Approximation of eigenvalue problems in mixed form — Part 1

Daniele Boffi

Dipartimento di Matematica “F. Casorati”, Università di Pavia
`http://www-dimat.unipv.it/boffi`

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Some initial computations

1D Laplacian

$$\begin{cases} -u''(t) = \lambda u(t) & t \in [0, \pi] \\ u(0) = u(\pi) = 0 \end{cases}$$

Find $\lambda \in \mathbb{R}$ and non-vanishing $u \in H_0^1(0, \pi)$ such that

$$\int_0^\pi u'(t)v'(t)\,dt = \lambda \int_0^\pi u(t)v(t)\,dt \quad \forall v \in H_0^1(0, \pi)$$

Exact solution:

$\lambda_k = k^2$
$u_k(t) = \sin(kt)$ $\qquad (k = 1, 2, 3, \dots)$

Discretization

Conforming approximation $V_h \subset V = H^1_0(0,\pi)$

Find $\lambda_h \in \mathbb{R}$ and non-vanishing $u_h \in V_h$ such that

$$\int_0^\pi u_h'(t)v'(t)\,dt = \lambda_h \int_0^\pi u_h(t)v(t)\,dt \quad \forall v \in V_h$$

$$\boxed{Ax = \lambda Mx}$$

Approximation with p/w linear finite elements

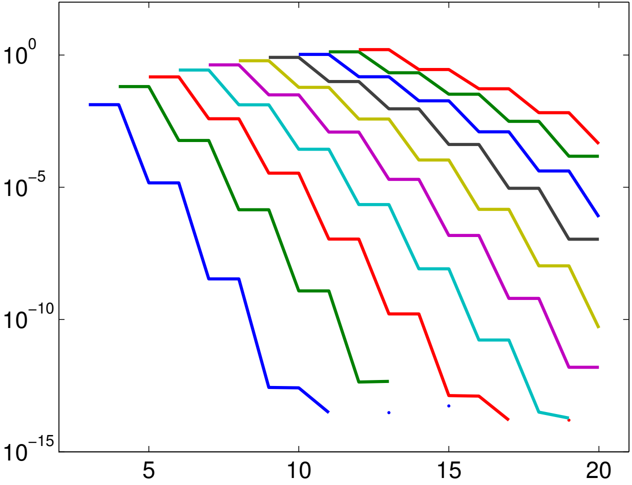
	$n = 8$	$n = 16$	$n = 32$
1	1.0129160450588	1.0032168743567	1.0008034482562
4	4.2095474481529	4.0516641802355	4.0128674974272
9	10.0802909335883	9.2631305555446	9.0652448637285
16	19.4536672593288	16.8381897926118	16.2066567209423
25	33.2628304890884	27.0649225609802	25.5059230069702

	$n = 64$	$n = 128$	$n = 256$
1	1.0002008137390	1.0000502004122	1.0000125499161
4	4.0032137930241	4.0008032549556	4.0002008016414
9	9.0162763381719	9.0040668861371	9.0010165838380
16	16.0514699897078	16.0128551720960	16.0032130198251
25	25.1257489536113	25.0313903532369	25.0078446408520

Approximation with quadratic finite elements

	Computed eigenvalue (rate)				
	$n = 8$	$n = 16$	$n = 32$	$n = 64$	$n = 128$
1	1.0000	1.0000 (4.0)	1.0000 (4.0)	1.0000 (4.0)	1.0000 (4.0)
4	4.0020	4.0001 (4.0)	4.0000 (4.0)	4.0000 (4.0)	4.0000 (4.0)
9	9.0225	9.0015 (3.9)	9.0001 (4.0)	9.0000 (4.0)	9.0000 (4.0)
16	16.1204	16.0082 (3.9)	16.0005 (4.0)	16.0000 (4.0)	16.0000 (4.0)
25	25.4327	25.0307 (3.8)	25.0020 (3.9)	25.0001 (4.0)	25.0000 (4.0)
36	37.1989	36.0899 (3.7)	36.0059 (3.9)	36.0004 (4.0)	36.0000 (4.0)
49	51.6607	49.2217 (3.6)	49.0148 (3.9)	49.0009 (4.0)	49.0001 (4.0)
64	64.8456	64.4814 (0.8)	64.0328 (3.9)	64.0021 (4.0)	64.0001 (4.0)
81	95.7798	81.9488 (4.0)	81.0659 (3.8)	81.0042 (4.0)	81.0003 (4.0)
100	124.9301	101.7308 (3.8)	100.1229 (3.8)	100.0080 (3.9)	100.0005 (4.0)
#	15	31	63	127	255

Exponential convergence



Two dimensional Laplacian

$$\begin{cases} -\Delta u = \lambda u & \text{in } \Omega =]0,\pi[\times]0,\pi[\\ u = 0 & \text{on } \partial\Omega \end{cases}$$

Find $\lambda \in \mathbb{R}$ and non-vanishing $u \in H_0^1(\Omega)$ such that

$$(\mathbf{grad}\, u \cdot \mathbf{grad}\, v) = \lambda(u,v) \quad \forall v \in H_0^1(\Omega)$$

Exact solution:

$\lambda_{m,n} = m^2 + n^2$
$u_{m,n}(x,y) = \sin(mx)\sin(ny)$ $\qquad (m,n = 1,2,3,\dots)$

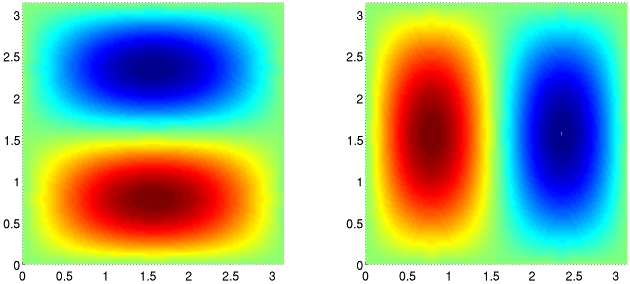
Approximation with p/w linear finite elements

Unstructured mesh

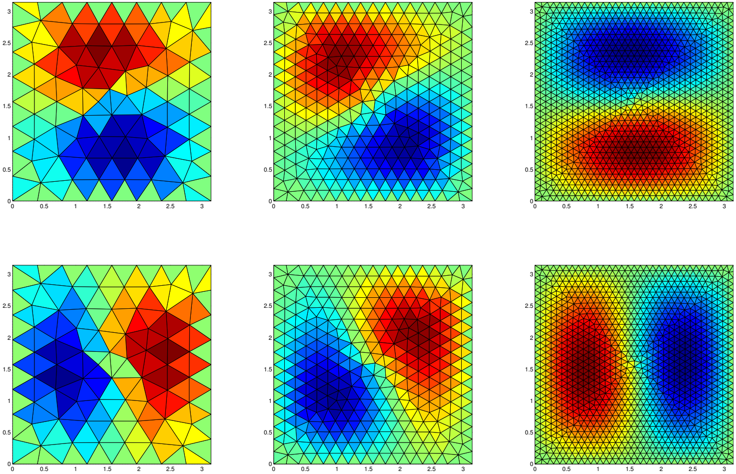
	Computed eigenvalue (rate)				
	$N = 4$	$N = 8$	$N = 16$	$N = 32$	$N = 64$
2	2.2468	2.0463 (2.4)	2.0106 (2.1)	2.0025 (2.1)	2.0006 (2.0)
5	6.5866	5.2732 (2.5)	5.0638 (2.1)	5.0154 (2.0)	5.0038 (2.0)
5	6.6230	5.2859 (2.5)	5.0643 (2.2)	5.0156 (2.0)	5.0038 (2.0)
8	10.2738	8.7064 (1.7)	8.1686 (2.1)	8.0402 (2.1)	8.0099 (2.0)
10	12.7165	11.0903 (1.3)	10.2550 (2.1)	10.0610 (2.1)	10.0152 (2.0)
10	14.3630	11.1308 (1.9)	10.2595 (2.1)	10.0622 (2.1)	10.0153 (2.0)
13	19.7789	14.8941 (1.8)	13.4370 (2.1)	13.1046 (2.1)	13.0258 (2.0)
13	24.2262	14.9689 (2.5)	13.4435 (2.2)	13.1053 (2.1)	13.0258 (2.0)
17	34.0569	20.1284 (2.4)	17.7468 (2.1)	17.1771 (2.1)	17.0440 (2.0)
17		20.2113	17.7528 (2.1)	17.1798 (2.1)	17.0443 (2.0)
#	9	56	257	1106	4573

Multiple eigenfunctions

Exact solutions $(5 = 1^2 + 2^2 = 2^2 + 1^2)$



Multiple eigenfunctions (discrete)

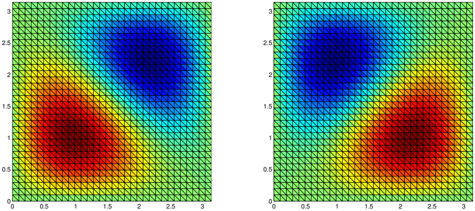


Uniform mesh

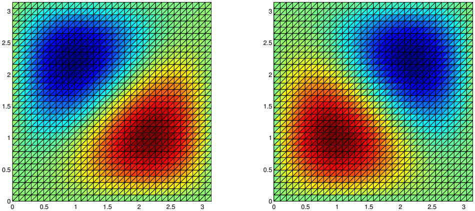
	Computed eigenvalue (rate)				
	$N = 4$	$N = 8$	$N = 16$	$N = 32$	$N = 64$
2	2.3168	2.0776 (2.0)	2.0193 (2.0)	2.0048 (2.0)	2.0012 (2.0)
5	6.3387	5.3325 (2.0)	5.0829 (2.0)	5.0207 (2.0)	5.0052 (2.0)
5	7.2502	5.5325 (2.1)	5.1302 (2.0)	5.0324 (2.0)	5.0081 (2.0)
8	12.2145	9.1826 (1.8)	8.3054 (2.0)	8.0769 (2.0)	8.0193 (2.0)
10	15.5629	11.5492 (1.8)	10.3814 (2.0)	10.0949 (2.0)	10.0237 (2.0)
10	16.7643	11.6879 (2.0)	10.3900 (2.1)	10.0955 (2.0)	10.0237 (2.0)
13	20.8965	15.2270 (1.8)	13.5716 (2.0)	13.1443 (2.0)	13.0362 (2.0)
13	26.0989	17.0125 (1.7)	13.9825 (2.0)	13.2432 (2.0)	13.0606 (2.0)
17	32.4184	21.3374 (1.8)	18.0416 (2.1)	17.2562 (2.0)	17.0638 (2.0)
17		21.5751	18.0705 (2.1)	17.2626 (2.0)	17.0653 (2.0)
#	9	49	225	961	3969

Multiple eigenfunctions (uniform meshes)

Uniform mesh



Uniform mesh (reversed)



Multiple eigenfunctions (symmetric mesh)

