; \mathbf{f}_N(\mu)$ where

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$$\underbrace{a(w,v;\mu)}_{} \;=\; \sum_{q=0}^{P} \underbrace{\kappa_q}_{} \; \underbrace{\int_{\Omega_q} \nabla w \cdot \nabla v \; d\Omega}_{}$$

$$\boldsymbol{v}^T \boldsymbol{\mathcal{A}}(\mu)\,\boldsymbol{w} \;=\; \sum_{q=0}^{P} \theta^q(\mu) \qquad \boldsymbol{v}^T \boldsymbol{\mathcal{A}}^q\,\boldsymbol{w}$$

Simplest Case

We thus obtain:

$$\mathbf{A}_N(\mu) \;=\; \boldsymbol{\mathcal{W}}_N^T \left(\sum_{q=1}^{\theta_a} \theta_a^q(\mu)\, \boldsymbol{\mathcal{A}}^q \right) \boldsymbol{\mathcal{W}}_N$$

Simplest Case

We thus obtain:

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Simplest Case

We thus obtain:

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Similarly, $\color{red}{\mathbf{f}_N}(\mu) = \sum_{q=1}^{Q} \theta_f^q(\mu)\, (\boldsymbol{\mathcal{W}}_N^T \boldsymbol{f}^q) = \sum_{q=1}^{Q} \theta_f^q\, (\mu)\mathbf{f}_N^q$

Simplest Case

We thus obtain:

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Offline stage:

- compute snapshot-basis $\boldsymbol{\mathcal{W}}_N$
- compute and store $\mathbf{A}_N^q,\ \mathbf{f}_N^q$ at cost $(\mathcal{N}^\bullet, N^\bullet)$

Online stage:

For any $\mu \in \mathcal{D}$

- assemble $\mathbf{A}_N(\mu),\ \mathbf{f}_N(\mu)$
- solve for $\mathbf{y}_N(\mu)$ at cost $(N^\bullet)$

Simplest Case

How can we compute the error bound efficiently?

Recall $$\Delta_N(\mu) = \frac{\|r(\cdot; y_N(\mu), \mu)\|_{\mathcal{Y}'}}{\alpha_a^{\mathrm{LB}}(\mu)}$$

where we assume we have $\alpha_a^{\mathrm{LB}}(\mu)$. SCM

Simplest Case

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Let $\|v\|_{\mathcal{Y}}^2 = \boldsymbol{v}^T \boldsymbol{\mathcal{Y}} \, \boldsymbol{v}$.

The dual norm of the residual is then $\boldsymbol{\mathcal{Y}}^{-1}$

$$\|r(\cdot; y_N(\mu), \mu)\|_{\mathcal{Y}'}^2 = (\boldsymbol{f}(\mu) - \boldsymbol{\mathcal{A}}(\mu)\boldsymbol{y}_N(\mu))^T \, \boldsymbol{\mathcal{Y}}^{-1}(\boldsymbol{f}(\mu) - \boldsymbol{\mathcal{A}}(\mu)\boldsymbol{y}_N(\mu))$$

Simplest Case

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$$\begin{aligned}
\|r(\cdot; y_N(\mu), \mu)\|_{\mathcal{Y}'}^2 &= (\boldsymbol{f}(\mu) - \boldsymbol{\mathcal{A}}(\mu)\boldsymbol{y}_N(\mu))^T \, \boldsymbol{\mathcal{Y}}^{-1}(\boldsymbol{f}(\mu) - \boldsymbol{\mathcal{A}}(\mu)\boldsymbol{y}_N(\mu)) \\
&= \boldsymbol{f}^T\boldsymbol{\mathcal{Y}}^{-1}\boldsymbol{f} - 2\boldsymbol{f}^T\boldsymbol{\mathcal{Y}}^{-1}\boldsymbol{\mathcal{A}}\,\boldsymbol{y}_N + (\boldsymbol{\mathcal{A}}\,\boldsymbol{y}_N)^T \, \boldsymbol{\mathcal{Y}}^{-1}\boldsymbol{\mathcal{A}}\,\boldsymbol{y}_N
\end{aligned}$$

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How can we compute the error bound efficiently?

Recall $$\Delta_N(\mu) = \frac{\|r(\cdot; y_N(\mu), \mu)\|_{\mathcal{Y}'}}{\alpha_a^{\mathrm{LB}}(\mu)}$$

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$$= \boldsymbol{f}^T\boldsymbol{\mathcal{Y}}^{-1}\boldsymbol{f} - 2\boldsymbol{f}^T\boldsymbol{\mathcal{Y}}^{-1}\boldsymbol{\mathcal{A}}\,\boldsymbol{y}_N + (\boldsymbol{\mathcal{A}}\,\boldsymbol{y}_N)^T\,\boldsymbol{\mathcal{Y}}^{-1}\boldsymbol{\mathcal{A}}\,\boldsymbol{y}_N$$

$$= \ldots + \left(\sum_{q=1}^{Q} \theta_a^q(\mu)\boldsymbol{\mathcal{A}}^q\boldsymbol{\mathcal{W}}_N\boldsymbol{y}_N(\mu)\right)^T \boldsymbol{\mathcal{Y}}^{-1}\left(\sum_{q'=1}^{Q} \theta_a^{q'}(\mu)\boldsymbol{\mathcal{A}}^{q'}\boldsymbol{\mathcal{W}}_N\boldsymbol{y}_N(\mu)\right)$$

which permits a similar offline-online decomposition.

Simplest Case

General Form

Find $y_N(\mu) \in \mathcal{Y}_N$ s.t.

$$a(y_N(\mu), v; \mu) = f(v; \mu) \quad \forall\, v \in \mathcal{Y}_N.$$

Matrix Form

Find $\mathbf{y}_N(\mu) \in \mathbb{R}^N$ s.t.

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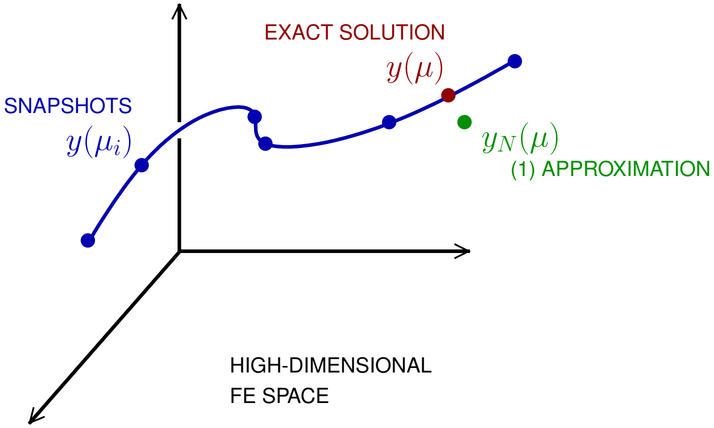
where $\mathbf{A}_N \in \mathbb{R}^{N\times N}$, $\mathbf{f}_N \in \mathbb{R}^N$.

Questions:



How can we compute the approximation?

How do we know the error is small?

How do we know what value of N to take?

How do we compute the approximations efficiently?

How do we choose the sample points optimally?

Simplest Case

How can we choose the snapshots optimally?

Greedy algorithm

Given the samples $S = \{\mu_1, ..., \mu_N\}$ and space of snapshots

$\mathcal{Y}_N = \mathrm{span}\{\ y(\mu_i)\ ,\ i = 1, \ldots, N\ \}$, we want to choose

$$\mu_{N+1} \;=\; \max_{\mu\in\mathcal{D}} \frac{\|y(\mu) - y_N(\mu)\|_{\mathcal{Y}}}{\|y(\mu)\|_{\mathcal{Y}}}\ .$$

Simplest Case

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$$\mu_{N+1} \;=\; \max_{\mu \in \mathcal{D}} \frac{\|y(\mu) - y_N(\mu)\|_{\mathcal{Y}}}{\|y(\mu)\|_{\mathcal{Y}}} \;.$$

In practice, we choose

$$\mu_{N+1} \;=\; \max_{\mu \in D} \frac{\Delta_N(\mu)}{\|y_N(\mu)\|_{\mathcal{Y}}}$$

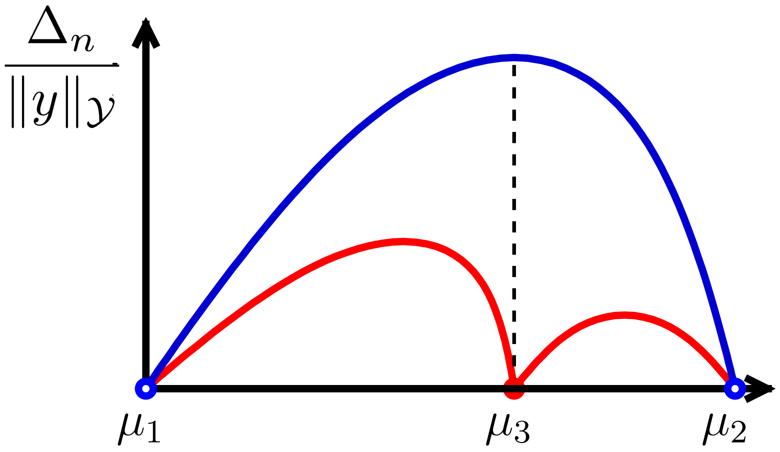
error bound

training sample

approximation

(Weak) Greedy Algorithm

Given $\mathcal{Y}_2 \;=\; \mathrm{span}\{y(\mu_1), y(\mu_2)\}$, how do we choose μ_3?



$$\mu_3 \;=\; \arg\max_{\mu \in D} \frac{\Delta_2(\mu)}{\|u_2(\mu)\|_{\mathcal{Y}}}$$

$$\mathcal{Y}_3 \;=\; \mathrm{span}\{u(\mu_1),\; u(\mu_2),\; u(\mu_3)\}$$

(see, e.g., [VEROY, et al., 2003], [BUFFA, et al. 2012],[BINEV, et al., 2011])

Key points:

- Assumes $\Delta_N(\mu)$ is sharp and inexpensive to compute (online)
- Error bounds enable choice of good approximation spaces

Simplest Case

Algorithm: Offline

Choose training sample $D \subset \mathcal{D}$ and first snapshot parameter $\mu_1 \in D$.

For $N = 1 \text{ to } N_{\max}$

Solve $a(y(\mu_N), v; \mu_N) = f(v; \mu_N), \quad \forall\, v \in \mathcal{Y}.$

Compute and store for $q, q' = 1, ..., Q$

$$\mathbf{A}_N^q = \boldsymbol{\mathcal{W}}_N^T \boldsymbol{\mathcal{A}}^q \boldsymbol{\mathcal{W}}_N, \qquad \boldsymbol{\Gamma}_N^{qq'} = (\boldsymbol{\mathcal{A}}^q \boldsymbol{\mathcal{W}}_N)^T\, \boldsymbol{\mathcal{Y}}^{-1} (\boldsymbol{\mathcal{A}}^{q'} \boldsymbol{\mathcal{W}}_N)$$

and other μ-independent quantities.

Find $\mu_{N+1} = \arg\max_{\mu \in D} \dfrac{\Delta_N(\mu)}{\|\boldsymbol{y}_N(\mu)\|_{\mathcal{Y}}}$.

Set $N = N + 1$.

end

Simplest Case

Algorithm: Online

For given $\mu \in \mathcal{D}$:

Assemble $\mathbf{A}_N(\mu) \; = \; \sum_{q=1}^{Q_a} \theta_a^q(\mu)\mathbf{A}_N^q$ and, similarly, $\mathbf{f}_N(\mu)$.

Solve $\mathbf{A}_N(\mu)\,\mathbf{y}_N(\mu) \; = \; \mathbf{f}_N(\mu)$.

Compute $\alpha_a^{\mathrm{LB}}(\mu)$,

$$\|r(\cdot; y_N(\mu), \mu)\|_{\mathcal{Y}'}^2 \; = \; \ldots + \sum_{q,q'}^{Q_a} \mathbf{y}_N^T(\mu)\,\mathbf{\Gamma}_N^{qq'}\,\mathbf{y}_N(\mu)\,.$$

and $\Delta_N(\mu) \; = \; \dfrac{\|r(\cdot; y_N(\mu), \mu)\|_{\mathcal{Y}'}}{\alpha_a^{\mathrm{LB}}(\mu)}$.

Simplest Case

What about the output of interest?

Keep $$s(\mu) = f(y(\mu))$$

Then $$s_N(\mu) = f(y_N(\mu)) = \boldsymbol{y}_N^T(\mu)\boldsymbol{f}$$

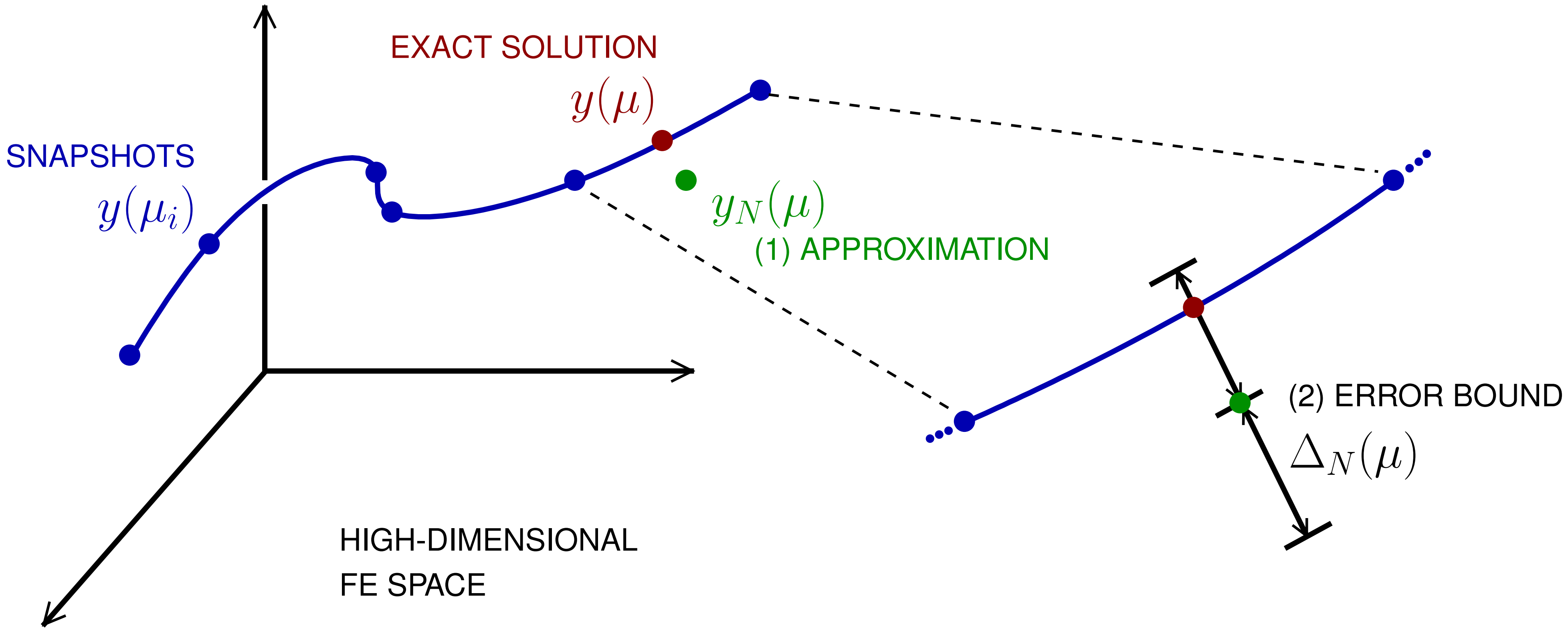
$$= \underbrace{\mathbf{y}_N^T(\mu)}_{\text{online}} \quad \underbrace{(\boldsymbol{\mathcal{W}}_N^T \boldsymbol{f})}_{\text{offline}}$$

One can show that

$$s(\mu) - s_N(\mu) \leq \frac{\|r(\cdot; y_N(\mu); \mu)\|_{\mathcal{Y}'}^2}{\alpha_a^{\mathrm{LB}}(\mu)},$$

which also permits an offline-online decomposition.

The Reduced Basis Method



EXACT SOLUTION

$y(\mu)$

SNAPSHOTS

$y(\mu_i)$

$y_N(\mu)$

(1) APPROXIMATION

(2) ERROR BOUND

$\Delta_N(\mu)$

HIGH-DIMENSIONAL
FE SPACE

(3) OFFLINE-ONLINE COMPUTATIONAL DECOMPOSITION

(4) GREEDY ALGORITHM

Simplest Case

Problem: Compute $s(\mu) = \ell(y(\mu);\mu)$ where

$$a(u(\mu), v; \mu) = f(v;\mu), \qquad \forall\, v \in \mathcal{Y},$$

where f is a bounded linear form, $\ell = f$.

a is a coercive, continuous bilinear form.

a, f are affine in μ .

What about more complex cases?

Introduction to the RB Method

- Simplest Case (Coercive, Compliant, Elliptic, Affine)
- **Non-compliant**
- **Parabolic**
- **Non-coercive**
- **Saddle Point**
- **Non-affine / Non-linear**

Noncompliant Case

Problem: Compute $s(\mu) \;=\; \ell(y(\mu);\mu)$ where

$$a(y(\mu), v; \mu) \;=\; f(v;\mu), \qquad \forall\, v \in \mathcal{Y}$$

with $\ell \neq f$.

Noncompliant Case

Problem: Compute $s(\mu) \; = \; \ell(y(\mu);\mu)$ where

$$a(y(\mu), v; \mu) \; = \; f(v;\mu), \qquad \forall\, v \in \mathcal{Y}$$

with $\ell \neq f$.

One can show that

$$\begin{aligned} |s(\mu) - s_N(\mu)| \; &= \; |\ell(y(\mu) - y_N(\mu);\mu)| \\ &\leq \|\ell(\cdot;\mu)\|_{\mathcal{Y}'} \|y(\mu) - y_N(\mu)\|_{\mathcal{Y}} \\ &\leq \|\ell(\cdot;\mu)\|_{\mathcal{Y}'} \Delta_N(\mu) \end{aligned}$$

Noncompliant Case

One can also show that

$$|s(\mu) - s_N(\mu)| \le \frac{1}{\alpha_a^{\mathrm{LB}}(\mu)} \, \|r^{\mathrm{pr}}(\cdot; y_N(\mu), \mu)\|_{\mathcal{Y}'} \, \|r^{\mathrm{du}}(\cdot; \psi_N(\mu), \mu)\|_{\mathcal{Y}'}$$

where r^{pr} is the primal residual (as before), and the dual residual is

$$r^{\mathrm{du}}(v; \mu) \;:=\; -\ell(v; \mu) - a(v, \psi_N(\mu); \mu) \qquad \forall\, v \in \mathcal{Y}.$$

Here, $\psi_N(\mu) \in \mathcal{Y}_N^{\mathrm{du}}$ approximates $\psi(\mu) \in \mathcal{Y}$ where

$$a(v, \psi(\mu); \mu) \;=\; -\ell(v; \mu), \qquad \forall\, v \in \mathcal{Y}.$$

Parabolic Case

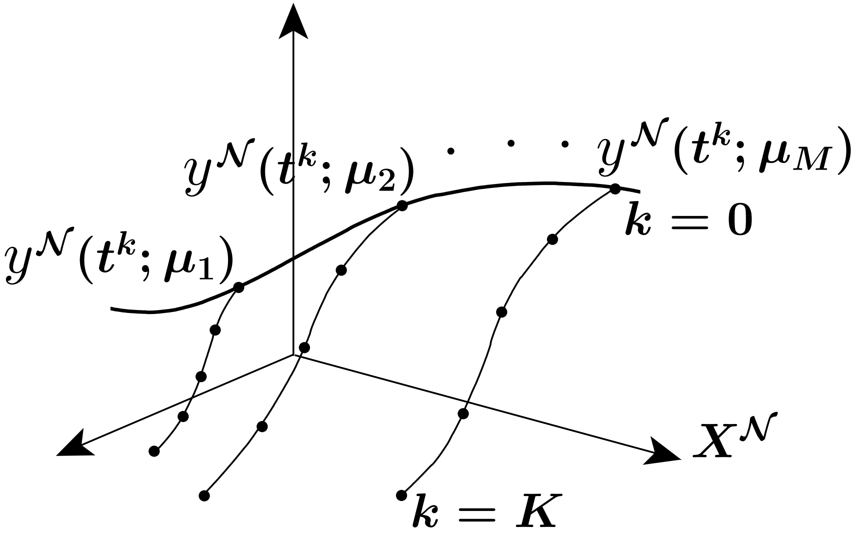
Problem: Given $\mu \in \mathcal{D}$, evaluate

$$s(t;\mu) = \ell(y(x,t;\mu);\mu)$$

where $y(x,t;\mu)$ satisfies

$$y^k = y(x,t^k;\mu)$$

$$m\left(\frac{y^k - y^{k-1}}{\Delta t}, v; \mu\right) + a(y,v;\mu) = f(v;\mu)g(t^k)$$



Assume that m and a are

- symmetric
- continuous
- coercive

bilinear forms for all $\mu \in \mathcal{D}$.

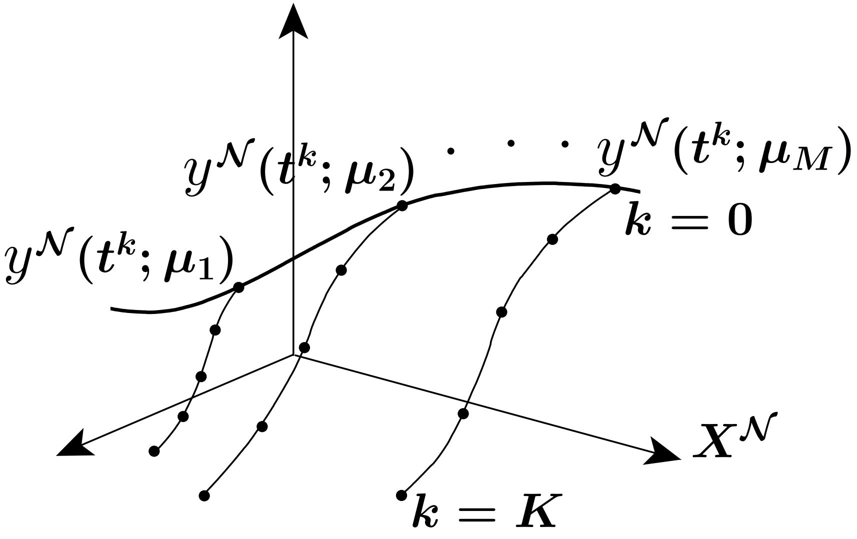
Parabolic Case

For a given $\mu \in \mathcal{D}$, let

$$P_R \; = \; \mathrm{POD}_{\mathcal{Y}}(\{y^k(\mu), 1 \leq k \leq K\}, R)$$

represent the R largest POD modes with respect to the $\mathcal{Y}$-inner product, s.t.

$$P_R \; = \; \arg\inf_{\mathcal{Y}_R \subset \mathrm{span}\{y^k,\, k=1\ldots K\}} \left(\frac{1}{k} \sum_{k=1}^{K} \inf_{v \in X_R} \|y^k(\mu) - v\|_{\mathcal{Y}}^2 \right)^{\frac{1}{2}}$$



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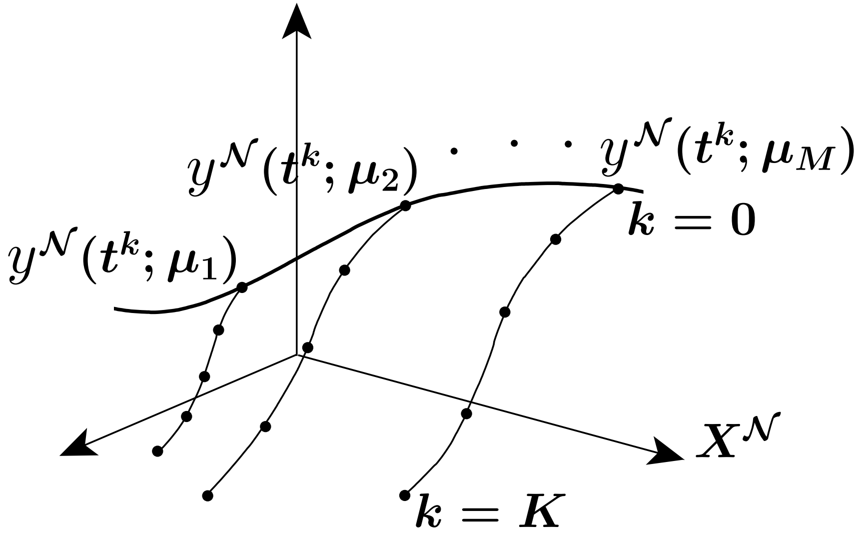

- Compute an $\mathrm{SVD}_{\mathcal{Y}}$.
- Choose largest mode(s).
- In practice, we do POD on the error instead of directly on the data.

Noncoercive Problems

Problem: Find $y(\mu) \in \mathcal{Y}$ such that

$$a(y(\mu), v; \mu) \;=\; f(v; \mu), \quad \forall\, v \in \mathcal{Y}.$$

Assume that

$$\beta(\mu) \;=\; \inf_{w \in \mathcal{Y}} \sup_{v \in \mathcal{Y}} \frac{a(w, v; \mu)}{\|w\|_{\mathcal{Y}} \|v\|_{\mathcal{Y}}} \geq \beta_0 > 0.$$

Noncoercive Problems

Problem: Find $y(\mu) \in \mathcal{Y}$ such that

$$a(y(\mu), v; \mu) \;=\; f(v; \mu), \quad \forall\, v \in \mathcal{Y}.$$

Assume that

$$\beta(\mu) \;=\; \inf_{w \in \mathcal{Y}} \sup_{v \in \mathcal{Y}} \frac{a(w, v; \mu)}{\|w\|_{\mathcal{Y}} \|v\|_{\mathcal{Y}}} \geq \beta_0 > 0.$$

RB Approximation: Find $y_N(\mu) \in \mathcal{Y}_N$ such that

$$a(y_N(\mu), v; \mu) \;=\; f(v; \mu), \quad \forall\, v \in \mathcal{Y}_N.$$

Is the RB Problem well-posed?

Noncoercive Problems

Problem: Find $y \in \mathcal{Y}_1$ such that

$$a(y, v) \;=\; f(v) \quad \forall\, v \in \mathcal{Y}_2$$

where $a: \; \mathcal{Y}_1 \times \mathcal{Y}_2 \to \mathbb{R}$ is a continuous bilinear form

$f: \; \mathcal{Y}_2 \to \mathbb{R}$ is a continuous linear functional.

Noncoercive Problems

Problem: Find $y \in \mathcal{Y}_1$ such that

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$f:\ \mathcal{Y}_2 \to \mathbb{R}$ is a continuous linear functional.

Banach-Nečas-Babuška Thm: The problem is well-posed if and only if:

$$\exists\, \beta_o > 0 \text{ such that } \inf_{w \in \mathcal{Y}_1} \sup_{v \in \mathcal{Y}_2} \frac{a(w,v)}{\|w\|_{\mathcal{Y}_1} \|v\|_{\mathcal{Y}_2}} \geq \beta_o$$

(BNB1)
$\mathrm{Ker}\{\boldsymbol{A}\} = 0$

$$\forall\, v \in \mathcal{Y}_2 \quad (a(w,v) = 0, \;\; \forall\, w \in \mathcal{Y}_1) \Rightarrow (v = 0).$$

(BNB2)
$\mathrm{Ker}\{\boldsymbol{A}^T\} = 0$

Noncoercive Problems

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$$a(y,v) \;=\; f(v) \quad \forall\, v \in \mathcal{Y}_2$$

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(BNB2)
$\mathrm{Ker}\{\boldsymbol{A}^T\} = 0$

Moreover,

$$\|y\|_{\mathcal{Y}_1} \leq \frac{1}{\beta} \|f\|_{\mathcal{Y}_2'} \quad \forall\, f \in \mathcal{Y}_2'.$$

Noncoercive Problems

Recall that

$$\alpha \;=\; \inf_{v \in \mathcal{Y}} \frac{a(v,v)}{\|v\|_{\mathcal{Y}}^2} \;=\; \min_{\boldsymbol{v} \in \mathbb{R}^{\mathcal{N}}} \frac{\boldsymbol{v}^T \boldsymbol{\mathcal{A}}\, \boldsymbol{v}}{\boldsymbol{v}^T \boldsymbol{\mathcal{Y}}\, \boldsymbol{v}}$$

In other words α is the minimum eigenvalue of

$$\boldsymbol{\mathcal{A}}\boldsymbol{\varphi} \;=\; \boldsymbol{\lambda}\boldsymbol{\mathcal{Y}}\boldsymbol{\varphi}\,.$$

How can we interpret the inf-sup constant β?

Noncoercive Problems

Riesz Representation Theorem

Let $\mathcal{Y}$ be a Hilbert space, and $f \in \mathcal{Y}'$. Then there exists a unique element $p \in \mathcal{Y}$ such that

$$f(v) \;=\; (v, p)_{\mathcal{Y}} \quad \forall\, v \in \mathcal{Y}.$$

Furthermore

$$\|f\|_{\mathcal{Y}'} \;=\; \sup_{v \in \mathcal{Y}} \frac{f(v)}{\|v\|_{\mathcal{Y}}} \;=\; \|p\|_{\mathcal{Y}}.$$

Noncoercive Problems

Riesz Representation Theorem

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Furthermore

$$\|f\|_{\mathcal{Y}'} \;=\; \sup_{v \in \mathcal{Y}} \frac{f(v)}{\|v\|_{\mathcal{Y}}} \;=\; \|p\|_{\mathcal{Y}}.$$

For given $w \in \mathcal{Y}$, let $f(v) = a(w, v)$ …

Noncoercive Problems

If a is continuous, then for a given $w \in \mathcal{Y}$,

- $a(w, \cdot) \in \mathcal{Y}'$,
- there exists a unique element $\mathcal{T}_w \in \mathcal{Y}$ s.t.

$$(\mathcal{T}_w, v)_{\mathcal{Y}} \; = \; a(w, v), \quad \forall v \in \mathcal{Y},$$

- where

$$\mathcal{T}_w \; = \; \arg \sup_{v \in \mathcal{Y}} \frac{a(w, v)}{\|v\|_{\mathcal{Y}}}$$

Noncoercive Problems

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$$\mathcal{T}_w = \arg \sup_{v \in \mathcal{Y}} \frac{a(w, v)}{\|v\|_{\mathcal{Y}}}$$

In matrix form, $\boldsymbol{\mathcal{Y}} \boldsymbol{\mathcal{T}}_w = \boldsymbol{\mathcal{A}} \boldsymbol{w}$ or $\boldsymbol{\mathcal{T}}_w = \boldsymbol{\mathcal{Y}}^{-1} \boldsymbol{\mathcal{A}} \boldsymbol{w}$.

Noncoercive Problems

Recall that

$$\beta(\mu) = \inf_{w \in \mathcal{Y}} \sup_{v \in \mathcal{Y}} \frac{a(w, v; \mu)}{\|w\|_{\mathcal{Y}} \|v\|_{\mathcal{Y}}}$$

Noncoercive Problems

Recall that

$$\beta(\mu) \;=\; \inf_{w \in \mathcal{Y}} \sup_{v \in \mathcal{Y}} \frac{a(w, v; w)}{\|w\|_{\mathcal{Y}} \|v\|_{\mathcal{Y}}} \;=\; \inf_{w \in \mathcal{Y}} \frac{1}{\|w\|_{\mathcal{Y}}} \left(\sup_{v \in \mathcal{Y}} \frac{a(w, v; \mu)}{\|v\|_{\mathcal{Y}}} \right)$$

Noncoercive Problems

Recall that

$$\beta(\mu) = \inf_{w \in \mathcal{Y}} \sup_{v \in \mathcal{Y}} \frac{a(w, v; w)}{\|w\|_{\mathcal{Y}} \|v\|_{\mathcal{Y}}} = \inf_{w \in \mathcal{Y}} \frac{1}{\|w\|_{\mathcal{Y}}} \left(\sup_{v \in \mathcal{Y}} \frac{a(w, v; \mu)}{\|v\|_{\mathcal{Y}}} \right)$$

$$= \inf_{w \in \mathcal{Y}} \frac{a(w, T_w; \mu)}{\|w\|_{\mathcal{Y}} \|T_w\|_{\mathcal{Y}}}$$

Noncoercive Problems

Recall that

$$\beta(\mu) = \inf_{w\in\mathcal{Y}} \sup_{v\in\mathcal{Y}} \frac{a(w,v;w)}{\|w\|_{\mathcal{Y}}\|v\|_{\mathcal{Y}}} = \inf_{w\in\mathcal{Y}} \frac{1}{\|w\|_{\mathcal{Y}}} \left(\sup_{v\in\mathcal{Y}} \frac{a(w,v;\mu)}{\|v\|_{\mathcal{Y}}} \right)$$

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Noncoercive Problems

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In matrix form,

$$\boldsymbol{\beta}^2(\mu) = \min_{w \in \mathbb{R}^{\mathcal{N}}} \frac{\boldsymbol{w}^T \boldsymbol{\mathcal{A}}(\mu)^T \boldsymbol{\mathcal{Y}}^{-1} \boldsymbol{\mathcal{A}}(\mu) \boldsymbol{w}}{\boldsymbol{w}^T \boldsymbol{\mathcal{Y}} \boldsymbol{w}}$$

Noncoercive Problems

Recall that

$$\beta(\mu) = \inf_{w \in \mathcal{Y}} \sup_{v \in \mathcal{Y}} \frac{a(w,v;w)}{\|w\|_{\mathcal{Y}} \|v\|_{\mathcal{Y}}} = \inf_{w \in \mathcal{Y}} \frac{1}{\|w\|_{\mathcal{Y}}} \left(\sup_{v \in \mathcal{Y}} \frac{a(w,v;\mu)}{\|v\|_{\mathcal{Y}}} \right)$$

$$= \inf_{w \in \mathcal{Y}} \frac{a(w, T_w; \mu)}{\|w\|_{\mathcal{Y}} \|T_w\|_{\mathcal{Y}}} = \frac{\|T_w\|_{\mathcal{Y}}}{\|w\|_{\mathcal{Y}}}$$

In matrix form,

$$\boldsymbol{\beta}^2(\mu) = \min_{w \in \mathbb{R}^{\mathcal{N}}} \frac{\boldsymbol{w}^T \boldsymbol{\mathcal{A}}(\mu)^T \boldsymbol{\mathcal{Y}}^{-1} \boldsymbol{\mathcal{A}}(\mu) \boldsymbol{w}}{\boldsymbol{w}^T \boldsymbol{\mathcal{Y}} \boldsymbol{w}}$$

In other words, $\boldsymbol{\beta}^2(\mu)$ is the minimum eigenvalue of

$$\boldsymbol{\mathcal{A}}(\mu)^T \boldsymbol{\mathcal{Y}}^{-1} \boldsymbol{\mathcal{A}}(\mu) \boldsymbol{\varphi} = \boldsymbol{\lambda} \boldsymbol{\mathcal{Y}} \boldsymbol{\varphi}$$

Noncoercive Problems

Problem: Find $y(\mu) \in \mathcal{Y}$ such that

$$a(y(\mu), v; \mu) \;=\; f(v; \mu), \quad \forall\, v \in \mathcal{Y}.$$

Assume that

$$\beta(\mu) \;=\; \inf_{w \in \mathcal{Y}} \sup_{v \in \mathcal{Y}} \frac{a(w, v; \mu)}{\|w\|_{\mathcal{Y}} \|v\|_{\mathcal{Y}}} \geq \beta_0 > 0.$$

RB Approximation: Find $y_N(\mu) \in \mathcal{Y}_N$ such that

$$a(y_N(\mu), v; \mu) \;=\; f(v; \mu), \quad \forall\, v \in \mathcal{Y}_N.$$

Is the RB Problem well-posed?

