Mixed Finite Element Methods

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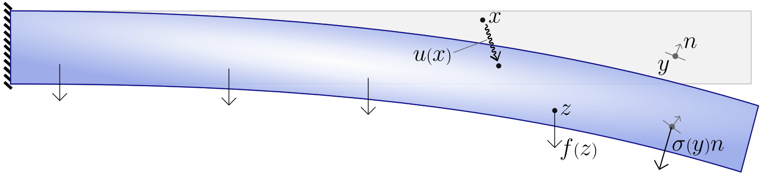
Linear elasticity



Given the load $f : \Omega \to \mathbb{R}^n$, find the displacement $u : \Omega \to \mathbb{R}^n$ and the stress $\sigma : \Omega \to \mathbb{S}^n$ satisfying the constitutive equation

sym grad

$$\sigma = C\,\epsilon\, u,$$

elasticity tensor is SPD from $\mathbb{S}^n \to \mathbb{S}^n$

the equilibrium equation

$$-\operatorname{div} \sigma = f,$$

and boundary conditions like $u = g$ on Γ_d, $\quad \sigma n = t$ on Γ_t.

Displacement formulation

Eliminating σ we get the displacement equation

$$-\operatorname{div} C\epsilon u = f, \quad u = 0 \text{ on } \partial\Omega$$

Multiplying by a test vector field and integrating over Ω by parts we get the weak form: $\quad u \in \mathring{H}^1(\Omega;\mathbb{R}^n)$ satisfies

$$B(u,v) := \int_\Omega C\epsilon u : \epsilon v \, dx = \int_\Omega f \cdot v \, dx \quad \forall v \in \mathring{H}^1(\Omega;\mathbb{R}^n)$$

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This is the Euler–Lagrange equation of a minimization:

$$u = \underset{u \in \mathring{H}^1(\Omega;\mathbb{R}^n)}{\arg\min} \, [\frac{1}{2}B(u,u) - (f,u)],$$

the variational form.

Finite element method

The discrete solution u_h is determined by Galerkin's method using a finite element subspace V_h of $\mathring{H}^1$: $u_h \in V_h$ satisfies

$$B(u_h, v) = (f, v) \quad \forall v \in V_h$$

Equivalently u_h minimizes the energy over V_h.

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Korn's inequality says that the bilinear form is coercive over $\mathring{H}^1$:

$$B(u, u) \geq \gamma \|u\|_1^2.$$

It follows that *for any choice of V_h Galerkin's method is stable* and so quasioptimal:

$$\|u - u_h\|_1 \leq \|B\|\gamma^{-1} \inf_{v \in V_h} \|u - v\|_1.$$

Dual variational principles

Finite elements based on the dual variational principle were advocated from the dawn of finite elements (Fraeijs de Veubeke '65).

Primal variational form

$$u = \operatorname*{arg\,min}_{u \in \mathring{H}^1(\Omega;\mathbb{R}^n)} \left[\frac{1}{2}(C\epsilon u, \epsilon u) - (f, u)\right]$$

Dual variational form

$$\sigma = \operatorname*{arg\,min}_{\substack{\sigma \in H(\operatorname{div};\mathbb{S}^n) \\ -\operatorname{div}\sigma = f}} \frac{1}{2}(A\sigma, \sigma)$$

$A = C^{-1}$

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Dual variational form

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It is not practical to find finite element subspaces that satisfy the constraint $-\mathrm{div}\,\sigma = f$, so we use a Lagrange multiplier:

$$(\sigma, u) = \underset{\substack{\sigma \in H(\mathrm{div};\mathbb{S}^n) \\ u \in L^2(\Omega;\mathbb{R}^n)}}{\arg\mathrm{crit}} \left[\frac{1}{2}(A\sigma, \sigma) + (u, \mathrm{div}\,\sigma + f)\right]$$