Criss-cross mesh

Exact	Computed eigenvalue (rate)				
	$N = 4$	$N = 8$	$N = 16$	$N = 32$	$N = 64$
2	2.0880	2.0216 (2.0)	2.0054 (2.0)	2.0013 (2.0)	2.0003 (2.0)
5	5.6811	5.1651 (2.0)	5.0408 (2.0)	5.0102 (2.0)	5.0025 (2.0)
5	5.6811	5.1651 (2.0)	5.0408 (2.0)	5.0102 (2.0)	5.0025 (2.0)
8	9.4962	8.3521 (2.1)	8.0863 (2.0)	8.0215 (2.0)	8.0054 (2.0)
10	12.9691	10.7578 (2.0)	10.1865 (2.0)	10.0464 (2.0)	10.0116 (2.0)
10	12.9691	10.7578 (2.0)	10.1865 (2.0)	10.0464 (2.0)	10.0116 (2.0)
13	17.1879	14.0237 (2.0)	13.2489 (2.0)	13.0617 (2.0)	13.0154 (2.0)
13	17.1879	14.0237 (2.0)	13.2489 (2.0)	13.0617 (2.0)	13.0154 (2.0)
17	25.1471	19.3348 (1.8)	17.5733 (2.0)	17.1423 (2.0)	17.0355 (2.0)
17	38.9073	19.3348 (3.2)	17.5733 (2.0)	17.1423 (2.0)	17.0355 (2.0)
18	38.9073	19.8363 (3.5)	18.4405 (2.1)	18.1089 (2.0)	18.0271 (2.0)
20	38.9073	22.7243 (2.8)	20.6603 (2.0)	20.1634 (2.0)	20.0407 (2.0)
20	38.9073	22.7243 (2.8)	20.6603 (2.0)	20.1634 (2.0)	20.0407 (2.0)
25	38.9073	28.7526 (1.9)	25.8940 (2.1)	25.2201 (2.0)	25.0548 (2.0)
25	38.9073	28.7526 (1.9)	25.8940 (2.1)	25.2201 (2.0)	25.0548 (2.0)
DOF	25	113	481	1985	8065

Mixed approximation of Laplacian

Find $\lambda \in \mathbb{R}$ and $u \in L^2(0,\pi)$ such that for some $s \in H^1(0,\pi)$

$$\begin{cases} (s,t) + (t',u) = 0 \quad \forall t \in H^1(0,\pi) & s = u' \\ (s',v) = -\lambda(u,v) \quad \forall v \in L^2(0,\pi) & s' = -\lambda u \end{cases}$$

After conforming discretization $\Sigma_h \subset \Sigma = H^1(0,\pi)$ and $U_h \subset U = L^2(0,\pi)$ the discrete problem has the following matrix form

$$\begin{bmatrix} A & B^\top \\ B & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \lambda \begin{bmatrix} 0 & 0 \\ 0 & -M \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

The good element

$P_1 - P_0$ scheme (in general, $P_{k+1} - P_k$)

Same eigenvalues as for the standard Galerkin P_1 scheme

$$\lambda_h^{(k)} = \frac{6}{h^2} \cdot \frac{1 - \cos kh}{2 + \cos kh}$$

$$u_h^{(k)}|_{]ih,(i+1)h[} = \frac{s_h^{(k)}(ih) - s_h^{(k)}((i+1)h)}{h\lambda_h^{(k)}}$$

$$s_h^{(k)}(ih) = k\cos(kih)$$

$$i = 0, \ldots, N \quad (N = \text{number of intervals})$$

$$k = 1, \ldots, N$$

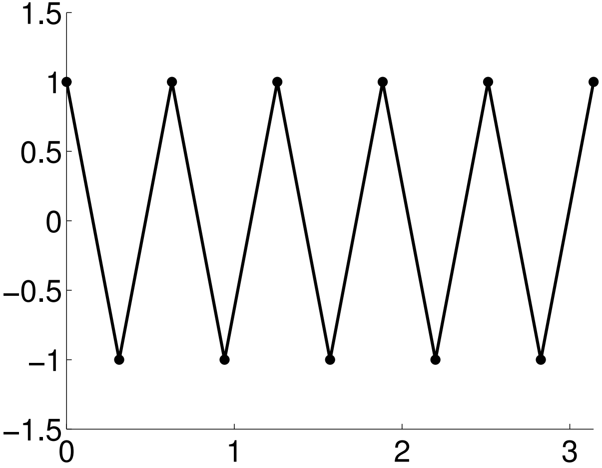
A troublesome element

$P_1 - P_1$ scheme

	Computed eigenvalue (rate)				
	$N = 8$	$N = 16$	$N = 32$	$N = 64$	$N = 128$
	0.0000	-0.0000	-0.0000	-0.0000	-0.0000
1	1.0001	1.0000 (4.1)	1.0000 (4.0)	1.0000 (4.0)	1.0000 (4.0)
4	3.9660	3.9981 (4.2)	3.9999 (4.0)	4.0000 (4.0)	4.0000 (4.0)
9	7.4257	8.5541 (1.8)	8.8854 (2.0)	8.9711 (2.0)	8.9928 (2.0)
9	8.7603	8.9873 (4.2)	8.9992 (4.1)	9.0000 (4.0)	9.0000 (4.0)
16	14.8408	15.9501 (4.5)	15.9971 (4.1)	15.9998 (4.0)	16.0000 (4.0)
25	16.7900	24.5524 (4.2)	24.9780 (4.3)	24.9987 (4.1)	24.9999 (4.0)
36	38.7154	29.7390 (-1.2)	34.2165 (1.8)	35.5415 (2.0)	35.8846 (2.0)
36	39.0906	35.0393 (1.7)	35.9492 (4.2)	35.9970 (4.1)	35.9998 (4.0)
49		46.7793	48.8925 (4.4)	48.9937 (4.1)	48.9996 (4.0)

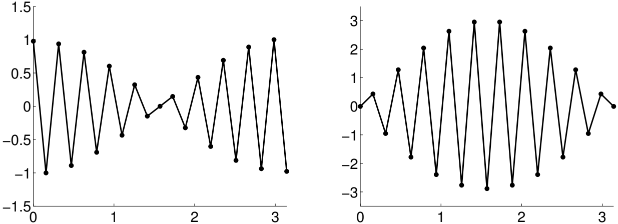
First spurious eigenfunction

$\lambda = 0$

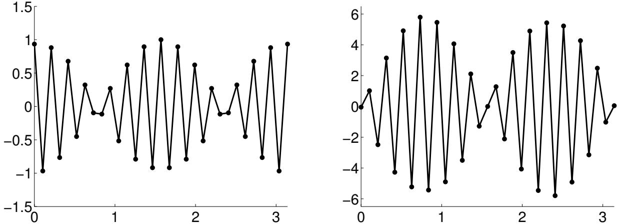


Higher order spurious eigenfunctions

$\lambda \simeq 9$



$\lambda \simeq 36$

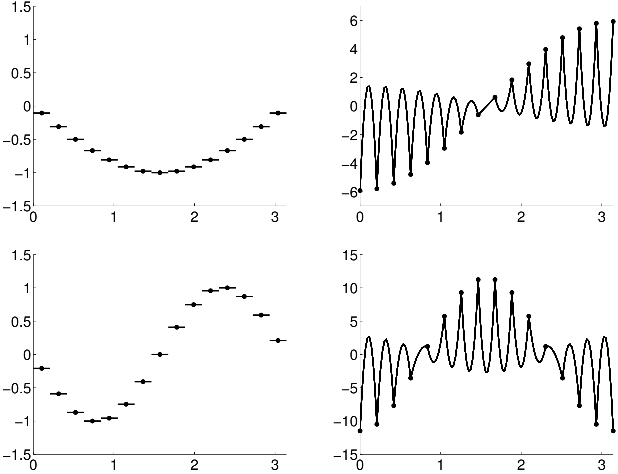


An intriguing element

$P_2 - P_0$ scheme

	Computed eigenvalue (rate with respect to 6λ)				
	$n = 8$	$n = 16$	$n = 32$	$n = 64$	$n = 128$
1	5.7061	5.9238 (1.9)	5.9808 (2.0)	5.9952 (2.0)	5.9988 (2.0)
4	19.8800	22.8245 (1.8)	23.6953 (1.9)	23.9231 (2.0)	23.9807 (2.0)
9	36.7065	48.3798 (1.6)	52.4809 (1.9)	53.6123 (2.0)	53.9026 (2.0)
16	51.8764	79.5201 (1.4)	91.2978 (1.8)	94.7814 (1.9)	95.6925 (2.0)
25	63.6140	113.1819 (1.2)	138.8165 (1.7)	147.0451 (1.9)	149.2506 (2.0)
36	71.6666	146.8261 (1.1)	193.5192 (1.6)	209.9235 (1.9)	214.4494 (2.0)
49	76.3051	178.6404 (0.9)	253.8044 (1.5)	282.8515 (1.9)	291.1344 (2.0)
64	77.8147	207.5058 (0.8)	318.0804 (1.4)	365.1912 (1.8)	379.1255 (1.9)
81		232.8461	384.8425 (1.3)	456.2445 (1.8)	478.2172 (1.9)
100		254.4561	452.7277 (1.2)	555.2659 (1.7)	588.1806 (1.9)
#	8	16	32	64	128

Eigenfunctions for the $P_2 - P_0$ element



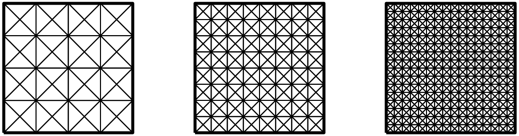
Another intriguing example in 2D

Neumann eigenvalue problem for the Laplacian

Find $\lambda \in \mathbb{R}$ and $u \in L^2_0(\Omega)$ such that for some $\boldsymbol{\sigma} \in \mathbf{H}_0(\mathrm{div};\Omega)$

$$\begin{cases} (\boldsymbol{\sigma}, \boldsymbol{\tau}) + (\mathrm{div}\,\boldsymbol{\tau}, u) = 0 & \forall \boldsymbol{\tau} \in \mathbf{H}_0(\mathrm{div};\Omega) \\ (\mathrm{div}\,\boldsymbol{\sigma}, v) = -\lambda(u, v) & \forall v \in L^2_0(\Omega) \end{cases}$$

Criss-cross mesh sequence, $P_1 - \mathrm{div}(P_1)$ scheme



	Computed eigenvalue (rate)				
	$N = 2$	$N = 4$	$N = 8$	$N = 16$	$N = 32$
1	1.0662	1.0170 (2.0)	1.0043 (2.0)	1.0011 (2.0)	1.0003 (2.0)
1	1.0662	1.0170 (2.0)	1.0043 (2.0)	1.0011 (2.0)	1.0003 (2.0)
2	2.2035	2.0678 (1.6)	2.0171 (2.0)	2.0043 (2.0)	2.0011 (2.0)
4	4.8634	4.2647 (1.7)	4.0680 (2.0)	4.0171 (2.0)	4.0043 (2.0)
4	4.8634	4.2647 (1.7)	4.0680 (2.0)	4.0171 (2.0)	4.0043 (2.0)
5	6.1338	5.3971 (1.5)	5.1063 (1.9)	5.0267 (2.0)	5.0067 (2.0)
5	6.4846	5.3971 (1.9)	5.1063 (1.9)	5.0267 (2.0)	5.0067 (2.0)
6	6.4846	5.6712 (0.6)	5.9229 (2.1)	5.9807 (2.0)	5.9952 (2.0)
8	11.0924	8.8141 (1.9)	8.2713 (1.6)	8.0685 (2.0)	8.0171 (2.0)
9	11.0924	10.2540 (0.7)	9.3408 (1.9)	9.0864 (2.0)	9.0217 (2.0)
9	11.1164	10.2540 (0.8)	9.3408 (1.9)	9.0864 (2.0)	9.0217 (2.0)
10		10.9539	10.4193 (1.2)	10.1067 (2.0)	10.0268 (2.0)
10		10.9539	10.4193 (1.2)	10.1067 (2.0)	10.0268 (2.0)
13		11.1347	13.7027 (1.4)	13.1804 (2.0)	13.0452 (2.0)
13		11.1347	13.7027 (1.4)	13.1804 (2.0)	13.0452 (2.0)
15		9.4537	13.9639 (2.1)	14.7166 (1.9)	14.9272 (2.0)
15		19.4537	13.9639 (2.1)	14.7166 (1.9)	14.9272 (2.0)
16		19.7860	17.0588 (1.8)	16.2722 (2.0)	16.0684 (2.0)
16		19.7860	17.0588 (1.8)	16.2722 (2.0)	16.0684 (2.0)
17		20.9907	18.1813 (1.8)	17.3073 (1.9)	17.0773 (2.0)
dof	11	47	191	767	3071

Source vs. eigenvalue problem in mixed form

⟨B.–Brezzi–Gastaldi ’00⟩

N.B.