Noncoercive Problems

One can show that for

$$\begin{aligned} \mathcal{Y}_N &:= \text{span}\{\, y(\mu_1), \ldots, y(\mu_N) \,\}, \\ \mathcal{V}_N^\mu &:= \text{span}\{\, T_\mu y(\mu_n),\ n = 1, \ldots, N \,\}, \end{aligned}$$

where

$$(\, T_\mu y(\mu_n), v \,)_{\mathcal{Y}} = a(y(\mu_n), v; , \mu), \quad \forall v \in \mathcal{Y}\,,$$

then the following reduced basis problem:

Find $y_N(\mu) \in \mathcal{Y}_N$ such that

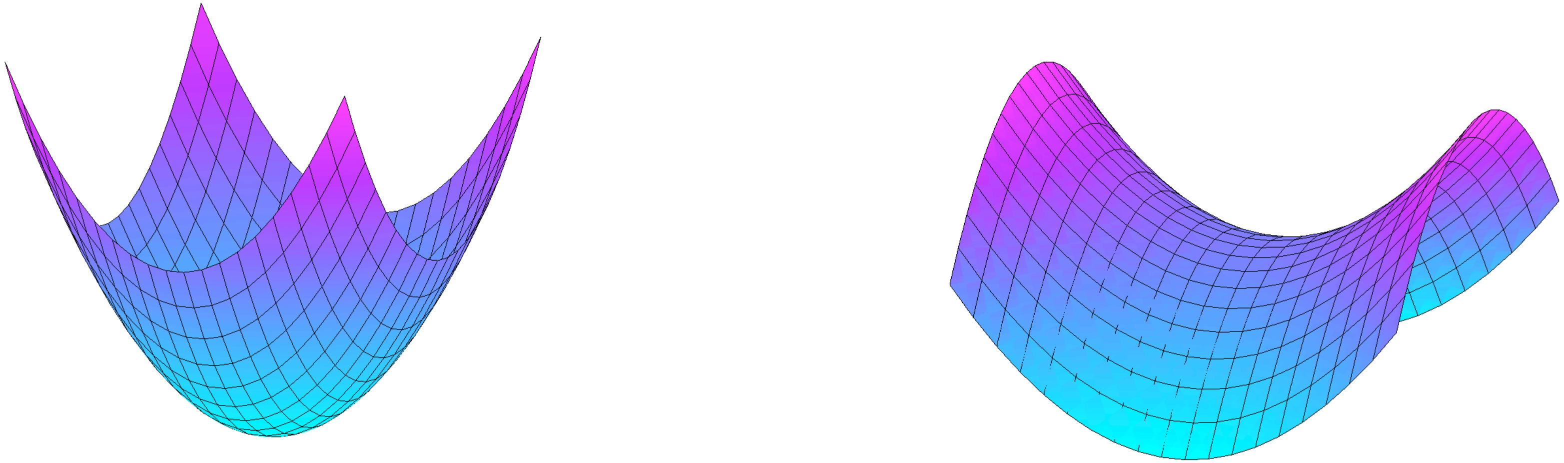
$$a(y_N(\mu), v; \mu) \;=\; f(v; \mu), \quad \forall\, v \in \mathcal{V}_N^\mu\,,$$

is well-posed with $\beta_N(\mu) \geq \beta(\mu)$.

Saddle Point Problems

Problem Structure

$$\mathbf{A}\,\mathbf{y} = \mathbf{f} \qquad \text{vs.} \qquad \underbrace{\begin{bmatrix}\mathbf{A} & \mathbf{B}^T\\ \mathbf{B} & \end{bmatrix}\begin{bmatrix}\mathbf{y}\\ \lambda\end{bmatrix} = \begin{bmatrix}\mathbf{f}\\ \mathbf{g}\end{bmatrix}}_{\boldsymbol{\mathcal{A}}\,\boldsymbol{u} = \boldsymbol{\mathcal{F}}}$$



Saddle Point Problems

RB Approximation

Find $(y_N, \lambda_N) \in \mathcal{Y}_N \times \mathcal{Z}_N$ such that $\qquad (\mu)$

$$\begin{aligned} a(y_N, v) \quad + \quad b(v, \lambda_N) \quad &= \quad f(v) \qquad && \forall\, v \in \mathcal{Y}_N \\ b(y_N, q) \quad &= \quad g(q) \qquad && \forall\, q \in \mathcal{Z}_N \end{aligned}$$

Issues:

- **Well-posedness** of the approximate problem
- Efficiently computable **bounds** for the errors

$$\|y - y_N\|_{\mathcal{Y}} \quad \text{and} \quad \|\lambda - \lambda_N\|_{\mathcal{Z}}$$

[BREZZI], [ROVAS, 2003], [ROZZA & VEROY, 2007], [GERNER & VEROY, 2012]

Saddle Point: Approximation

The spaces $\mathcal{Y}_N, \mathcal{Z}_N$ constitute a stable pair if for all $\mu \in \mathcal{D}$

$$\beta_N(\mu) := \inf_{q \in \mathcal{Z}_N} \sup_{v \in \mathcal{Y}_N} \frac{\langle B(\mu)v, q\rangle}{\|v\|_{\mathcal{Y}} \|q\|_{\mathcal{Z}}} > 0$$ [BREZZI]

For any $q \in \mathcal{Z}_N, \mathcal{Y}_N$ must contain “supremizing" functions.

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For any $q \in \mathcal{Z}_N, \mathcal{Y}_N$ must contain “supremizing" functions.

Pressure Space:

For $\mu_i \in \mathcal{D}$, $i = 1, \ldots, N$, and

$$\mathcal{Z}_N := \mathrm{span}\{\lambda(\mu_i),\ i = 1 \text{ to } N\}$$

Saddle Point: Approximation

The spaces $\mathcal{Y}_N, \mathcal{Z}_N$ constitute a stable pair if for all $\mu \in \mathcal{D}$

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For $\mu_i \in \mathcal{D}$, $i = 1, \ldots, N$, and

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Velocity Space:

Option 0: The Naive Choice

$$\mathcal{Y}_N^0 := \mathrm{span}\{y(\mu_i),\ i = 1 \text{ to } N\}$$

Saddle Point: Approximation

The spaces $\mathcal{Y}_N, \mathcal{Z}_N$ constitute a stable pair if for all $\mu \in \mathcal{D}$

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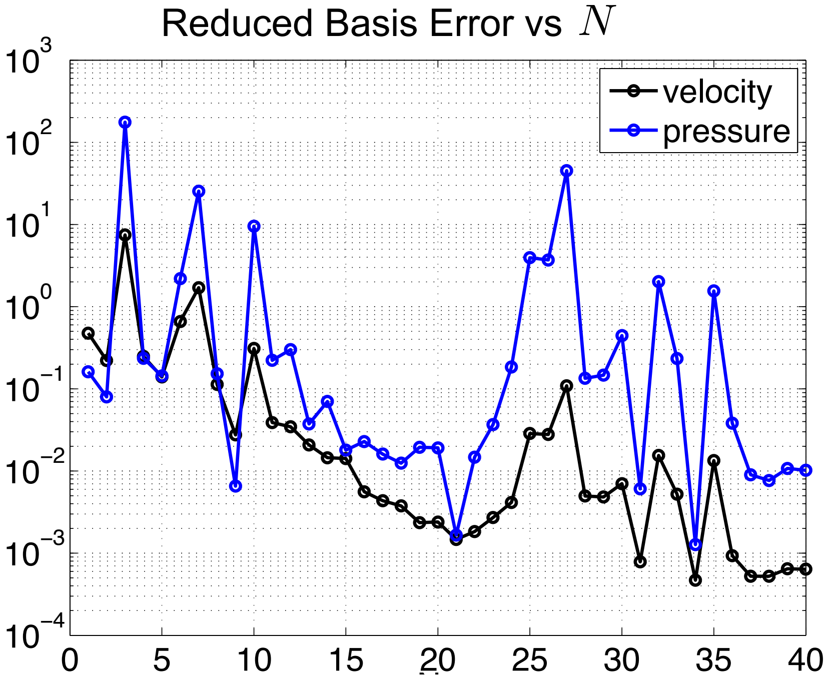

Saddle Point: Approximation

Velocity Space

For $i = 1, \dots, N$ and $q = 1, \dots, Q_b$

- **Option 1*** $\Rightarrow$ provably stable

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*[ROVAS, 2003], [ROZZA & VEROY, 2007] †[GERNER & VEROY, 2012]

Saddle Point: Approximation

Velocity Space

For $i = 1, \ldots, N$ and $q = 1, \ldots, Q_b$

- **Option 1*** $\Rightarrow$ provably stable

$$\mathcal{Y}_N^1 := \operatorname{span}\{\ y(\mu_i)\ ,\ T^q\lambda(\mu_i)\}$$

where

$$T^q p = \arg \sup_{v \in \mathcal{Y}_N} \frac{\langle B^q v, p\rangle}{\|v\|_{\mathcal{Y}}}$$

and

$$B(\mu) = \sum_{q=1}^{Q_b} \theta_b^q(\mu) B^q$$

---

*[ROVAS, 2003], [ROZZA & VEROY, 2007] †[GERNER & VEROY, 2012]

Saddle Point: Approximation

Velocity Space

For $i = 1, \ldots, N$ and $q = 1, \ldots, Q_b$

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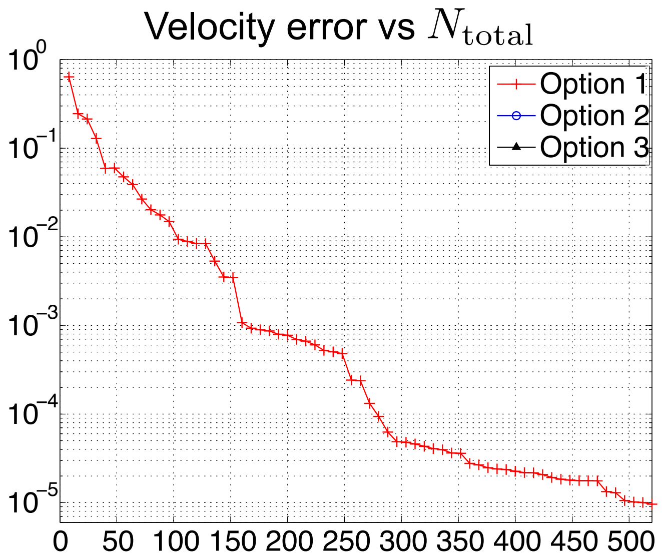
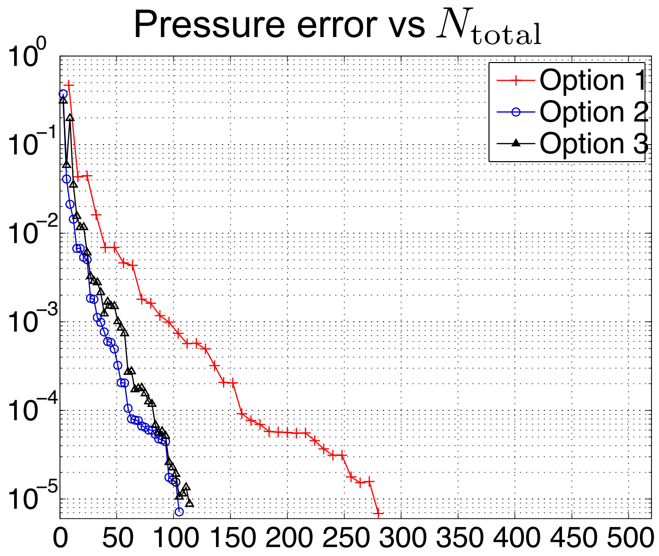
$$\mathcal{Y}_N^1 := \text{span}\{\ y(\mu_i)\ ,\ \underbrace{T^q \lambda(\mu_i)}_{Q_b \text{ SUPREMIZERS}}\ \}$$





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*[ROVAS, 2003], [ROZZA & VEROY, 2007] †[GERNER & VEROY, 2012]

Saddle Point: Approximation

Velocity Space

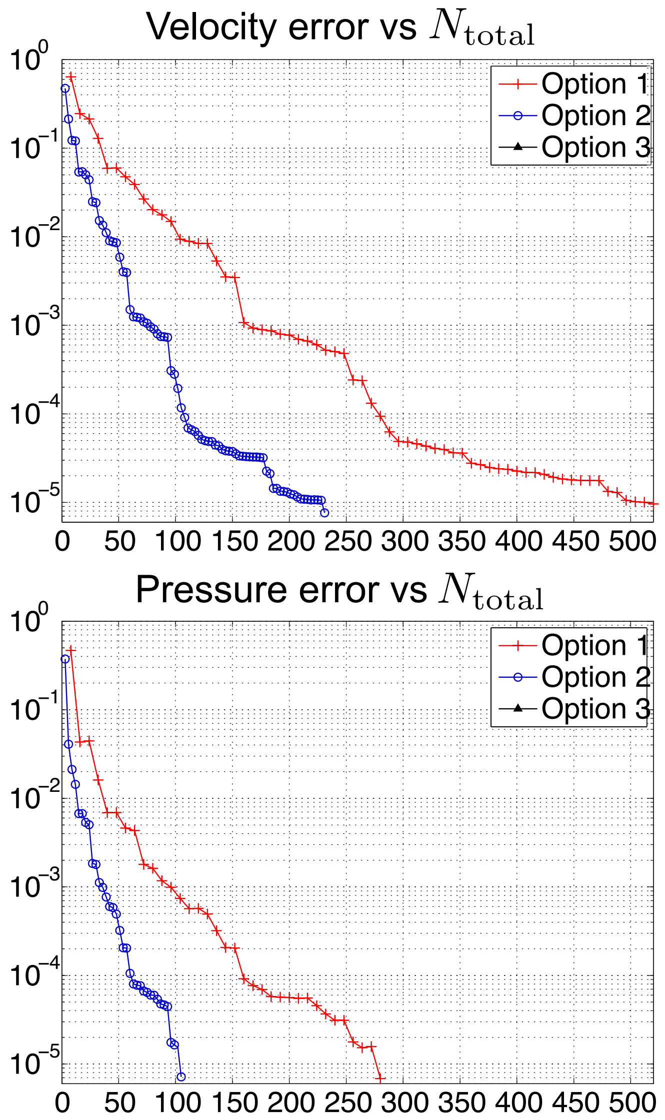
For $i = 1, \ldots, N$ and $q = 1, \ldots, Q_b$

- **Option 1*** $\Rightarrow$ provably stable

$$\mathcal{Y}_N^1 := \text{span}\{\ y(\mu_i)\ ,\ \underbrace{T^q \lambda(\mu_i)}_{Q_b \text{ SUPREMIZERS}}\ \}$$

- **Option 2**[†] $\Rightarrow$ justifiably stable

$$\mathcal{Y}_N^2 := \text{span}\{\ y(\mu_i)\ ,\ \underbrace{T_{\mu_i} \lambda(\mu_i)}_{\text{SUPREMIZER SNAPSHOTS}}\ \}$$


Velocity error vs N_{total}

Pressure error vs N_{total}

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*[ROVAS, 2003], [ROZZA & VEROY, 2007] †[GERNER & VEROY, 2012]

Saddle Point: Approximation

Velocity Space

For $i = 1, \ldots, N$ and $q = 1, \ldots, Q_b$

- **Option 1*** $\Rightarrow$ provably stable

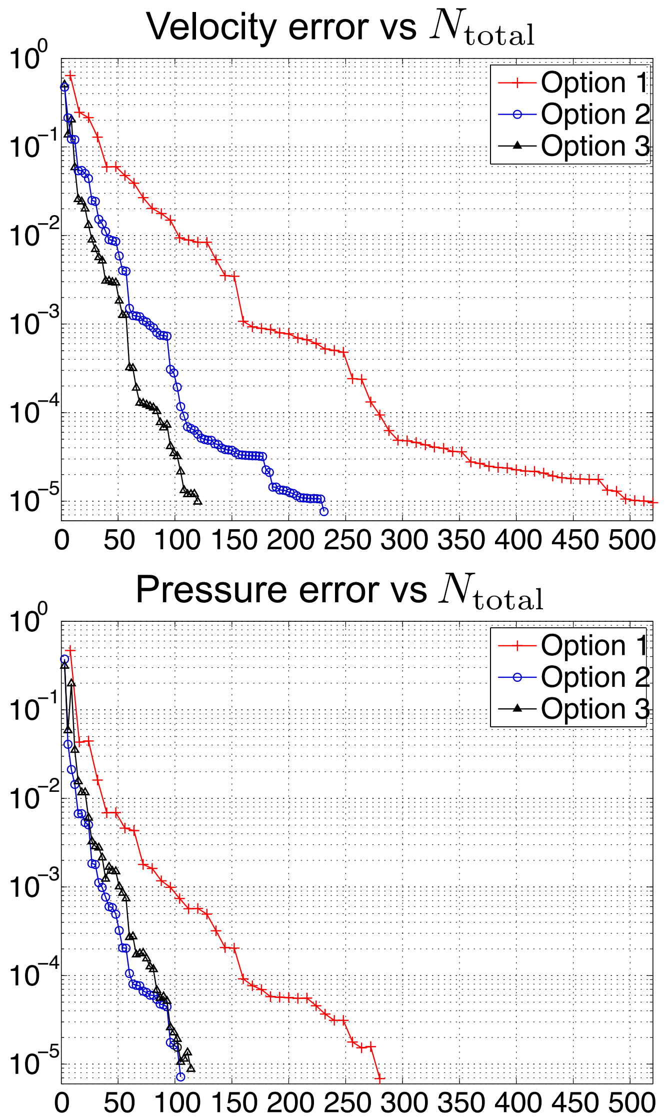
$$\mathcal{Y}_N^1 := \text{span}\{\ y(\mu_i)\ ,\ \underbrace{T^q \lambda(\mu_i)}_{Q_b \text{ SUPREMIZERS}}\ \}$$

- **Option 2**† $\Rightarrow$ justifiably stable

$$\mathcal{Y}_N^2 := \text{span}\{\ y(\mu_i)\ ,\ \underbrace{T_{\mu_i} \lambda(\mu_i)}_{\text{SUPREMIZER SNAPSHOTS}}\ \}$$

- **Option 3**† $\Rightarrow$ empirically stable

$$\mathcal{Y}_N^3 := \text{span}\{y(\mu_i),\ \underbrace{y(\mu_i')}_{\text{VELOCITY SNAPSHOTS}}\ \}$$


Velocity error vs N_{total}

Pressure error vs N_{total}

---

*[ROVAS, 2003], [ROZZA & VEROY, 2007] †[GERNER & VEROY, 2012]

Saddle Point: Error Estimation

1. Treat entire system as a general noncoercive problem

$$\mathcal{A}(U(\mu), V; \mu) = \mathcal{F}(V; \mu), \quad \forall\, V \in \mathcal{X}\,,$$

Let $\mathcal{R}(V;\mu)$ be the residual, [BANACH-NECAS-BABUSKA]

$$\|U(\mu) - U_N(\mu)\|_{\mathcal{X}} \leq \frac{\|\mathcal{R}(\,\cdot\,;\mu)\|_{\mathcal{X}'}}{\beta_{\mathrm{LB}}^{\mathrm{A}}(\mu)} =: \Delta_N^U(\mu)\,.$$

[VEROY, PRUD’HOMME, ROVAS & PATERA, 2003]
and, e.g., [ROZZA, HUYNH & MANZONI, 2013]

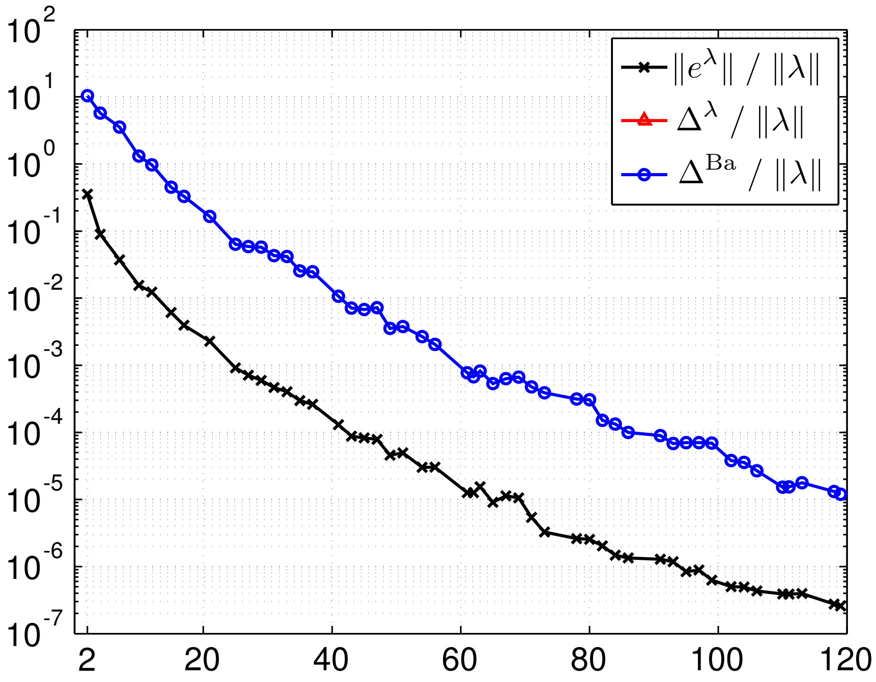
2. Treat the system as a saddle point problem [BREZZI]

$$\begin{aligned} \|y - y_N\|_{\mathcal{X}} &\leq \frac{\|r_N^1\|_{\mathcal{X}'}}{\alpha_{\mathrm{LB}}} + \left(1 + \frac{\gamma_{\mathrm{UB}}}{\alpha_{\mathrm{LB}}}\right)\frac{\|r_N^2\|_{\mathcal{X}'}}{\beta_{\mathrm{LB}}^b} &&=: \Delta_N^y \\ \|\lambda - \lambda_N\|_{\mathcal{X}} &\leq \frac{\|r_N^1\|_{\mathcal{X}'}}{\beta_{\mathrm{LB}}^b} + \frac{\gamma_{\mathrm{UB}}}{\beta_{\mathrm{LB}}^b}\Delta_N^y &&=: \Delta_N^\lambda \end{aligned}$$

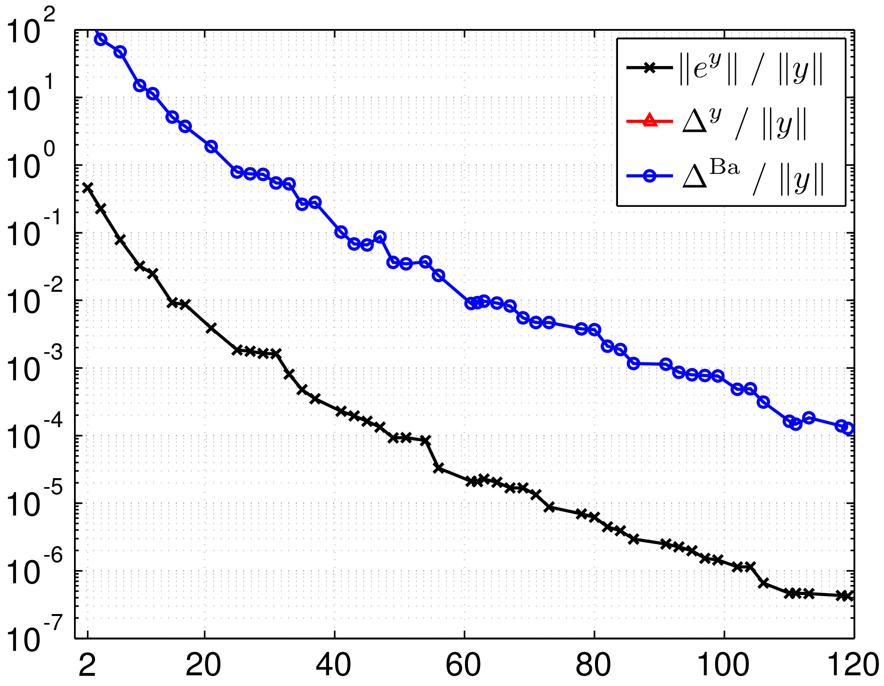
[GERNER & VEROY, 2012]

Saddle Point: Error Estimation

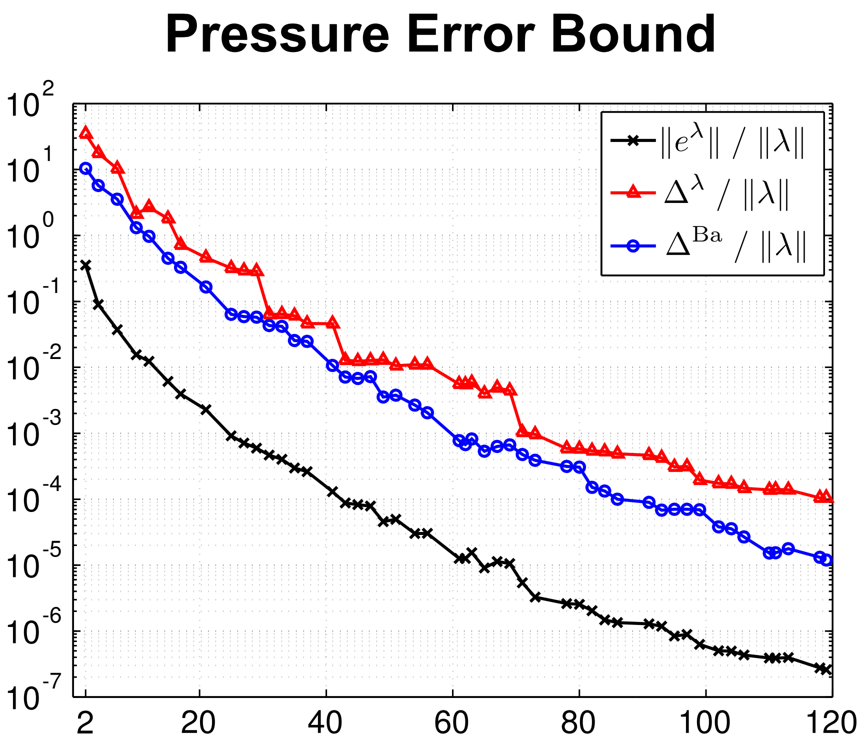
Pressure Error Bound

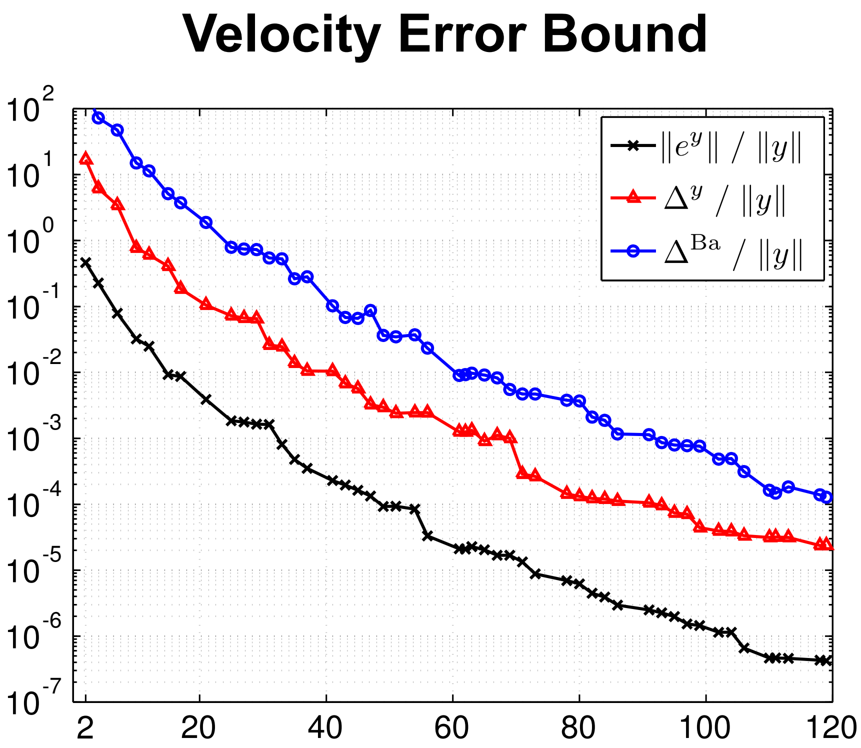


Velocity Error Bound

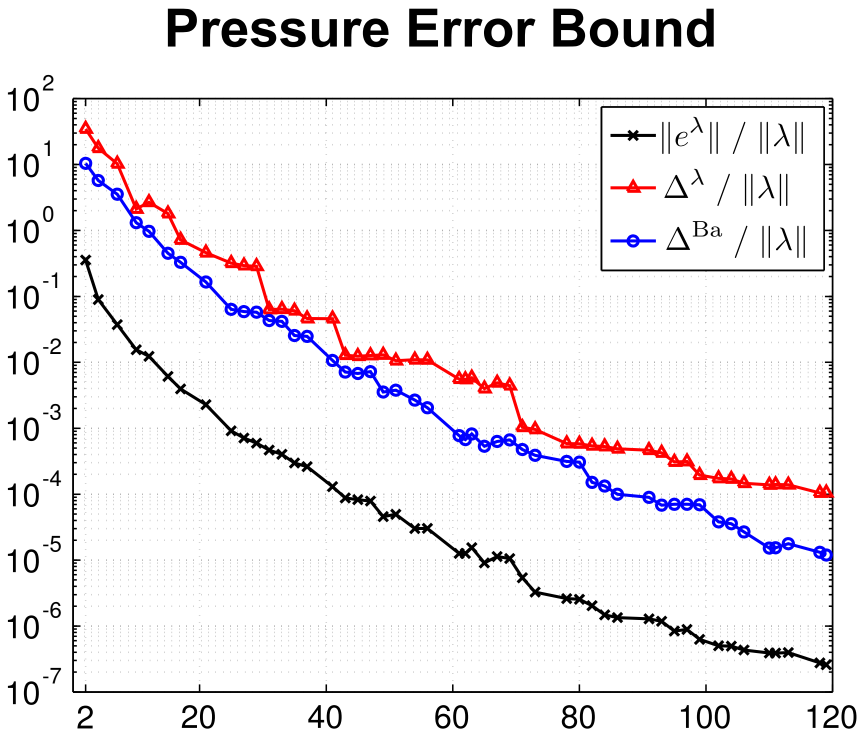


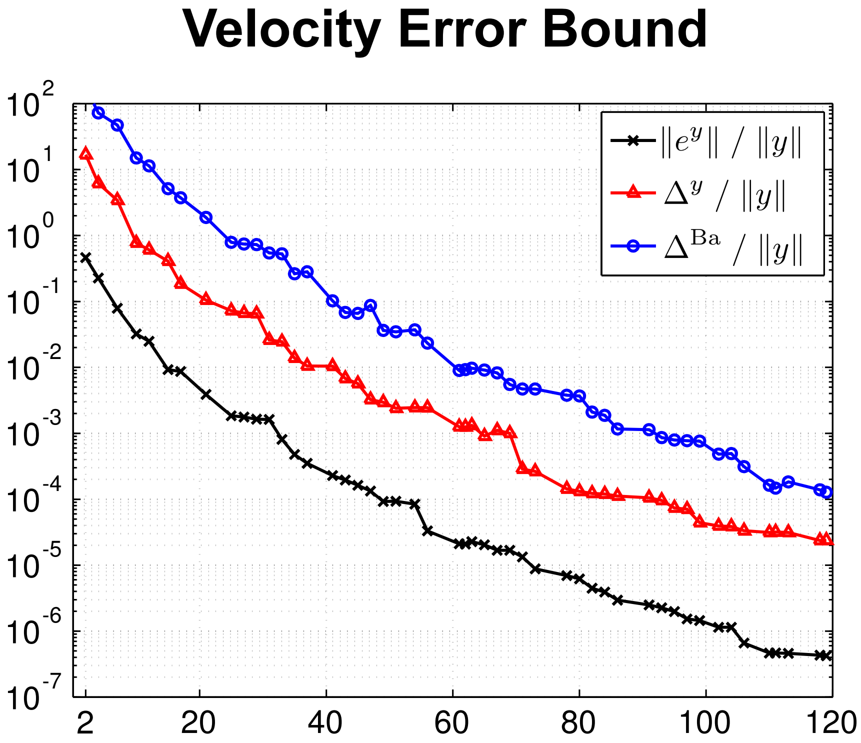
Saddle Point: Error Estimation





Saddle Point: Error Estimation





Key Result:

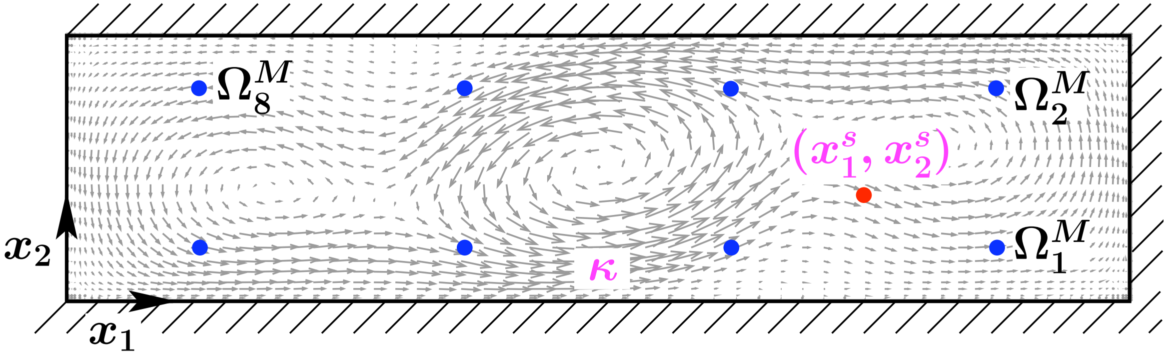
We provide separate error bounds for y_N, λ_N

that depend only on β^b, not on $\beta^{\mathcal{A}}$.