The saddle-point problem

$$(\sigma, u) = \underset{\substack{\sigma \in H(\mathrm{div};\mathbb{S}^n) \\ u \in L^2(\Omega;\mathbb{R}^n)}}{\arg\operatorname{crit}} \underbrace{[\frac{1}{2}(A\sigma, \sigma) + (u, \operatorname{div}\sigma + f)]}_{L(\sigma,u)}$$

- *Weak formulation:* Find $(\sigma, u) \in H(\mathrm{div}, \mathbb{S}^n) \times L^2(\Omega; \mathbb{R}^n)$ s.t.

$$(A\sigma, \tau) + (u, \operatorname{div}\tau) = 0 \quad \forall \tau \in H(\mathrm{div}, \mathbb{S}^n),$$
$$(\operatorname{div}\sigma, v) = -(f, v) \quad \forall v \in L^2(\Omega; \mathbb{R}^n)$$

- *Euler–Lagrange equations:* $A\sigma - \epsilon\, u = 0, \quad -\operatorname{div}\sigma = f.$
- Lagrange multiplier is the displacement.
- Critical point is a *saddle point*:

$$L(\sigma, v) \le L(\sigma, u) \le L(\tau, v) \quad \forall \sigma \in H(\mathrm{div}, \mathbb{S}^n),\ v \in L^2(\Omega; \mathbb{R}^n)$$

- Displacement boundary conditions are *natural*, not essential.
- The bilinear form
  $B(\sigma, u; \tau, v) = (A\sigma, \tau) + (u, \operatorname{div}\tau) + (\operatorname{div}\sigma, v)$
  is symmetric, but *not coercive.* $\begin{pmatrix} \mathcal{A} & \mathcal{B}^* \\ \mathcal{B} & 0 \end{pmatrix}$
- Finding stable finite elements is a major challenge.

A motivation: Poisson locking

$\mathcal{P}_1$ Lagrange, 88,374 triangles, dim $V_h = 89,972$, $E = 10$, $\nu = 0.2$

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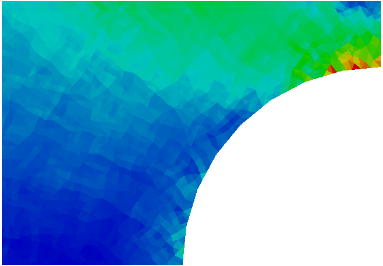
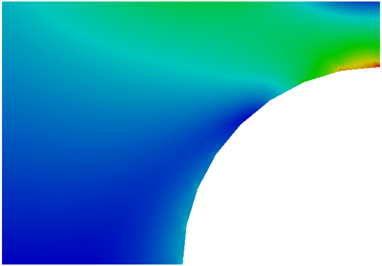
Mixed methods are robust



Displacement method, Lagrange $\mathcal{P}_1$



Mixed method, lowest order AFW

Detail of stress computed for $\nu = 0.4999$

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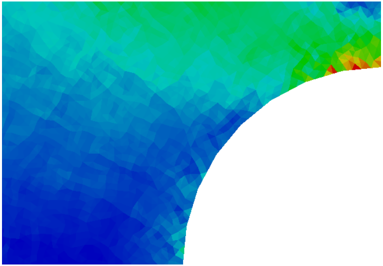

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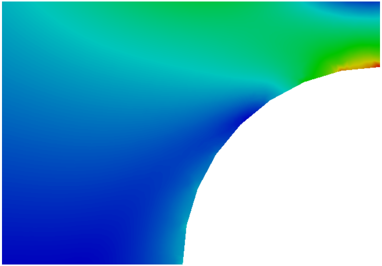

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The problem is that $C \to \infty$, even though A has a perfectly nice limit.