The criss-cross $P_1 - \mathrm{div}(P_1)$ element is a good element for the *source* problem (inf-sup condition OK!)

The $Q_1 - P_0$ scheme

⟨B.–Durán–Gastaldi '99⟩

The discrete eigenvalues can be explicitly computed:

$$\lambda_h^{(mn)} = \frac{4}{h^2} \frac{\sin^2(\frac{mh}{2}) + \sin^2(\frac{nh}{2}) - 2\sin^2(\frac{mh}{2})\sin^2(\frac{nh}{2})}{1 - \frac{2}{3}(\sin^2(\frac{mh}{2}) + \sin^2(\frac{nh}{2})) + \frac{4}{9}\sin^2(\frac{mh}{2})\sin^2(\frac{nh}{2})}$$

$$\boldsymbol{\sigma}_h^{(mn)} = (\sigma_1^{(mn)}, \sigma_2^{(mn)})$$

$$\sigma_1^{(m,n)}(x_i, y_j) = \frac{2}{h}\sin\left(\frac{mh}{2}\right)\cos\left(\frac{nh}{2}\right)\sin(mx_i)\cos(ny_j)$$

$$\sigma_2^{(m,n)}(x_i, y_j) = \frac{2}{h}\cos\left(\frac{mh}{2}\right)\sin\left(\frac{nh}{2}\right)\cos(mx_i)\sin(ny_j)$$

Does it converge?

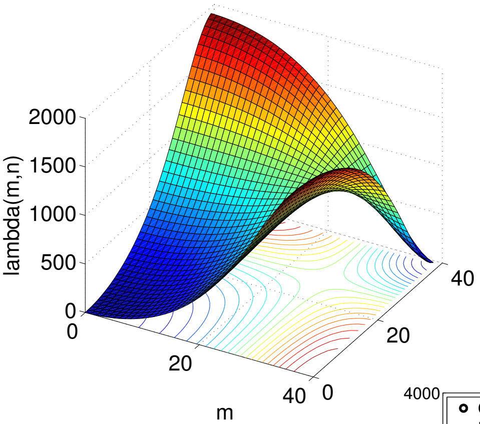
	Computed eigenvalue (rate)				
	$N = 4$	$N = 8$	$N = 16$	$N = 32$	$N = 64$
1	1.0524	1.0129 (2.0)	1.0032 (2.0)	1.0008 (2.0)	1.0002 (2.0)
1	1.0524	1.0129 (2.0)	1.0032 (2.0)	1.0008 (2.0)	1.0002 (2.0)
2	1.9909	1.9995 (4.1)	2.0000 (4.0)	2.0000 (4.0)	2.0000 (4.0)
4	4.8634	4.2095 (2.0)	4.0517 (2.0)	4.0129 (2.0)	4.0032 (2.0)
4	4.8634	4.2095 (2.0)	4.0517 (2.0)	4.0129 (2.0)	4.0032 (2.0)
5	5.3896	5.1129 (1.8)	5.0288 (2.0)	5.0072 (2.0)	5.0018 (2.0)
5	5.3896	5.1129 (1.8)	5.0288 (2.0)	5.0072 (2.0)	5.0018 (2.0)
8	7.2951	7.9636 (4.3)	7.9978 (4.1)	7.9999 (4.0)	8.0000 (4.0)
9	8.7285	10.0803 (-2.0)	9.2631 (2.0)	9.0652 (2.0)	9.0163 (2.0)
9	11.2850	10.0803 (1.1)	9.2631 (2.0)	9.0652 (2.0)	9.0163 (2.0)
10	11.2850	10.8308 (0.6)	10.2066 (2.0)	10.0515 (2.0)	10.0129 (2.0)
10	12.5059	10.8308 (1.6)	10.2066 (2.0)	10.0515 (2.0)	10.0129 (2.0)
13	12.5059	13.1992 (1.3)	13.0736 (1.4)	13.0197 (1.9)	13.0050 (2.0)
13	12.8431	13.1992 (-0.3)	13.0736 (1.4)	13.0197 (1.9)	13.0050 (2.0)
16	12.8431	14.7608 (1.3)	16.8382 (0.6)	16.2067 (2.0)	16.0515 (2.0)
16		17.5489	16.8382 (0.9)	16.2067 (2.0)	16.0515 (2.0)
17		19.4537	17.1062 (4.5)	17.1814 (-0.8)	17.0452 (2.0)
17		19.4537	17.7329 (1.7)	17.1814 (2.0)	17.0452 (2.0)
18		19.9601	17.7329 (2.9)	17.7707 (0.2)	17.9423 (2.0)
18		19.9601	17.9749 (6.3)	17.9985 (4.0)	17.9999 (4.0)
20		21.5584	20.4515 (1.8)	20.1151 (2.0)	20.0289 (2.0)
20		21.5584	20.4515 (1.8)	20.1151 (2.0)	20.0289 (2.0)

Wrong proof?

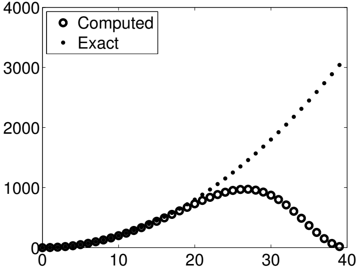
$$\lambda_h^{(mn)} = \frac{4}{h^2} \frac{\sin^2(\frac{mh}{2}) + \sin^2(\frac{nh}{2}) - 2\sin^2(\frac{mh}{2})\sin^2(\frac{nh}{2})}{1 - \frac{2}{3}(\sin^2(\frac{mh}{2}) + \sin^2(\frac{nh}{2})) + \frac{4}{9}\sin^2(\frac{mh}{2})\sin^2(\frac{nh}{2})}$$

Indeed, if $h = \pi/N$, we have:

$$\lim_{N\to\infty} \lambda_h^{(N-1,N-1)} = 18$$



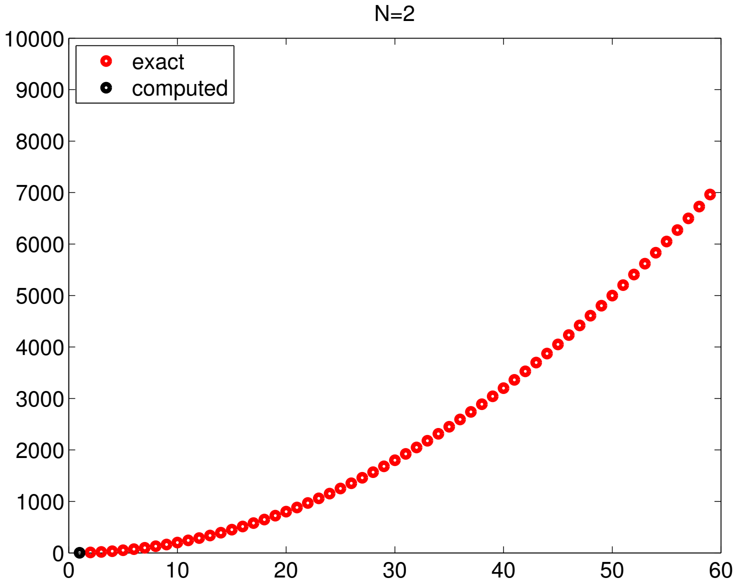
2000
1500
1000
500
0
lambda(m,n)
0
20
40
m
0
20
40



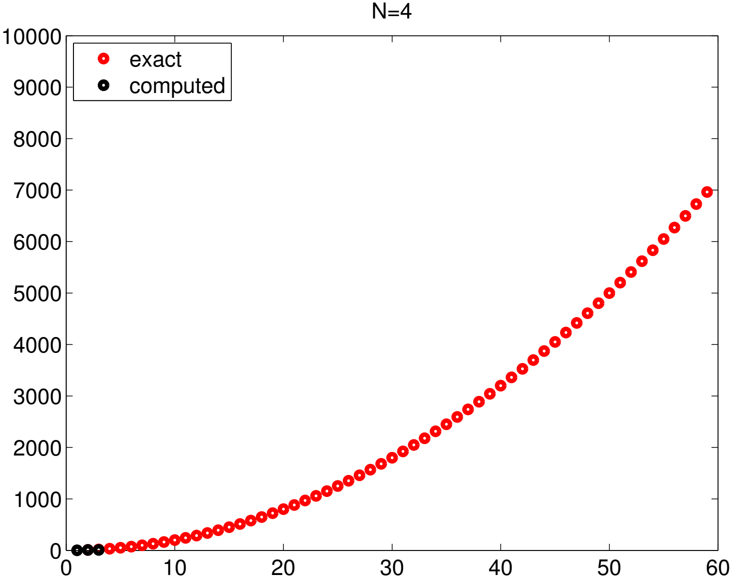
4000
3000
2000
1000
0
0
10
20
30
40
Computed
Exact

