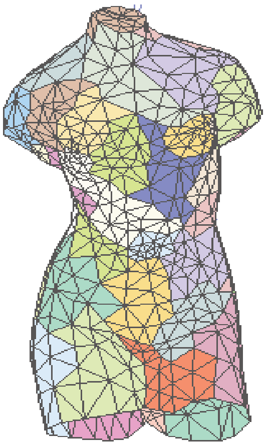
Domain Decomposition solvers (FETI)

a random walk in history
and some current trends



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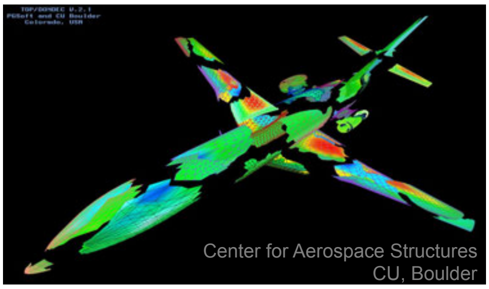
39th Woudschoten Conference,

organised by the *Werkgemeenschap Scientific Computing (WSC)*

DIVIDE ET IMPERA



When splitting the problem in parts and asking different cpu's (or threads) to take care of subproblems, will the problem be solved faster ?

FETI, Primal Schur (Balancing) method

around 1990 …….. basic methods, mesh decomposer technology

1990-2001 ……… improvements

- preconditioners, coarse grids
- application to Helmholtz, dynamics, non-linear ...

Here the concepts are outlined using some mechanical interpretation.
For mathematical details, see lecture of Axel Klawonn.

"

…

Pure mathematicians sometimes are satisfied with showing that the non-existence of a solution implies a logical contradiction, while engineers might consider a numerical result as the only reasonable goal.

Such one sided views seem to reflect human limitations rather than objective values.

In itself mathematics is an indivisible organism uniting theoretical contemplation and active application.

… "

R. Courant

in *Variational Methods for the solution of problems of equilibrium and vibrations*
Bulletin of American Mathematical Society, 49, pp.1-23, 1943

Here the concepts are outlined using some mechanical interpretation.
For mathematical details, see lecture of Axel Klawonn.

Content

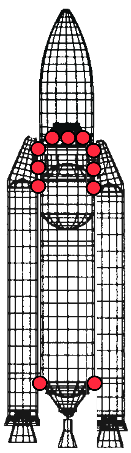
Part 1

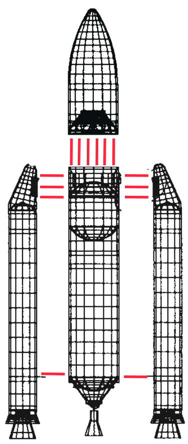
1. Non-overlapping DD: Primal and Dual approaches
2. The FETI saga
   - FETI-1 (natural coarse grid – lumped preconditioner)
   - FETI-2 and FETI-DP (auxiliary coarse grids)

Part 2

3. FETI for heterogeneous problems
   - Scaled preconditioners
   - FETI-Geneo (bad modes)
   - FETI-Simultaneous
4. FETI for concurrent multiscale

Domain Decomposition: Primal / Dual Assembly



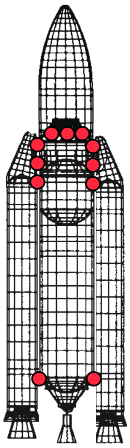


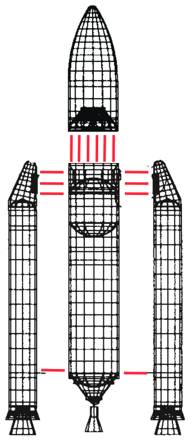
$$\begin{bmatrix} \boldsymbol{K}^{(1)} & & 0 \\ & \ddots & \\ 0 & & \boldsymbol{K}^{(N_s)} \end{bmatrix} \begin{bmatrix} \boldsymbol{u}^{(1)} \\ \vdots \\ \boldsymbol{u}^{(N_s)} \end{bmatrix} = \begin{bmatrix} \boldsymbol{f}^{(1)} \\ \vdots \\ \boldsymbol{f}^{(N_s)} \end{bmatrix}$$

signed Boolean matrices

$$\begin{bmatrix} \boldsymbol{K}^{(1)} & & 0 & \boldsymbol{B}^{(1)T} \\ & \ddots & & \vdots \\ 0 & & \boldsymbol{K}^{(N_s)} & \boldsymbol{B}^{(N_s)T} \\ \boldsymbol{B}^{(1)} & \dots & \boldsymbol{B}^{(N_s)} & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{u}^{(1)} \\ \vdots \\ \boldsymbol{u}^{(N_s)} \\ \lambda \end{bmatrix} = \begin{bmatrix} \boldsymbol{f}^{(1)} \\ \vdots \\ \boldsymbol{f}^{(N_s)} \\ 0 \end{bmatrix}$$