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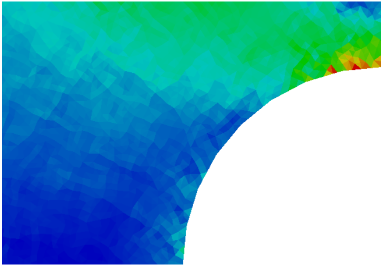

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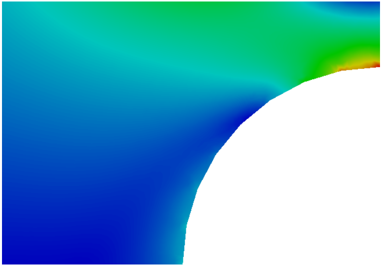

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Other issues with the displacement approach: thin domains, rough coefficients, loss of accuracy for σ, inapplicability to some materials, …

Mixed formulations for various problems

	$\mathcal{A}\sigma + \mathcal{B}^* u = f$	$\mathcal{B}\sigma = g$
elasticity	$A\sigma - \epsilon\, u = 0$	$-\operatorname{div} \sigma = f$
Stokes	$-\Delta u + \operatorname{grad} p = f$	$\operatorname{div} u = 0$
Poisson eq	$u - \operatorname{grad} p = 0$	$-\operatorname{div} u = f$
Poisson-like	$Au - \operatorname{grad} p = 0$	$-\operatorname{div} u + \alpha p = f$
biharmonic	$\sigma - \operatorname{grad} \operatorname{grad} u = 0$	$\operatorname{div} \operatorname{div} \sigma = f$

Brezzi's theorem

Let $a : V \times V \to \mathbb{R}$ and $b : V \times W \to \mathbb{R}$ be bdd bilinear forms on H-spaces. Define $Z = \{ v \in V \mid b(v,w) = 0 \quad \forall w \in W \}$.

Suppose that there exist $\gamma_1, \gamma_2 > 0$ such that

B1: $a(z,z) \geq \gamma_1 \|z\|_V^2 \quad \forall z \in Z$ coercivity over kernel

B2: $\forall w \in W \ \exists 0 \neq v \in V$ s.t. $b(v,w) \geq \gamma_2 \|v\|_V \|w\|_W$ inf-sup condition

Then, for any $F \in V^*, G \in W^*$, $\exists!\, u \in V$, $p \in W$ such that

$$\begin{aligned} a(u,v) + b(v,p) &= F(v), \quad \forall v \in V, \\ b(u,q) &= G(q), \quad \forall q \in q \in W. \end{aligned}$$

$$\begin{pmatrix} \mathcal{A} & \mathcal{B}^* \\ \mathcal{B} & 0 \end{pmatrix} \begin{pmatrix} u \\ p \end{pmatrix} = \begin{pmatrix} F \\ G \end{pmatrix}$$

Moreover,

$$\|u\|_V + \|p\|_W \leq c(\|F\|_V^* + \|G\|_W^*)$$

with c depending only on γ_1, γ_2, and $\|a\|$.

Proof of Brezzi's theorem

We want to show that $\begin{pmatrix} \mathcal{A} & \mathcal{B}^* \\ \mathcal{B} & 0 \end{pmatrix} : V \times W \to V^* \times W^*$ is *invertible*,
where $\mathcal{A} : V \to V^*$, $\mathcal{B} : V \to W^*$ are the linear ops associated to a and b.

If we decompose $V = Z \oplus Z^\perp$ the big operator becomes

$$\begin{pmatrix} \mathcal{A}_{ZZ} & \mathcal{A}_{Z\perp} & 0 \\ \mathcal{A}_{\perp Z} & \mathcal{A}_{\perp\perp} & \mathcal{B}_\perp^* \\ 0 & \mathcal{B}_\perp & 0 \end{pmatrix} : Z \times Z^\perp \times W \to Z^* \times Z^{\perp *} \times W^*.$$

B1 implies that $\mathcal{A}_{ZZ}$ is invertible, and B2 implies that $\mathcal{B}_\perp$ is invertible.
Therefore $\mathcal{B}_\perp^*$ is invertible as well.

Move the last column to the first and interchange the first two rows,
so the operator becomes

$$\begin{pmatrix} \mathcal{B}_\perp^* & \mathcal{A}_{\perp Z} & \mathcal{A}_{\perp\perp} \\ 0 & \mathcal{A}_{ZZ} & \mathcal{A}_{Z\perp} \\ 0 & 0 & \mathcal{B}_\perp \end{pmatrix} : W \times Z \times Z^\perp \to Z^{\perp *} \times Z^* \times W^*.$$

Triangular with invertible diagonal elements $\implies$ invertible.