Plot for $m = n$

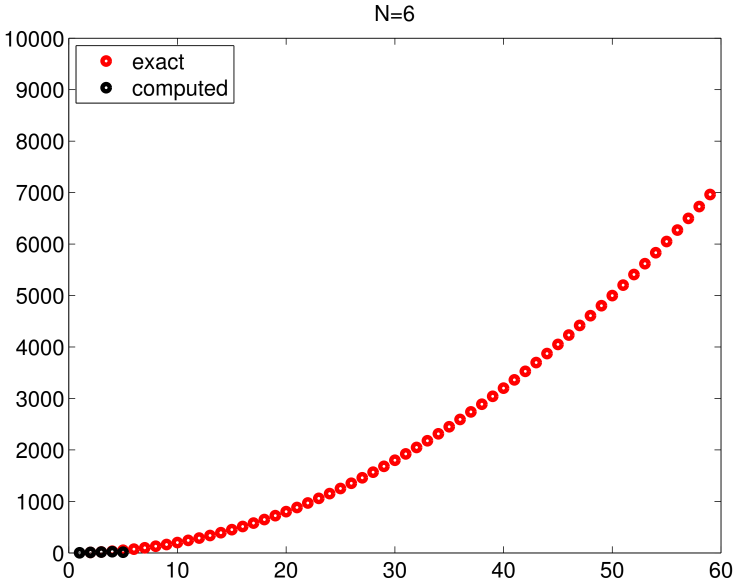
Pointwise vs. uniform convergence



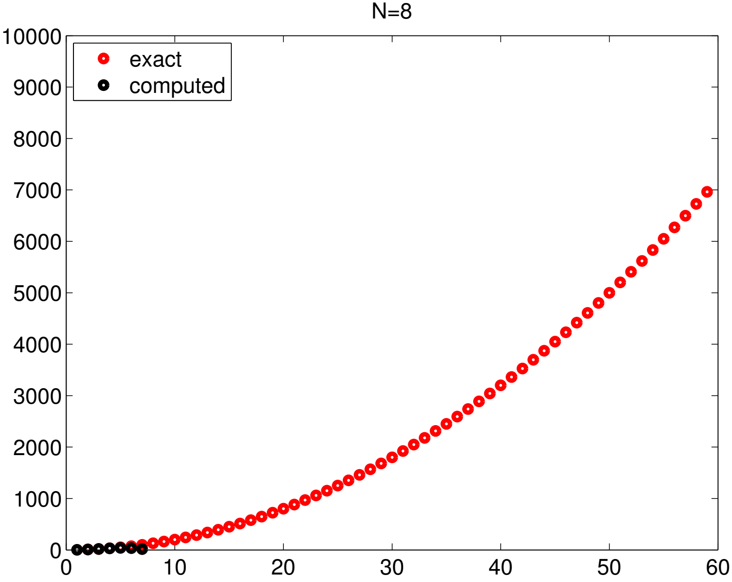
Pointwise vs. uniform convergence



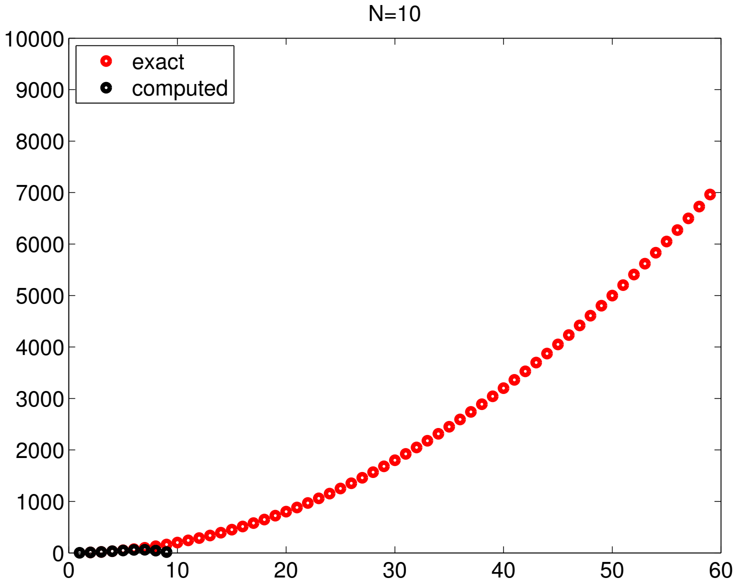
Pointwise vs. uniform convergence



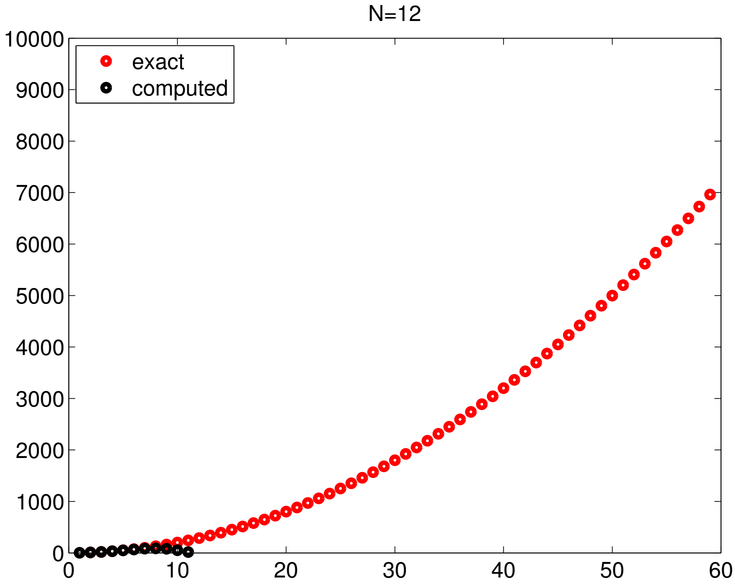
Pointwise vs. uniform convergence



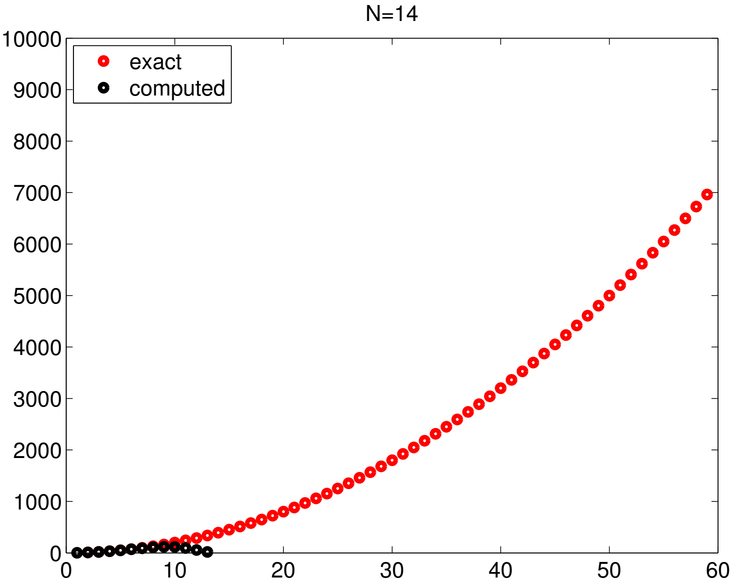
Pointwise vs. uniform convergence



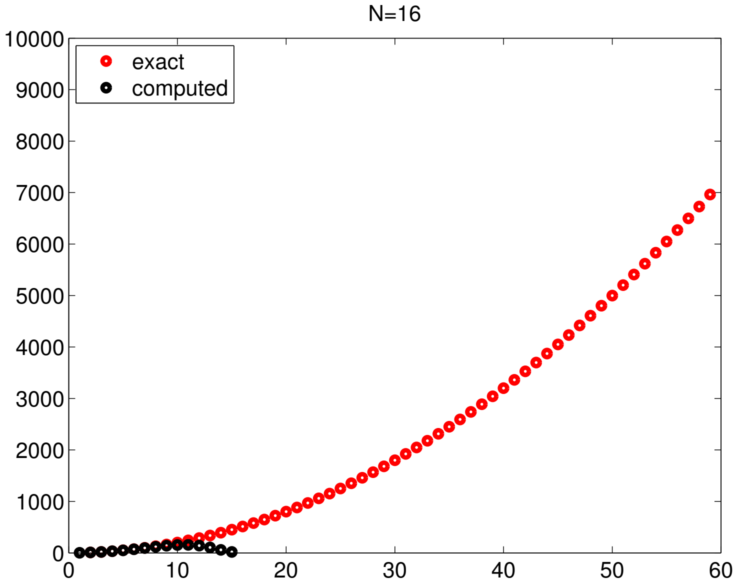
Pointwise vs. uniform convergence



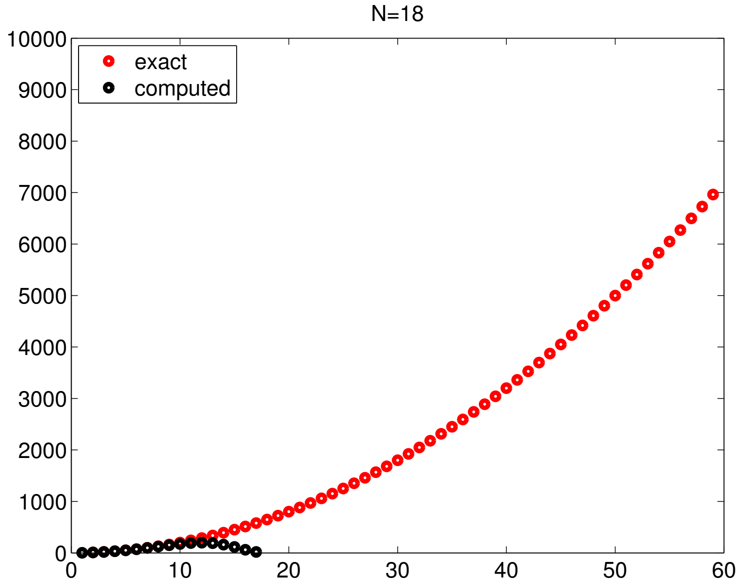
Pointwise vs. uniform convergence



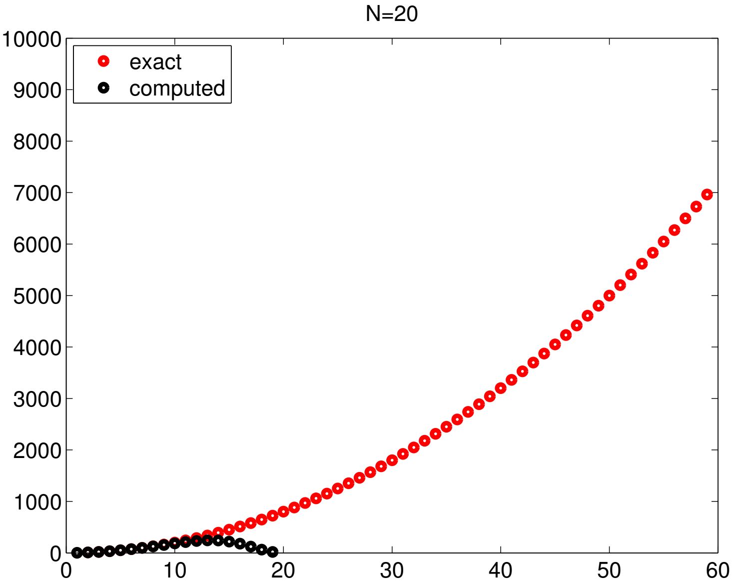
Pointwise vs. uniform convergence



Pointwise vs. uniform convergence



Pointwise vs. uniform convergence



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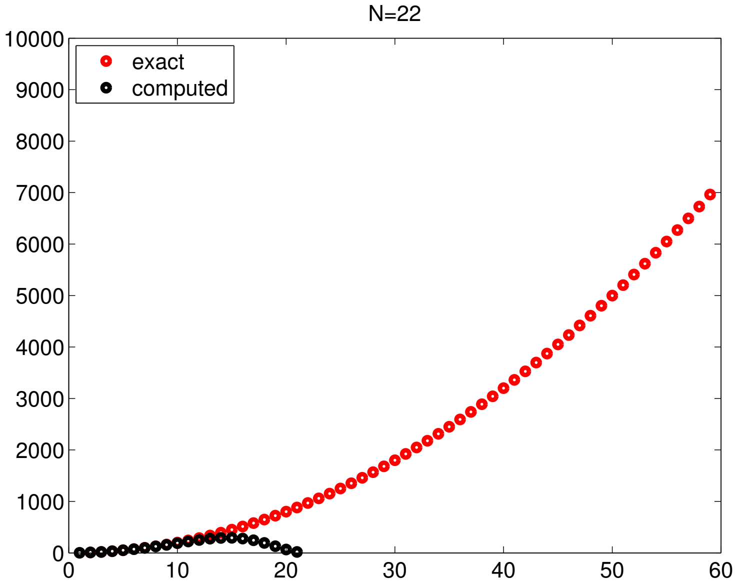
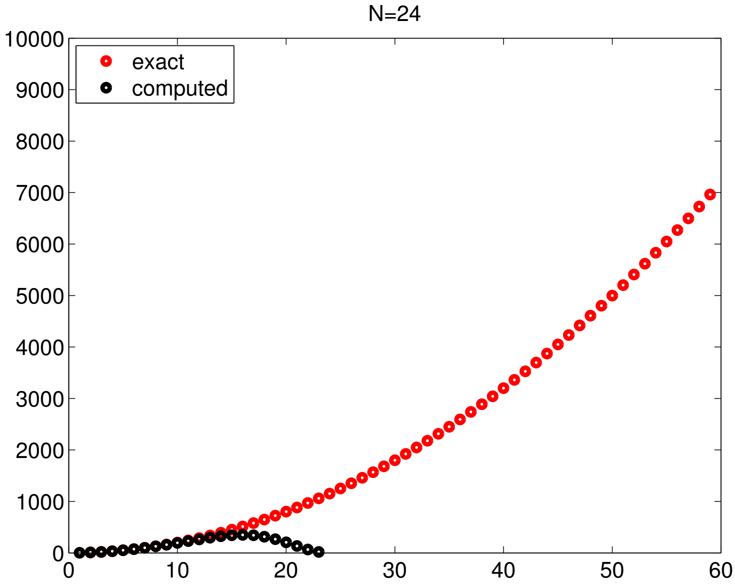
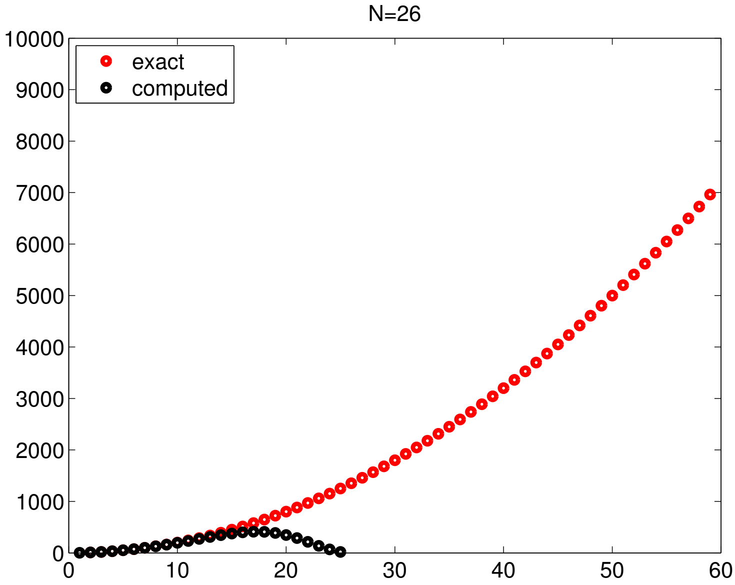