Domain Decomposition: Primal / Dual Assembly

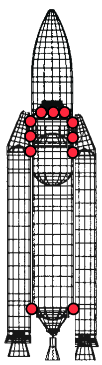


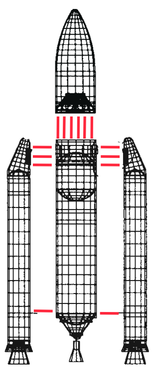


Block-diagonal notation

$$\begin{bmatrix} K^{(1)} & & 0 \\ & \ddots & \\ 0 & & K^{(N_s)} \end{bmatrix} \begin{bmatrix} u^{(1)} \\ \vdots \\ u^{(N_s)} \end{bmatrix} = \begin{bmatrix} f^{(1)} \\ \vdots \\ f^{(N_s)} \end{bmatrix} \qquad \begin{bmatrix} K & B^T \\ B & \mathbf{0} \end{bmatrix} \begin{bmatrix} u \\ \lambda \end{bmatrix} = \begin{bmatrix} f \\ \mathbf{0} \end{bmatrix}$$

Iterative solution of interface problem





Iterate on interface dofs $\mathbf{u}_b$

solve for internal dofs $\mathbf{u}_i$

end *(when interface in equilibrium)*

Primal Schur and BDD
[Letallec *et al.*, 91]

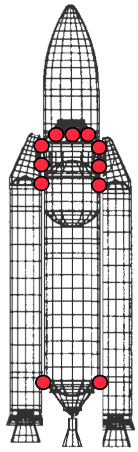
Iterate on interface forces $\boldsymbol{\lambda}$

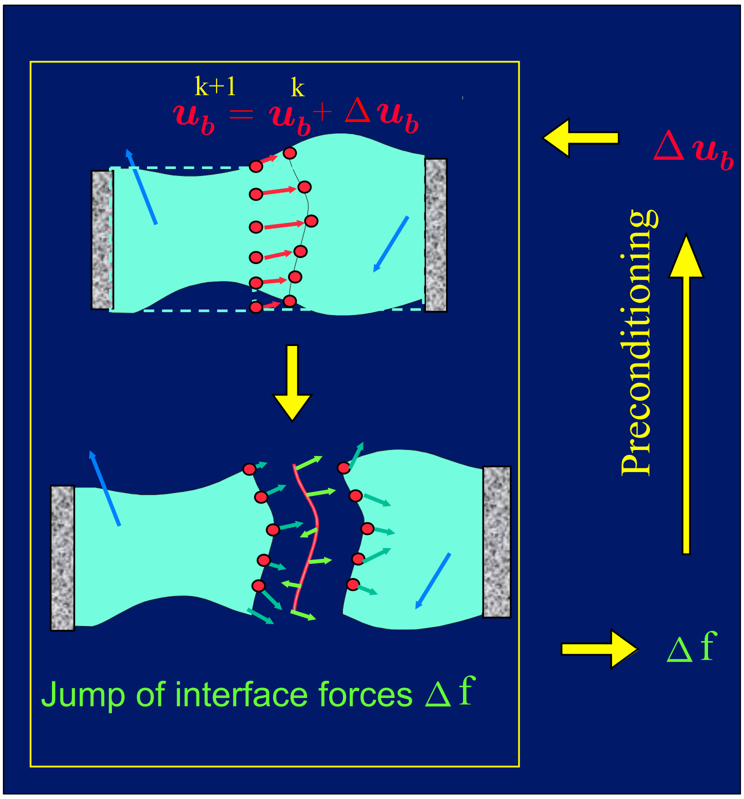
solve for domain dofs $\mathbf{u}$

end *(when interface compatible)*

FETI
(Finite Element Tearing and Interconnecting)
[Farhat- Roux, 91]