Discretization

Now suppose we do Galerkin's method with $V_h \subset V$ and $W_h \subset W$.

In order to insure that the discrete system is nonsingular we need Brezzi's B1 and B2 on the discrete level. For stability we need that the constants γ_1 and γ_2 do not degenerate with h.

Define $Z_h = \{ v \in V_h \mid b(v,w) = 0 \quad \forall w \in W_h \}$.

Suppose that there exist $\gamma_1, \gamma_2 > 0$ *independent of h* such that

B1$_h$: $a(z,z) \geq \gamma_1 \|z\|_V^2 \quad \forall z \in Z_h$

B2$_h$: $\forall w \in W_h \; \exists 0 \neq v \in V_h \quad \text{s.t.} \quad b(v,w) \geq \gamma_2 \|v\|_V \|w\|_W$

Then $\exists!\, u_h \in V_h, p_h \in W_h$ solving the Galerkin equations. Moreover,

$$\|u - u_h\|_V + \|p - p_h\|_W \leq c \left(\inf_{v \in V_h} \|u - v\|_V + \inf_{q \in W_h} \|p - q\|_W \right).$$

starting point for other estimates . . .

The rub: finding stable elements

The greatest difficulty will often be the verification of the abstract hypotheses proposed here.

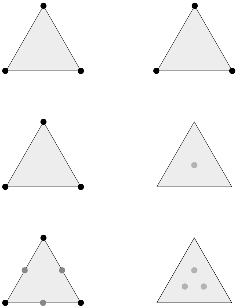
– F. Brezzi, RAIRO 8(2) 1974

B1 and B2 are in opposition, and generally not easy to satisfy simultaneously. Naive choices of elements for mixed formulations are rarely stable.

Things are easier, though not easy, if the bilinear form a is coercive over all of V, since then it is coercive over Z_h with the same constant, and any choice of spaces satisfies B1. The main example of this situation is the Stokes system.

2D Stokes elements: naive elements are unstable

$u \in H^1(\Omega; \mathbb{R}^2)$ $\quad$ $p \in L^2(\Omega)$



$\mathcal{P}_1$–$\mathcal{P}_1$: singular!

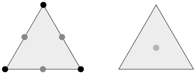
$\mathcal{P}_1$–$\mathcal{P}_0$: unstable!

$\mathcal{P}_2$–$\mathcal{P}_1^{\mathrm{d}}$: unstable except for special meshes

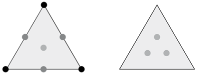
pressure from $\mathcal{P}_2$–$\mathcal{P}_1^{\mathrm{d}}$
uniform mesh of 7,024 elements

Stable 2D Stokes elements

$\mathcal{P}_2$–$\mathcal{P}_0$



$\mathcal{P}_2$ bubble–$\mathcal{P}_1^{\mathrm{d}}$



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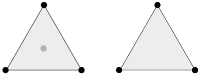
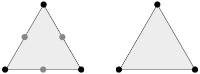


Taylor–Hood



pressure from $\mathcal{P}_2$ bubble–$\mathcal{P}_1^{\mathrm{d}}$
uniform mesh of 7,024 elements

Mixed Laplacian (=elasticity) in 1D

$$u - p' = 0, \quad -u' = f \qquad \text{on } (-1,1)$$

$H(\text{div})$ in 1D

$$B(u,p;v,q) := \int_{-1}^{1} (u\,v + p\,v' + u'q)\,dx = -\int_{-1}^{1} f\,q\,dx \quad \forall v \in H^1,\ q \in L^2$$

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P_1-P_1: singular

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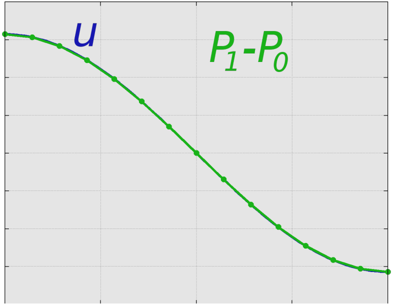
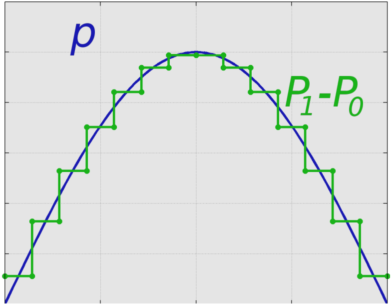
P_1-P_1: singular $\qquad$ P_1-P_0: stable!