Non-Affine Problems

Nonaffine problems

Contaminant Transport



Concentration of pollutant governed by scalar convection-diffusion equation

$$\frac{\partial}{\partial t} y(t;\mu) + \mathbf{U} \cdot \nabla y(t;\mu) = \kappa \nabla^2 y(t;\mu) + g^{\mathrm{PS}}(x;\mu)\, g(t), \qquad y(x, t=0;\mu) = 0$$

with source term modeled by

$$g^{\mathrm{PS}}(x;\mu) = \frac{50}{\pi}\, e^{-50((x_1 - x_1^s)^2 + (x_2 - x_2^s)^2)}.$$

Goal: Identify source location $\Rightarrow$ parameter $\mu \equiv (\kappa, x_1^s, x_2^s)$.

Nonaffine Problems

Contaminant Transport - Truth Problem Statement

Given $\mu \in \mathcal{D} \subset \mathbb{R}^P$, evaluate $\forall\, k \in \mathbb{K}$

$$s(t^k;\mu) = \ell(y(t^k;\mu))$$

where $y(t^k;\mu) \in X$ satisfies $y(t^0;\mu) = 0$

$$m\left(\frac{y(t^k;\mu) - y(t^{k-1};\mu)}{\Delta t}, v; \mu\right) + \frac{1}{2}\, a(y(t^k;\mu) + y(t^{k-1};\mu), v; \mu)$$

$$= b(v;\mu)\, \tfrac{1}{2}\, (g(t^k) + g(t^{k-1})),\ \ \forall v \in X,$$

for $b(v;\mu) = \int_\Omega g^{\mathrm{PS}}(x;\mu)\, v\, d\Omega$ with g^{PS} nonaffine.

$$g^{\mathrm{PS}}(x;\mu) = \frac{50}{\pi}\, e^{-50((x_1 - x_1^s)^2 + (x_2 - x_2^s)^2)}.$$

Nonaffine Problems

Nonaffine Source Term

Evaluation of RB quantities ($v = \zeta_i, \; 1 \leq i \leq N_{\max}$):

$$\begin{aligned} b(\zeta_i; \mu) &= \int_\Omega g^{\mathrm{PS}}(x; \mu)\, \zeta_i \\ &= \frac{50}{\pi} \int_\Omega e^{-50((x_1 - \mu_2)^2 + (x_2 - \mu_3)^2)}\, \zeta_i \end{aligned}$$

requires even in the online stage

$\mathcal{O}(\mathcal{N} N)$ operations.

Difficulty:

There is no ($\mathcal{N}$ -independent) affine representation of $g^{\mathrm{PS}}(x; \mu)$.

Nonlinear Problems

Model Problem:

Given $\mu = (\mu_1, \mu_2) \in \mathcal{D} \equiv [0.01, 10]^2$, evaluate

$$s^k(\mu) = \int_\Omega y^k_{N,M}(\mu)$$

where $y^k_{N,M}(\mu) \in Y$, $1 \leq k \leq K$, satisfies $y^0(\mu) = 0$

$$\frac{1}{\Delta t} m(y^k_{N,M}(\mu) - y^{k-1}_{N,M}(\mu), v) + a(y^k_{N,M}(\mu), v)$$

$$+ \int_\Omega g^{\mathrm{nl}}(y^k(\mu); x; \mu)\, v = b(v)\, \sin(2\pi t^k), \quad \forall\, v \in Y,$$

with $g^{\mathrm{nl}}(y^k(\mu); x; \mu) = \mu_1 \dfrac{e^{\mu_2\, y^k(\mu)} - 1}{\mu_2}$.

Nonlinear Problems

Given $\mu \in \mathcal{D}$ evaluate $\quad\quad \forall\, k \in \mathbb{K}$

$$s^k(\mu) = \ell(y^k(\mu))$$

where $y^k(\mu) \in \mathcal{Y}, 1 \leq k \leq K,$ satisfies $\quad\quad y^0(\mu) = 0$

$$\frac{1}{\Delta t}\, m(y^k(\mu) - y^{k-1}(\mu), v) + a(y^k(\mu), v; \mu)$$

$$+ \int_\Omega g^{\mathrm{nl}}(y^k(\mu); x; \mu)\, v = b(v) y(t^k), \ \forall v \in \mathcal{Y}.$$

Note:

- Standard RB-Galerkin recipe suffices for (at most) quadratic nonlinearities: $\mathcal{O}(N^4)$ online cost ([VPP03, VP05, NVP05]...)
- Higher order or nonpolynomial nonlinearities $\Rightarrow$ EIM

Empirical Interpolation Method

Main Idea

$$g^{\mathrm{PS}}(x;\mu) \approx g_M^{\mathrm{PS}}(x;\mu) = \sum_{m=1}^{M} \underbrace{\varphi_{Mm}(\mu)}_{\text{EIM}} \qquad \underbrace{q_m(x)}_{\text{Collateral RB}}$$

Recall: $b(\zeta_i;\mu) \quad = \quad \int_\Omega g^{\mathrm{PS}}(x;\mu)\,\zeta_i \approx \int_\Omega g_M^{\mathrm{PS}}(x;\mu)\,\zeta_i$

$$= \quad \sum_{m=1}^{M} \varphi_{Mm}(\mu) \int_\Omega q_m(x)\,\zeta_i \ ,$$

If we can calculate the $\varphi_{Mm}(\mu)$ efficiently, we can again follow an offline-online computational procedure, but

- how do we calculate the $q_m(x)$ and the $\varphi_{Mm}(\mu)$?
- (what is the interpolation error introduced?)

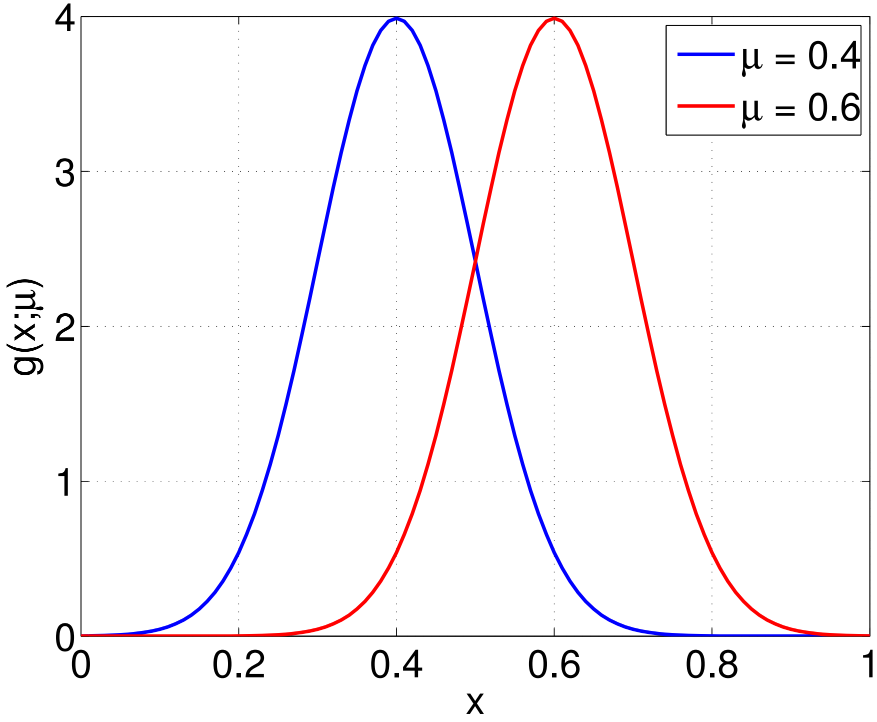
[BARRAULT, MADAY, NGUYEN & PATERA, 2004], [GREPL, MADAY, NGUYEN & PATERA, 2007]

Empirical Interpolation Method: Example / Demo

We consider the nonaffine function

$$g(x;\mu) \equiv \frac{10}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{0.1}\right)^2}$$

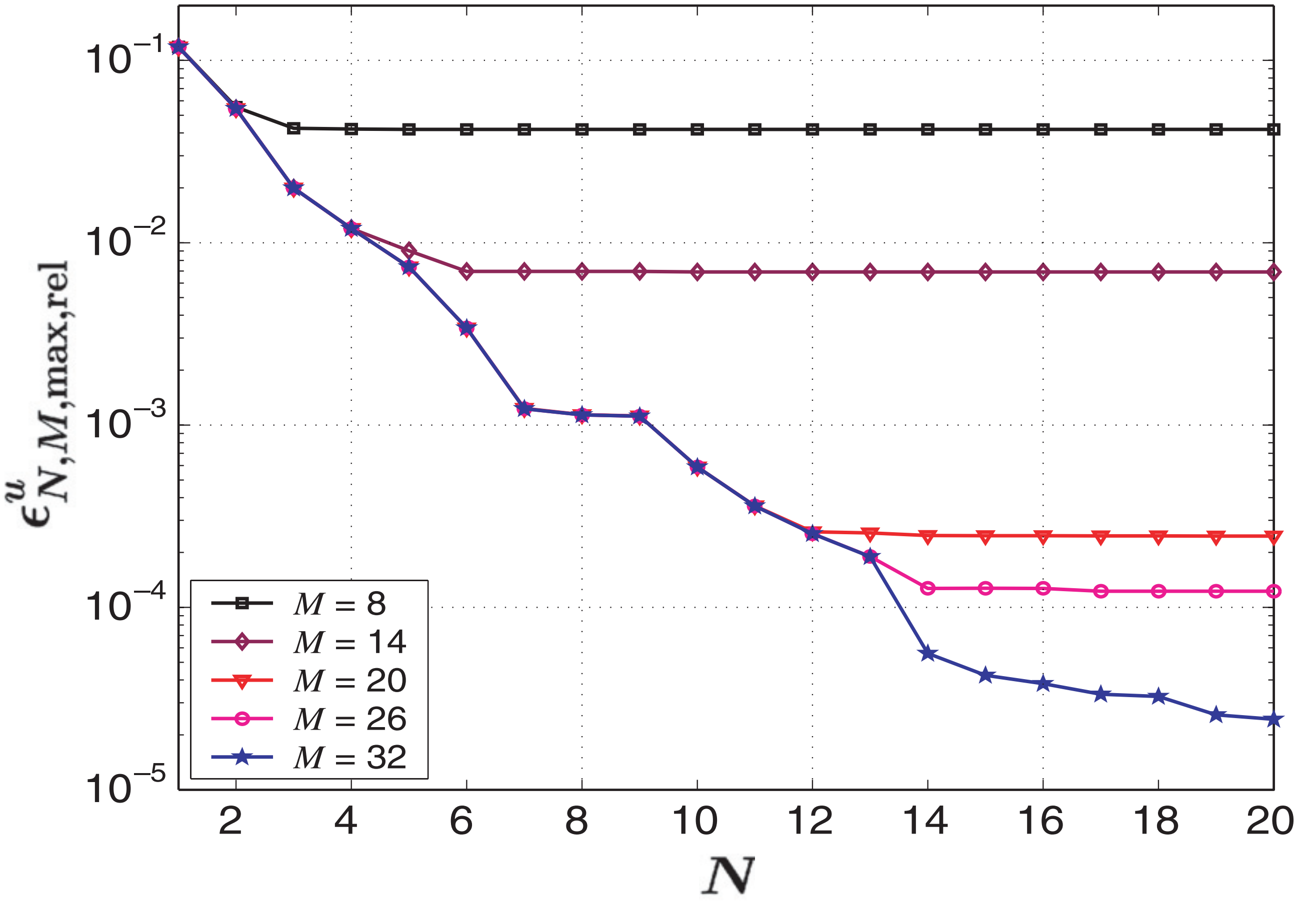
for $x \in \Omega \equiv [0,1]$ and $\mu \in \mathcal{D} \equiv [0.4, 0.6]$.



Empirical Interpolation Method: Example / Demo

Empirical Interpolation Method

Maximum relative error in the field variable



The Reduced Basis Method

Part I: Theoretical Introduction

Motivation

RB for the Simplest Case

Extensions

Part II: Applications

Data Assimilation and Experimental Design

Cancer Treatment Planning

Introduction to Reduced Basis Methods: Theory and Applications

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Advanced Study in
Computational
Engineering Science

RWTH AACHEN UNIVERSITY

PART II

Overview

Part I: Introduction to the Reduced Basis Method

- Motivation
- RB for the Simplest Case
- Extensions
  - EIM for Non-Affine and Nonlinear Problems

Part II: Applications

Overview

Part I: Introduction to the Reduced Basis Method

Part II: Applications

Theoretical: State Estimation

Generalized EIM and PBDW method

Optimal Control + Data Assimilation (4DVAR)

Data Assimilation (3DVAR) + Sensor Placement

Industrial: Cancer Treatment Planning

Deutsche Forschungsgemeinschaft
DFG

MARIE CURIE ACTIONS

European Commission | Horizon 2020
European Union funding
for Research & Innovation

Recall: EIM

Function space interpolation

Approximate the function

A set of functions, e.g. solutions from different models with different parameters

$$\varphi \in F$$

with its interpolation

Interpolation functions

$$\mathcal{I}_M[\varphi] := \sum_{i=1}^{M} \tilde{\alpha}_j^M(\varphi)\tilde{q}_j$$

Interpolation coefficients

where the coefficients $\tilde{\alpha}_j^M(\varphi)$ are chosen such that

$$\mathcal{I}_M[\varphi](x_i) := \varphi(x_i), \qquad i = 1, \ldots, M$$

Interpolation points

Data

Generalized EIM

Function space interpolation

[MADAY & MULA, 2013]

Approximate the function

$$\varphi \in F$$

A set of functions, e.g.
solutions from different models
with different parameters

with its interpolation

$$\mathcal{I}_M[\varphi] := \sum_{i=1}^{M} \tilde{\alpha}_j^M(\varphi)\tilde{q}_j$$

Interpolation functions

Interpolation coefficients

where the coefficients $\tilde{\alpha}_j^M(\varphi)$ are chosen such that

$$\sigma_i(\mathcal{I}_M[\varphi]) := \sigma_i(\varphi)$$

Linear functionals

Data, e.g., measurements

Generalized EIM

State Estimation

Given measurements $d_i = \sigma_i(y_{\text{true}}),\ i = 1, ..., M$, of some unknown state y_{true},

Assume y_{true} can be expected to be close to a set F of candidate states.

Approximate y_{true} using

$$y_{\text{true}} \approx \sum_{i=1}^{M} \tilde{\alpha}_j^M(\varphi)\tilde{q}_j \qquad \text{where} \quad d_i = \sum_{j=1}^{M} \tilde{\alpha}_j^M(\varphi)\sigma_i(\tilde{q}_j), \quad \forall\ i = 1, \ldots, M.$$

This corresponds to: $y_{\text{true}} \approx y$ where

$$y \in \text{span}\{\ \tilde{q}_i \in F,\ i = 1, \ldots, M\ \} \quad \text{such that} \quad \sigma_i(y) = d_i, \quad i = 1, \ldots, M.$$

[MADAY & MULA, 2013]

Generalized EIM

Initialization

$$\tilde{\varphi}_1 := \arg\max_{\varphi \in F} ||\varphi||$$

Generating function

$$\sigma_1 := \arg\max_{\sigma \in \Sigma} |\sigma(\tilde{\varphi}_1)|$$

First measurement functional

$$\tilde{q}_1 := \frac{\tilde{\varphi}_1}{\sigma_1(\tilde{\varphi}_1)}$$

First interpolating basis function

[MADAY & MULA, 2013]

Generalized EIM

Iterative Procedure

Suppose $\{\tilde{q}_1, \ldots, \tilde{q}_{M-1}\}$ and $\{\sigma_1, \ldots, \sigma_{M-1}\}$ have been constructed.

$$\tilde{\varphi}_M := \arg\max_{\varphi \in F} ||\varphi - \mathcal{I}_{M-1}[\varphi]||$$

Function that is currently worst approximated

$$\sigma_M := \arg\sup_{\sigma \in \Sigma} |\sigma(\tilde{\varphi}_M - \mathcal{I}_{M-1}[\varphi])|$$

Next measurement functional

$$\tilde{q}_M := \frac{\tilde{\varphi}_M - \mathcal{I}_{M-1}[\varphi]}{\sigma_M(\tilde{\varphi}_M - \mathcal{I}_{M-1}[\varphi])}$$

Next interpolation function

[MADAY & MULA, 2013]

Generalized EIM

Iterative Procedure

Suppose $\{\tilde{q}_1, \ldots, \tilde{q}_{M-1}\}$ and $\{\sigma_1, \ldots, \sigma_{M-1}\}$ have been constructed.

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Next measurement functional

$$\tilde{q}_M := \frac{\tilde{\varphi}_M - \mathcal{I}_{M-1}[\varphi]}{\sigma_M(\tilde{\varphi}_M - \mathcal{I}_{M-1}[\varphi])}$$

Next interpolation function

[MADAY & MULA, 2013]

Parametrized-Background Data-Weak (PBDW) Formulation

[Maday, Patera, Penn & Yano 2014, 2015], [Taddei 2017], [Maday & Taddei 2017(p)], [Taddei & Patera 2018], [Hammond, Chaqir, Bourquin & Maday 2018(p)]

Parametrized-Background Data-Weak (PBDW) Formulation

Unlimited-observations PBDW statement

Find $y_N^* \in \mathcal{Y},\ z_N^* \in \mathcal{Z}_N,\ d_N^* \in \mathcal{Y} \quad \text{s.t.}$

state prediction **model prior** **data misfit**

$$(y_N^*, z_N^*, d_N^*) = \arg \inf_{\substack{y_N \in \mathcal{Y} \\ z_N \in \mathcal{Z}_N \\ d_N \in \mathcal{Y}}} \|d_N\|^2$$

Subject to

misfit with data **prediction of background model**

$$(y_N, v) = (d_N, v) + (z_N, v) \quad \forall v \in \mathcal{Y},$$

$$(y_N, \phi) = (y^{\text{true}}, \phi) \quad \forall \phi \in \mathcal{Y}.$$

[Maday, Patera, Penn & Yano 2014, 2015], [Taddei 2017], [Maday & Taddei 2019], [Taddei & Patera 2018], [Hammond, Chaqir, Bourquin & Maday 2019]

Parametrized-Background Data-Weak (PBDW) Formulation

Limited-observations PBDW statement

Introduce library of observation functionals

$$\mathcal{L} = \{\ell \in \mathcal{Y}' \mid \ell = \ell^o_m\}$$

where (for example) $\ell^\circ_m(v) = \mathrm{Gauss}\,(v; x^c_m, r_m)$

Let $\mathcal{T}_M = \mathrm{span}\,\{R_{\mathcal{Y}}\, \ell^\circ_m\}_{m=1}^{M}\,, \quad M = 1, \ldots, M_{\max}$

Riesz representation

where $(v,\ R_{\mathcal{Y}}\, \ell^\circ_m) = \ell^\circ_m(v) \quad \forall v \in \mathcal{Y}.$

[Maday, Patera, Penn & Yano 2014, 2015], [Taddei 2017], [Maday & Taddei 2017(p)], [Taddei & Patera 2018], [Hammond, Chaqir, Bourquin & Maday 2018(p)]

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[Maday, Patera, Penn & Yano 2014, 2015], [Taddei 2017], [Maday & Taddei 2019], [Taddei & Patera 2018], [Hammond, Chaqir, Bourquin & Maday 2019]

Parametrized-Background Data-Weak (PBDW) Formulation

Limited-observations PBDW statement

Find $y^*_{N,M} \in \mathcal{Y},\ z^*_{N,M} \in \mathcal{Z}_N,\ d^*_{N,M} \in \mathcal{Y} \quad \text{s.t.}$

state prediction **model prior** **data misfit**

$$(y^*_{N,M},\ z^*_{N,M},\ d^*_{N,M}) = \arg \inf_{\substack{y_{N,M} \in \mathcal{Y} \\ z_{N,M} \in \mathcal{Z}_N \\ d_{N,M} \in \mathcal{Y}}} \|d_{N,M}\|^2$$

Subject to

misfit with data **prediction of background model**

$$(y_{N,M}, v) = (d_{N,M}, v) + (z_{N,M}, v) \quad \forall v \in \mathcal{Y},$$

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[Maday, Patera, Penn & Yano 2014, 2015], [Taddei 2017], [Maday & Taddei 2019], [Taddei & Patera 2018], [Hammond, Chaqir, Bourquin & Maday 2019]

PBDW Formulation

Limited-observations PBDW

The PBDW approximation error satisfies

$$\left\| d_N^* - d_{N,M}^* \right\| \leqslant \inf_{q \in \mathcal{T}_M \cap \mathcal{Z}_N^\perp} \inf_{z \in \mathcal{Z}_N} \left\| y^{\text{true}} - z - q \right\|,$$

$$\left\| z_N^* - z_{N,M}^* \right\| \leqslant \frac{1}{\beta_{N,M}} \inf_{q \in \mathcal{T}_M \cap \mathcal{Z}_N^\perp} \inf_{z \in \mathcal{Z}_N} \left\| y^{\text{true}} - z - q \right\|,$$

$$\left\| y^{\text{true}} - y_{N,M}^* \right\| \leqslant \left(1 + \frac{1}{\beta_{N,M}} \right) \inf_{q \in \mathcal{T}_M \cap \mathcal{Z}_N^\perp} \inf_{z \in \mathcal{Z}_N} \left\| y^{\text{true}} - z - q \right\|,$$

where the stability constant $\beta_{N,M}$ is defined by

$$\beta_{N,M} \equiv \inf_{z \in \mathcal{Z}_N} \sup_{q \in \mathcal{T}_M} \frac{(z, q)}{\|z\| \|q\|}.$$

[MPPY14,15], [T17], [MT19]
[TP18], [HCBM19]

PBDW Formulation

Limited-observations PBDW

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[MPPY14,15], [T17], [MT19]
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PBDW Formulation

Limited-observations PBDW statement

Find $(y^*_{N,M} \in \mathcal{Y},\ z^*_{N,M} \in \mathcal{Z}_N,\ d^*_{N,M} \in \mathcal{Y})$

$$(y^*_{N,M},\ z^*_{N,M},\ d^*_{N,M}) = \arg \inf_{\substack{y_{N,M} \in \mathcal{Y} \\ z_{N,M} \in \mathcal{Z}_N \\ d_{N,M} \in \mathcal{Y}}} \|d_{N,M}\|^2$$

subject to

misfit with data **prediction of background model**

$$(y_{N,M}, v) = (d_{N,M}, v) + (z_{N,M}, v) \quad \forall v \in \mathcal{Y},$$

$$(y_{N,M}, \phi) = (y^{\text{true}}, \phi) \quad \forall \phi \in \mathcal{T}_M .$$

[Maday, Patera, Penn & Yano 2014, 2015], [Taddei 2017], [Maday & Taddei 2017(p)], [Taddei & Patera 2018], [Hammond, Chaqir, Bourquin & Maday 2018(p)]

Optimal Control

with

M. Kärcher and M. Grepl

Optimal Control

Problem Statement

$$\min_{u\in U} \; J(y,u) = \frac{\lambda}{2}\|y - y_d\|^2_{L^2(D)} + \frac{1}{2}\|u\|^2_U$$

$$\text{s.t.} \quad a(y,v) = f(v) + b(u,v), \quad \forall v \in Y,$$

Desired state	$y_d(x),\ x \in \Omega$
Distributed control	$u(x),\ x \in \Omega$
PDE-constraint	state y is governed by a μ-PDE

Optimal Control

Reduced Basis Approximation

$$\min_{u_N \in U_N} J(y_N, u_N) = \frac{\lambda}{2}\|y_N - y_d\|^2_{L^2(D)} + \frac{1}{2}\|u_N\|^2_U$$
$$\text{s.t.} \quad a(y_N, v) = b(u_N, v) + f(v), \quad \forall v \in Y_N,$$

RB approximation as surrogate $y_N, \; u_N$

Error estimation $\|u^{\mathrm{o}} - u^{\mathrm{o}}_N\|_U \leq \Delta^u_N$

$$|J(u^{\mathrm{o}}) - J(u^{\mathrm{o}}_N)| \leq \Delta^J_N$$

Optimal Control

Background

- **Scalar Optimal Control**
  - sharp (POD) error bounds, but requires FE-solves

    [TRÖLTZSCH & VOLKWEIN, 2009]
  - online-efficient error estimates

    [DEDÈ, 2010a], [DEDÈ, 2012]
  - online-efficient, error bounds

    [GREPL & KÄRCHER, 2011], [KÄRCHER & GREPL, 2014]

Optimal Control

Background: Distributed optimal control

- **Perturbation Bound** in [KÄRCHER, 2011]
  - based on [TV09], [GK11], [KG14]
  - online-efficient, separate error bounds for state, control, and adjoint

$$\|u^* - u_N^*\|_U \leq \quad \Delta_N^u = \frac{1}{\tau}\|\tau(u_N^* - u_d) - B^* p_N^*\|_U + \frac{1}{\tau}\gamma_c \Delta_N^p$$

$$\|p^* - p_N^*\|_Y \leq \quad \Delta_N^p \equiv \frac{1}{\alpha_a}\left(\|r_p\|_{Y'} + C_D^2 \Delta_N^y\right)$$

$$\|y^* - y_N^*\|_Y \leq \quad \Delta_N^y \equiv \frac{1}{\alpha_a}\|r_y\|_{Y'}$$

  - depends only on $\alpha_a, \gamma_c,$ and C_D
  - bound for error in u contains terms which scale as

$$\sim \lambda \|r_y\|_{Y'}$$

Optimal Control

Status: **Distributed optimal control**

- **BNB Bound** in [NEGRI, ROZZA, MANZONI & QUARTERONI, 2013]
  - based on the Banach-Nečas-Babuška Theorem and RB for general non-coercive problems
  - consider the entire optimality system

$$\begin{bmatrix} \underline{M} & \underline{0} & \underline{A} \\ \underline{0} & \frac{1}{\lambda}\underline{D} & -\underline{B}^T \\ \underline{A} & -\underline{B} & \underline{0} \end{bmatrix} \begin{bmatrix} \underline{y}^* \\ \underline{u}^* \\ \underline{p}^* \end{bmatrix} = \begin{bmatrix} \underline{M}\,\underline{y}_d \\ \underline{f} \\ \underline{0} \end{bmatrix}$$

and introduce $x = (y, u, p) \in \mathcal{Z}$ to obtain

$$\|x^* - x_N^*\|_{\mathcal{Z}} \leq \frac{1}{\beta_{\mathrm{Ba}}} \|r_x\|_{\mathcal{Z}'}$$

  - online-efficient error bounds that depends on β_{Ba}
  - provides only combined bounds for state, control, adjoint

Optimal Control

Motivation

- Analyze the optimal control problem as a saddle point problem
- Saddle point results not directly applicable:
  - “A-block” is coercive only on kernel of the “B-block”
  - online-efficient, rigorous error bounds on (y, u)
- But perhaps we can use some elements of the proof …

Optimal Control

- **Alternative Bound** in [KÄRCHER, TOKOUTSI, GREPL & VEROY, 2018]
  - by direct manipulation of the error residual equations, we obtain

$$\|u^* - u_N^*\|_U \leq \frac{\lambda}{2}\left(\|r_u\|_{U'} + \frac{\gamma_c}{\alpha_a}\|r_p\|_{\mathcal{Y}'}\right) + \frac{\lambda}{2}\left[\left(\|r_u\|_{U'} + \frac{\gamma_c}{\alpha_a}\|r_p\|_{\mathcal{Y}'}\right)^2 + \frac{8}{\alpha_a\lambda}\|r_y\|_{\mathcal{Y}'}\|r_p\|_{\mathcal{Y}'} + \frac{C_D^2}{\alpha_a^2\lambda}\|r_y\|^2_{\mathcal{Y}'}\right]^{\frac{1}{2}}$$

  - bounds for error in $y_N^*,\ p_N^*$ as in perturbation bound
  - online-efficient, separate error bounds for state, control, and adjoint
  - depends only on $\alpha_a, \gamma_c,$ and C_D
  - bound for error in u contains terms which scale as

$$\sim \sqrt{\lambda}\|r_y\|_{\mathcal{Y}'}$$

Data Assimilation

with

S. Boyaval, M. Grepl, and M. Kärcher

Background

(Variational) Data Assimilation

3D-/4D-VAR	[Lorenc '81], [Le Dimet '81], [Courtier '85], … [Le Dimet & Talagrand '86], … [Navon et al] …
+ Kalman Filter, Bayesian Methods	[Law & Stuart '15], [Reich '15], …

MOR + Data Assimilation (+Sensor Placement)

Gappy-POD	[Everson & Sirovich ’95], [Willcox '06] …
GEIM	[Maday & Mula '13] …
PGD (+ EIM)	[Nadal, Chinesta, Diez, Fuenmayor & Denia '15] …
PBDW	[**Maday, Patera, Penn & Yano ’14**, ’15], [Taddei ’17], [**Maday & Taddei ’19**], [Taddei & Patera ’18], [Hammond, Chaqir, Bourquin & Maday ’18(p)]
OMP	… [Binev, Cohen, Mula & Nichols ’18] …

MOR + Optimal Control

RB + OC	[Negri, Rozza, Manzoni, Quarteroni '13], [Tröltzsch & Volkwein '09], [**Kärcher, Tokoutsi, Grepl & V. '18**]

Data Assimilation

4DVAR

$$\min_{\mu\in\mathcal{D}} \min_{u\in\mathcal{U}} \quad \frac{1}{2}\|u-u_b\|^2_{\mathcal{U}} \quad + \quad \frac{\lambda}{2}\sum_{k=1}^{K}\Delta t \big\|Cy^k - C\tilde{y}^k_{\mathrm{true}}\big\|^2_D$$

$$\text{s.t.} \quad m(y^k,\nu) = m(y^{k-1},\nu) - \Delta t\, a(y^k,\nu;\mu) + \Delta t\ f(\nu), \qquad \forall\, \nu \in Y,\ k=1,\dots,K$$

$$y^0 = u$$

Solve for μ^* and the estimate $(u^*(\mu^*), y^*(\mu^*))$.

Data Assimilation

Reduced Order 4DVAR

$$\min_{\mu \in \mathcal{D}} \min_{u_N \in \mathcal{U}_N} \quad \frac{1}{2} \| u_N - u_b \|_{\mathcal{U}}^2 \quad + \quad \frac{\lambda}{2} \sum_{k=1}^{K} \Delta t \left\| C y_N^k - C y_{\text{true}}^k \right\|_D^2$$

$$\text{s.t.} \quad m(y_N^k, \nu) = m(y_N^{k-1}, \nu) - \Delta t \; a(y_N^k, \nu; \mu) + \Delta t \; f(\nu),$$

$$\forall \, \nu \in Y_N, \; k = 1, \ldots, K$$

$$y_N^0 = u_N$$

Data Assimilation

Reduced Order 4DVAR

$$\min_{\mu \in \mathcal{D}} \min_{u_N \in \mathcal{U}_N} \quad \frac{1}{2} \| u_N - u_b \|_{\mathcal{U}}^2 \quad + \quad \frac{\lambda}{2} \sum_{k=1}^{K} \Delta t \left\| C y_N^k - C y_{\text{true}}^k \right\|_D^2$$

$$\text{s.t.} \quad m(y_N^k, \nu) = m(y_N^{k-1}, \nu) - \Delta t \; a(y_N^k, \nu; \mu) + \Delta t \; f(\nu),$$

$$\forall \, \nu \in Y_N, \; k = 1, \ldots, K$$

$$y_N^0 = u_N$$

Solve for μ^* and the estimate $(u^*(\mu^*), y^*(\mu^*))$.