Primal Schur: iteration



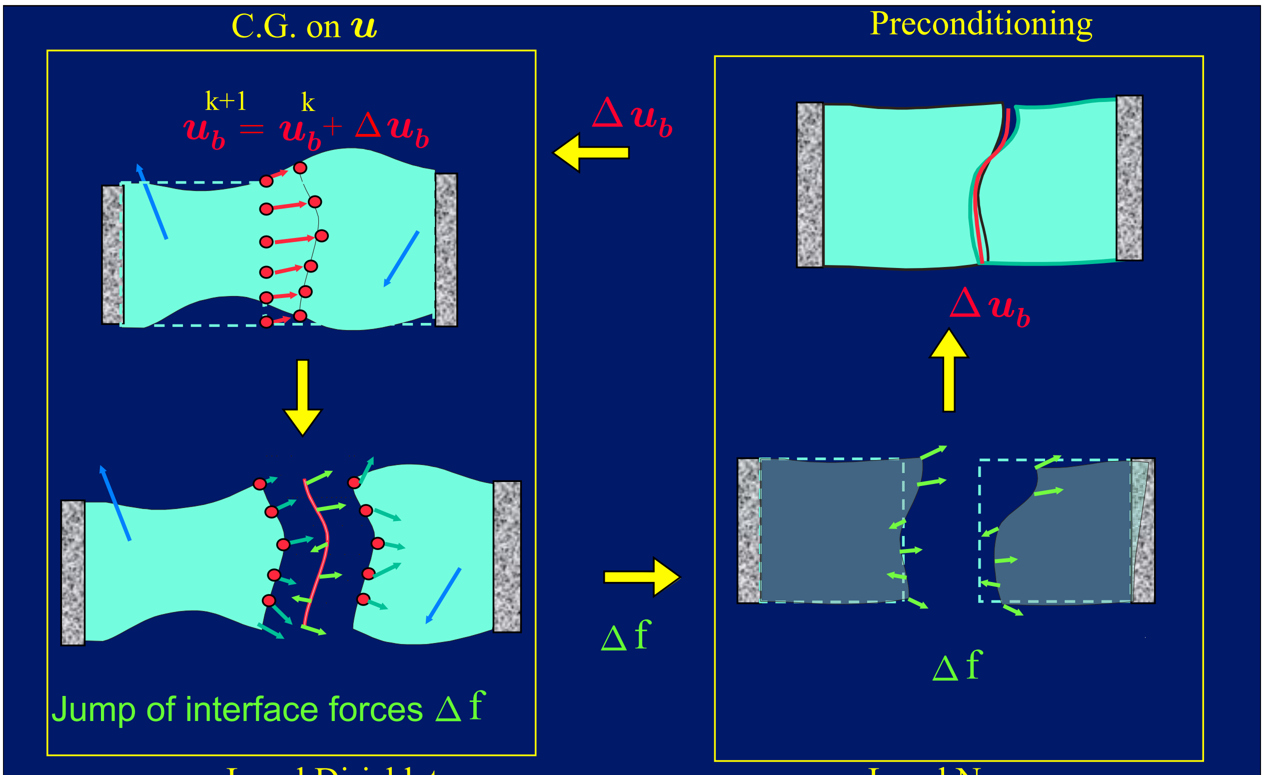


Iterate on interface dof $\boldsymbol{u}_b$

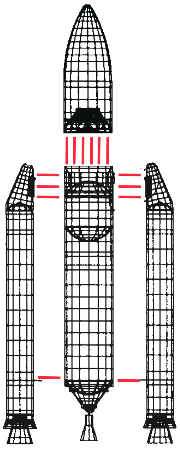
| Solve for internal dof $\boldsymbol{u}_i$