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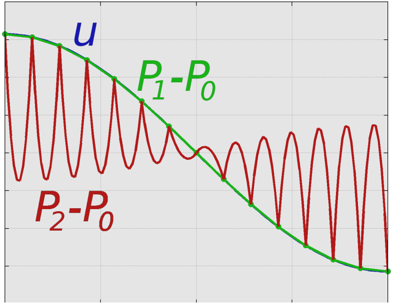
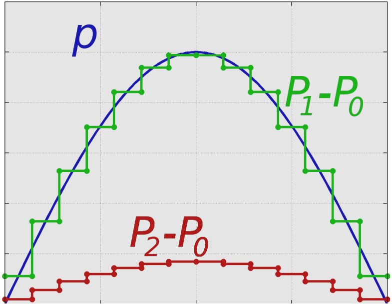
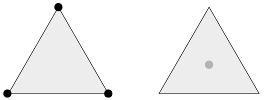
P_1-P_1: singular $\qquad$ P_1-P_0: stable! $\qquad$ P_2-P_0: unstable!



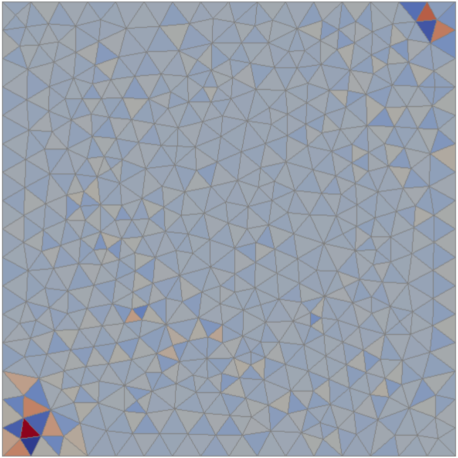


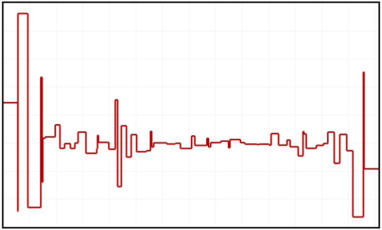
Mixed Laplacian (Darcy flow) in 2D computed with $\mathcal{P}_1$-$\mathcal{P}_0$

$$u = \frac{k}{\mu}\operatorname{grad} p, \quad \operatorname{div} u = f$$



unstable!



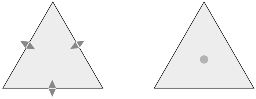


pressure field

Darcy flow computed with Raviart–Thomas elements

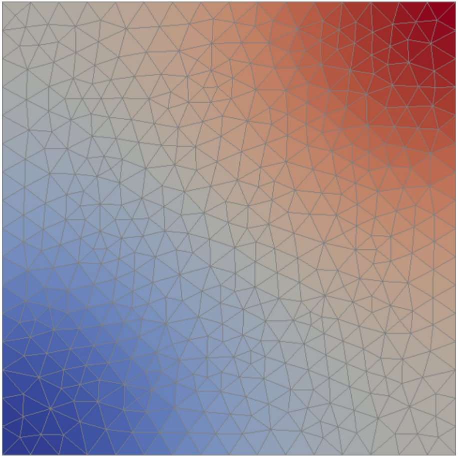
Velocity shape functions: $V(T) = \{ (a_1 + bx_1, a_2 + bx_2) \mid a_1, a_2, b \in \mathbb{R} \}$