Order Reduction for

- PDE governing model dynamics
- Optimization space

[Robert, Durbiano, Blayo, Verron, Blum, Le Dimet 2005], [Chen, Navon, Fang 2009],
[Dimitriu, Apreutesei, Stefanescu 2010], [Nadal, Chinesta, Diez, Fuenmayor & Denia '15] …

Data Assimilation

4D-Var

Solve

$$\min_{\mu\in\mathcal{D}}\ \min_{u\in\mathcal{U}} \frac{1}{2}\|u(\mu)-u_b\|_{\mathcal{U}}^2 + \frac{\lambda}{2}\sum_{k=1}^{K}\Delta t\left\|Cy^k - C\tilde{y}^k_{\mathrm{true}}\right\|^2$$

$$\text{s.t.}\quad m(y^{k+1},v) = m(y^k,v) - \Delta t a(y^k,v;\mu) + \Delta t f(v), \qquad \forall\, v\in Y,\ 1\le k\le K$$

$$y^{\mathrm{o}} = u$$

for μ^* and the corresponding $(u^*(\mu^*), y^*(\mu^*))$.

Lagrangian

$$\mathcal{L}(y,p,u;\mu) = \frac{1}{2}\|u-u_b\|_{\mathcal{U}}^2 + \frac{\lambda}{2}\sum_{k=1}^{K}\Delta t\|Cy^k - y_d^k\|_D^2 + \sum_{k=1}^{K} m(y^k,p^k) - m(y^{k-1},p^k) + \Delta t a(y^k,p^k) - \Delta t f(p^k),$$

Data Assimilation

Reduced Optimality Conditions

$$
\begin{aligned}
f(\phi) - a(y_N^k, \phi) - \frac{1}{\Delta t} m(y_N^k - y_N^{k-1}, \phi) &= \quad 0 && \mathcal{L}_p \\
\lambda(C y_N^k - C y_{\text{true}}^k, C\varphi)_D - \frac{1}{\Delta t} m(\varphi, p_N^k - p_N^{k+1}) + a(\varphi, p_N^k; \mu) &= \quad 0 && \mathcal{L}_y \\
m(\psi, p_N^1) - (u_N - u_{\text{b}}, \psi)_U &= \quad 0 && \mathcal{L}_u
\end{aligned}
$$

for all $\phi \in \mathcal{Y}_N, \ \varphi \in \mathcal{Y}_N, \ \psi \in \mathcal{U}_N$, where

CONTROL $\mathcal{U}_N = \text{span}\{ \ u^*(\mu_i), \ i = 1, \ldots, N \}$

STATE/ADJOINT $\mathcal{Y}_N = \text{span}\{ \ \text{POD}_{\mathcal{Y}}(y^*(\mu_i)), \ \text{POD}_{\mathcal{Y}}(p^*(\mu_i)), \ i = 1, \ldots, N \}$

Data Assimilation

Reduced-Order 4DVAR

Solve $\min_{\mu \in \mathcal{D}} \min_{u_N \in \mathcal{U}_N} \quad \frac{1}{2} \| u_N - u_b \|_{\mathcal{U}}^2 \quad + \quad \frac{\lambda}{2} \sum_{k=1}^{K} \Delta t \left\| C y_N^k - C y_{\text{true}}^k \right\|_D^2$

$$\text{s.t.} \quad m(y_N^k, \nu) = m(y_N^{k-1}, \nu) - \Delta t\; a(y_N^k, \nu; \mu) + \Delta t\; f(\nu),$$

$$\forall\, \nu \in Y_N, \;\; k = 1, \ldots, K$$

$$y_N^0 = u_N$$

for μ^* and the estimate $(u^*(\mu^*), y^*(\mu^*))$.

Can we quantify the error?

CONTROL $\quad ||u^*(\mu) - u_N^*(\mu)||_{\mathcal{U}} \leq \Delta_N^u(\mu)$

STATE $\quad ||y^*(\mu) - y_N^*(\mu)||_{\mathcal{U}} \leq \Delta_N^y(\mu)$

Data Assimilation

Error

STATE $e_y^k(\mu) := y^{*k}(\mu) - y_N^{*k}(\mu)$

ADJOINT $e_p^k(\mu) := p^{*k}(\mu) - p_N^{*k}(\mu)$

CONTROL $e_u^k(\mu) := u^{*k}(\mu) - u_N^{*k}(\mu)$

Data Assimilation

Error

STATE $e_y^k(\mu) := y^{*k}(\mu) - y_N^{*k}(\mu)$

ADJOINT $e_p^k(\mu) := p^{*k}(\mu) - p_N^{*k}(\mu)$

CONTROL $e_u^k(\mu) := u^{*k}(\mu) - u_N^{*k}(\mu)$

Error Residual Equations

STATE $r_y^k(\phi;\mu) := a(e_y^k, \phi; \mu) + \frac{1}{\Delta t} m(e_y^k - e_y^{k-1}, \phi)$

ADJOINT $r_p^k(\varphi, \mu) := \lambda(Ce_y^k, C\varphi)_D + \frac{1}{\Delta t} m(\varphi, e_p^k - e_p^{k+1}) + a(\varphi, e_p^k; \mu)$

CONTROL $r_u(\mu) := (e_u, \psi)_{\mathcal{U}} - m(\psi, e_p^1)$

Data Assimilation

A Posteriori Error Estimation

We can show that

$$||u^*(\mu) - u_N^*(\mu)||_{\mathcal{U}} \leq \Delta_N^u(\mu) = c_1(\mu) + \sqrt{c_1(\mu)^2 + c_2(\mu)}$$

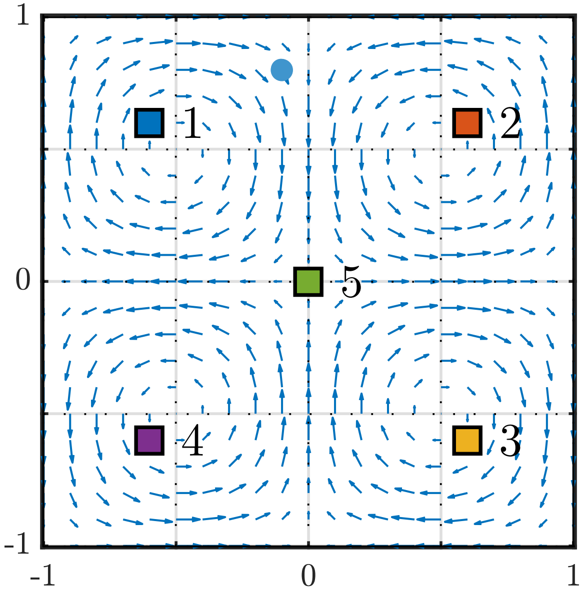
with non-negative terms

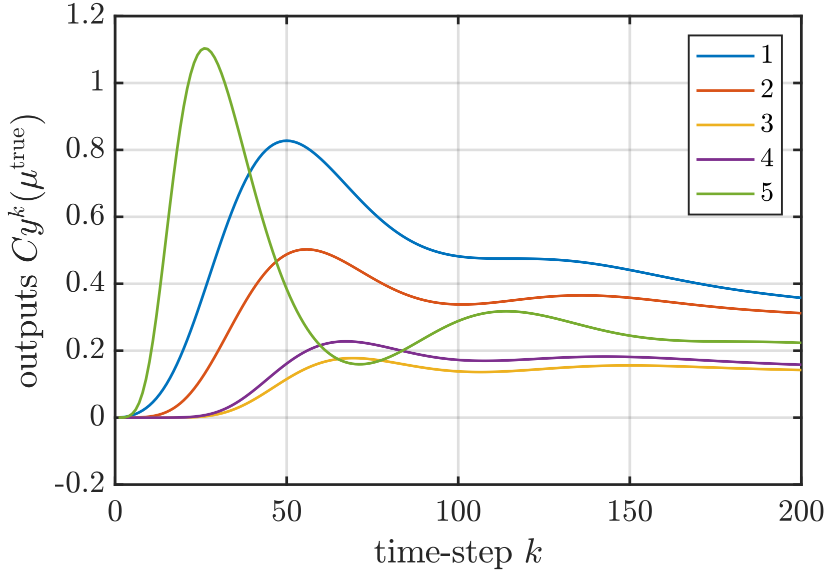
$$c_1 := \frac{1}{2}||r_u||_{\mathcal{U}'} + \frac{1}{\sqrt{\alpha_a^{\mathrm{LB}}}} R_p \qquad c_2 := \left(\frac{1+\sqrt{2}}{\alpha_a^{\mathrm{LB}}} R_y R_p + \frac{\lambda \gamma_C^2}{2(\alpha_a^{\mathrm{LB}})^2} R_y^2 \right)$$

where $R_{y,p} = \left(\tau \sum_{k=1}^{K} ||r_{y,p}^k||_{Y'}^2 \right)^{1/2}$, and r_y^k, r_p^k, r_u are the residuals in the state, adjoint, and control equations.

[Kärcher, Boyaval, Grepl, V., 2018]

Data Assimilation





Model Problem

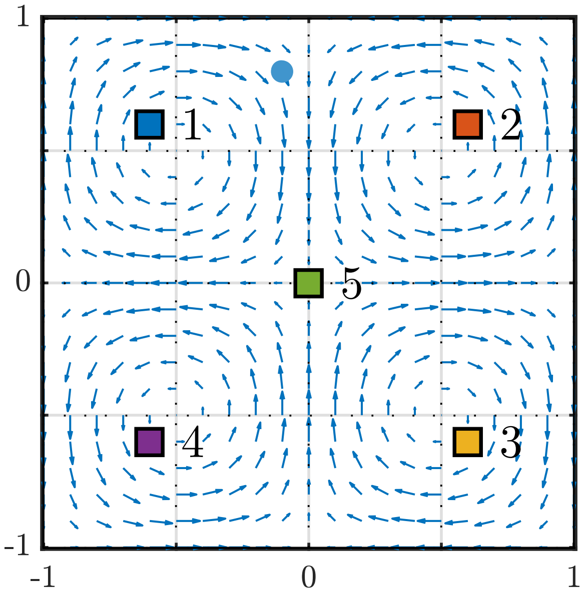
Contaminant transport

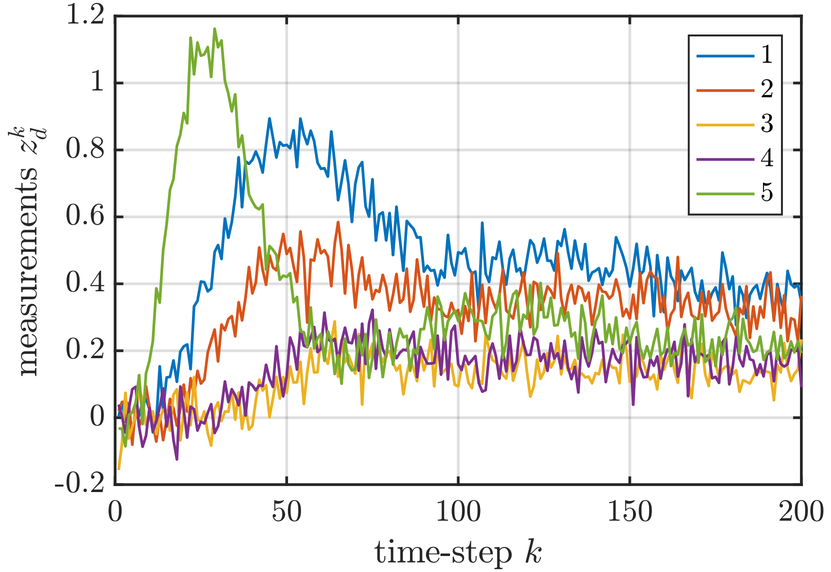
- “Gaussian” initial condition, $\bar{u}_0$
- Known Taylor-Green vortex velocity field
- Parameter $\mu = \mathrm{Pe} \in [10, 50], \bar{\mu} = 30$
- FE dimension $(\mathcal{N} = 13000, K = 200)$

Assumptions:

- Data generated with true initial condition
- Uncertainty due only to noise and “unknown” parameter
- Prior is exact

Data Assimilation





Model Problem

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- Data generated with true initial condition
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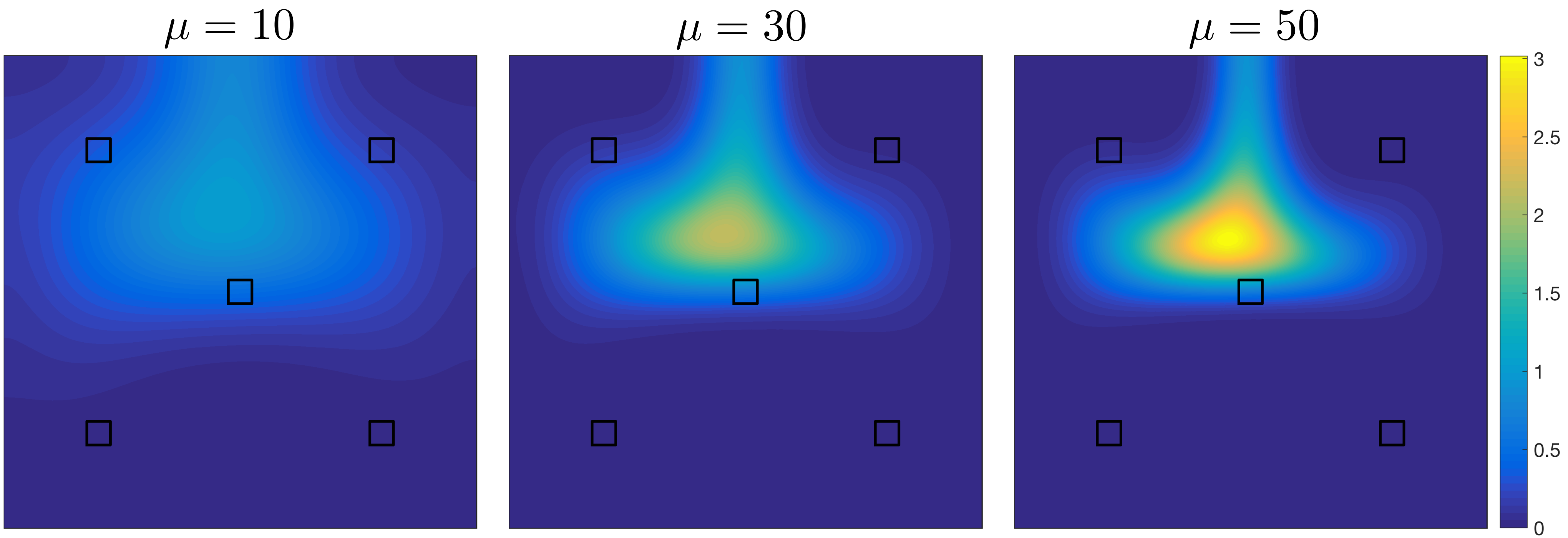
Data Assimilation

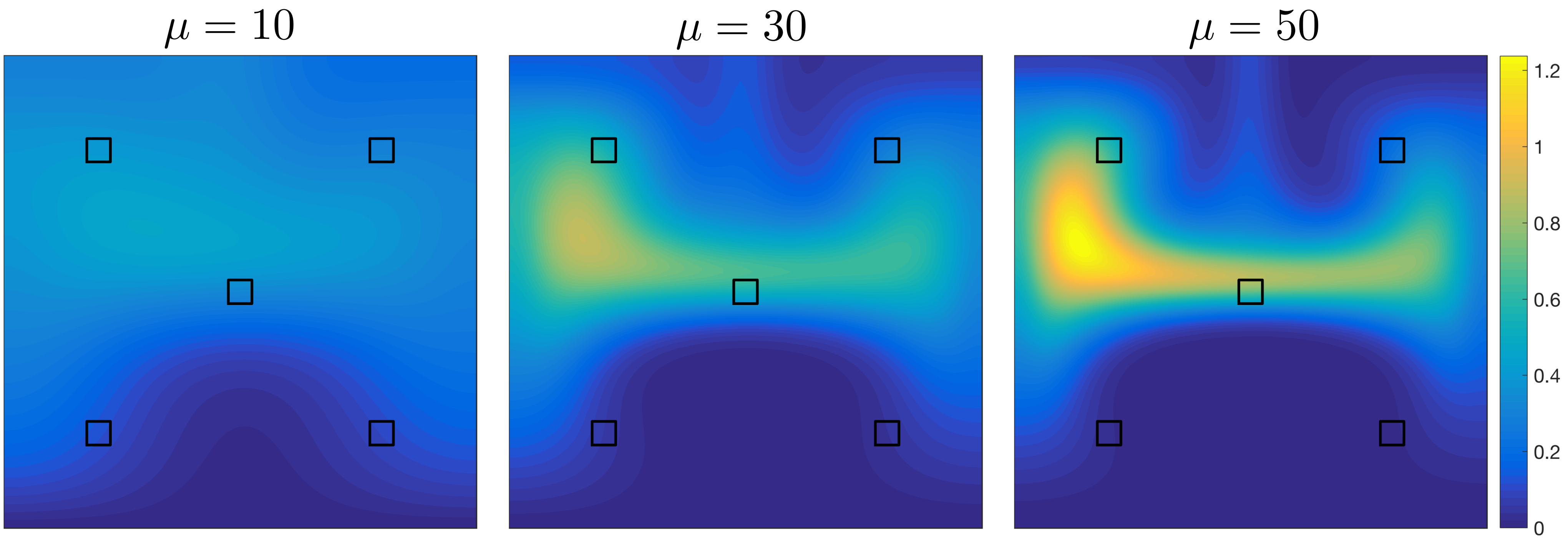
State variable $y(\mu)$



$k = 20$

Data Assimilation

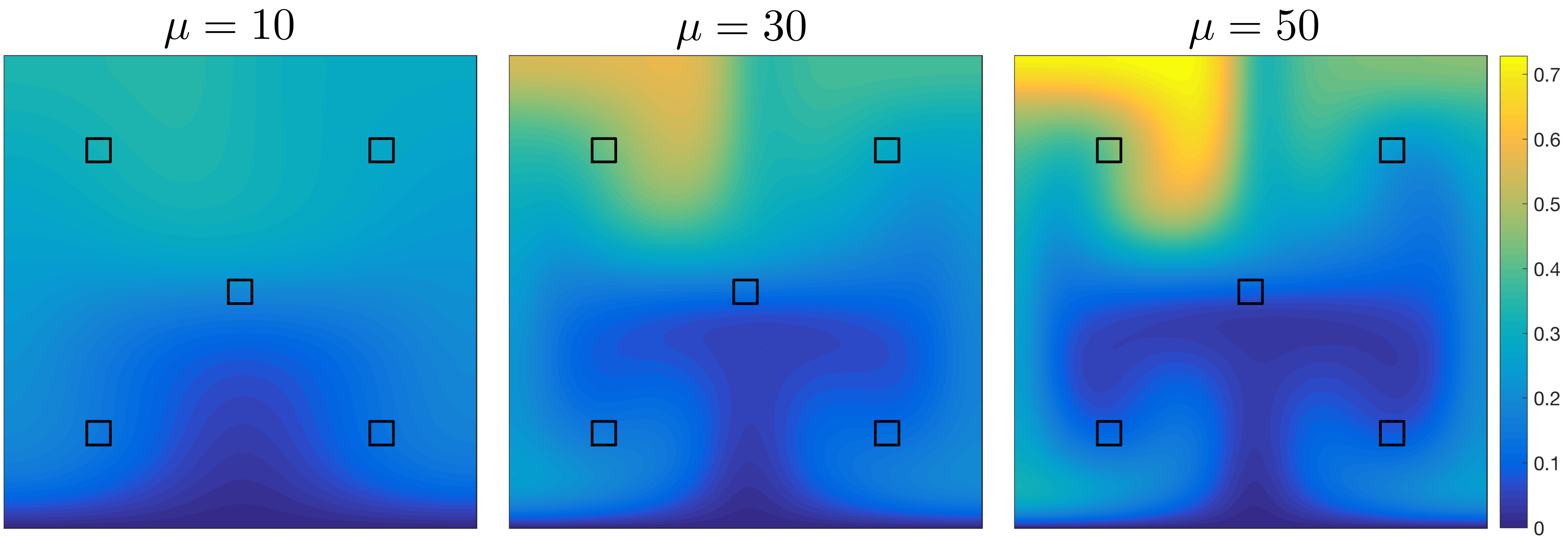
State variable $y(\mu)$



$k = 40$

Data Assimilation

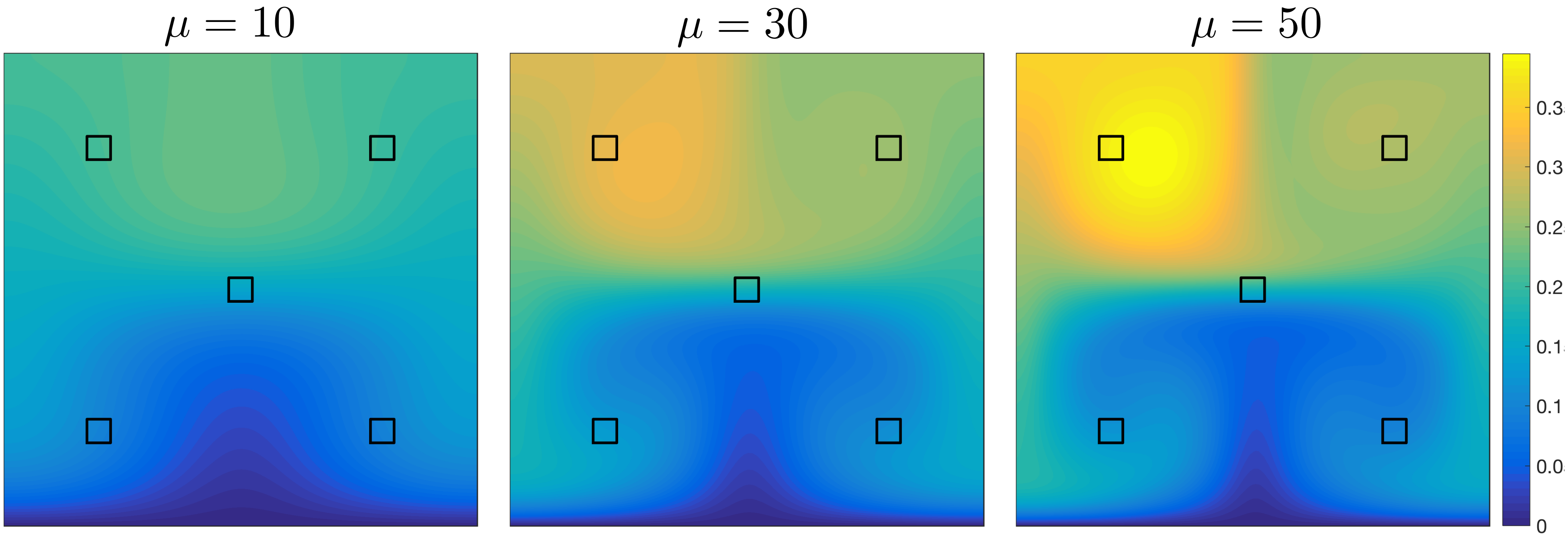
State variable $y(\mu)$



$k = 80$

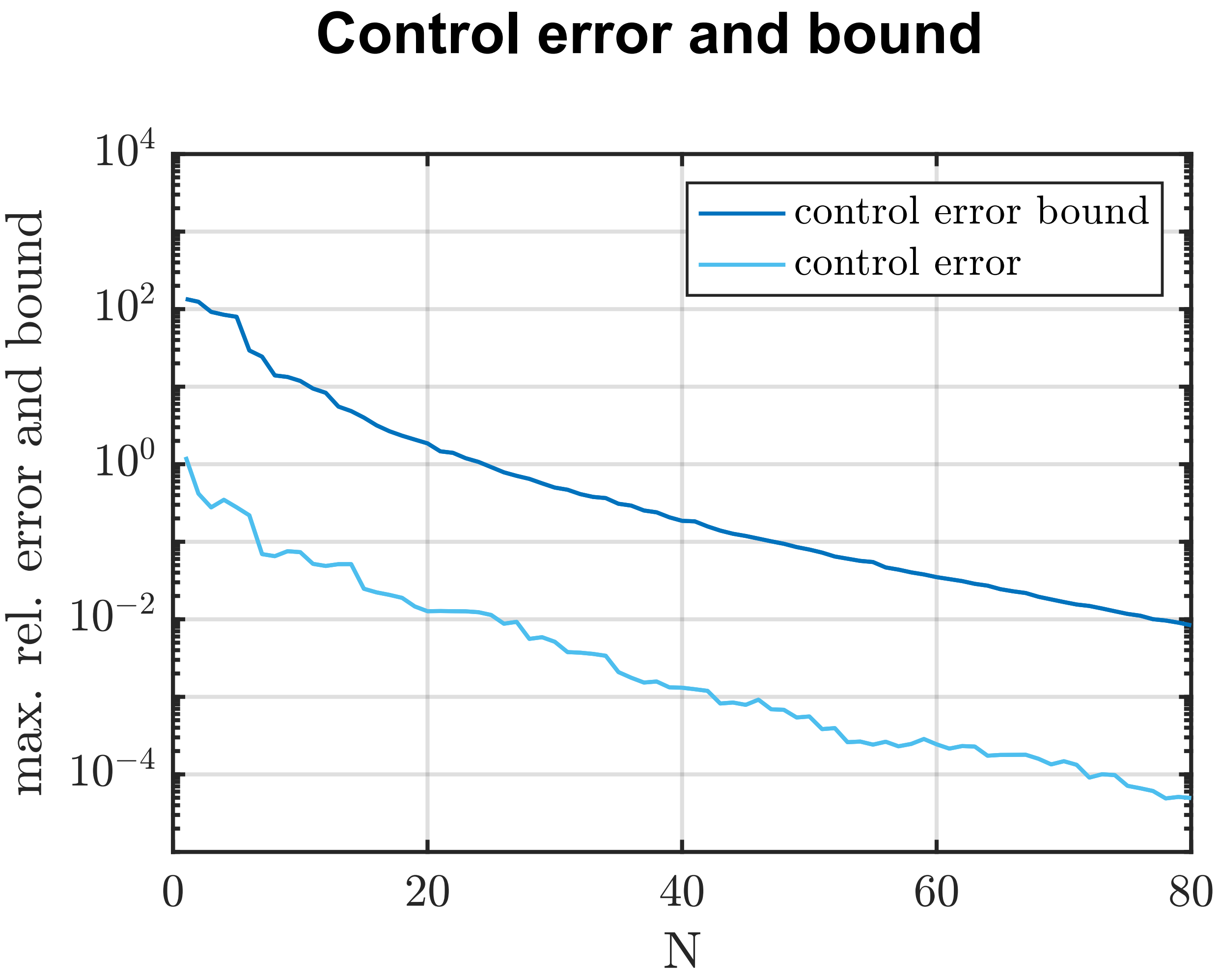
Data Assimilation

State variable $y(\mu)$

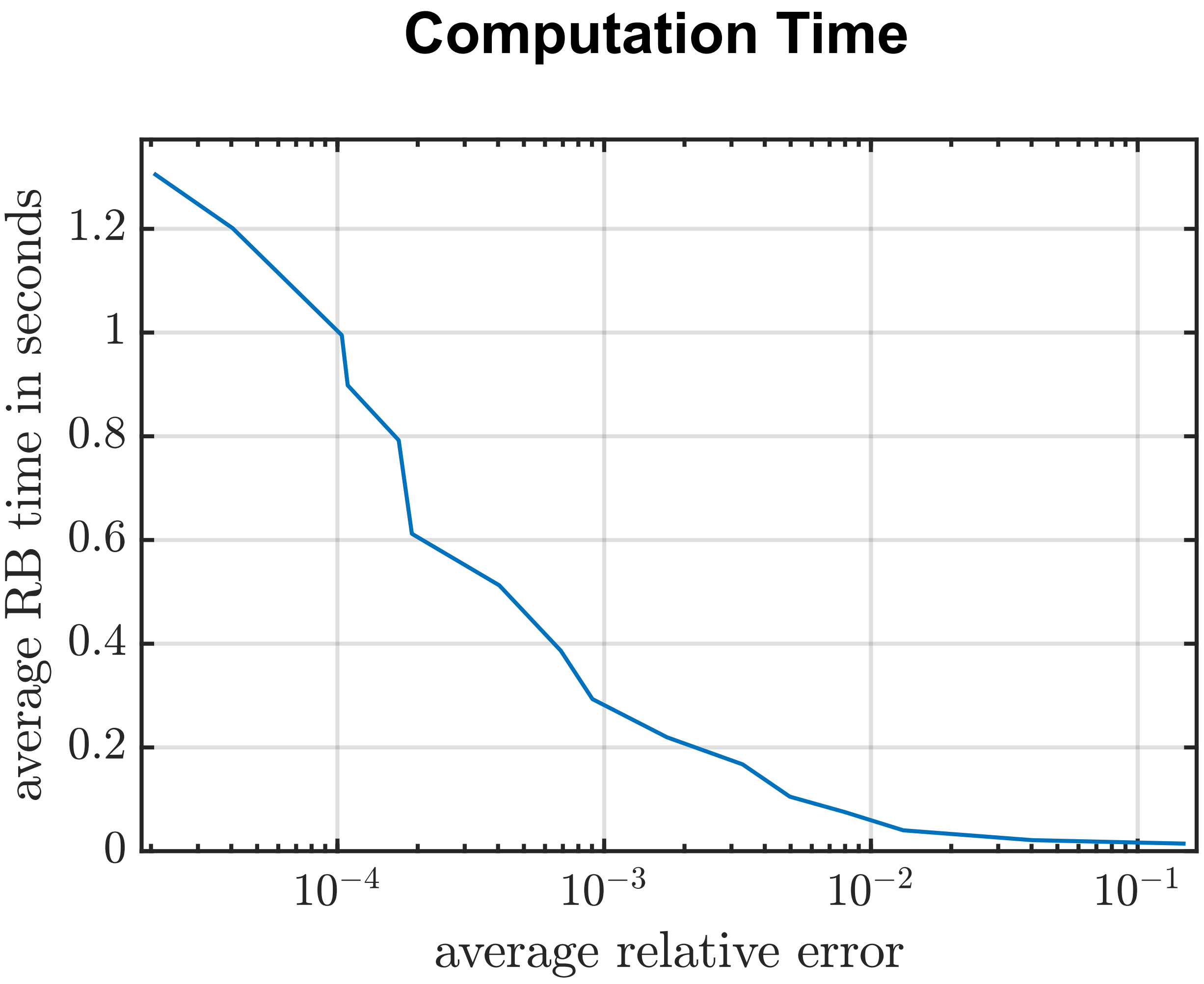


$k = 160$

Data Assimilation



Data Assimilation



4DVAR Summary

Key points

- Approximated solutions of the parametrized 4D-VAR problem using reduced basis methods
- Developed *a posteriori* error bounds for the error in the control (initial condition or model error) as well as state and adjoint
- Applied proposed methods to a simple parametrized convection-diffusion problem
- Estimated unknown parameter, initial condition, and model error

Issues and Perspectives

- Convergence and error estimates for the parameter estimation problem
- Introduce uncertainty in prior
- **Sensor placement**

Data Assimilation + Sensor Placement

with

N. Aretz-Nellesen and M. Grepl

Background

(Variational) Data Assimilation

3D-/4D-VAR	[Lorenc '81], [Le Dimet '81], [Courtier '85], … [Le Dimet & Talagrand '86], … [Navon et al] …
+ Kalman Filter, Bayesian Methods	[Law & Stuart '15], [Reich '15], …

MOR + Data Assimilation (+Sensor Placement)

Gappy-POD	[Everson & Sirovich ’95], [Willcox '06] …
GEIM	[Maday & Mula '13] …
PGD (+ EIM)	[Nadal, Chinesta, Diez, Fuenmayor & Denia '15] …
PBDW	[**Maday, Patera, Penn & Yano ’14**, ’15], [Taddei ’17], [**Maday & Taddei ’17(p)**], [Taddei & Patera ’18], [Hammond, Chaqir, Bourquin & Maday ’18(p)]
OMP	[**Binev, Cohen, Mula & Nichols ’18**]

MOR + Optimal Control

RB + OC	[Negri, Rozza, Manzoni, Quarteroni '13], [Tröltzsch & Volkwein '09], [Kärcher, Tokoutsi, Grepl & V. '18]

Data Assimilation + Sensor Placement

4DVAR

$$\min_{\mu \in \mathcal{D}} \min_{u \in \mathcal{U}} \quad \frac{1}{2}\|u - u_b\|_{\mathcal{U}}^2 \quad + \quad \frac{\lambda}{2}\sum_{k=1}^{K} \Delta t \left\| C y^k - y_d^k \right\|_D^2$$

$$\text{s.t.} \quad m(y^k, \nu) = m(y^{k-1}, \nu) - \Delta t \; a(y^k, \nu; \mu) + \Delta t \; f(\nu), \qquad \forall\, \nu \in Y, \; k = 1, \ldots, K$$

$$y^0 = u$$

Solve for μ^* and the estimate $(u^*(\mu^*), y^*(\mu^*))$.