end *(when interface in equilibrium)*

Primal Schur: iteration



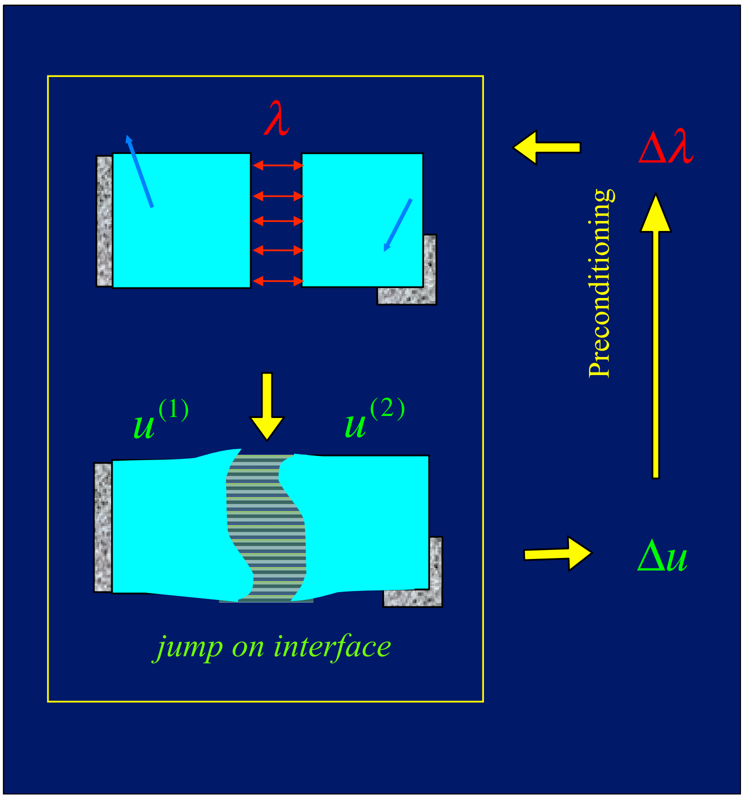
Dual Schur (FETI): iteration



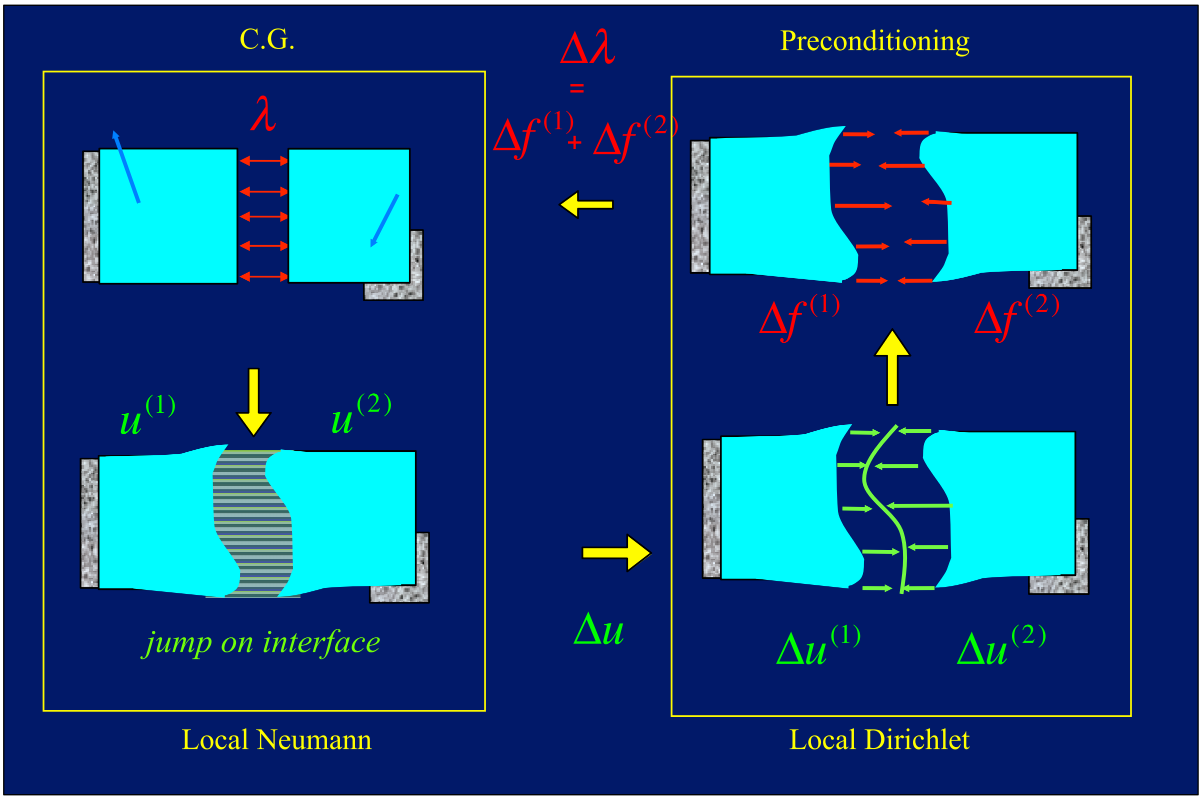
Iterate on interface forces $\boldsymbol{\lambda}$

solve for domain dofs $\mathbf{u}$

end *(when interface compatible)*



Dual Schur (FETI): iteration



The primal and dual Schur complement approaches are very similar.

Most of the "tricks" used in the one method can be applied for the other.
So which method to use is nearly a matter of religion …

There exists some subtle differences such as

- treatment of cross-points (interface nodes on more then 2 domains)
- determination of an initial estimate

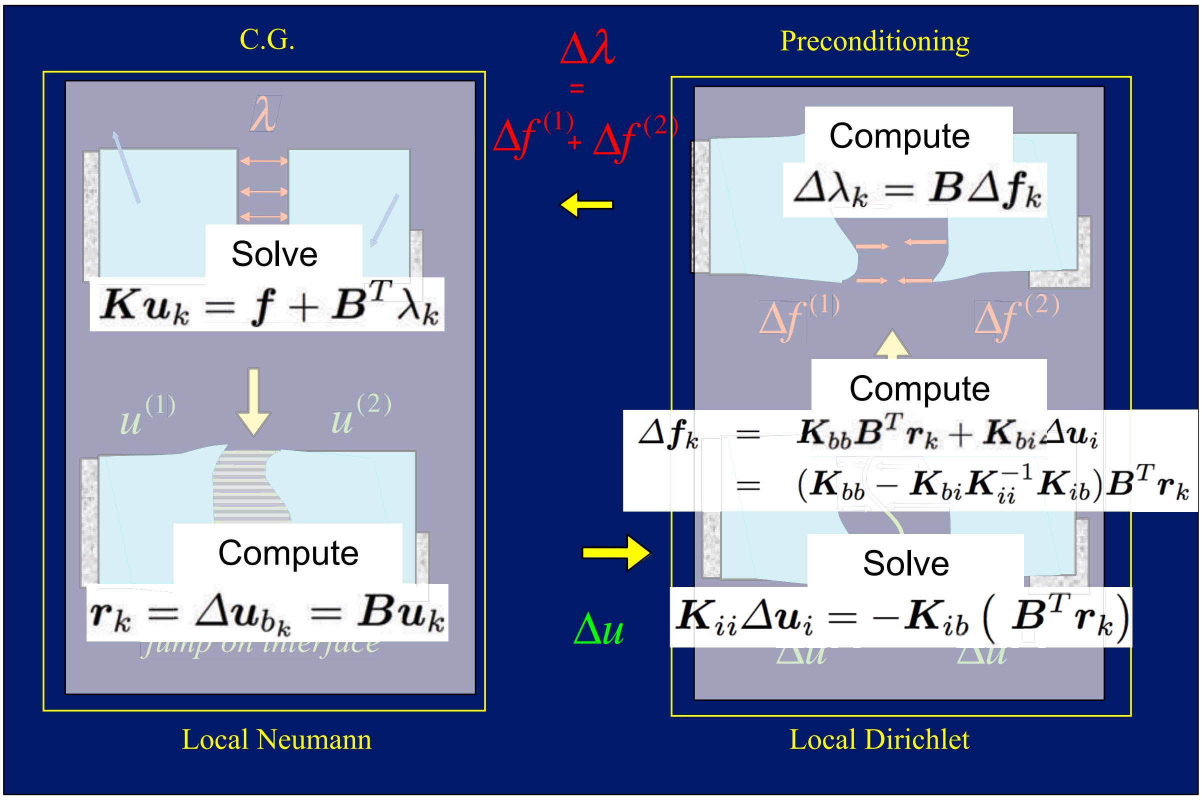
[Klawonn 02, Gosselet *et al.* 03, Gosselet-Rey 06]

Here we outline only the dual approach (FETI)
but one finds many publications on similar developments for the primal approach.