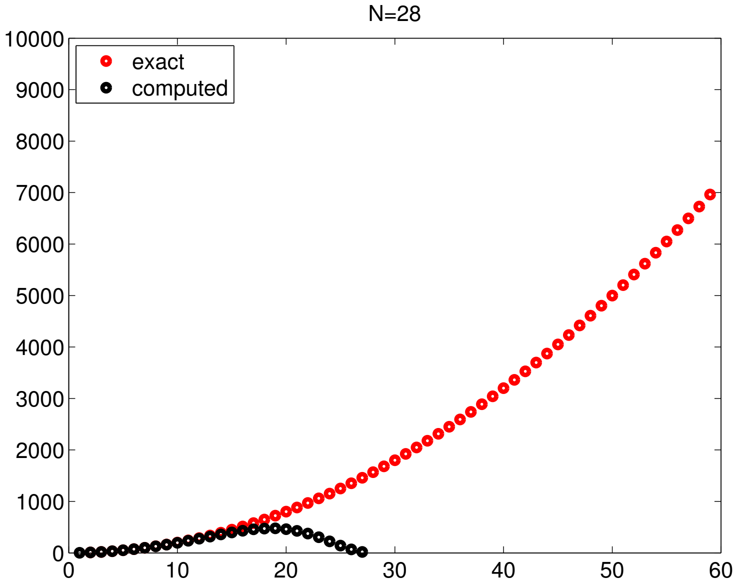
Pointwise vs. uniform convergence



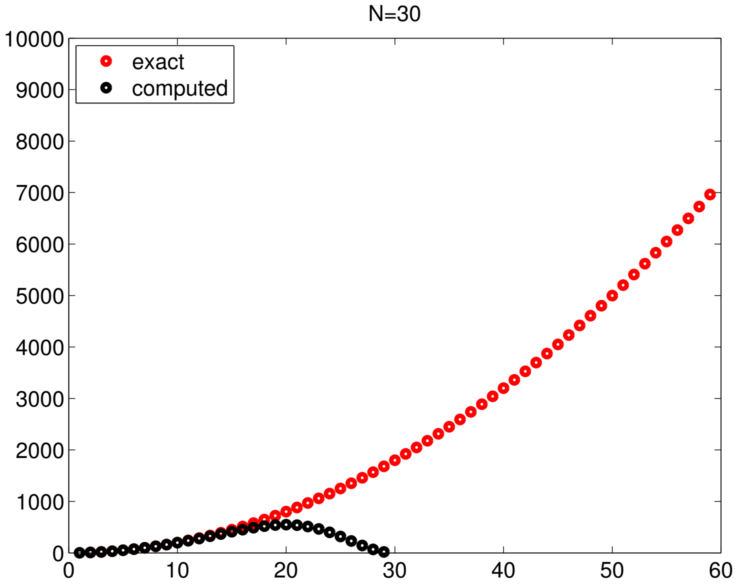
Pointwise vs. uniform convergence



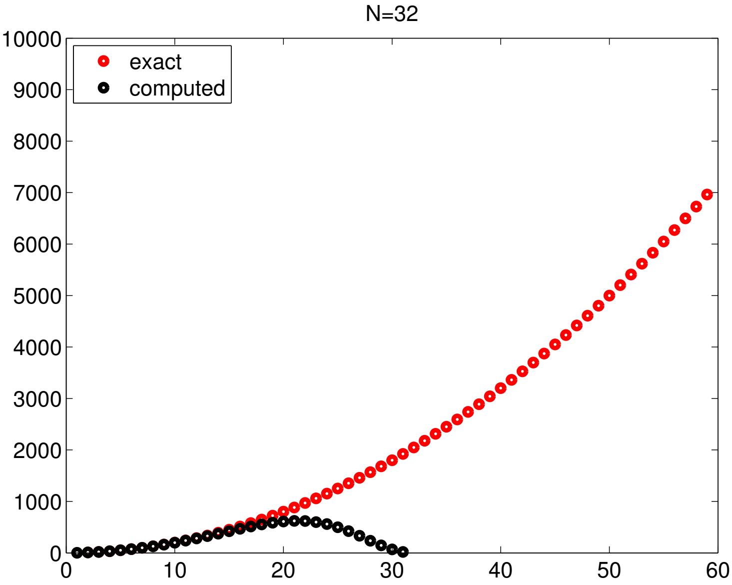
Pointwise vs. uniform convergence



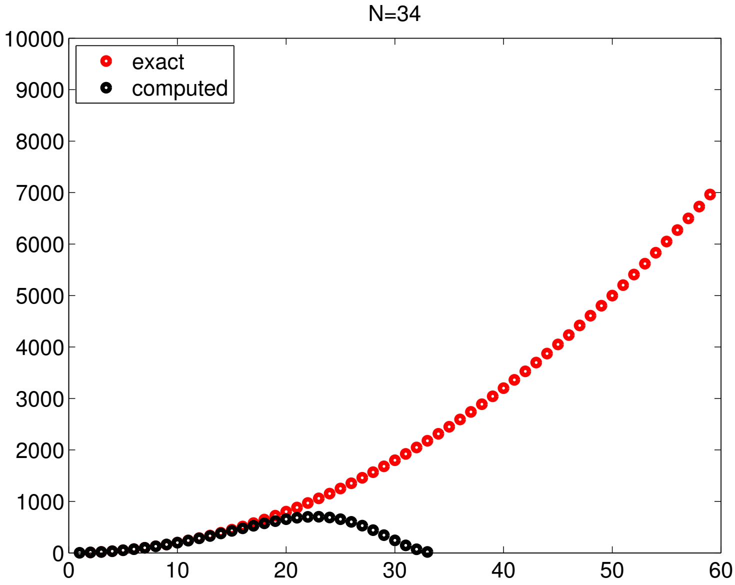
Pointwise vs. uniform convergence



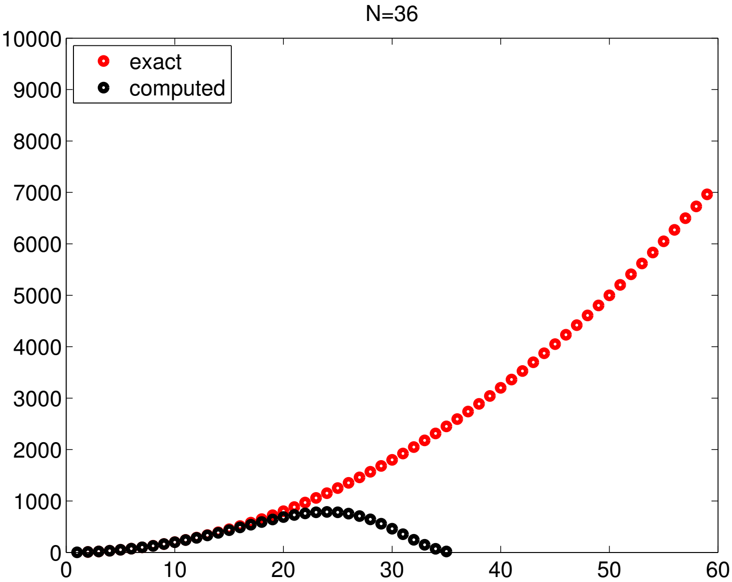
Pointwise vs. uniform convergence



Pointwise vs. uniform convergence



Pointwise vs. uniform convergence



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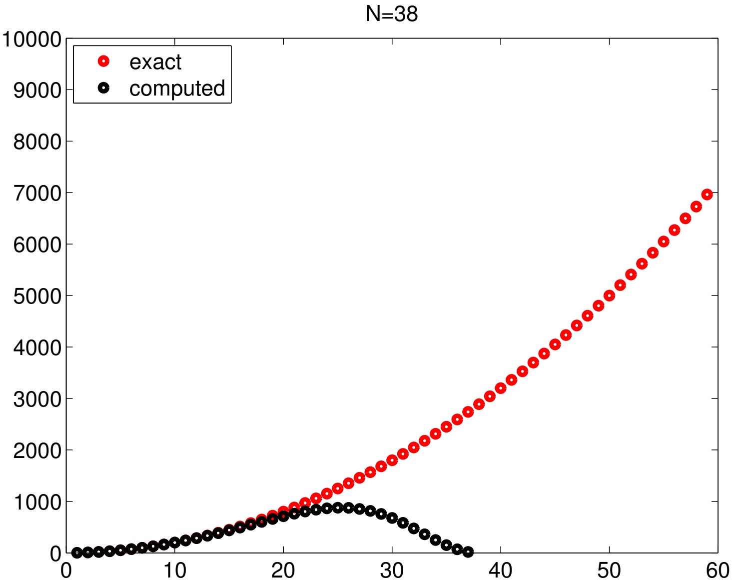