Data Assimilation

Modified 3D-VAR Formulation

$$\min_{u \in \mathcal{U}} \frac{1}{2}\|u\|_{\mathcal{U}}^2 + \frac{\lambda}{2}\|d\|_{\mathcal{Y}}^2$$

$$\text{s.t.} \quad a(y,v) = f(v) + b(u,v) \quad \forall v \in \mathcal{Y} \qquad (\mathcal{M})$$

$$(y + d, \tau)_{\mathcal{Y}} = (y_{\mathrm{d}}, \tau)_{\mathcal{Y}} \quad \forall \tau \in \mathcal{T} \subset \mathcal{Y}$$

where

u	model bias
d	misfit between state and “data"
y	state
y_d	"data"
λ	regularisation parameter
$\mathcal{M}$	is the best- knowledge model of the physics.

Data Assimilation

Modified 3D-VAR Formulation

$$\min_{u \in \mathcal{U}} \frac{1}{2}\|u\|_{\mathcal{U}}^2 + \frac{\lambda}{2}\|d\|_{\mathcal{Y}}^2$$

$$\text{s.t.} \quad a(y,v) = f(v) + b(u,v) \quad \forall v \in \mathcal{Y} \qquad (\mathcal{M})$$

$$(y + d, \tau)_{\mathcal{Y}} = (y_{\mathrm{d}}, \tau)_{\mathcal{Y}} \quad \forall \tau \in \mathcal{T} \subset \mathcal{Y}$$

Variational Data Assimilation

- Prevalent in meteorology and oceanography

[Law & Stuart 2015], [Reich 2015],…

- Given a best knowledge model and data

  find (allowed) perturbations u to the model

  such that u and the misfit d are as small as possible.

Data Assimilation

Modified 3D-VAR Formulation

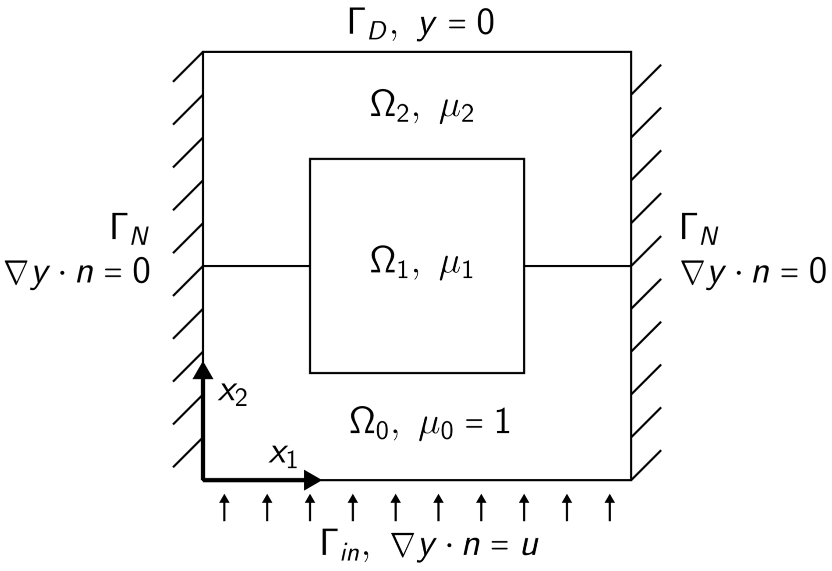
$$\left(\min_{\mu\in\mathcal{D}}\right)\min_{u\in\mathcal{U}}\frac{1}{2}\|u(\mu)\|_{\mathcal{U}}^2+\frac{\lambda}{2}\|d(\mu)\|_{\mathcal{Y}}^2$$

$$\text{s.t.}\quad a(y(\mu),v;\mu)=f(v;\mu)+b(u(\mu),v)\qquad\forall v\in\mathcal{Y}$$

$$(y(\mu)+d(\mu),\tau)_{\mathcal{Y}}=(y_{\rm d},\tau)_{\mathcal{Y}}\qquad\forall\tau\in\mathcal{T}\subset\mathcal{Y}$$

Issues

- Bias in boundary conditions
- Error in model form
- Unknown or uncertain parameters
- Noisy data



Data Assimilation

Modified 3D-VAR Formulation

$$\left(\min_{\mu \in \mathcal{D}}\right) \min_{u \in \mathcal{U}} \frac{1}{2}\|u(\mu)\|_{\mathcal{U}}^2 + \frac{\lambda}{2}\|d(\mu)\|_{\mathcal{Y}}^2$$

$$\text{s.t.} \quad a(y(\mu), v; \mu) = f(v; \mu) + b(u(\mu), v) \quad \forall v \in \mathcal{Y}$$

$$(y(\mu) + d(\mu), \tau)_{\mathcal{Y}} = (y_{\mathrm{d}}, \tau)_{\mathcal{Y}} \quad \forall \tau \in \mathcal{T} \subset \mathcal{Y}$$

Issues

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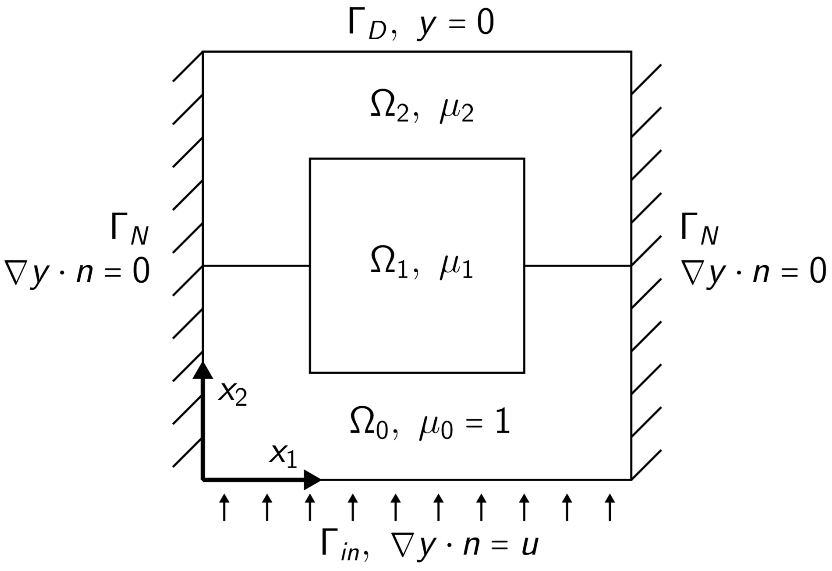

Data Assimilation

Modified 3D-VAR Formulation

$$\left(\min_{\mu\in\mathcal{D}}\right) \min_{u\in\mathcal{U}} \frac{1}{2}\|u(\mu)\|_{\mathcal{U}}^2 + \frac{\lambda}{2}\|d(\mu)\|_{\mathcal{Y}}^2$$

$$\text{s.t.} \quad a(y(\mu), v; \mu) = f(v;\mu) + b(u(\mu), v) \qquad \forall v \in \mathcal{Y}$$

$$(y(\mu) + d(\mu), \tau)_{\mathcal{Y}} = (y_{\mathrm{d}}, \tau)_{\mathcal{Y}} \qquad \forall \tau \in \mathcal{T} \subset \mathcal{Y}$$

Relation to Optimal Control

[Nellesen 2018(MS)]

- Distributed optimal control (+ minimization problem)
- $b(u, v)$ represents permitted corrections to the model
- Optimality leads to saddle point structure
- Use RB approximation, error bounds in [Kärcher, Tokoutsi, Grepl & V. 2017]
- Difference: Measurement space $\mathcal{T} \subset \mathcal{Y}$

Data Assimilation

Modified 3D-VAR Formulation

$$\left(\min_{\mu\in\mathcal{D}}\right) \min_{u_N\in\mathcal{U}_N} \frac{1}{2}\|u_N(\mu)\|^2_{\mathcal{U}} + \frac{\lambda}{2}\|d_N(\mu)\|^2_{\mathcal{Y}}$$

$$\begin{aligned} \text{s.t.} \quad & a(y_N(\mu), v; \mu) = f(v;\mu) + b(u_N(\mu), v) && \forall v \in \mathcal{Y}_N \\ & (y_N(\mu) + d_N(\mu), \tau)_{\mathcal{Y}} = (y_{\mathrm{d}}, \tau)_{\mathcal{Y}} && \forall \tau \in \mathcal{T} \end{aligned}$$

Reduced Basis Approximation

[Nellesen '18(MS)]

- Introduce reduced spaces for control, state, and adjoints
- Galerkin projection onto reduced basis spaces
- A posteriori error bounds for control, state, adjoint, and misfit
- Offline / online decomposition
- Greedy algorithm to construct approximation spaces

Data Assimilation

Residuals

To obtain an *a posteriori* error bound for each error term

$$e_u := u^* - u_N^*, \qquad e_y := y^* - y_N^*, \qquad e_d := d^* - d_N^*, \qquad e_p := p^* - p_N^*,$$

(control) (state) (misfit) (adjoint)

we define the residuals

$$\begin{aligned} r_u : \mathcal{U} \to \mathbb{R} \quad r_u(\phi) &:= b + \mu(\phi, p_N^*) - (u_N^*, \phi)_{\mathcal{U}} \\ r_p : \mathcal{Y} \to \mathbb{R} \quad r_p(\psi) &:= \lambda(\psi, d_N^*)_{\mathcal{Y}} - a_\mu(\psi, p_N^*) \\ r_y : \mathcal{Y} \to \mathbb{R} \quad r_y(\psi) &:= f_\mu(\psi) + b_\mu(u_N^*, \psi) - a_\mu(y_N^*, \psi) \end{aligned}$$

whose norms can be computed in an *offline-online* procedure.

Data Assimilation

A Posteriori Error Bounds

Define further

$$g_u := \|r_u\|_{\mathcal{U}'} + \frac{1}{\alpha_\mu}\|b_\mu\|\|r_p\|_{\mathcal{Y}'} \qquad g_d := \frac{1}{\alpha_\mu}\|r_y\|_{\mathcal{Y}'}$$

$$h_u := \frac{2}{\alpha_u}\|r_p\|_{\mathcal{Y}'}\|r_y\|_{\mathcal{Y}'} + \frac{\lambda}{4\alpha_\mu^2}\|r_y\|_{\mathcal{Y}'}^2 \qquad h_d := \frac{2}{\lambda\alpha_\mu}\|r_p\|_{\mathcal{Y}'}\|r_y\|_{\mathcal{Y}'} + \frac{1}{4\lambda}g_u^2$$

Then

$$\|e_u\|_{\mathcal{Y}} \leqslant \tfrac{1}{2}g_u + \sqrt{\tfrac{1}{4}g_u^2 + h_u} \qquad \|e_y\|_{\mathcal{U}} \leqslant \frac{1}{\alpha_\mu}\|r_y\|_{\mathcal{Y}'} + \frac{\|b_\mu\|}{\alpha_\mu}\|e_u\|_{\mathcal{U}}$$

$$\|e_d\|_{\mathcal{Y}} \leqslant \tfrac{1}{2}g_d + \sqrt{\tfrac{1}{4}g_d^2 + h_d} \qquad \|e_p\|_{\mathcal{U}} \leqslant \frac{1}{\alpha_\mu}\|r_p\|_{\mathcal{Y}'} + \frac{\lambda}{\alpha_\mu}\|e_d\|_{\mathcal{U}}$$

Similar *a posteriori* error bounds in [Kärcher, Tokoutsi, Grepl & V. '18]

Data Assimilation

Modified 3D-VAR Formulation

$$\left(\min_{\mu\in\mathcal{D}}\right) \min_{u\in\mathcal{U}} \frac{1}{2}\|u(\mu)\|_{\mathcal{U}}^2 + \frac{\lambda}{2}\|d(\mu)\|_{\mathcal{Y}}^2$$

$$\text{s.t.} \quad a(y(\mu), v; \mu) = f(v;\mu) + b(u(\mu), v) \qquad \forall v \in \mathcal{Y}$$

$$(y(\mu) + d(\mu), \tau)_{\mathcal{Y}} = (y_{\mathrm{d}}, \tau)_{\mathcal{Y}} \qquad \forall \tau \in \mathcal{T} \subset \mathcal{Y}$$

How do we optimally select the measurements?

Data Assimilation

Modified 3D-VAR Formulation

$$\left(\min_{\mu\in\mathcal{D}}\right)\min_{u\in\mathcal{U}}\frac{1}{2}\|u(\mu)\|_{\mathcal{U}}^2+\frac{\lambda}{2}\|d(\mu)\|_{\mathcal{Y}}^2$$

$$\begin{aligned}\text{s.t.}\quad & a(y(\mu),v;\mu)=f(v;\mu)+b(u(\mu),v) && \forall v\in\mathcal{Y}\\ & (y(\mu)+d(\mu),\tau)_{\mathcal{Y}}=(y_{\mathrm{d}},\tau)_{\mathcal{Y}} && \forall\tau\in\mathcal{T}\subset\mathcal{Y}\end{aligned}$$

How do we optimally select the measurement space $\mathcal{T}$

where $\mathcal{T}$ is the space spanned by the Riesz representation

of the measurement functionals?

→ Stability analysis

[Maday, Patera, Penn & Yano 2014]

Data Assimilation

One can show that

$$\|(u^*_\mu, y^*_\mu)(\lambda)\|_{\mathcal{U}\times\mathcal{Y}} \leqslant C^\mu_1(\lambda)\|y_{\mathrm{d}}\|_{\mathcal{Y}} + C^\mu_2(\lambda)\|f_{\mathrm{bk},\mu}\|_{\mathcal{Y}'}$$

$$\|p^*_\mu(\lambda)\|_{\mathcal{Y}} \leqslant C^\mu_3(\lambda)\|y_{\mathrm{d}}\|_{\mathcal{Y}} + C^\mu_4(\lambda)\|f_{\mathrm{bk},\mu}\|_{\mathcal{Y}'}$$

with positive stability constants.

The stability constants are “better-behaved” for [Aretz-Nellesen et al. 2019]

$$\underline{\eta}(\mu) := \inf_{(u,y)\in\mathcal{H}^0(\mu)} \frac{\|y\|_{\mathcal{Y}}}{\|u\|_{\mathcal{U}}} \overset{!}{>} 0 \qquad \beta_{\mathcal{T}}(\mu) := \inf_{y\in\mathcal{Y}_\mu} \sup_{\tau\in\mathcal{T}} \frac{(y,\tau)_{\mathcal{Y}}}{\|y\|_{\mathcal{Y}}\|\tau\|_{\mathcal{Y}}} \overset{!}{>} 0$$

as large as possible. Here,

$$\begin{aligned}
\mathcal{H}^0(\mu) \quad &:= \{\ (u,y) \in \mathcal{U}\times\mathcal{Y} : a_\mu(y,\psi) = b_\mu(u,\psi) \qquad\quad \forall\psi\in\mathcal{Y}\ \},\\
\mathcal{Y}_\mu \quad &:= \{\ y\in\mathcal{Y} : \exists\ u\in\mathcal{U} \text{ s.t. } a_\mu(y,\psi) = b_\mu(u,\psi) \quad \forall\psi\in\mathcal{Y}\ \}.
\end{aligned}$$

Data Assimilation

One can show that

$$\|(u_\mu^*, y_\mu^*)(\lambda)\|_{\mathcal{U}\times\mathcal{Y}} \leqslant C_1^\mu(\lambda)\|y_{\mathrm{d}}\|_{\mathcal{Y}} + C_2^\mu(\lambda)\|f_{\mathrm{bk},\mu}\|_{\mathcal{Y}'}$$

$$\|p_\mu^*(\lambda)\|_{\mathcal{Y}} \leqslant C_3^\mu(\lambda)\|y_{\mathrm{d}}\|_{\mathcal{Y}} + C_4^\mu(\lambda)\|f_{\mathrm{bk},\mu}\|_{\mathcal{Y}'}$$

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$$\begin{aligned} \mathcal{H}^0(\mu) \quad &:= \{\ (u,y) \in \mathcal{U}\times\mathcal{Y} : a_\mu(y,\psi) = b_\mu(u,\psi) \qquad \forall\psi\in\mathcal{Y}\ \}, \\ \mathcal{Y}_\mu \quad &:= \{\ y\in\mathcal{Y} : \exists\ u\in\mathcal{U} \text{ s.t. } a_\mu(y,\psi) = b_\mu(u,\psi) \quad \forall\psi\in\mathcal{Y}\ \}. \end{aligned}$$

Data Assimilation

Modified 3D-VAR Formulation

$$
\begin{aligned}
\left(\min_{\mu\in\mathcal{D}}\right) \min_{u\in\mathcal{U}} \ & \frac{1}{2}\|u(\mu)\|_{\mathcal{U}}^2 + \frac{\lambda}{2}\|d(\mu)\|_{\mathcal{Y}}^2 \\
\text{s.t.} \quad & a(y(\mu), v; \mu) = f(v;\mu) + b(u(\mu), v) && \forall v \in \mathcal{Y} \\
& (y(\mu) + d(\mu), \tau)_{\mathcal{Y}} = (y_{\mathrm{d}}, \tau)_{\mathcal{Y}} && \forall \tau \in \mathcal{T} \subset \mathcal{Y}
\end{aligned}
$$

Data Assimilation

Modified 3D-VAR Formulation

$$\left(\min_{\mu\in\mathcal{D}}\right)\min_{u\in\mathcal{U}}\frac{1}{2}\|u_N(\mu)\|_{\mathcal{U}}^2+\frac{\lambda}{2}\|d_N(\mu)\|_{\mathcal{Y}}^2$$

$$\text{s.t.}\quad a(y_N(\mu),v;\mu)=f(v;\mu)+b(u_N(\mu),v)\qquad\forall v\in\mathcal{Y}_N$$

$$(y_N(\mu)+d_N(\mu),\tau)_{\mathcal{Y}}=(y_{\mathrm{d}},\tau)_{\mathcal{Y}}\qquad\forall\tau\in\mathcal{T}$$

Assume that [Aretz-Nellesen, Grepl & V. 2019]

$$\|y-y_N\|_{\mathcal{Y}}\leqslant\varepsilon_\mu\|y\|_{\mathcal{Y}}\quad\text{where }0\leqslant\varepsilon_\mu\ll 1$$

Then

$$\beta_{\mathcal{T}}(\mu)\geqslant(1-\varepsilon_\mu)\beta_{\mathcal{T},N}(\mu)-\varepsilon_\mu$$

Data Assimilation

Construction of RB Spaces

Recall optimality conditions [Aretz-Nellesen, Grepl & V. 2019]

$$
\begin{array}{rclll}
(u^*_\mu, \phi)_{\mathcal{U}} - b_\mu(\phi, p^*_\mu) & = & 0 & \forall \quad \phi \in \mathcal{U} & \text{control} \\
a_\mu(\psi, p^*_\mu) - \lambda(\psi, d^*_\mu)_{\mathcal{Y}} & = & 0 & \forall \quad \psi \in \mathcal{Y} & \text{adjoint} \\
a_\mu(y^*_\mu, \psi) - b_\mu(u^*_\mu, \psi) & = & f_{\mathrm{bk},\mu}(\psi) & \forall \quad \psi \in \mathcal{Y} & \text{state} \\
(y^*_\mu + d^*_\mu, \tau)_{\mathcal{Y}} & = & (y_\mathrm{d}, \tau)_{\mathcal{Y}} & \forall \quad \tau \in \mathcal{T}. & \text{misfit}
\end{array}
$$

Data Assimilation

Construction of RB Spaces

Recall optimality conditions [Aretz-Nellesen, Grepl & V. 2019]

$$
\begin{aligned}
(u^*_\mu, \phi)_{\mathcal{U}} - b_\mu(\phi, p^*_\mu) &= 0 & \forall \quad & \phi \in \mathcal{U} & & \text{control} \\
a_\mu(\psi, p^*_\mu) - \lambda(\psi, d^*_\mu)_{\mathcal{Y}} &= 0 & \forall \quad & \psi \in \mathcal{Y} & & \text{adjoint} \\
a_\mu(y^*_\mu, \psi) - b_\mu(u^*_\mu, \psi) &= f_{\mathrm{bk},\mu}(\psi) & \forall \quad & \psi \in \mathcal{Y} & & \text{state} \\
(y^*_\mu + d^*_\mu, \tau)_{\mathcal{Y}} &= (y_{\mathrm{d}}, \tau)_{\mathcal{Y}} & \forall \quad & \tau \in \mathcal{T}. & & \text{misfit}
\end{aligned}
$$

$$\begin{tikzcd}
\mathcal{U}_N \arrow[r] & \mathcal{Y}_{y,N} \arrow[r] & \mathcal{T} \arrow[r] & \mathcal{Y}_{p,N} \arrow[r] & \mathcal{Y}_N = \mathcal{Y}_{y,N} + \mathcal{Y}_{p,N}
\end{tikzcd}$$

Given a low dimensional approximation to the space
of model corrections (i.e., control)

Data Assimilation

Construction of RB Spaces

Recall optimality conditions [Nellesen, Grepl & V. 2018]

$$(u^*_\mu, \phi)_\mathcal{U} - b_\mu(\phi, p^*_\mu) = 0 \qquad \forall \quad \phi \in \mathcal{U} \qquad \text{control}$$

$$a_\mu(\psi, p^*_\mu) - \lambda(\psi, d^*_\mu)_\mathcal{Y} = 0 \qquad \forall \quad \psi \in \mathcal{Y} \qquad \text{adjoint}$$

$$\boxed{a_\mu(y^*_\mu, \psi) - b_\mu(u^*_\mu, \psi) = f_{\mathrm{bk},\mu}(\psi) \qquad \forall \quad \psi \in \mathcal{Y} \qquad \text{state}}$$

$$(y^*_\mu + d^*_\mu, \tau)_\mathcal{Y} = (y_\mathrm{d}, \tau)_\mathcal{Y} \qquad \forall \quad \tau \in \mathcal{T}. \qquad \text{misfit}$$

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Construct an RB space for the state
Note that $\mathcal{T}$ is not yet required!

Data Assimilation

Construction of RB Spaces

Recall optimality conditions [Nellesen, Grepl & V. 2018]

$$\begin{aligned}
(u_\mu^*, \phi)_{\mathcal{U}} - b_\mu(\phi, p_\mu^*) &= 0 && \forall \quad \phi \in \mathcal{U} && \text{control} \\
a_\mu(\psi, p_\mu^*) - \lambda(\psi, d_\mu^*)_{\mathcal{Y}} &= 0 && \forall \quad \psi \in \mathcal{Y} && \text{adjoint} \\
a_\mu(y_\mu^*, \psi) - b_\mu(u_\mu^*, \psi) &= f_{\mathrm{bk},\mu}(\psi) && \forall \quad \psi \in \mathcal{Y} && \text{state}
\end{aligned}$$

$$\boxed{(y_\mu^* + d_\mu^*, \tau)_{\mathcal{Y}} = (y_{\mathrm{d}}, \tau)_{\mathcal{Y}} \quad \forall \quad \tau \in \mathcal{T}. \qquad \text{misfit}}$$

$$\mathcal{U}_N \longrightarrow \mathcal{Y}_{y,N} \longrightarrow \mathcal{T} \longrightarrow \mathcal{Y}_{p,N} \longrightarrow \mathcal{Y}_N = \mathcal{Y}_{y,N} + \mathcal{Y}_{p,N}$$

Select optimal measurements via
greedy algorithm in the parameter domain +
orthogonal matching pursuit [Binev et al. 2018]

Data Assimilation

Construction of RB Spaces

Recall optimality conditions [Nellesen, Grepl & V. 2018]

$$\begin{array}{rclcll}
(u^*_\mu, \phi)_{\mathcal{U}} - b_\mu(\phi, p^*_\mu) & = & 0 & \forall & \phi \in \mathcal{U} & \text{control} \\
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\end{array}$$

$$\begin{tikzcd}
\mathcal{U}_N \arrow[r] & \mathcal{Y}_{y,N} \arrow[r] & \mathcal{T} \arrow[r] & \mathcal{Y}_{p,N} \arrow[r] & \mathcal{Y}_N = \mathcal{Y}_{y,N} + \mathcal{Y}_{p,N}
\end{tikzcd}$$

Construct an RB space for the adjoint

Data Assimilation

Numerical Experiment: Thermal Block

- State space

$\mathcal{Y}$ FE-discretization of

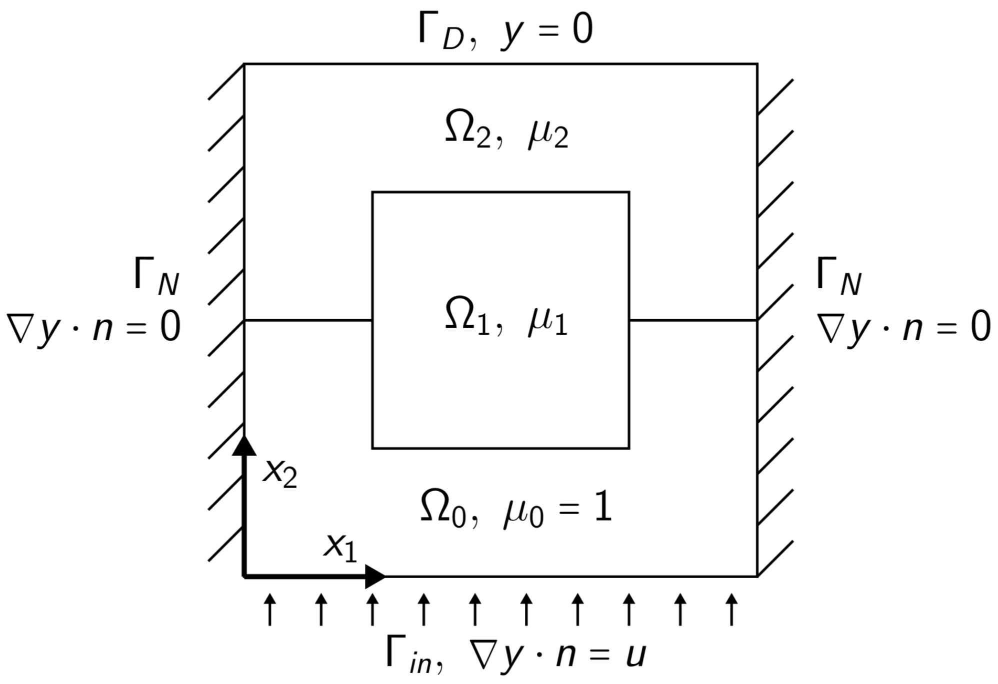
$$\mathcal{Y}_\mathrm{e} := \{y \in H^1(\Omega) : y|_{\Gamma_\mathrm{D}} = 0\}$$

- Bilinear form

$$a_\mu(y,w) := \sum_{i=0}^{2} \mu_i \int_{\Omega_i} \nabla y \cdot \nabla w \; dx$$

- Parameter domain

$$\mathcal{C} := [0.1, 10]^2$$



Data Assimilation

Numerical Experiment: Thermal Block

- Source term

$$b(u,\cdot) \in \mathcal{Y}', \; b(u,w) := \int_{\Gamma_{\text{in}}} uw \; dS,$$

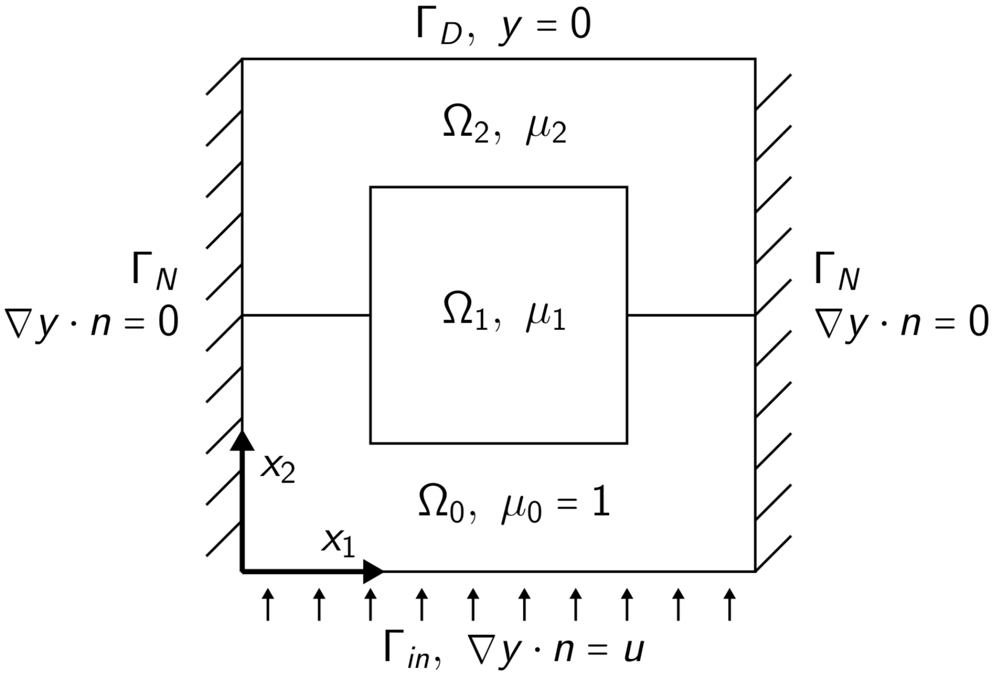
where $u \in L^2(\Gamma_{\text{in}})$

- BK-model source term

$$f_{\text{bk},\mu} = b(u_{\text{bk}},\cdot), \; u_{\text{bk}} \equiv 1$$

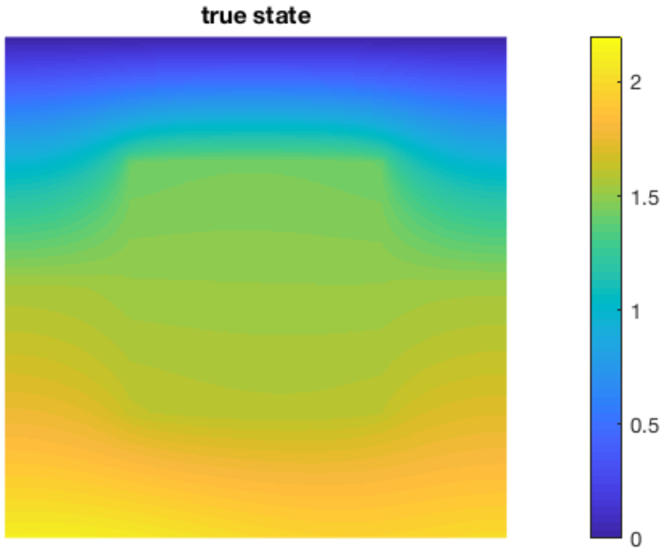
- Model correction

$\mathcal{U} = \mathbb{P}_3$ (polynomial space)



Data Assimilation

Parameter Estimation Problem Statement



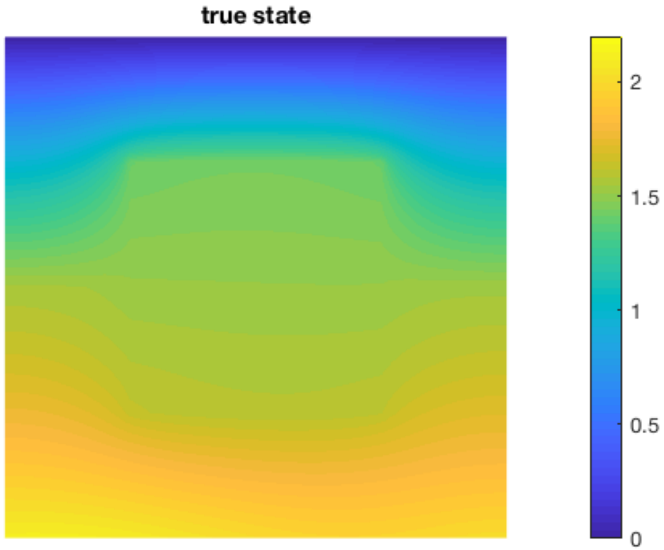
Approximate the unknown variables

- $\mu_{\text{true}} = (7, 0.3) \in \mathcal{C}$
- $u_{\text{true}}(x) \approx 1.5 + 0.3\sin(2\pi x), \;\; x \in \Gamma_{\text{in}}$
- $y_{\text{true}} = y_{\mu_{\text{true}}}(u_{\text{true}})$

with the 3D-VAR solution.

Data Assimilation

Parameter Estimation Problem Statement



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- $y_{\mathrm{true}} = y_{\mu_{\mathrm{true}}}(u_{\mathrm{true}})$

with the 3D-VAR solution.