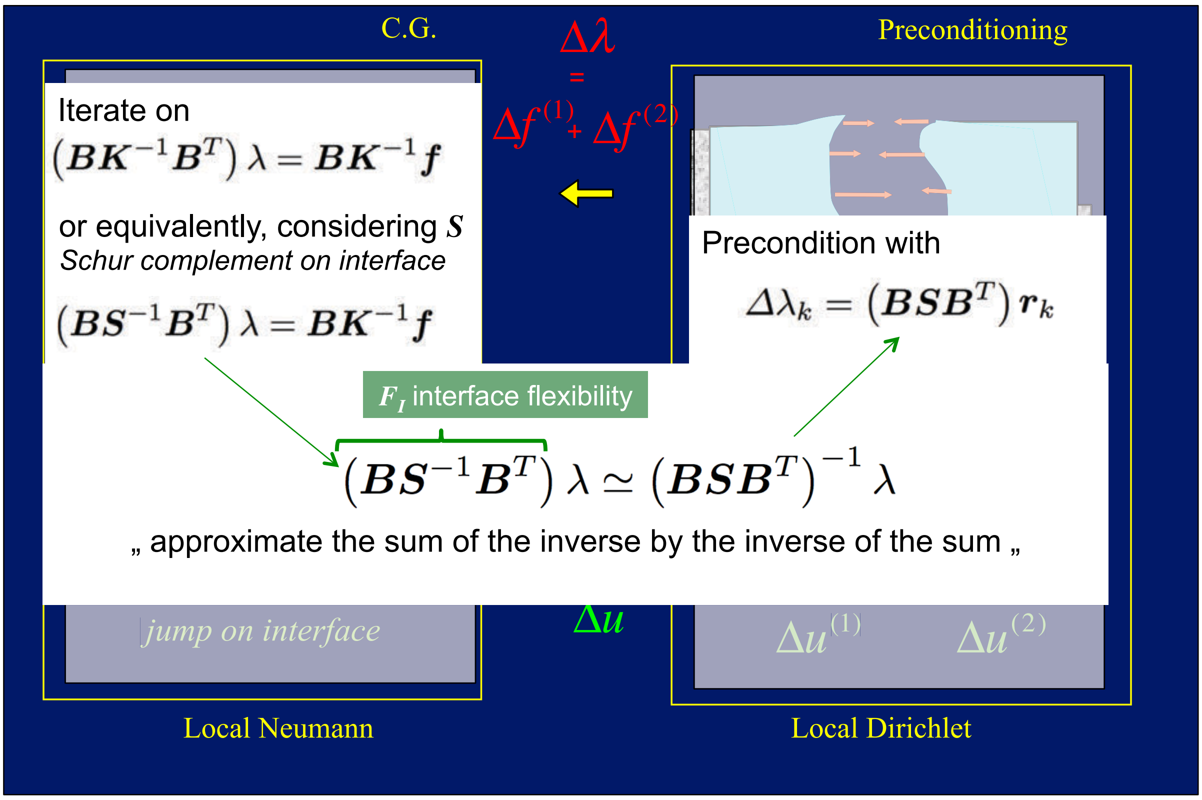
Content

Part 1

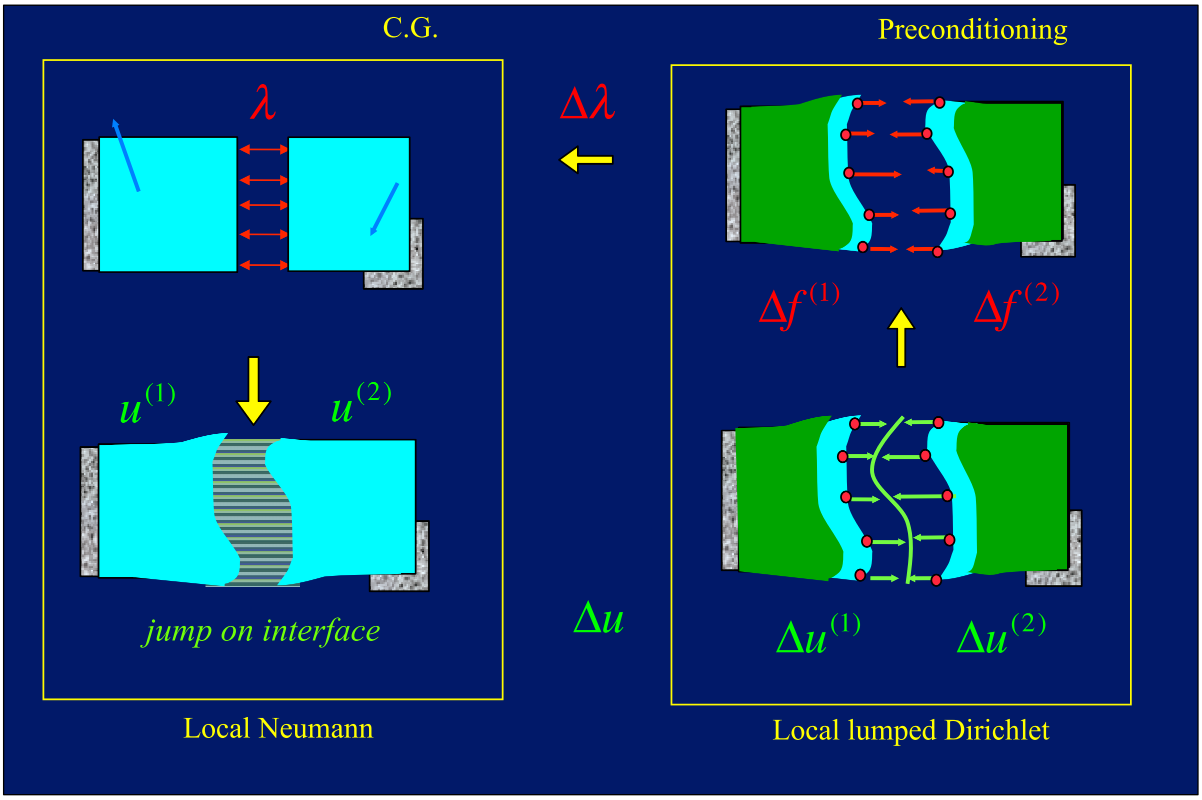
Dual Schur (FETI): iteration