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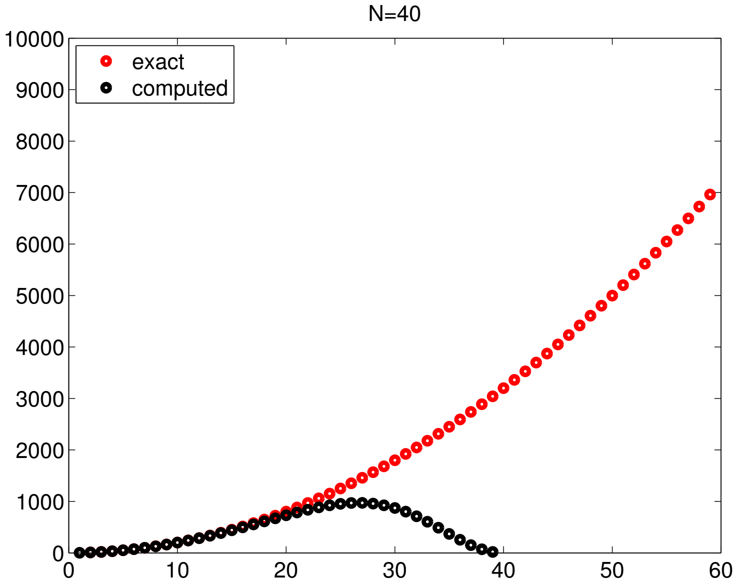
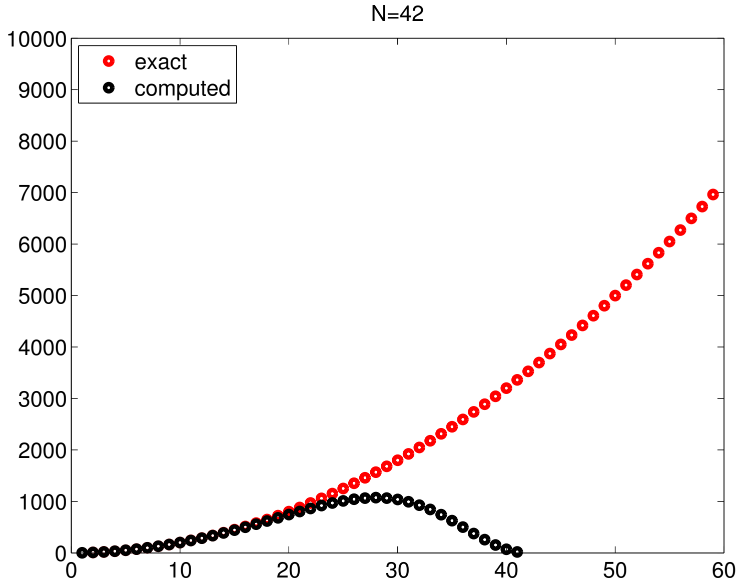


Pointwise vs. uniform convergence



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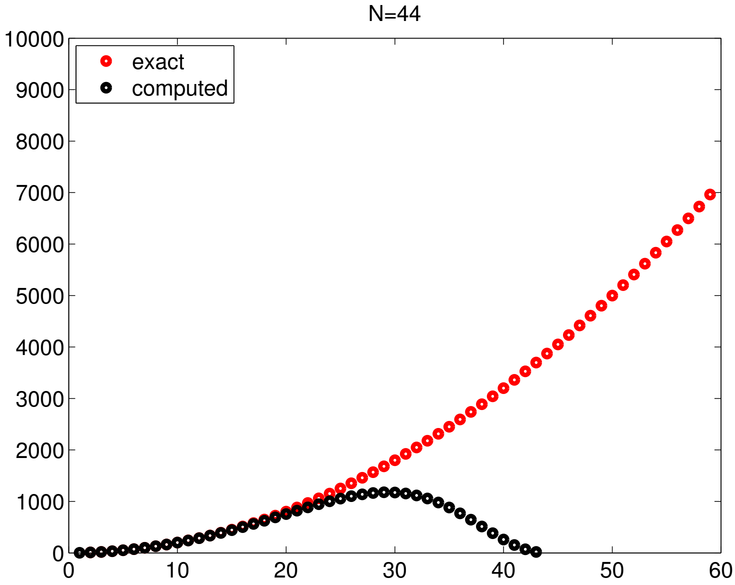

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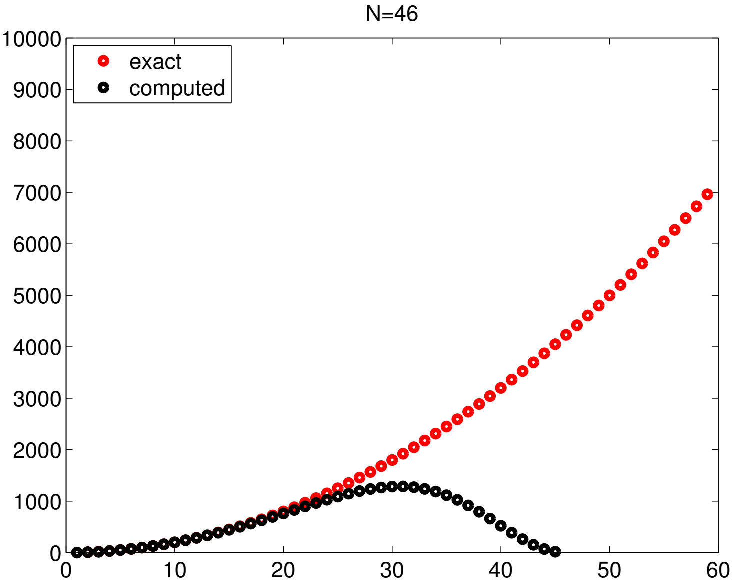

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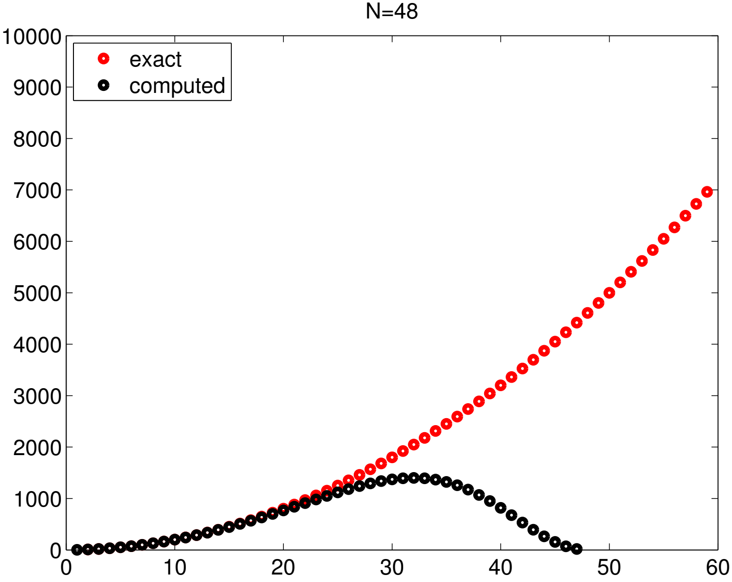
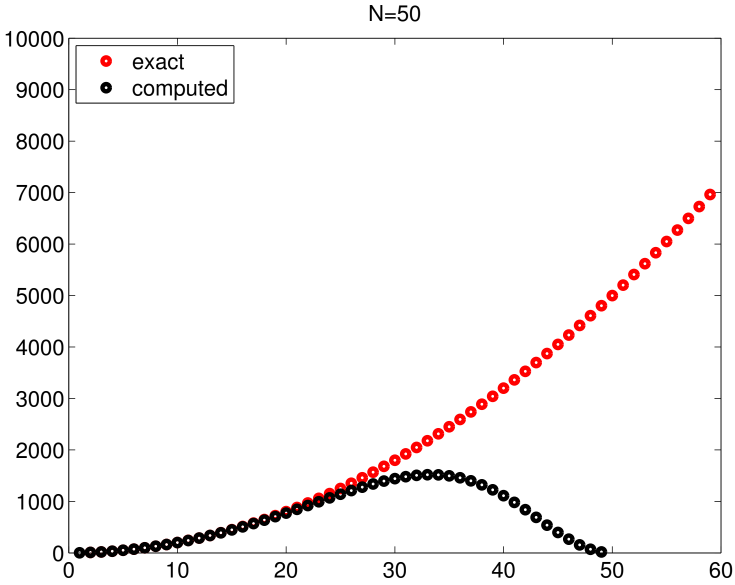
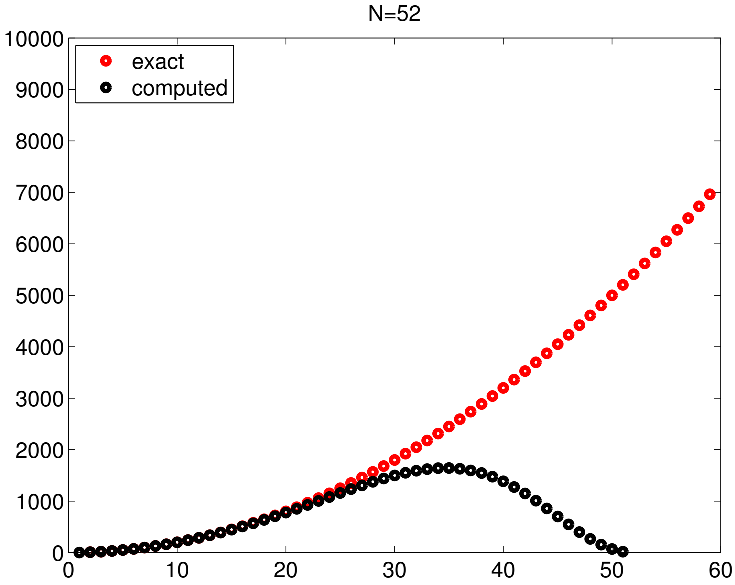