Prior Knowledge:

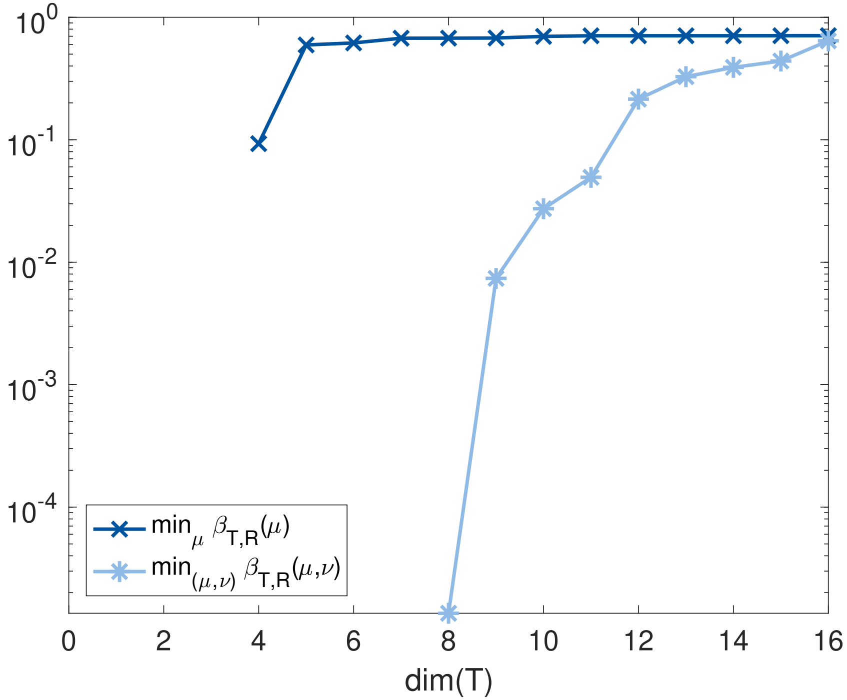
u_{true} can be approximated in $\mathcal{U} := \mathcal{P}_3$ sufficiently

Measurement Space:

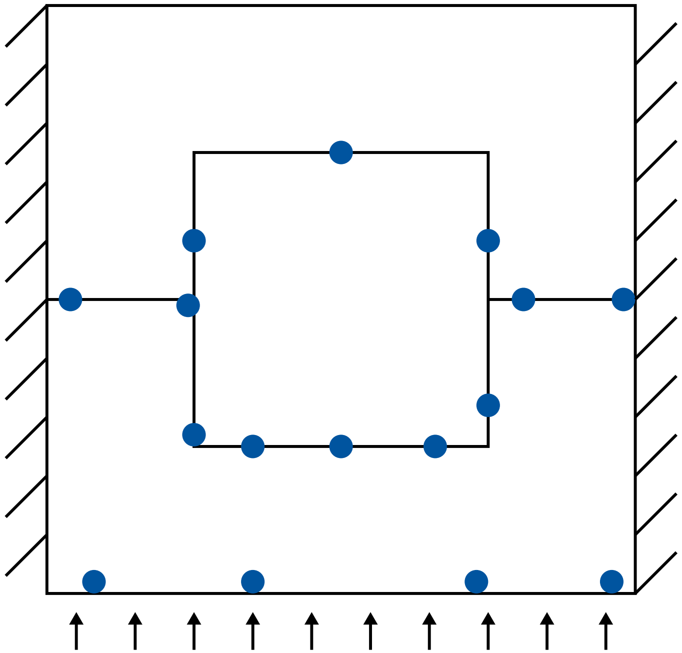
A small number of measurements may be chosen from a library of gaussian functionals with std. dev. 0.01 and centers within $(0.02, 0.98)^2 \subset \Omega$

Data Assimilation

Selection of the Measurement Space



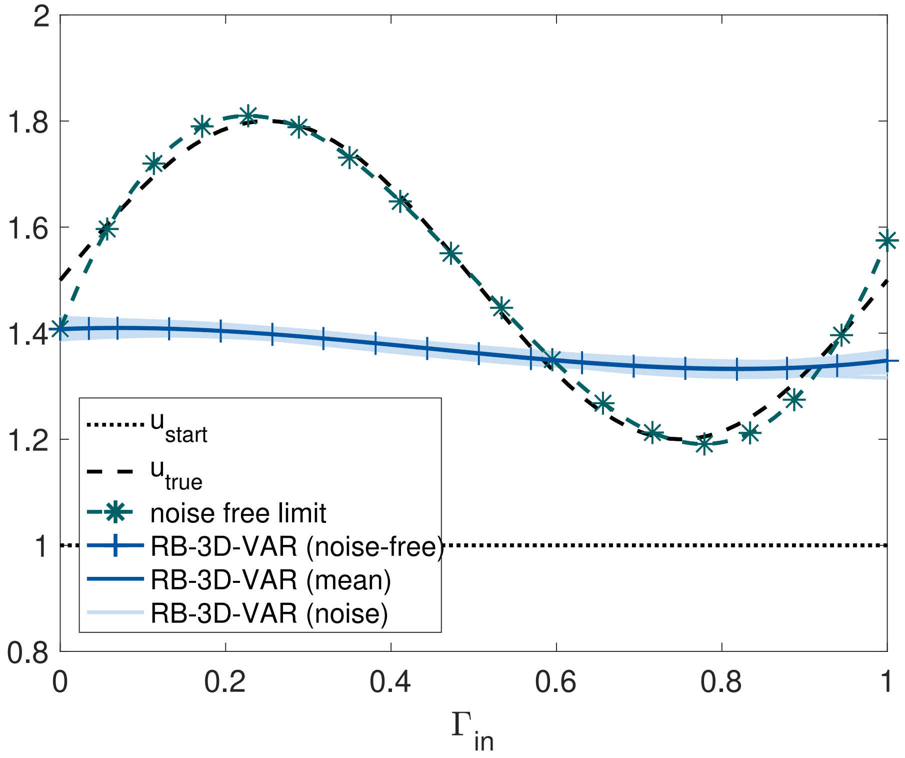
inf-sup constants during measurement selection



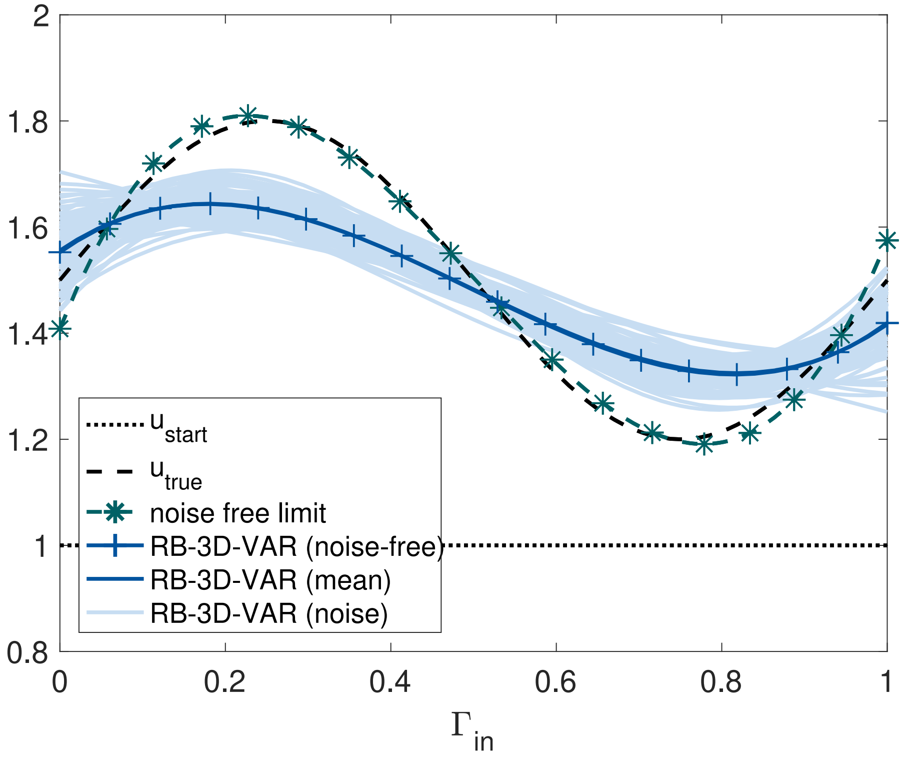
chosen measurement centers

Data Assimilation

3D- VAR model correction



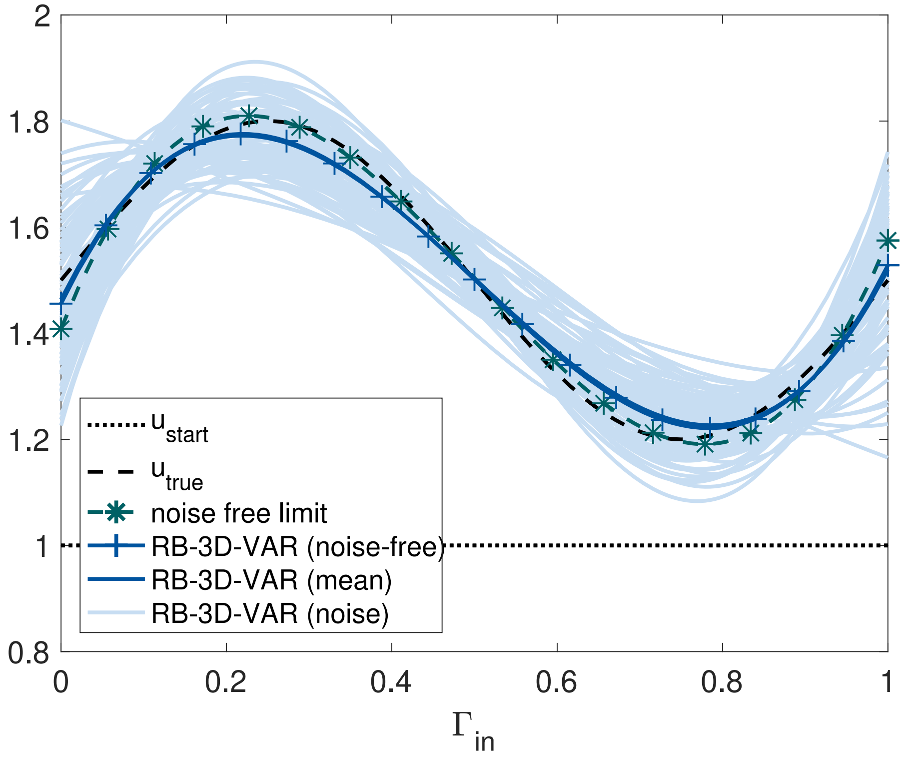
$\lambda = 1$



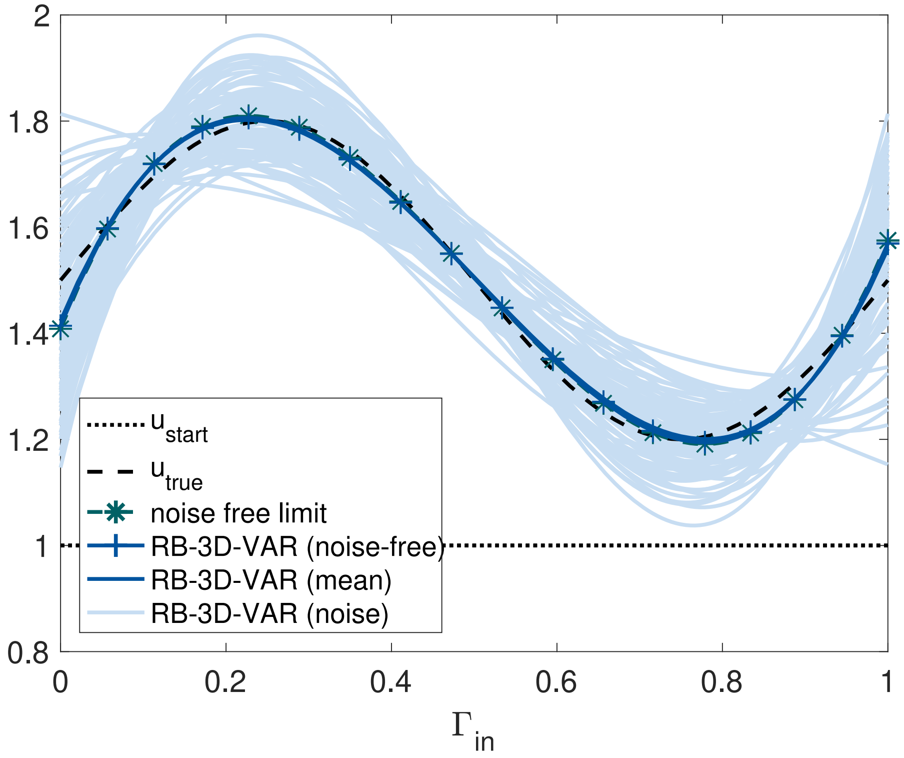
$\lambda = 10$

Data Assimilation

3D- VAR model correction



λ = 100



λ = 1000

Data Assimilation

Reduced Basis Spaces

$$\mathcal{U}_N \longrightarrow \mathcal{Y}_{y,N} \longrightarrow \mathcal{T} \longrightarrow \mathcal{Y}_{p,N} \longrightarrow \mathcal{Y}_N = \mathcal{Y}_{y,N} + \mathcal{Y}_{p,N}$$

Space dimensions:

	$\mathcal{U}$	$\mathcal{Y}_y$	$\mathcal{Y}_y$	$\mathcal{Y}_R$	$\mathcal{T}$
dim	4	64	95	159	16

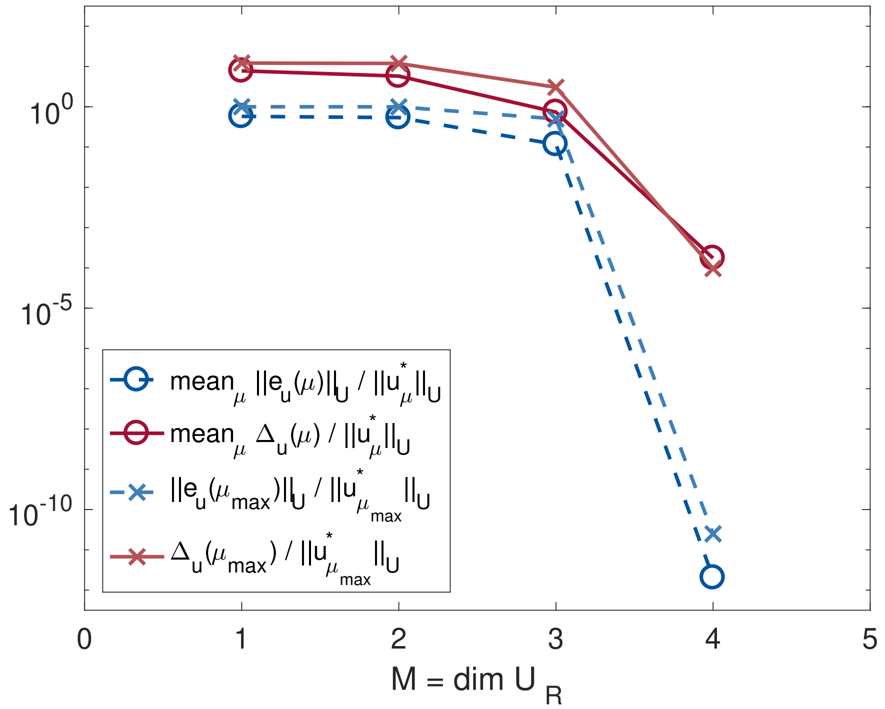
Computational time:

FE-3D-VAR	RB-3D-VAR			speedup
	offline	online	error bound	
7.08 s	463 s	4.2 ms	1.3 ms	1,276

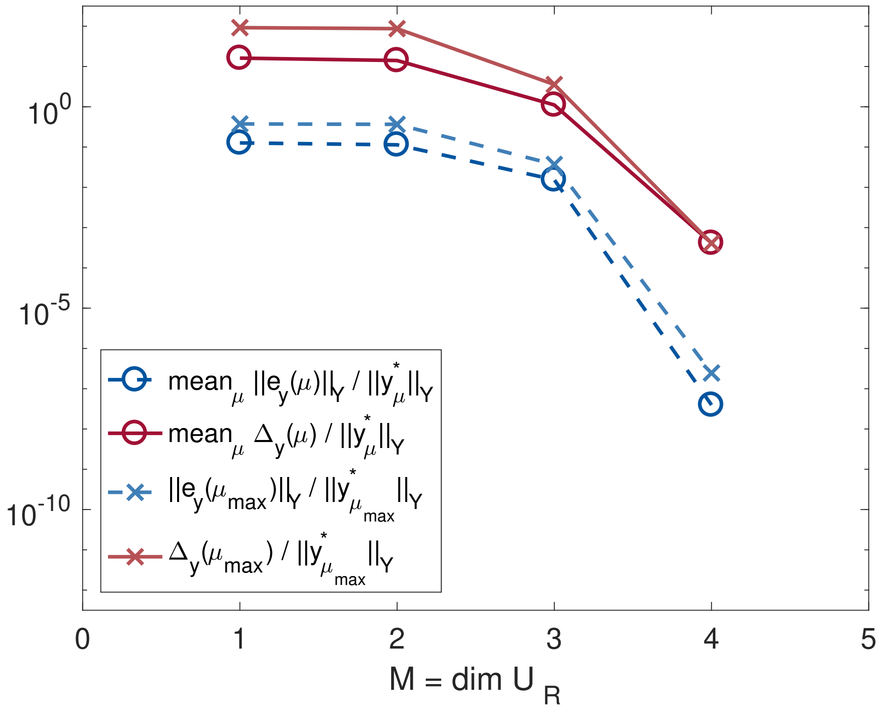
Parameter estimation: roughly 25-28 mins.

Data Assimilation

A Posteriori Error Bounds



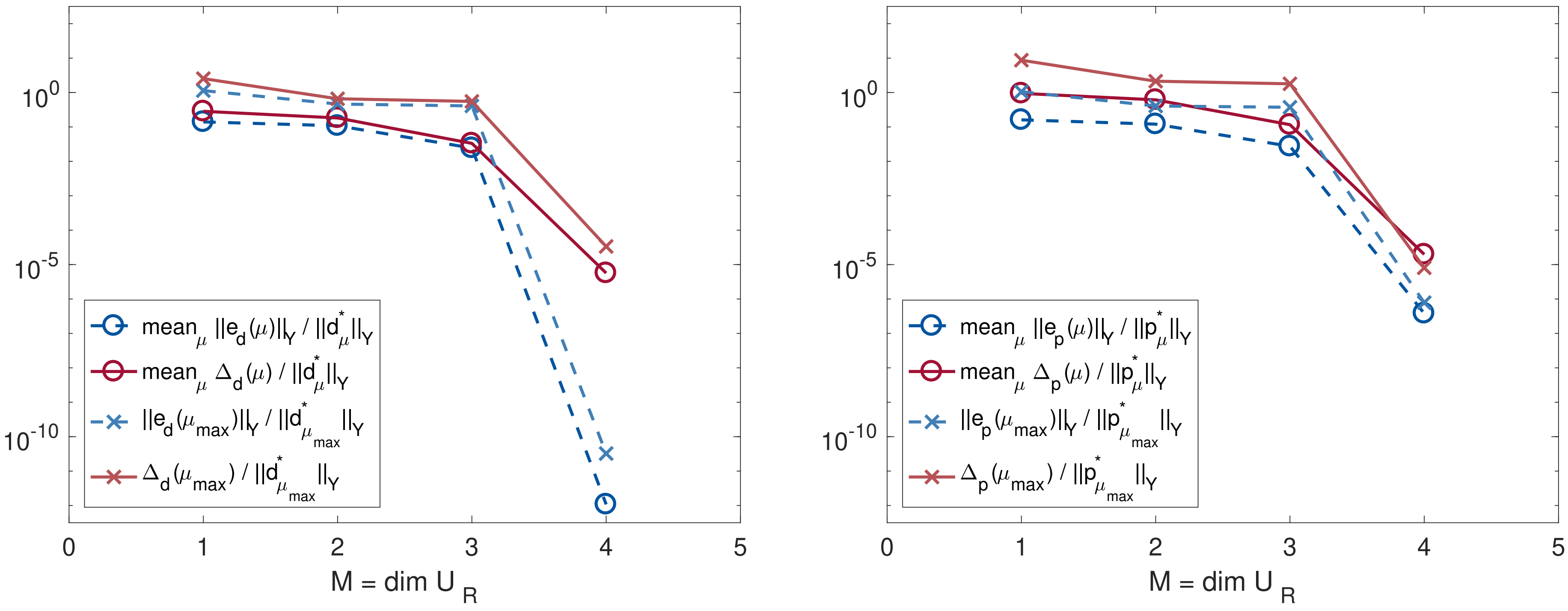
Model Correction



State

Data Assimilation

A Posteriori Error Bounds



Observable Difference

Adjoint

200 random measurements, λ =100.

Summary

We developed a certified RB method for 3D & 4D variational data assimilation

- model order reduction for state, adjoint, and control variables
- a posteriori bounds for error in RB approximation
- determination of unknown parameters
- estimation of model bias

Here, we focused on:

- Selection of measurements through stability-based greedy-OMP algorithm
- Reduce sensitivity to experimental noise
- Step-wise construction of RB spaces
- Application to 3D-VAR

Next steps:

- Extension to 4D-VAR
- Selection of model modifications
- Application to large-scale problems

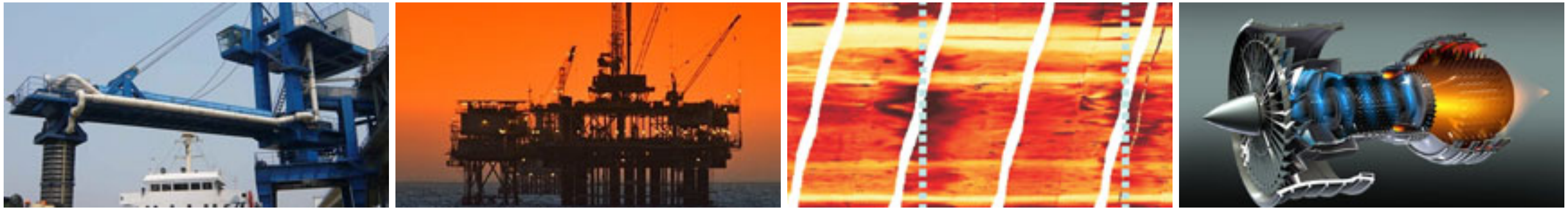
Applications and Future Work

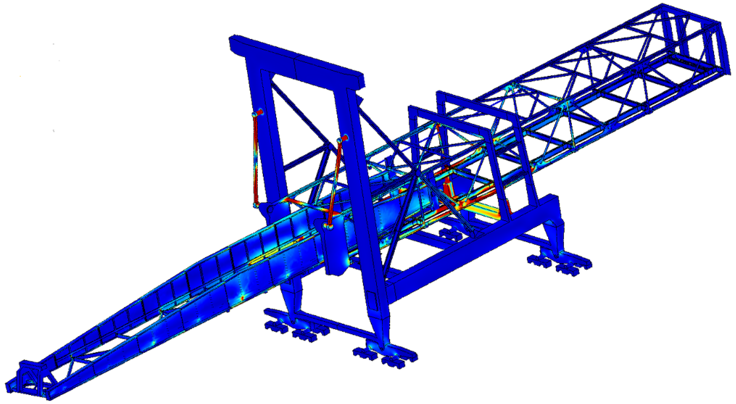
with

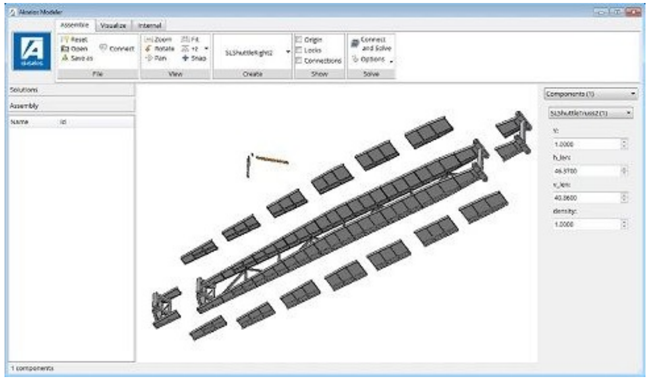
D. Degen, M. Grepl, F. Wellmann (RWTH)

M. Baragona, V. Lavezzo, R. Maessen, Z. Tokoutsi, N. Vaidya (Philips)

Industrial Applications







Y: 1.0000
h_len: 46.5700
v_len: 40.0600
density: 1.0000

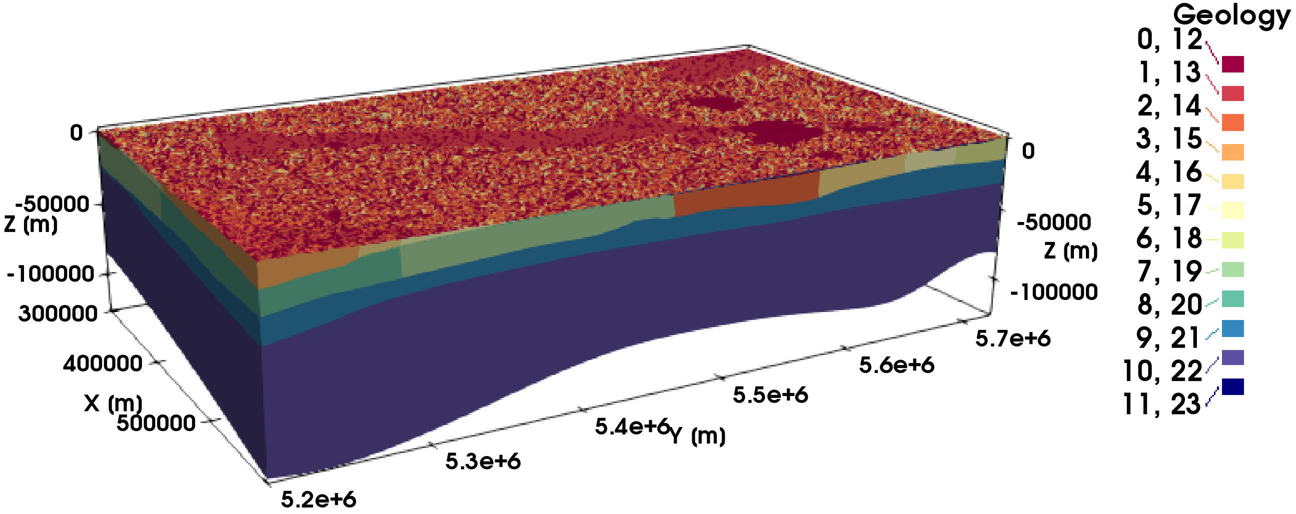
Source: akselos.com

The reduced basis method is useful for the

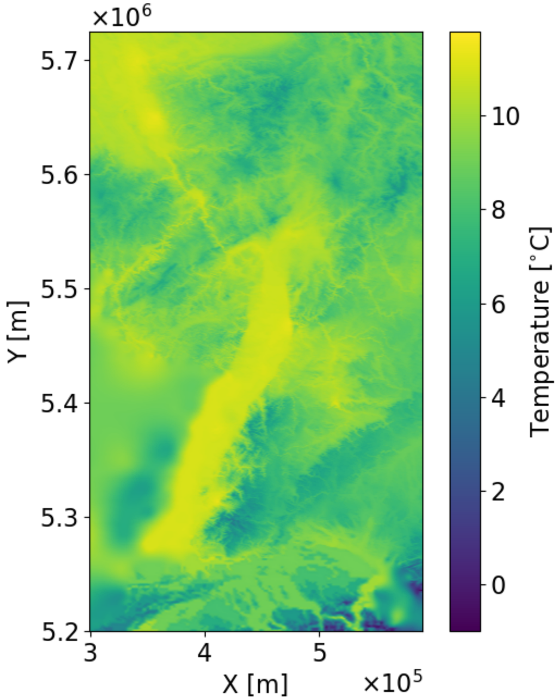
many-query, real-time, and slim-computing contexts.

Application: Geosciences

Upper Rhine Graben (Germany)



Courtesy of Prof. Scheck-Wenderoth, GFZ Postdam.



Upper boundary condition

Model

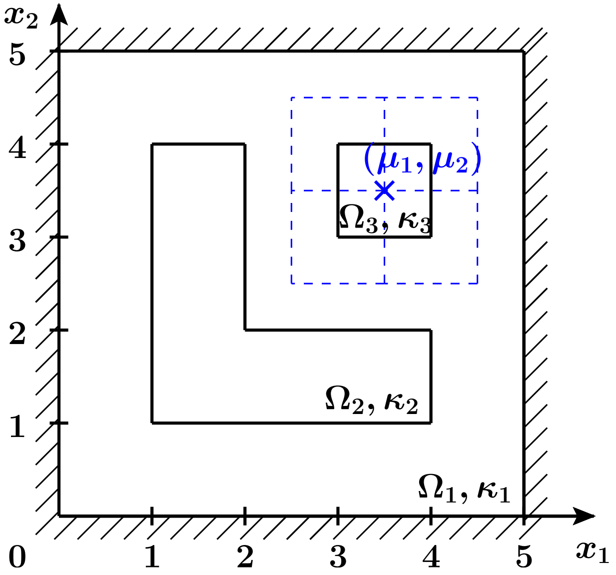
Thermal diffusion with radiogenic heat production

$$\nu\nabla^2 T + S = 0$$

- Parameter Estimation
- Model Calibration
- Inverse Problems
- Sensitivity Analysis
- Data Assimilation

Optimal Control

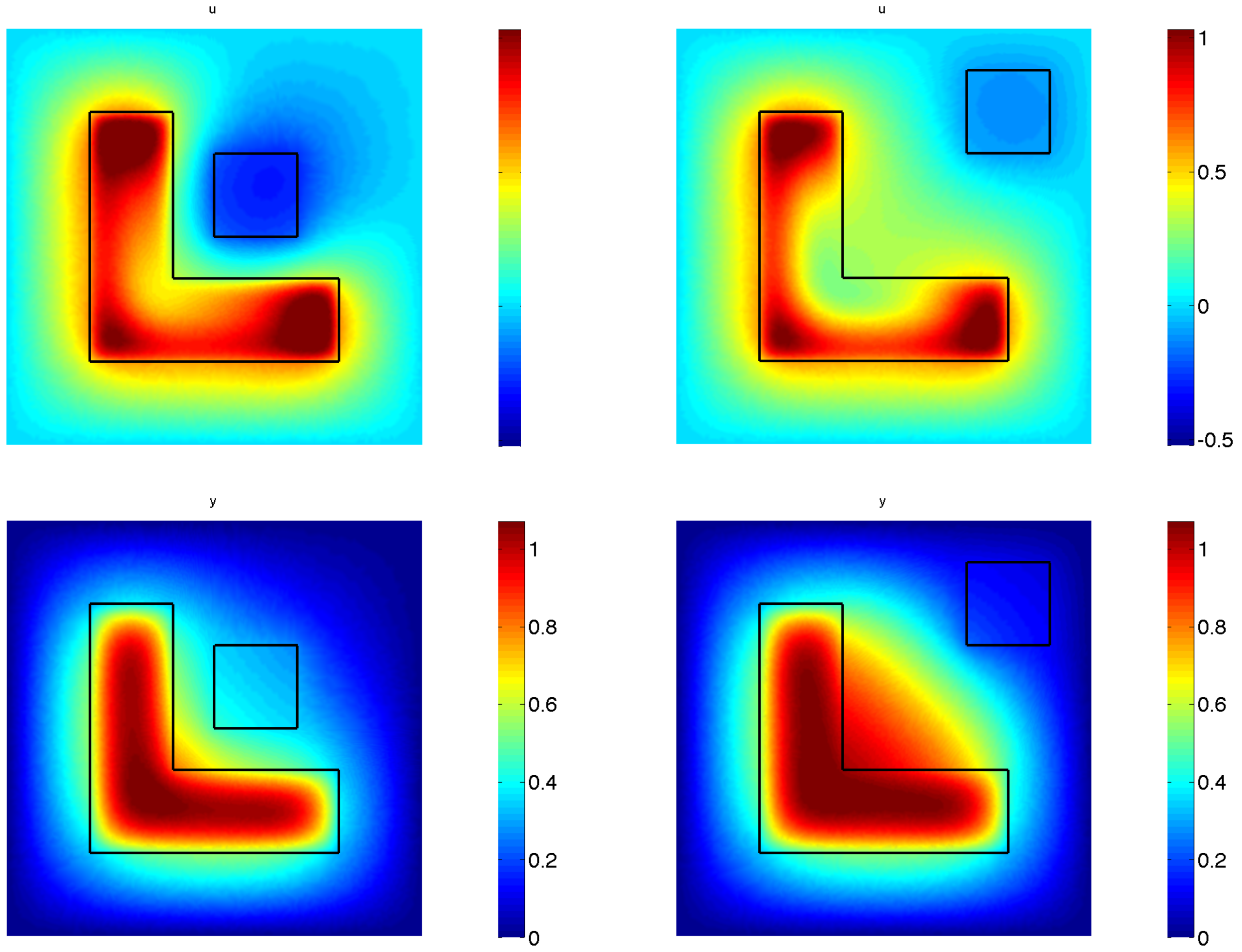
Model Problem



- Steady heat conduction with conductivities $(\kappa_1, \kappa_2, \kappa_3)$
- FE Dimension $\dim(Y) \approx 18.000$
- State $y_d = 1$ in Ω_2 and $y_d = 0$ in Ω_3,
- Regularization parameter: $\lambda = 10$
- Input parameter: $\mu = (\mu_1, \mu_2) \in \mathcal{D} \equiv [3,4]^2$.