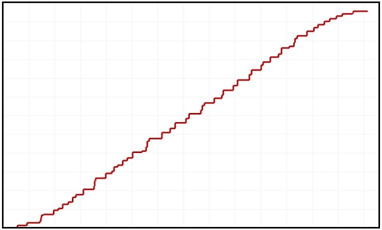
Degrees of freedom: $u \mapsto \int_e u \cdot n$



stable

$V_h \times S_h \subset H(\mathrm{div}) \times L^2$



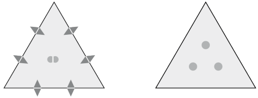


pressure field

Higher order Raviart–Thomas elements

Velocity shape functions: $V(T) = \{ (a_1 + bx_1, a_2 + bx_2) \mid a_1, a_2, b \in \mathcal{P}_1 \}$

Degrees of freedom:



stable

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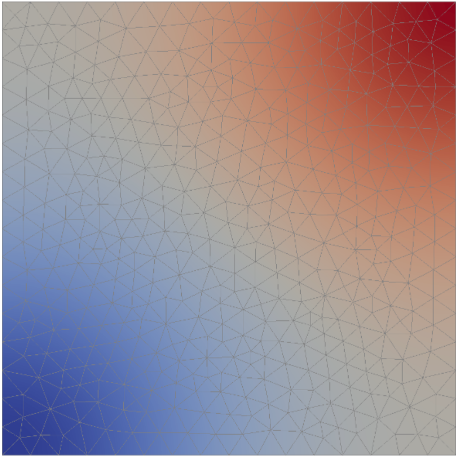

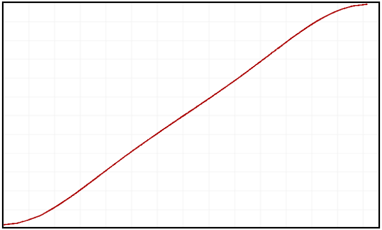

pressure field

Summary of the Raviart–Thomas elements

Let $r \geq 1$. For any triangle T define

homogeneous polys

- Shape functions: $\mathcal{P}_r^-(T) := \mathcal{P}_{r-1}(T, \mathbb{R}^n) \oplus \mathcal{H}_{r-1}(T)(x_1, x_2)$
- DOFs: $\tau \mapsto \int_e \tau \cdot n\, p\, ds, \quad p \in \mathcal{P}_{r-1}(e),$ for each edge e
  $\tau \mapsto \int_T \tau \cdot \rho\, dx, \quad \rho \in \mathcal{P}_{r-2}(T, \mathbb{R}^2)$
- # DOFs $= \dim \mathcal{P}_r^-(T)$ and the DOFs are unisolvent
- Let V_h be the assembled FE space for some mesh. The DOFs enforce normal continuity, so $V_h \subset H(\mathrm{div})$
- Let $W_h = \mathcal{P}_{r-1}^{\mathrm{disc}}$. Then $\mathrm{div}\, V_h \subset W_h$. It follows that $Z_h \subset Z$ and so B1 holds uniformly in h.
- The projection operator coming from the DOFs satisfies

$$\mathrm{div}\, \pi_h \tau = P_{W_h} \mathrm{div}\, \tau, \quad \tau \in H^1(\Omega; \mathbb{R}^2).$$

$$\begin{array}{ccc} H^1 & \xrightarrow{\mathrm{div}} & L^2 \\ \downarrow \pi_h & & \downarrow P_{W_h} \\ V_h & \xrightarrow{\mathrm{div}} & W_h \end{array}$$

  From this B2 follows. Stability!
- The following estimates hold:

$$\|\sigma - \sigma\| \leq ch^r \|\sigma\|_r, \quad \|\mathrm{div}(\sigma - \sigma_h)\| \leq ch^r \|\mathrm{div}\, \sigma\|_r, \quad \|u - u_h\| \leq ch^r \|u\|_{r+1}$$

- Using duality: $\|u - u_h\| \leq ch\|u\|_r$ if Ω is convex and $r > 1$
- Carries over to n dimensions