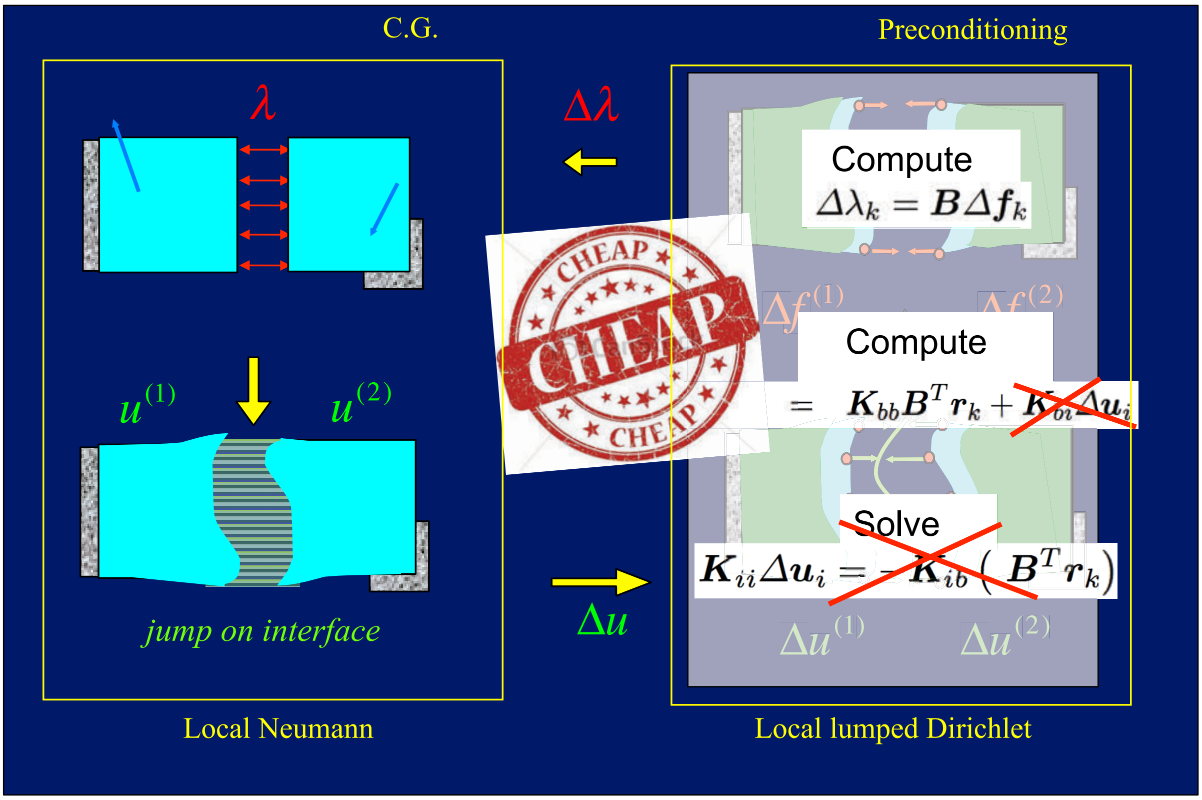
Dual Schur (FETI): iteration



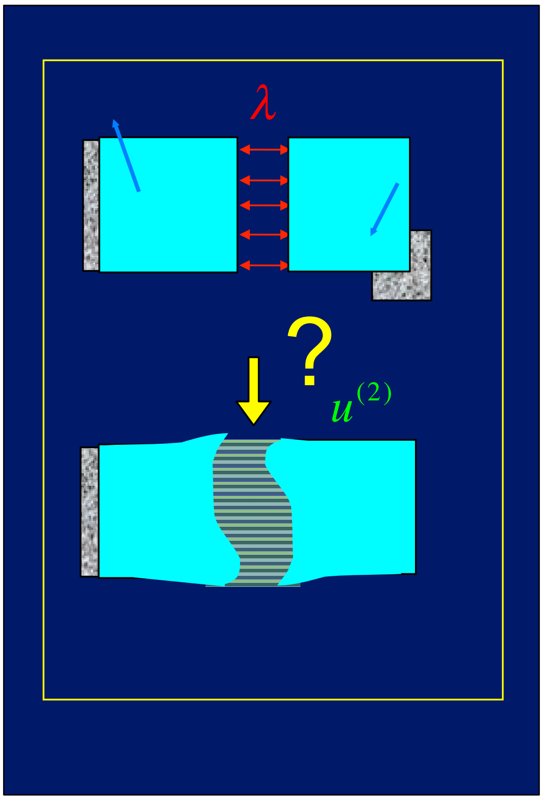
Dual Schur (FETI): lumped preconditioner