Optimal Control

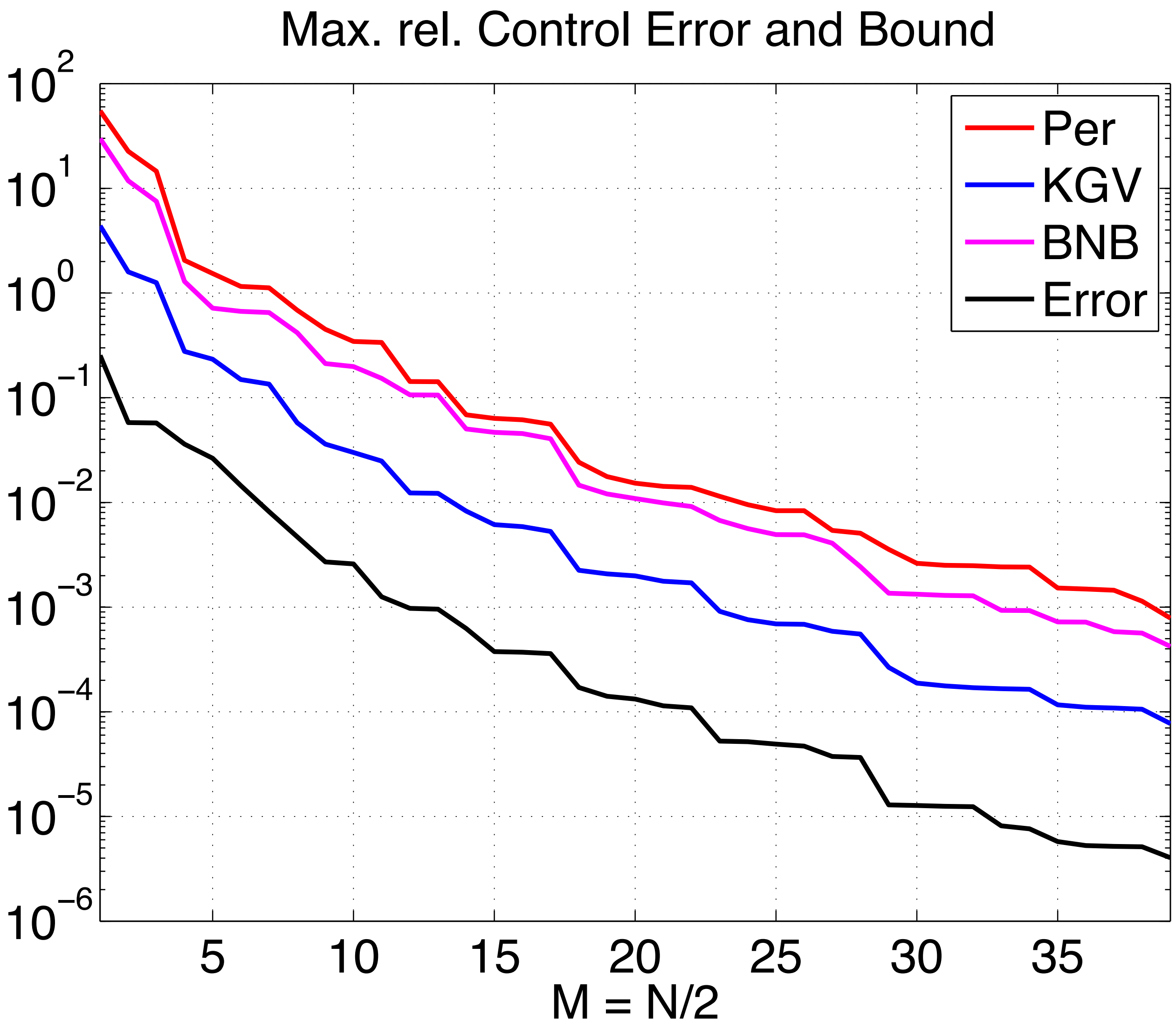
Sample Solutions $(\lambda = 10)$



control

state

Optimal Control



Timings:

- $t_{\mathrm{FE}} = 1.23s$
- $t_{\mathrm{RB}} \in [1.2, 4.8]\mathrm{ms}$
- $t_{\mathrm{RB},\Delta} \in [2, 7]\mathrm{ms}$

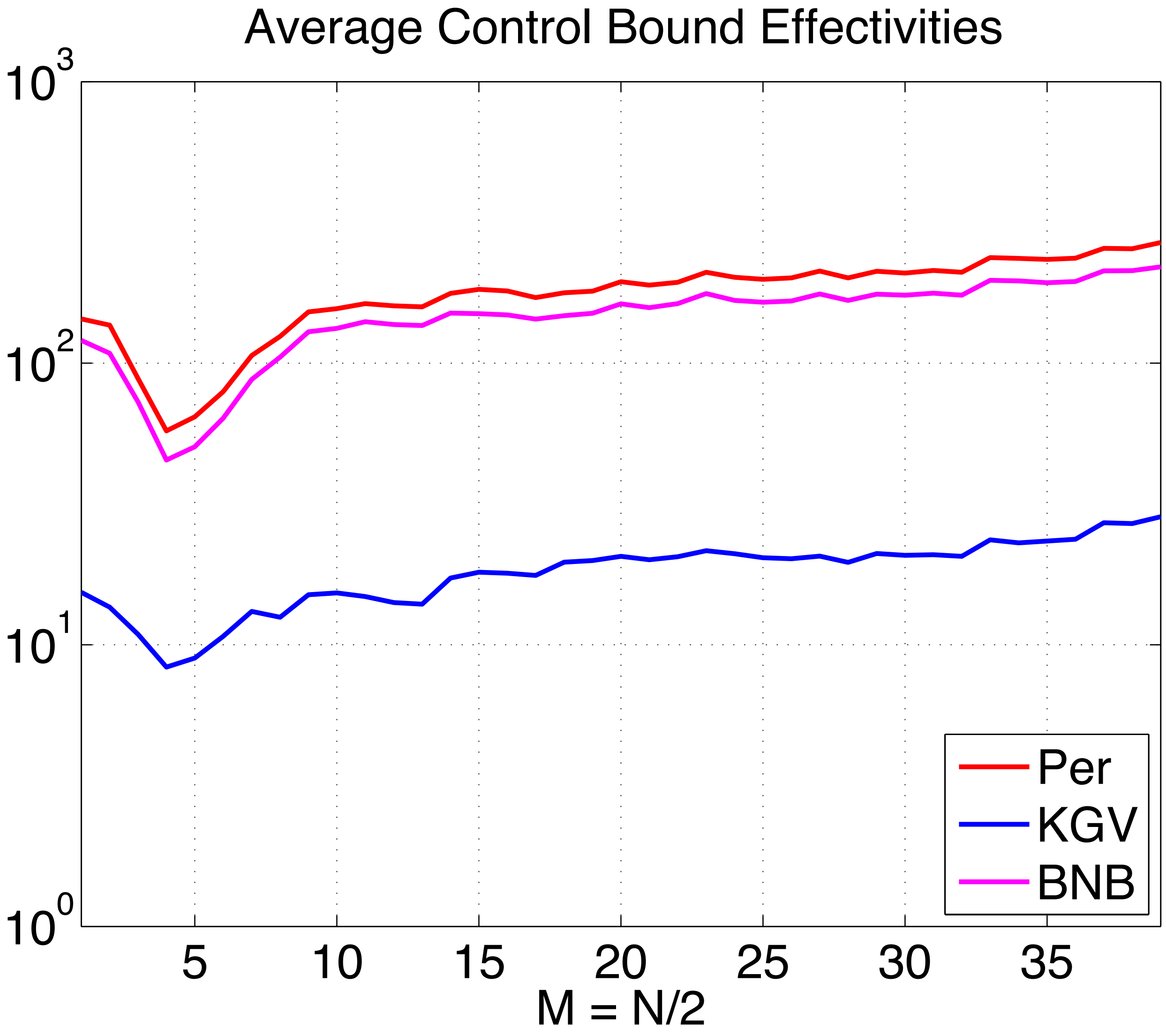
Speedups:

- RB: 256-1025
- RB+Bound: 176-615

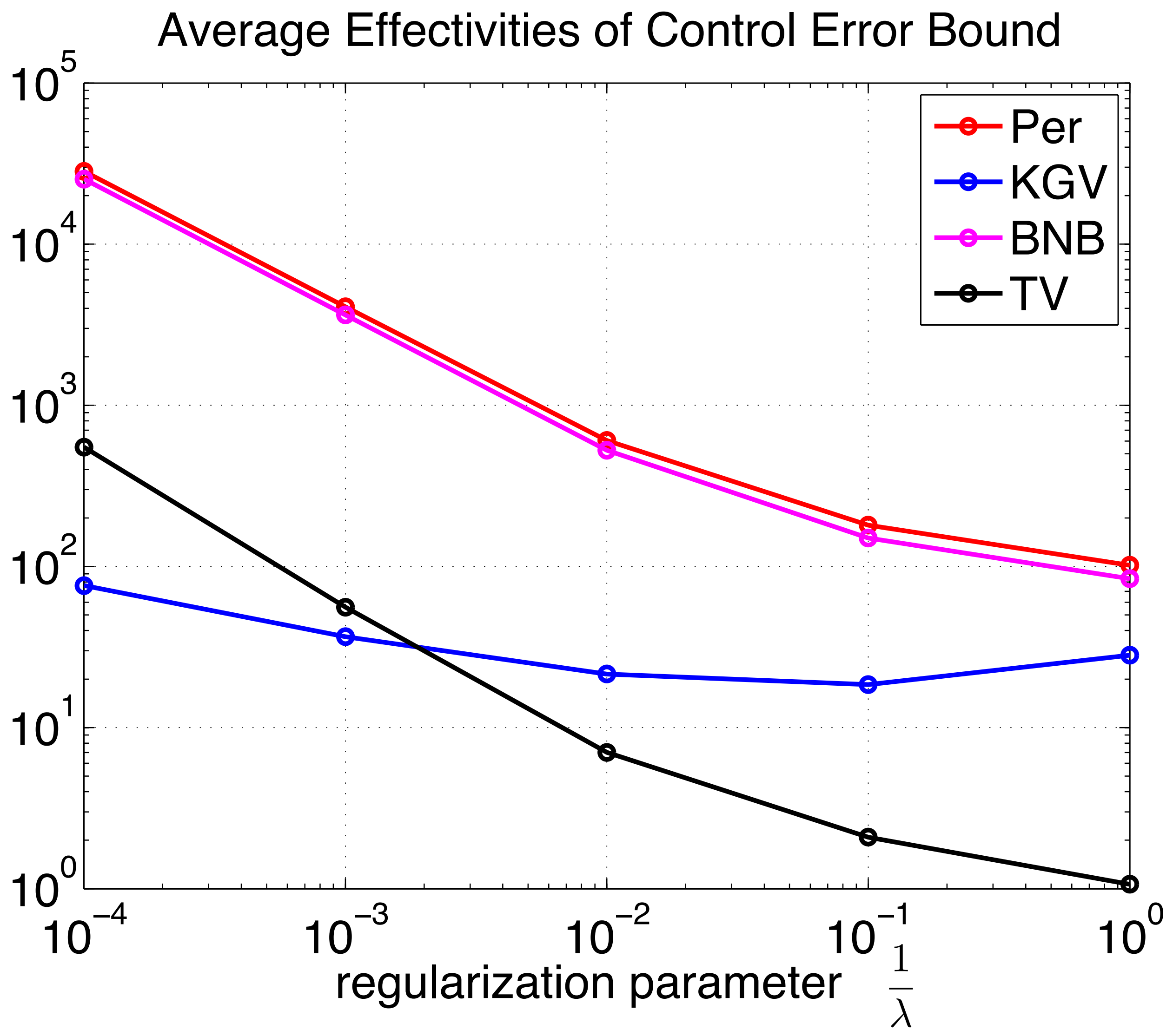
Test set:

- $|\Xi_{\mathrm{test}}| = 20$

Optimal Control



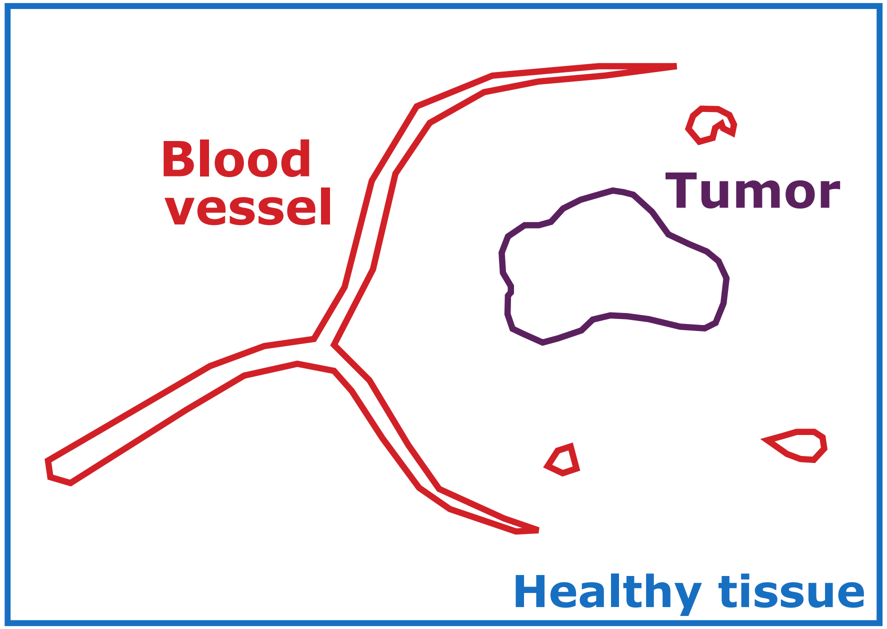
Optimal Control



Application: Cancer Treatment

Motivation: Thermal Ablation Treatment Planning

- **Thermal Ablation:** destroy target tissue by increasing temperature above threshold. [Chu and Dupuy 2014]



Application: Cancer Treatment

Collaborators:

- **M. Grepl**, IGPM, RWTH University Aachen
- **Z. Tokoutsi**, Philips Research Eindhoven
- **M. Baragona**, Philips Research Eindhoven
- **R. Maessen**, Philips Research Eindhoven

Funding:

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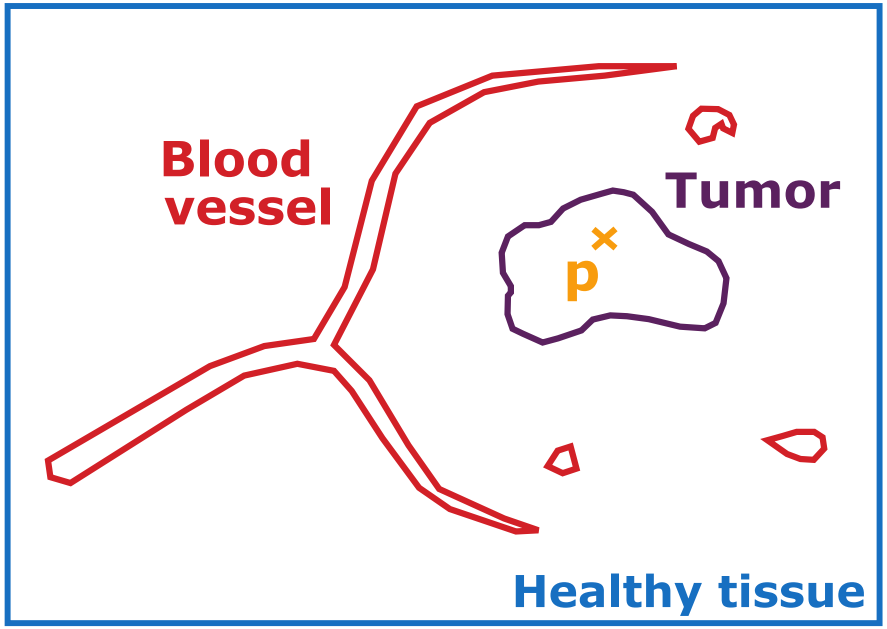
- **Treatment Planning:**
  - Determine placement parameters for ablation probes.
  - Determine device power settings.



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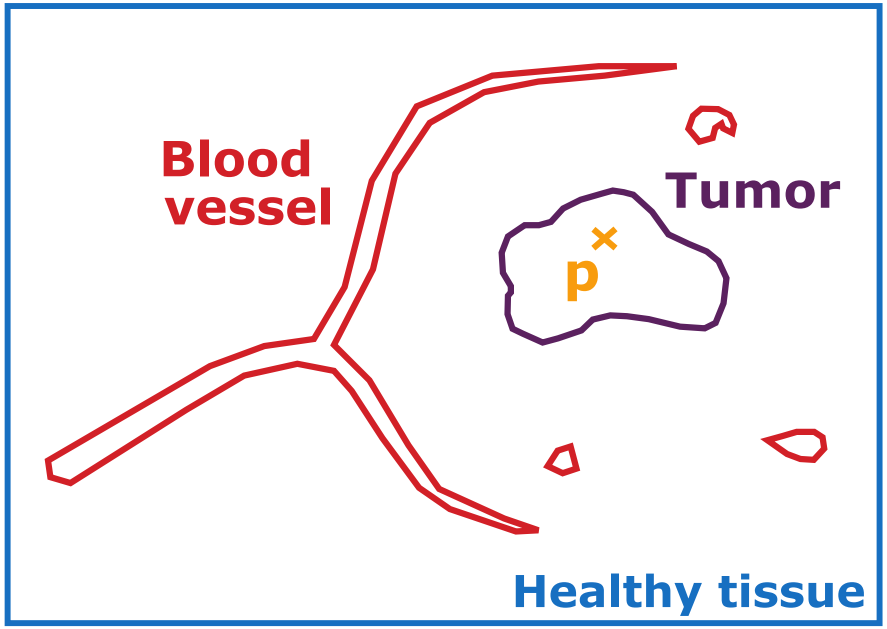

Bioheat Equation [Pennes 1948]

Heat Diffusion in living tissue following the Pennes Bioheat model

$$\begin{aligned} -k\Delta T + \rho C w(T - T_{\text{core}}) &= Q, \qquad \text{in } \Omega \\ k\nabla_\nu T + h(T - T_{\text{core}}) &= 0 \qquad \text{on } \Gamma \end{aligned}$$

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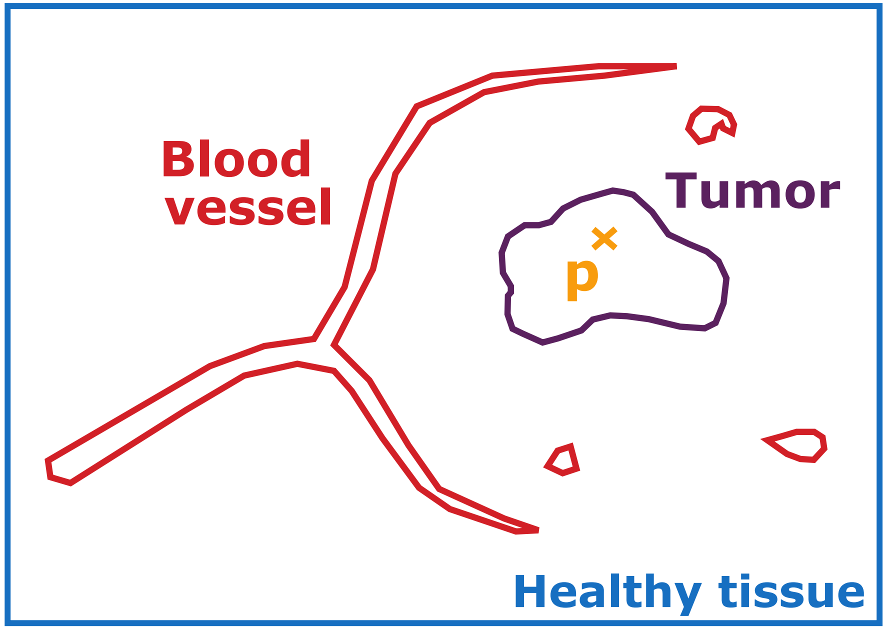


Bioheat Equation [Pennes 1948]

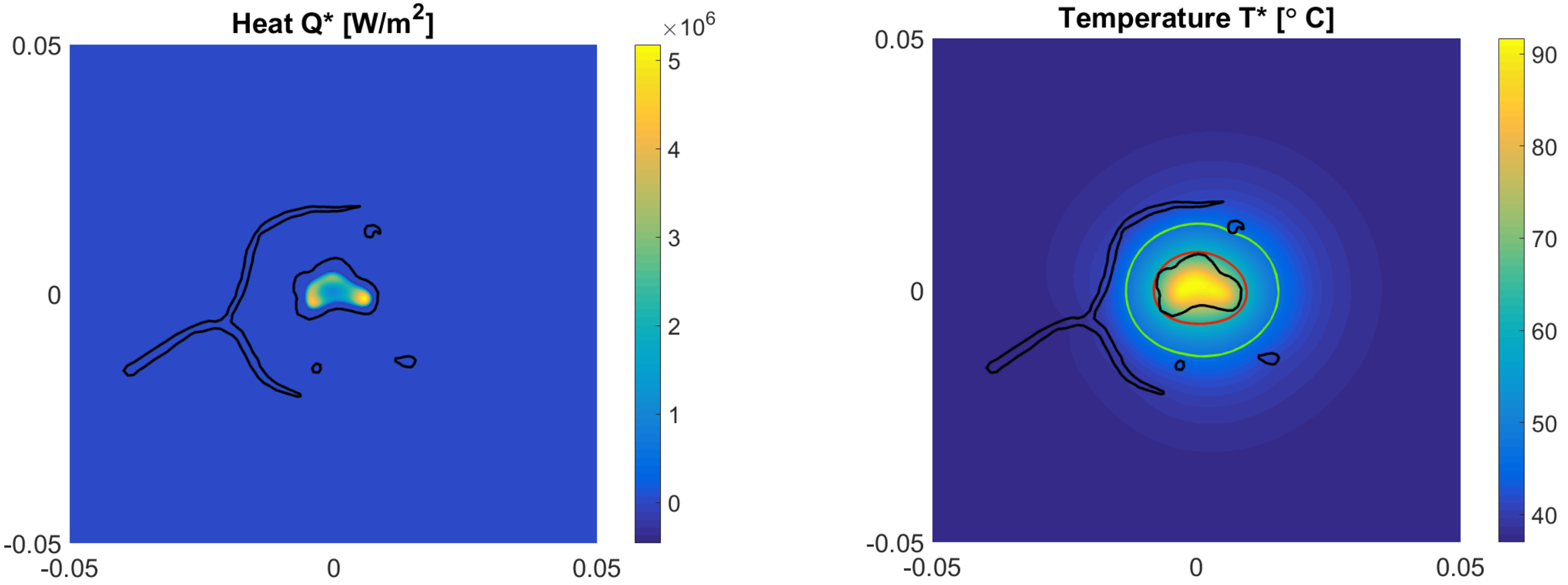
Non-dimensional Bioheat Equation

$$\begin{aligned} -k\Delta y + cy &= u, && \text{in } \Omega \\ k\nabla_\nu y + hy &= 0 && \text{on } \Gamma \end{aligned}$$

Application: Cancer Treatment

Part I: Optimal Heat Source

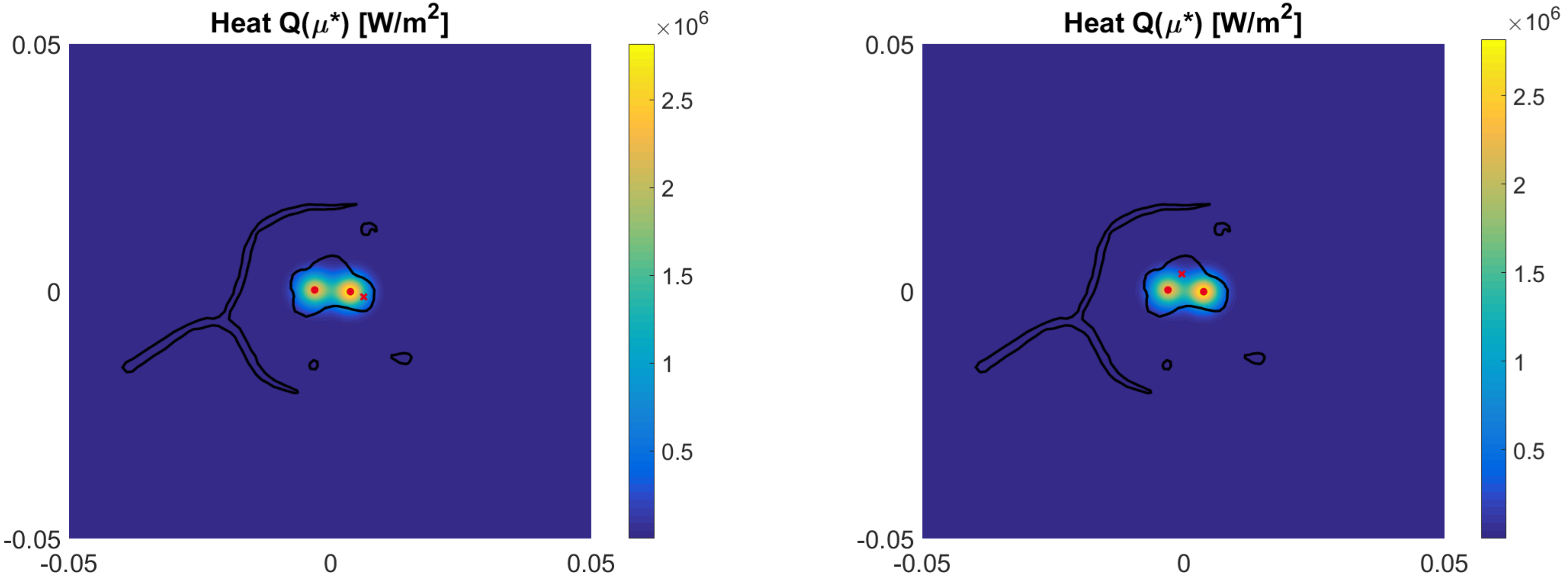
$$\min_{u \in U_{ad}} J_{\text{heat}}(y, u; \mu) := \sum_{i=1}^{3} \frac{\lambda_i}{2} \|y - y_d\|^2_{L^2(\Omega_i)} + \frac{\lambda}{2} \|u\|^2_{L^2(\Omega)}$$



Application: Cancer Treatment

Part II: Optimize Placement and Power

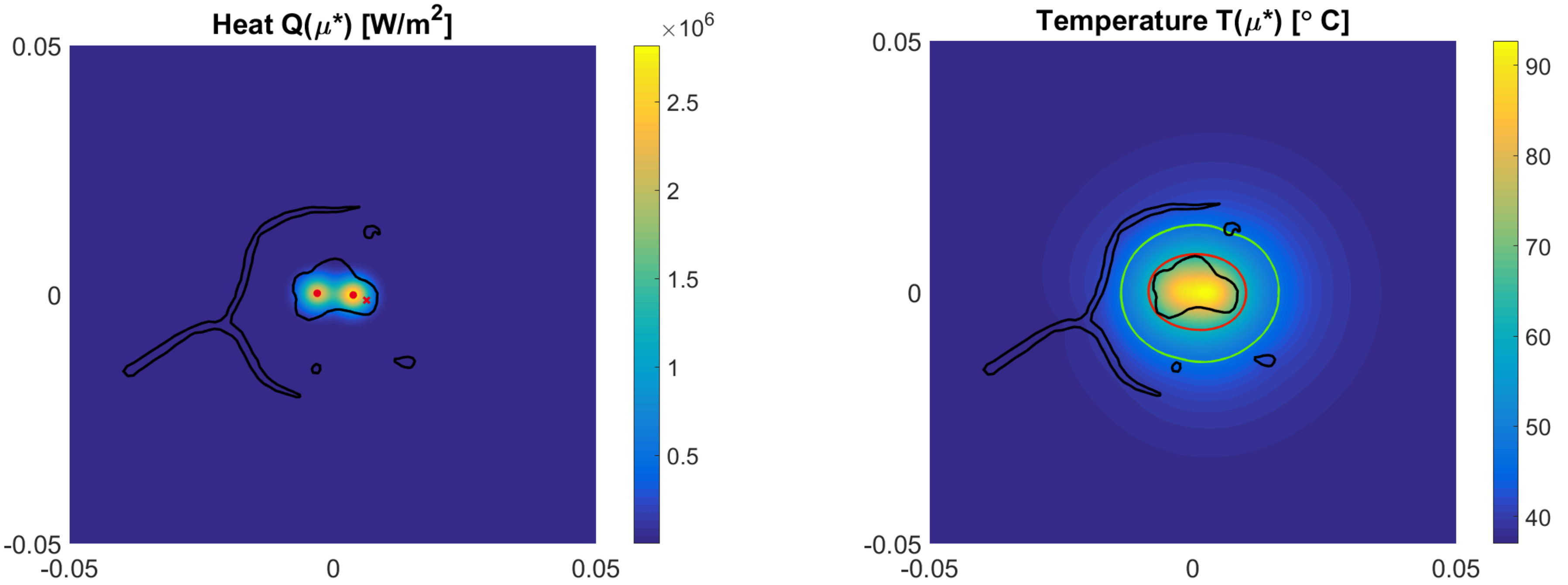
$$\min_{(p,P)} J_{\text{plac}}(p,P) := \frac{1}{2}\|Q_G(x;p,P) - u^*(x)\|^2_{L^2(\Omega)}.$$



Application: Cancer Treatment

Part III: Power Control

$$\min_{P \geq 0} J_{\text{power}}(P_1, \cdots, P_{n_P}) := \sum_{i=1}^{3} \frac{\lambda_i}{2} \left\| y - y_d \right\|^2_{L^2(\Omega_i)}$$



Application: Cancer Treatment

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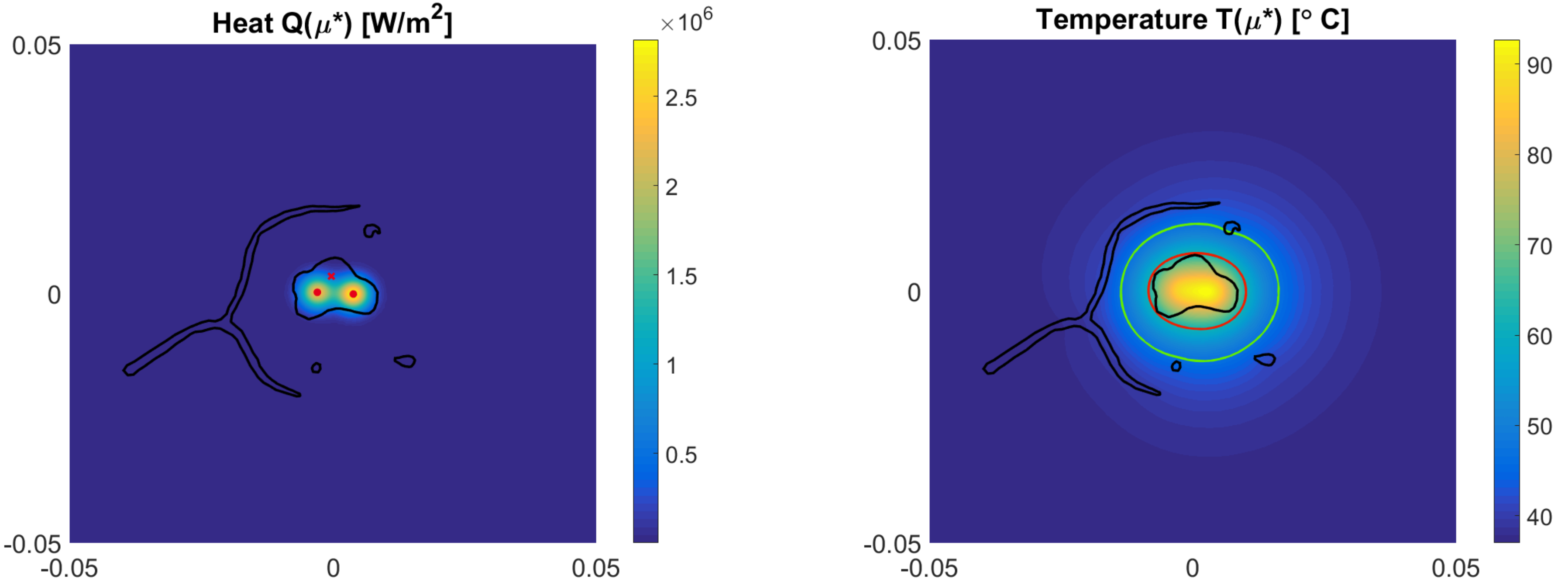


Application: Cancer Treatment

The RB Method for Real-Time Updates:

- Adjust regularization weights
- Adjust models with patient specific parameters
- Update solution w.r.t. geometric parameters
  - shifted tumor location
  - power control w.r.t. final probe placement

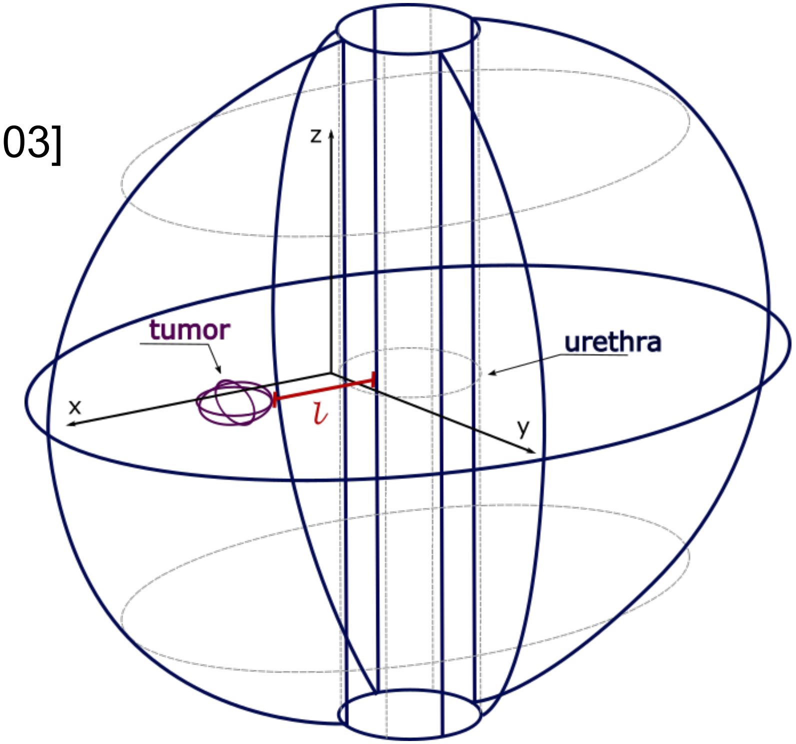
Application: Cancer Treatment

Problem Set Up

Bioheat Equation:

Heat Diffusion in living tissue following the Pennes Bioheat model [Pennes 1948], [Davidson and Sherar 2003]

$$\begin{aligned} -k\Delta y + cy &= u, && \text{in } \Omega(l) \\ k\nabla_\nu y + \mathrm{Bi}(y - y_{\mathrm{cool}}) &= 0 && \text{on } \Gamma_C \\ y &= 0, && \text{on } \Gamma_D \end{aligned}$$



Domain and mesh were created using Gmsh [Geuzaine and Remacle 2009]

Application: Cancer Treatment

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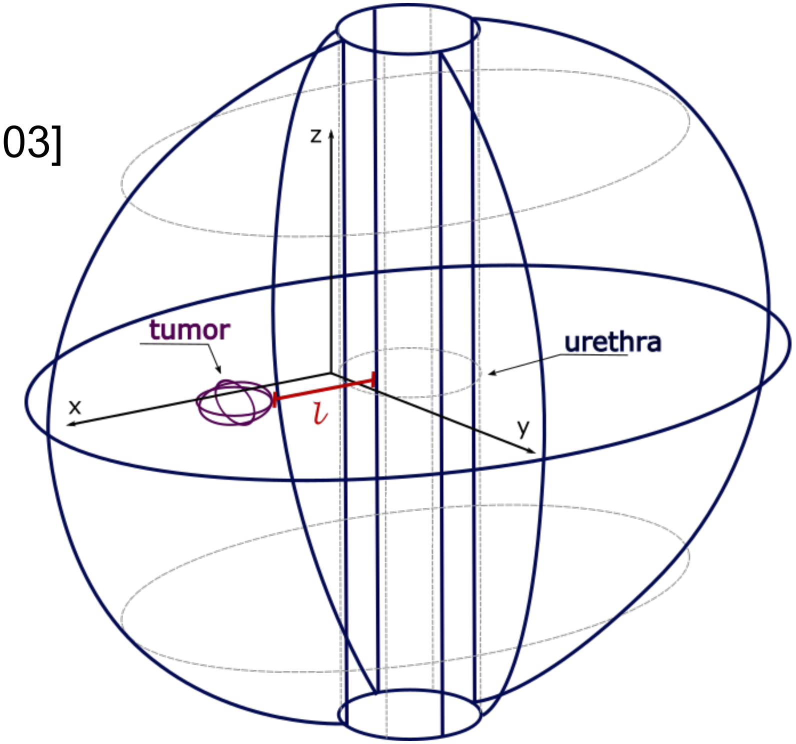


Parametrization:

- Blood perfusion rate c
- Distance from urethra l
- cooling temperature y_{cool}

Domain and mesh were created using Gmsh [Geuzaine and Remacle 2009]

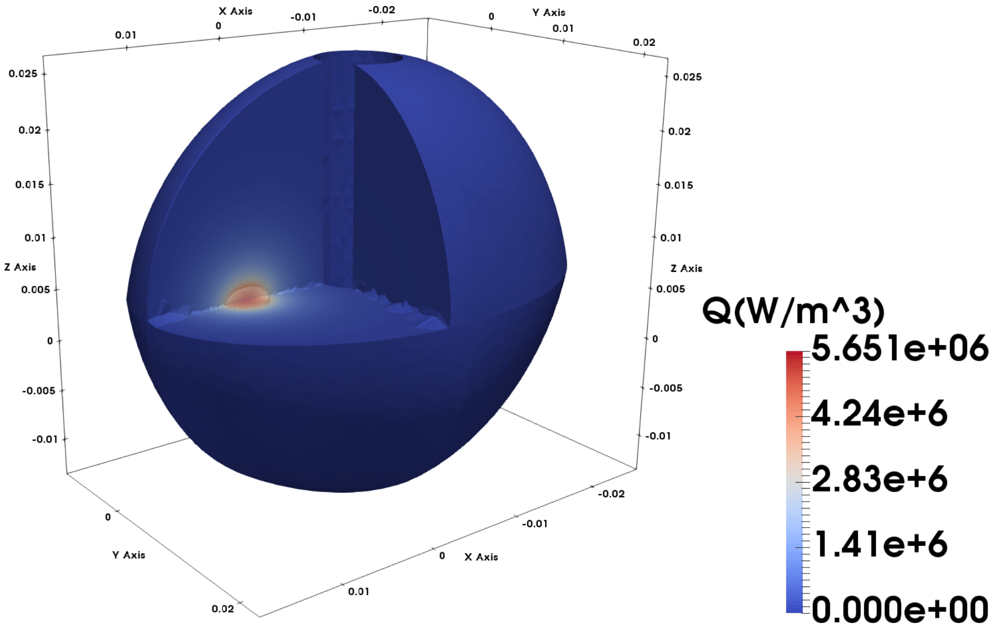
Application: Cancer Treatment

Optimal Control Problem

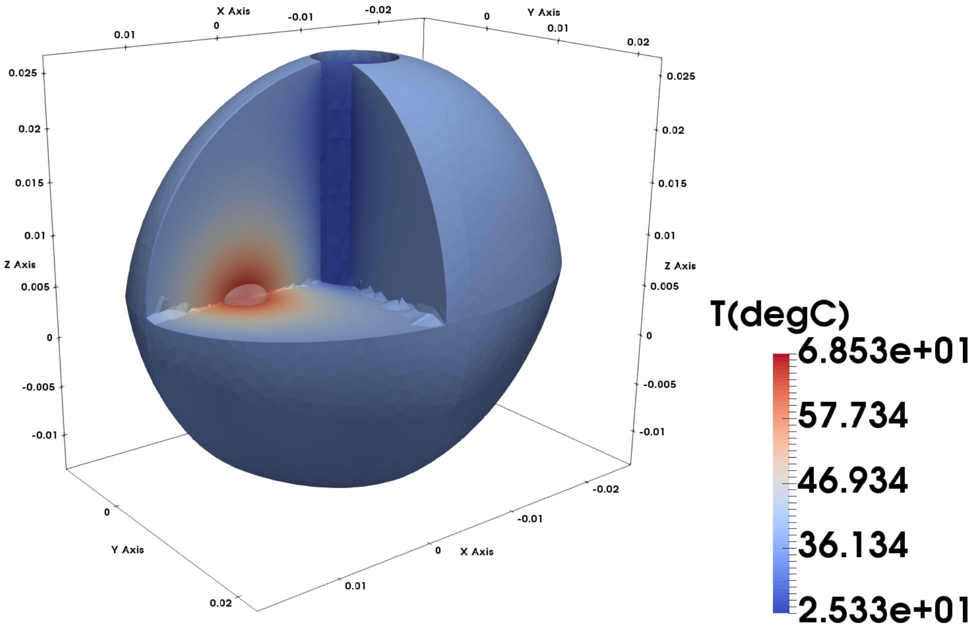
For any $\mu \in \mathcal{D}$ solve

$$\min_{y \in Y, u \in U} J_{\text{heat}}(y, u; \mu) = \frac{1}{2} |y - y_d|^2_{D(\mu)} + \frac{\lambda}{2} \|u\|^2_{U(\mu)}$$

Optimal Heat u^*



Optimal Temperature y^*



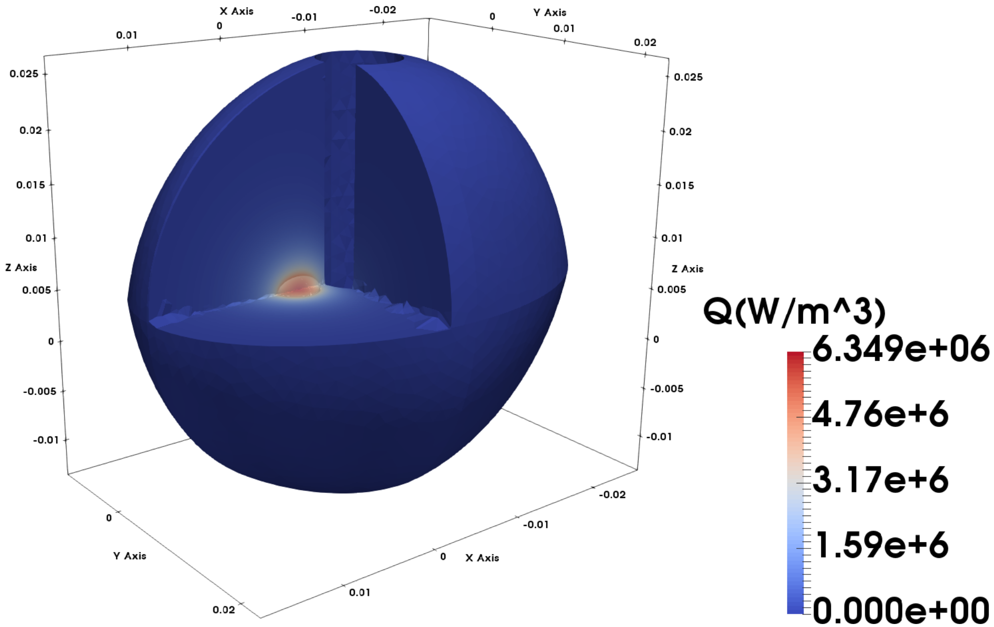
Application: Cancer Treatment

Optimal Control Problem

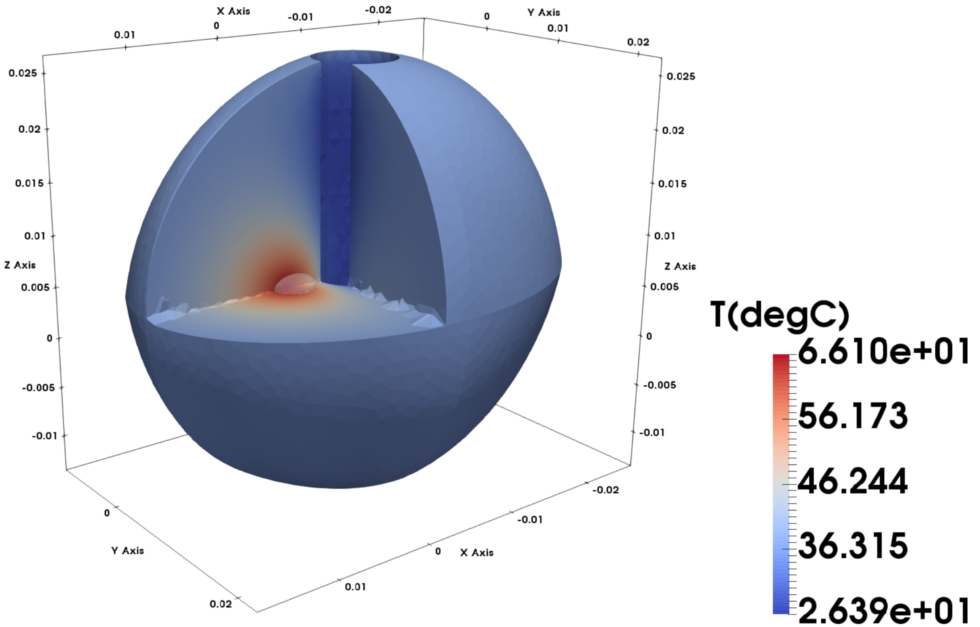
For any $\mu \in \mathcal{D}$ solve

$$\min_{y \in Y, u \in U} J_{\text{heat}}(y, u; \mu) = \frac{1}{2} |y - y_d|^2_{D(\mu)} + \frac{\lambda}{2} \|u\|^2_{U(\mu)}$$

Optimal Heat u^*

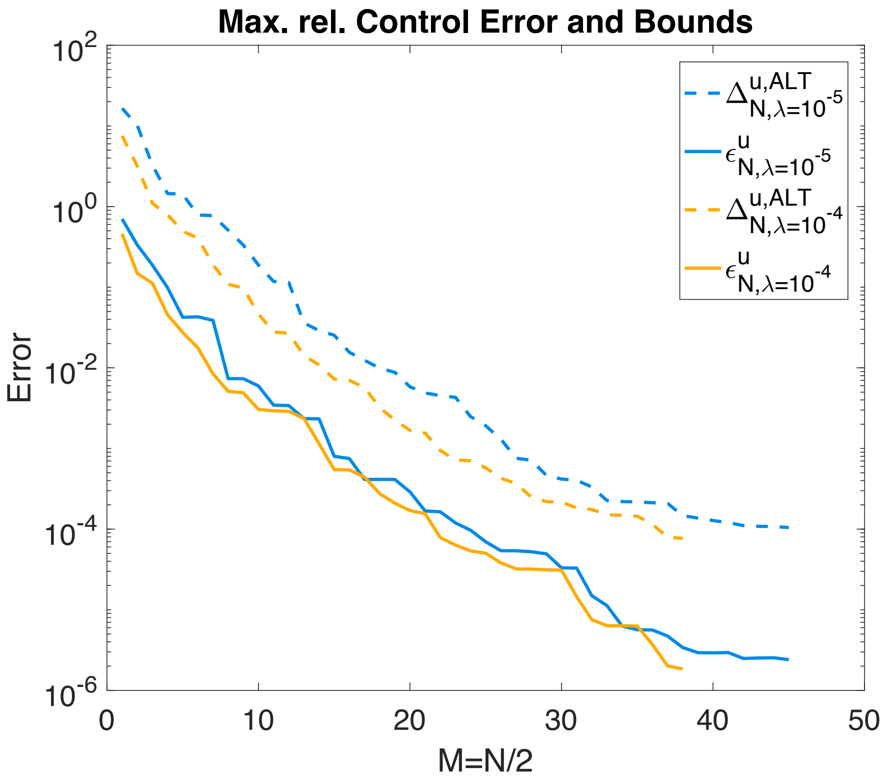


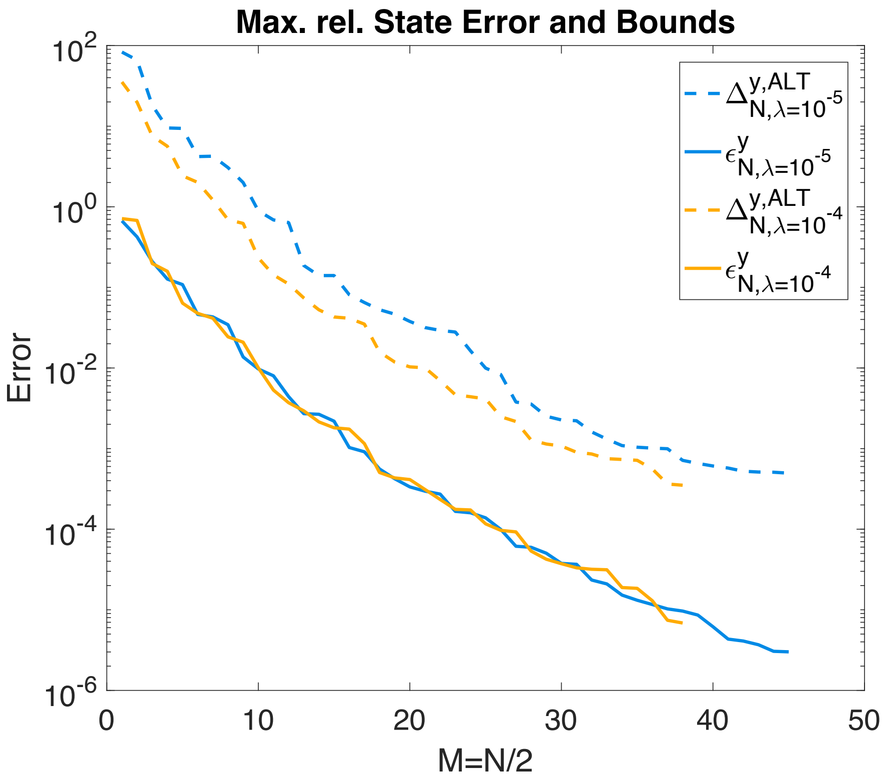
Optimal Temperature y^*



Application: Cancer Treatment

Error and Bounds





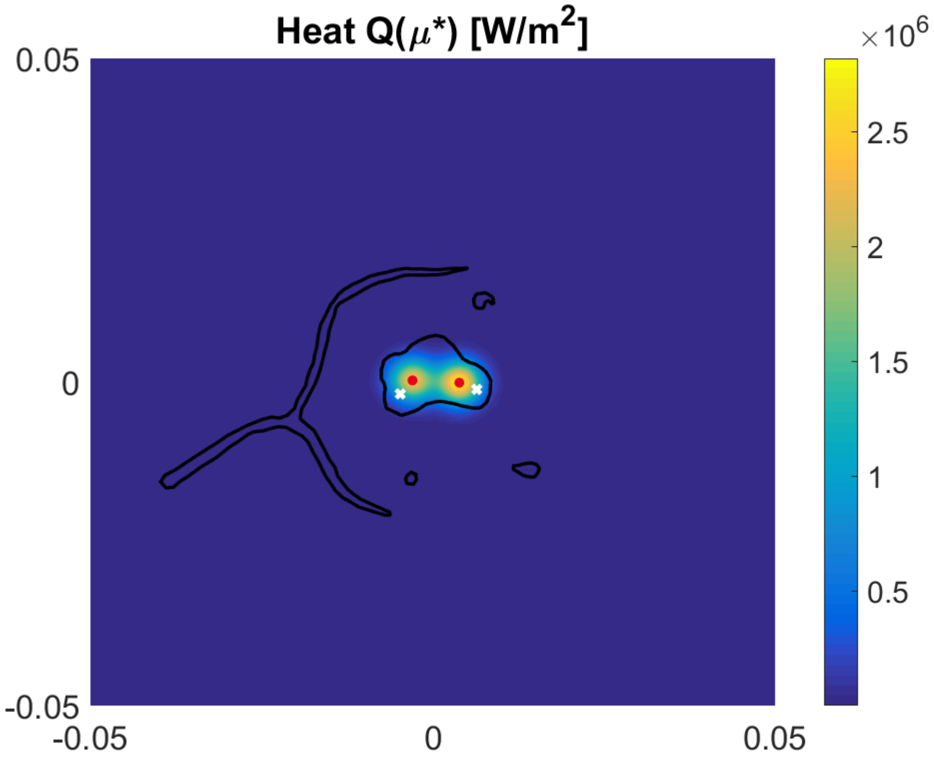
λ	FEM	RB
10^{-4}	400s	$[0.3, 1.6]$ms
10^{-5}	500s	$[0.3, 1.4]$ms

[T. et al. 2018]

Application: Cancer Treatment

Motivation: Non-Affine Problems

Can we update the device power control in real time for different ablation probe placements?



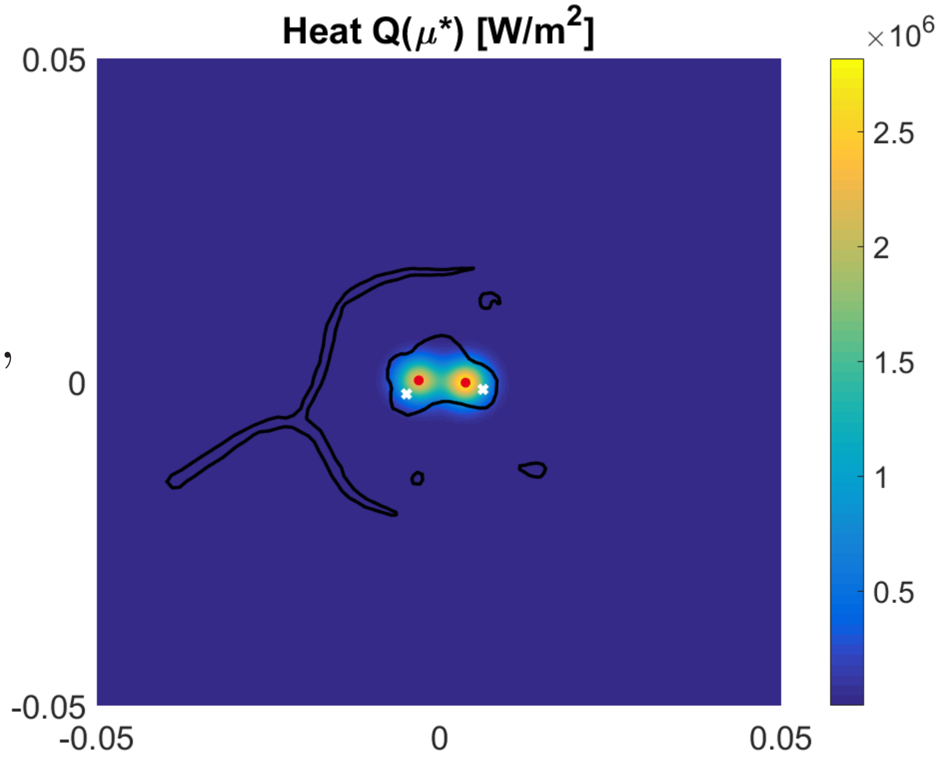
Application: Cancer Treatment

Motivation: Non-Affine Problems

Can we update the device power control in real time for different ablation probe placements?

For any $\mu \in \mathcal{D}$

$$\min_{y \in Y, P \in \mathbb{R}^{n_P}} J_{\text{power}}(y, P; \mu) := \frac{1}{2} \left| y - y_d \right|_D^2 + \frac{\lambda}{2} \| P - P_d \|_2^2,$$



Application: Cancer Treatment

Motivation: Non-Affine Problems

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For any $\mu \in \mathcal{D}$

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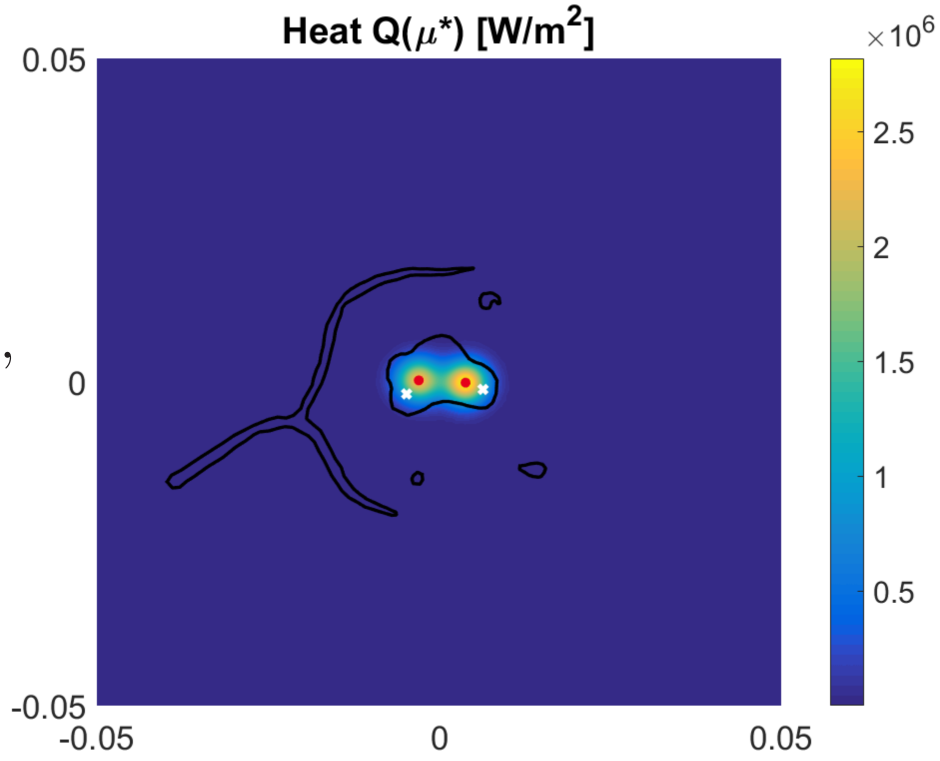
$\text{s.t.} (y, u) \in Y \times \mathbb{R}^{n_P}$ solves

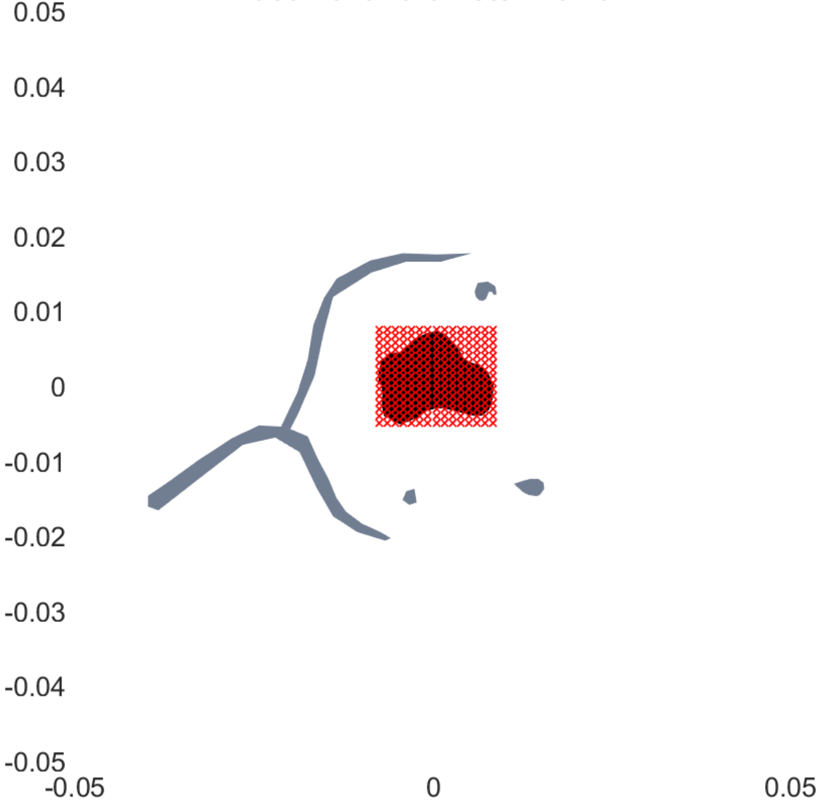
$$-k\Delta y + cy = \sum_{i=1}^{n_P} P_i \, g(x; p_i), \quad \text{in } \Omega$$

$$k\nabla_\nu y + hy = 0, \quad \text{on } \Gamma$$

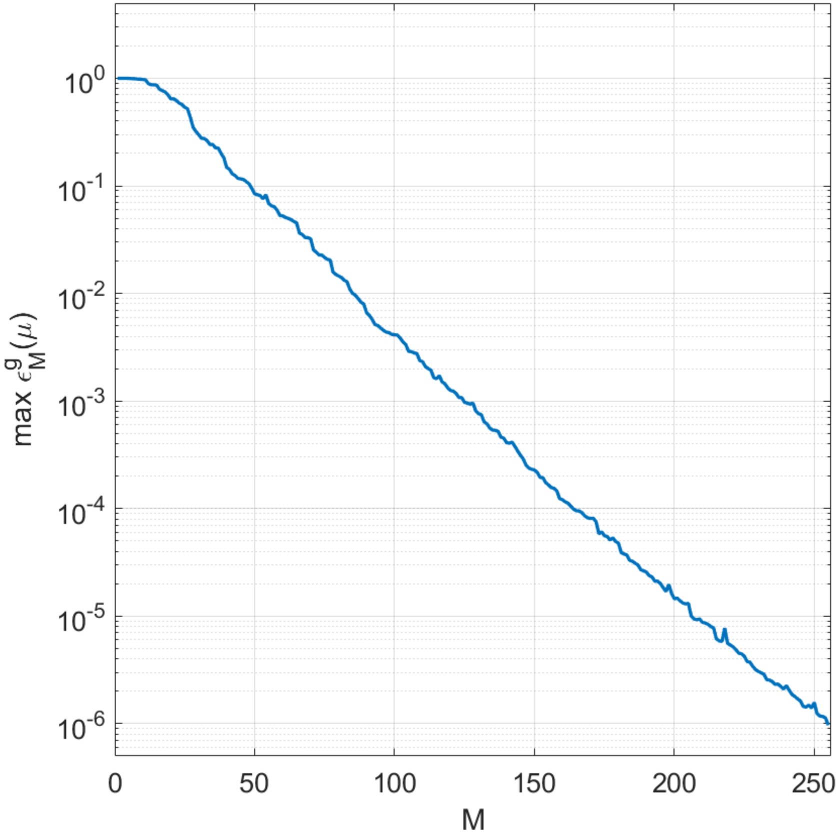


Application: Cancer Treatment

EIM



Placement Parameter Domain

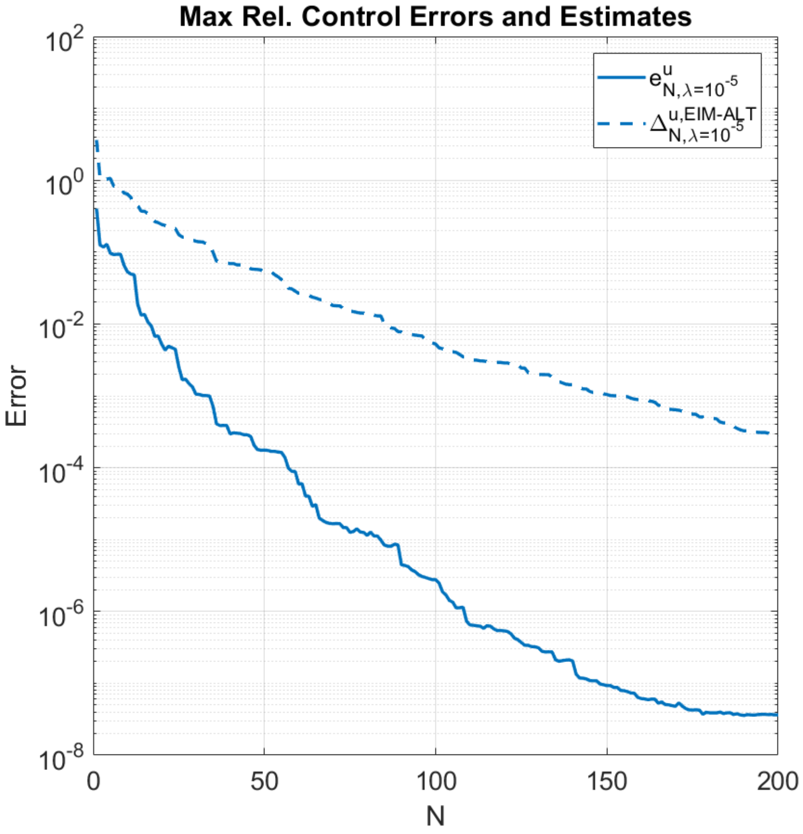


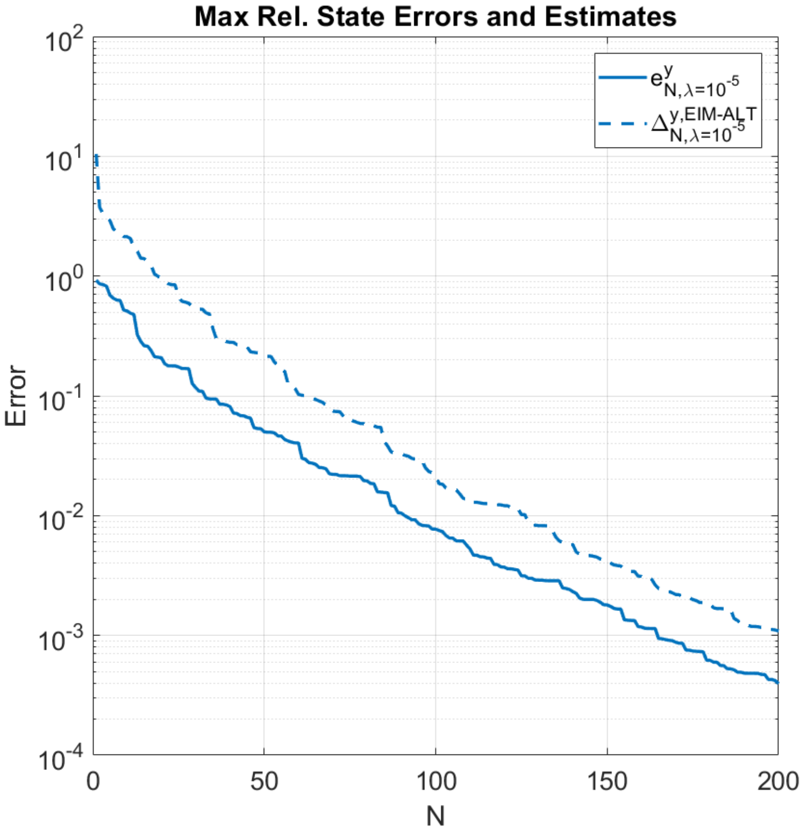
EIM Greedy Sampling Error

[Barrault, Maday, Nguyen, Patera 2004]

Application: Cancer Treatment

Relative Errors and Error Estimates





EIM tol	M	FE time	RB time
$1e-6$	256	1.1 s	0.1 – 10 ms

Outlook and Conclusions

Overview:

- Introduction to treatment planning problems.
- Discussed an algorithm for thermal treatment planning.
- Motivated the need for real-time responsive simulations.
- Applied the reduced basis method and reviewed test cases.

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Ongoing Work:

- Apply our work to model reduction for time dependent power control.

Outlook and Conclusions

Literature

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Discussion

Questions or comments?

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