Pointwise vs. uniform convergence



Pointwise vs. uniform convergence



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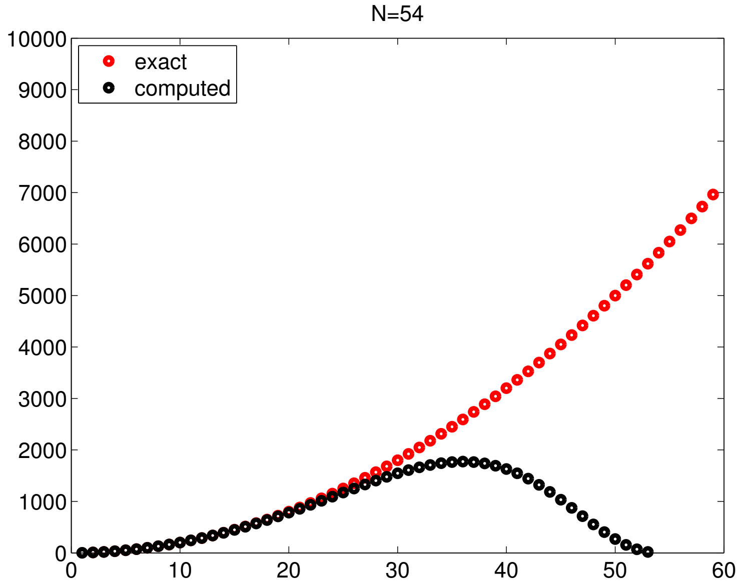

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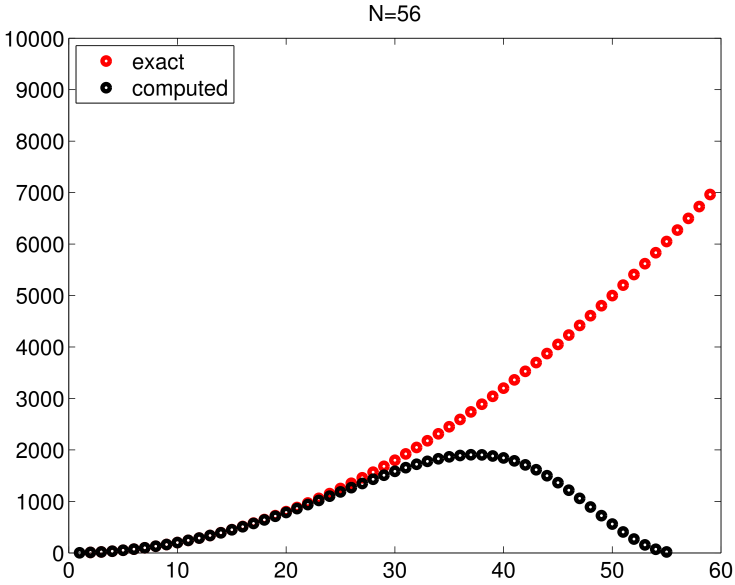

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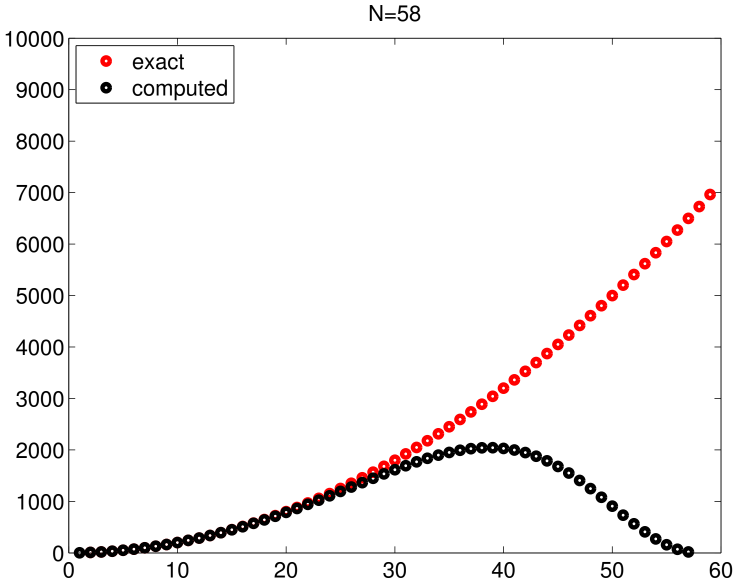

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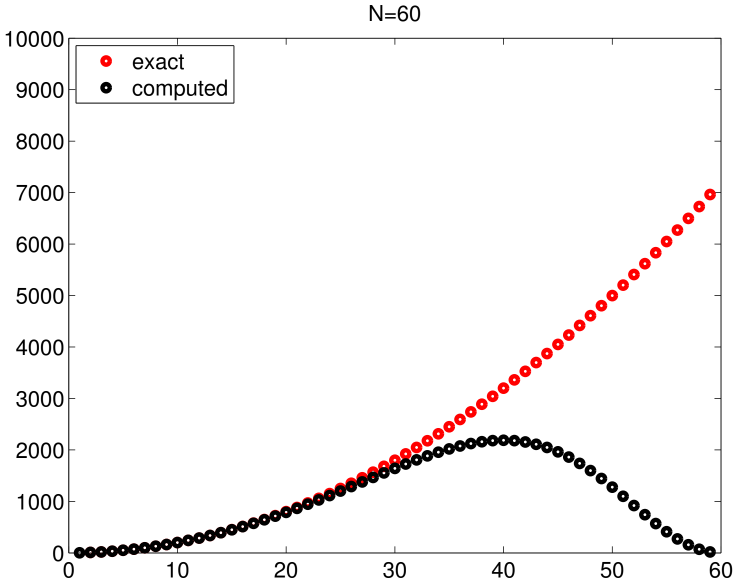

Some theoretical framework

⟨Babuška–Osborn '91⟩
⟨B. '00⟩

Variationally posed eigenproblem (Laplace operator)

Abstract framework

$H \quad (= L^2(\Omega)) \qquad V \quad (= H^1_0(\Omega)) \qquad \subset H$
Hilbert spaces, V compactly embedded in H

$a(u,v) \quad (= \int_\Omega \mathbf{grad}\, u \cdot \mathbf{grad}\, v\, d\mathbf{x}) \qquad V \times V \to \mathbb{R}$
bilinear, continuous, symmetric, coercive

$b(u,v) \quad (= (u,v)) \qquad H \times H \to \mathbb{R}$
bilinear, continuous, symmetric

Eigenvalue problem

Find $\lambda \in \mathbb{R}$ such that for some $u \in V$ with $u \not\equiv 0$ it holds
$a(u,v) = \lambda b(u,v) \quad \forall v \in V$

Standard Laplace eigenvalue problem

Strong form

$$\begin{aligned} -\Delta u &= \lambda u && \text{in } \Omega \\ u &= 0 && \text{on } \partial\Omega \end{aligned}$$

Weak form

$\lambda \in \mathbb{R}$, $u \in V$, $u \not\equiv 0$:
$a(u,v) = \lambda b(u,v) \quad \forall v \in V$

Solution operator

$T : H \to H$, $\qquad T(H) \subset V$ implies T is compact
$a(Tf, v) = b(f, v) \quad \forall v \in V$

$\lambda_1 \le \lambda_2 \le \cdots \le \lambda_i \le \cdots$
$E_i = \operatorname{span}(u_i)$, normalization $b(u_i, u_i) = 1$
$V = \bigoplus_{i=1}^{\infty} E_i$

Discrete problem

$V_h \subset V$, $\dim V_h = N(h)$

Find $\lambda_h \in \mathbb{R}$ such that for some $u_h \in V_h$ with $u_h \not\equiv 0$ it holds

$$a(u_h, v) = \lambda_h b(u_h, v) \quad \forall v \in V_h$$

Discrete (compact) solution operator

$$T_h : H \to H$$
$$a(T_h f, v) = b(f, v) \quad \forall v \in V_h$$

$$\lambda_{1,h} \leq \lambda_{2,h} \leq \cdots \leq \lambda_{i,h} \leq \cdots \leq \lambda_{N(h),h}$$

$E_{i,h} = \mathrm{span}(u_{i,h})$, normalization $b(u_{i,h}, u_{i,h}) = 1$

$$V_h = \bigoplus_{i=1}^{N(h)} E_{i,h}$$

Some notation

$m : \mathbb{N} \to \mathbb{N}$ such that for $N \in \mathbb{N}$

$\lambda_{m(1)} < \lambda_{m(2)} < \cdots < \lambda_{m(N)} < \ldots$

$\hat{\delta}(E,F) = \max(\delta(E,F), \delta(F,E))$, where E, F subspaces of H

$\delta(E,F) = \sup_{u \in E,\ ||u||_H = 1} \inf_{v \in F} ||u - v||_H$

Definition of convergence

$\forall \varepsilon > 0,\ \forall N \in \mathbb{N},\ \exists h_0 > 0$ such that $\forall h \le h_0$

$$\max_{i=1,\ldots,m(N)} |\lambda_i - \lambda_{i,h}| \le \varepsilon$$

$$\hat{\delta}\left(\bigoplus_{i=1}^{m(N)} E_i, \bigoplus_{i=1}^{m(N)} E_{i,h}\right) \le \varepsilon$$

Uniform convergence

Convergence in norm

$$||T - T_h||_{\mathcal{L}(H,H)} \to 0$$

Theorem

If T is selfadjoint and compact

Uniform convergence $\Longleftrightarrow$ *Eigenmodes convergence*

Strategy

1) prove uniform convergence,
2) estimate the order of convergence

Galerkin approximation of compact operators

⟨Bramble–Osborn '73⟩
⟨Osborn '75⟩
⟨Kolata '78⟩

Céa's Lemma

$T_h = P_h T$, with P_h projection w.r.t. bilinear form a

$$T - T_h = (I - P_h)T$$

Consequence of $a(Tf - T_h f, v_h) = 0 \quad \forall v_h \in V_h$

If $I - P_h$ converges to zero *pointwise* and T is *compact*, then $T - T_h$ converges to zero *uniformly* (consequence of Banach–Steinhaus uniform boundedness principle)

Crucial proof

First we show that $\{\|I - P_h\|_{\mathcal{L}(V,H)}\}$ is bounded

Define $c(h,u)$ by $\|(I - P_h)u\|_H = c(h,u)\|u\|_V$
For each u we have $c(h,u) \to 0$ (pointwise convergence)

$M(u) = \max_h c(h,u) < \infty$ implies $\|I - P_h\|_{\mathcal{L}(V,H)} \le C$ uniformly

Take $\{f_h\}$ s.t. $\|f_h\|_H = 1$ and $\|T - T_h\|_{\mathcal{L}(H)} = \|(T - T_h)f_h\|_H$

Extract subsequence with $Tf_h \to w$ in V

$$\begin{aligned}\|(I - P_h)Tf_h\|_H &\le \|(I - P_h)(Tf_h - w)\|_H + \|(I - P_h)w\|_H \\ &\le C\|Tf_h - w\|_V + \|(I - P_h)w\|_H \le \varepsilon\end{aligned}$$

Comment on the norms

1. $T : H \to V$ compact + p/w convergence $V \to H$ $\qquad \mathcal{L}(H)$
2. $T : V \to V$ compact + p/w convergence $V \to V$ $\qquad \mathcal{L}(V)$

Standard Galerkin formulation are OK for eigenvalues

Important conclusion

Standard Galerkin formulation: all finite element schemes providing good approximation to the source problem can be successfully applied to the corresponding eigenvalue problem

Is the same true for eigenvalue problems in mixed form?