Dual Schur (FETI): lumped preconditioner



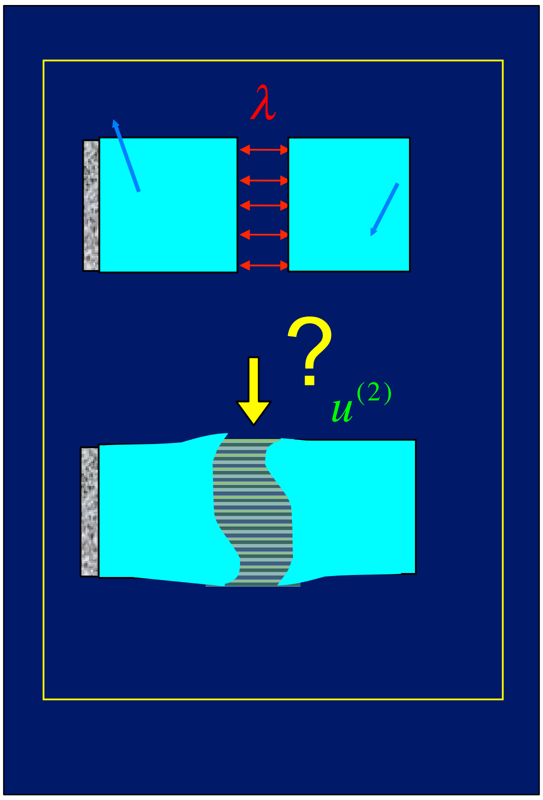
The basic FETI and its natural coarse grid



$$\begin{bmatrix} \mathbf{K} & \mathbf{B}^T \\ \mathbf{B} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ \boldsymbol{\lambda} \end{bmatrix} = \begin{bmatrix} \mathbf{f} \\ \mathbf{0} \end{bmatrix}$$

Not enough constraints
Badly defined local problems
Singular K

The basic FETI and its natural coarse grid



$$\begin{bmatrix} \mathbf{K} & \mathbf{B}^T \\ \mathbf{B} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ \boldsymbol{\lambda} \end{bmatrix} = \begin{bmatrix} \mathbf{f} \\ \mathbf{0} \end{bmatrix}$$

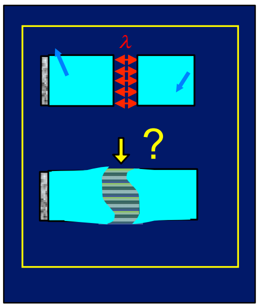
Force the inner problem to have a bit of compatibility to make it regular: at every iteration *enforce a weak compatibility*

$$\begin{bmatrix} \boldsymbol{K} & \boldsymbol{B}^T\boldsymbol{T} & \boldsymbol{B}^T \\ \boldsymbol{T}^T\boldsymbol{B} & 0 & 0 \\ \boldsymbol{B} & 0 & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{u} \\ \beta \\ \lambda \end{bmatrix} = \begin{bmatrix} \boldsymbol{f} \\ \mathbf{o} \\ \mathbf{o} \end{bmatrix}$$

$\boldsymbol{T}$ such that local problems are well posed
→ weak coupling of domains
→ small coarse grid

This interpretation of FETI explained in [Rixen *et al.* 01]

The basic FETI and its natural coarse grid



$$\begin{bmatrix} K & B^T T & B^T \\ T^T B & 0 & 0 \\ B & 0 & 0 \end{bmatrix} \begin{bmatrix} u \\ \beta \\ \lambda \end{bmatrix} = \begin{bmatrix} f \\ \mathbf{o} \\ \mathbf{o} \end{bmatrix}$$

How to choose T such that the inner problem is well-posed?

If $T=0$, the problem is singular: nullspace = Rigid Body Modes of the floating domains

$$\begin{bmatrix} K^{(1)} & 0 & & \\ & K^{(2)} & & \\ & & \ddots & \\ 0 & & & K^{(N_s)} \end{bmatrix} \begin{bmatrix} 0 & 0 & \cdots \\ R^{(2)} & 0 & \\ 0 & R^{(3)} & \\ \vdots & & \end{bmatrix} = KR = 0$$

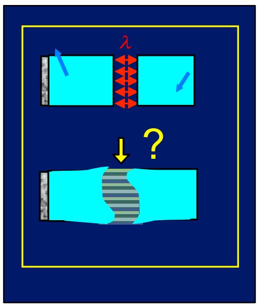
So if there exists a nullspace for the inner problem, it must have the form: $u_{null} = R\gamma$

How to choose $\mathbf{T}$? such that
imposing $T^T B u_{null} = T^T B R \gamma = 0$
implies $u_{null} = 0$