Laplace eigenproblem in mixed form

⟨Mercier–Osborn–Rappaz–Raviart '81⟩
⟨B.–Brezzi–Gastaldi '97-'00⟩

Find $\lambda \in \mathbb{R}$ and $u \in U$ with $u \not\equiv 0$ such that for some $\sigma \in \Sigma$

$$\begin{cases} (\sigma, \tau) + (\operatorname{div} \tau, u) = 0 & \forall \tau \in \Sigma \qquad \boldsymbol{\sigma} = \mathbf{grad}\, u \\ (\operatorname{div} \sigma, v) = -\lambda(u, v) & \forall v \in U \qquad \operatorname{div} \boldsymbol{\sigma} = -\lambda u \end{cases}$$

Matrix form ($\Sigma_h \subset \Sigma$, $U_h \subset U$)

$$\begin{bmatrix} A & B^\top \\ B & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = -\lambda \begin{bmatrix} 0 & 0 \\ 0 & M_U \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Similarly, one could deal with problems of the type

$$\begin{bmatrix} A & B^\top \\ B & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \lambda \begin{bmatrix} M_\Sigma & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Definition of the solution operator

Source problem

$$\begin{cases} (\sigma,\tau)+(\operatorname{div}\tau,u)=0 & \forall\tau\in\Sigma \\ (\operatorname{div}\sigma,v)=-(g,v) & \forall v\in U \end{cases} \qquad \begin{aligned} \boldsymbol{\sigma} &= \mathbf{grad}\, u \\ -\operatorname{div}\boldsymbol{\sigma} &= g \end{aligned}$$

A first natural (but wrong) definition

$T_1 : U \to \Sigma\times U$
$T_1(g) = (\sigma,u)$

One would like to compute eigenvalues...

$T_2 : (\Sigma\times U)' \to \Sigma\times U$
$T_2(f,g) = (\sigma,u)$ with

$$\begin{cases} (\sigma,\tau)+(\operatorname{div}\tau,u)=\langle f,v\rangle & \forall\tau\in\Sigma \\ (\operatorname{div}\sigma,v)=-(g,v) & \forall v\in U \end{cases}$$

$$T_{\Sigma U}\ \boxed{\begin{matrix}(f,g) \xmapsto{\text{cutoff}} (0,g) \xmapsto{T_2} (\sigma,u) \\ L^2\times L^2 \longrightarrow L^2\times L^2\end{matrix}}$$ is compact

Uniform convergence?

Let's try to follow Kolata's argument

$T_{\Sigma U} - T_{\Sigma U,h} = (I - Q_h) T_{\Sigma U}$

✓ $||(I - Q_h)(\sigma, u)||_{\Sigma \times U} \to 0$ for all $(\sigma, u) \in \Sigma \times U$

✗ $T_{\Sigma U} : L^2 \times L^2 \to \Sigma \times U$ is not compact

✗ $T_{\Sigma U} : \Sigma \times U \to \Sigma \times U$ is not compact either

Standard mixed estimates don't help

$$||\sigma - \sigma_h||_\Sigma + ||u - u_h||_U \leq C \inf_{\tau_h, v_h} \big(\underbrace{||\sigma - \tau_h||_\Sigma}_{O(1)} + \underbrace{||u - v_h||_U}_{O(h)} \big)$$

$$\inf_{\tau_h} \|\sigma - \tau_h\|_{H(\mathrm{div})} \leq C h^s (\|\sigma\|_{H^s} + \|\operatorname{div} \sigma\|_{H^s})$$

Better definition of the solution operator

⟨B.–Brezzi–Gastaldi '97⟩

$T_U : U \to U$

$\sigma \in \Sigma$, $T_U g \in U$ such that

$$\begin{cases} (\sigma, \tau) + (\operatorname{div} \tau, T_U g) = 0 & \forall \tau \in \Sigma \\ (\operatorname{div} \sigma, v) = -(g, v) & \forall v \in U \end{cases}$$

Operator is now compact, but standard mixed estimates don't help again

$$||\sigma - \sigma_h||_\Sigma + ||u - u_h||_U \leq C \inf_{\tau_h, v_h} (\underbrace{||\sigma - \tau_h||_\Sigma}_{O(1)} + \underbrace{||u - v_h||_U}_{O(h)})$$

Fundamental comment

We need an estimate for u_h which does not involve $\operatorname{div} \sigma$

Uniform convergence $||T_U - T_{U,h}|| \to 0$

- Ellipticity in the kernel

$$||\tau_h||^2_{L^2} \geq \alpha ||\tau_h||^2_\Sigma$$

  for all $\tau_h \in \Sigma_h$ s.t. $\{(\operatorname{div} \tau_h, v) = 0,\ \forall v \in U_h\}$
- Fortin operator $\Pi_h : \Sigma^+ \to \Sigma_h$ s.t.

$$(\operatorname{div}(\sigma - \Pi_h \sigma), v) = 0 \quad \forall v \in U_h$$
$$||\Pi_h \sigma||_\Sigma \leq C||\sigma||_{\Sigma^+}$$

Theorem

$$||\sigma - \sigma_h||_{L^2} \leq C\left(||\sigma - \Pi_h\sigma||_{L^2} + (1/\sqrt{\alpha}) \inf_{v_h \in U_h} ||u - v_h||_U\right)$$
$$||u - u_h||_U \leq C\left(\inf_{v_h \in U_h} ||u - v_h||_U + ||\sigma - \sigma_h||_{L^2}\right)$$

Proof

$P = L^2$-projection onto U_h

$$\begin{aligned}
\|\Pi_h\sigma - \sigma_h\|_{L^2}^2 &= (\Pi_h\sigma - \sigma, \Pi_h\sigma - \sigma_h) + (\sigma - \sigma_h, \Pi_h\sigma - \sigma_h) \\
&= (\Pi_h\sigma - \sigma, \Pi_h\sigma - \sigma_h) - (\operatorname{div}(\Pi_h\sigma - \sigma_h), u - Pu) \\
&\le \|\Pi_h\sigma - \sigma\|_{L^2}\|\Pi_h\sigma - \sigma_h\|_{L^2} + \|\operatorname{div}(\Pi_h\sigma - \sigma_h)\|_{L^2}\|u - Pu\|_U \\
&\le \|\Pi_h\sigma - \sigma_h\|_{L^2}\left(\|\Pi_h\sigma - \sigma\|_{L^2} + (1/\sqrt{\alpha})\|u - Pu\|_U\right)
\end{aligned}$$

$$\begin{aligned}
\|Pu - u_h\|_U &\le C\sup_{\tau_h}\frac{(Pu - u_h, \operatorname{div}\tau_h)}{\|\tau_h\|_\Sigma} \\
&\le C\sup_{\tau_h}\frac{(Pu - u, \operatorname{div}\tau_h) + (u - u_h, \operatorname{div}\tau_h)}{\|\tau_h\|_\Sigma} \\
&\le C\left(\|Pu - u\|_U + \sup_{\tau_h}\frac{-(\sigma - \sigma_h, \tau_h)}{\|\tau_h\|_\Sigma}\right) \\
&\le C\left(\|Pu - u\|_U + \|\sigma - \sigma_h\|_{L^2}\right)
\end{aligned}$$

Fortid condition

Definition

The spaces Σ_h, U_h satisfy the Fortid condition if there exists a Fortin operator which converges strongly to the identity operator, namely

$\Pi_h : \Sigma^+ \to \Sigma_h$ s.t.

$$(\mathrm{div}(\sigma - \Pi_h \sigma), v) = 0 \quad \forall v \in U_h$$
$$||\Pi_h \sigma||_\Sigma \le C||\sigma||_{\Sigma^+}$$

$$\boxed{||I - \Pi_h||_{\mathcal{L}(\Sigma^+, L^2)} \to 0}$$

Final convergence result

Theorem

Assume ellipticity in the kernel and Fortid condition
For any $N \in \mathbb{N}$ *define* $\rho_N(h) :]0,1] \to \mathbb{R}$ *as*

$$\rho_N(h) = \sup_{u \in \bigoplus_{i=1}^{m(N)} E_i} \left(\inf_{v_h} ||u - v_h||_U + ||\,\mathbf{grad}\,u - \Pi_h\,\mathbf{grad}\,u||_{L^2} \right)$$

Then $||T_U - T_{U,h}||_{\mathcal{L}(U,U)} \to 0$ *and the following estimates hold true*

$$\sum_{i=1}^{m(N)} |\lambda_i - \lambda_{i,h}| \leq C(\rho_N(h))^2$$

$$\hat{\delta}\left(\bigoplus_{i=1}^{m(N)} E_i, \bigoplus_{i=1}^{m(N)} E_{i,h} \right) \leq C\rho_N(h)$$

Back to the criss-cross (counter)-example

Crisscross mesh
$\Sigma_h = \{\text{continuous p/w linears (componentwise)}\}$
$U_h = \operatorname{div} \Sigma_h \subset \{\text{p/w constants}\}$

Theorem

With the above choice of spaces, there exists a sequence $\{g_h\} \subset U$ *with* $\|g_h\|_0 = 1$ *s.t.*

$$\boxed{||u - u_h||_U \not\to 0}$$

that is $\|T_U - T_{U,h}\|_{\mathcal{L}(U,U)} \not\to 0$

Proof.

Estimate by Qin '94 based on idea of Boland–Nicolaides '85 □

Raviart–Thomas scheme

General mesh (triangles, parallelograms, tetrahedrons, parallelepipeds)

Σ_h : Raviart–Thomas space of order k
U_h : $\mathcal{P}_{k-1}$ or tensor product polynomials $\mathcal{Q}_{k-1}$

Fortid

The interpolant is a Fortin operator

See also Falk–Osborn ’80

Convergence: $O(h^{2k})$ eigenvalues, $O(h^k)$ eigenfunctions

Conclusions for Part 1

- Source problems involve *pointwise* convergence of operators
- Eigenvalue problems involve *uniform* convergence of operators
- Standard Galerkin formulations for elliptic problems have nice compactess properties: pointwise convergence implies uniform convergence
- Mixed formulations may have weaker compactness properties: stronger assumptions for uniform convergence may be required