$\Rightarrow$

$T^T BR$
must be full column rank

The basic FETI and its natural coarse grid



$$\begin{bmatrix} K & B^T T & B^T \\ T^T B & 0 & 0 \\ B & 0 & 0 \end{bmatrix} \begin{bmatrix} u \\ \beta \\ \lambda \end{bmatrix} = \begin{bmatrix} f \\ \mathbf{o} \\ \mathbf{o} \end{bmatrix}$$

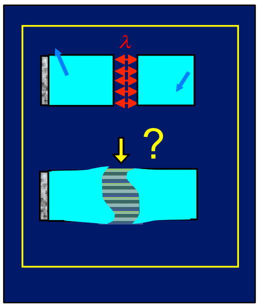
$T^T BR$
must be full column rank

Note: BR is full column rank
(otherwise singular global problem)
[Rixen 98]

A „natural“ choice for a *minimum weak compatibilty* is $T = BR$

$$\begin{bmatrix} K & B^T (BR) & B^T \\ (R^T B^T)\, B & 0 & 0 \\ B & 0 & 0 \end{bmatrix} \begin{bmatrix} u \\ \beta \\ \lambda \end{bmatrix} = \begin{bmatrix} f \\ \mathbf{o} \\ \mathbf{o} \end{bmatrix}$$

The FETI and its natural coarse grid



$$
\begin{bmatrix} K & B^T(BR) & B^T \\ (R^T B^T)\,B & 0 & 0 \\ B & 0 & 0 \end{bmatrix}
\begin{bmatrix} u \\ \beta \\ \lambda \end{bmatrix}
=
\begin{bmatrix} f \\ o \\ o \end{bmatrix}
$$

One requires at each iteration on the interface forces λ that the compatibility is satisfied on average (trace of the rigid-body modes) – *„Natural coarse grid“*

The average compatibility is enforced by determining the interface forces in *Image(**BR**)* such that the interface forces are orhtogonal to *null(**K**)* – *„self-equilibrated“*

enforced by projecting the iterates such that $R^T\left(B^T\lambda_k - f\right) = 0$

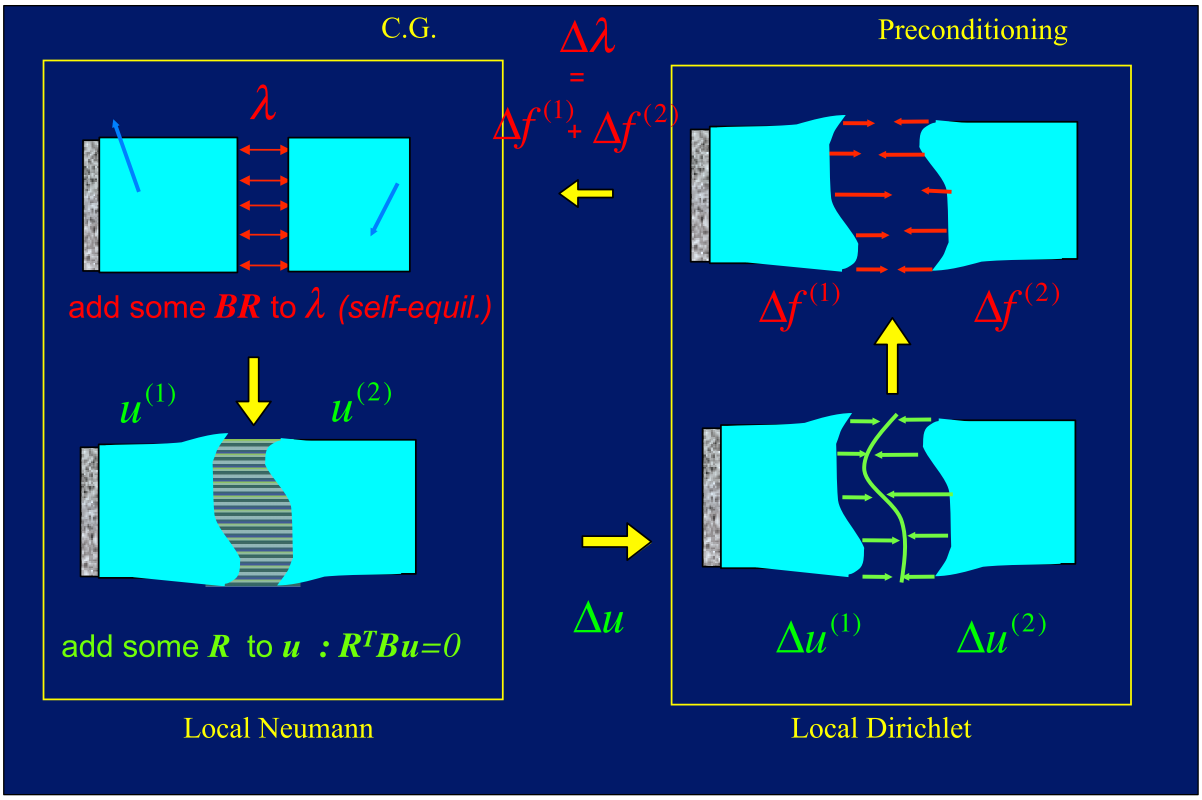
The compatibility must be satisfied in the subspace ***BR*** at every iteration

→ „coarse grid“

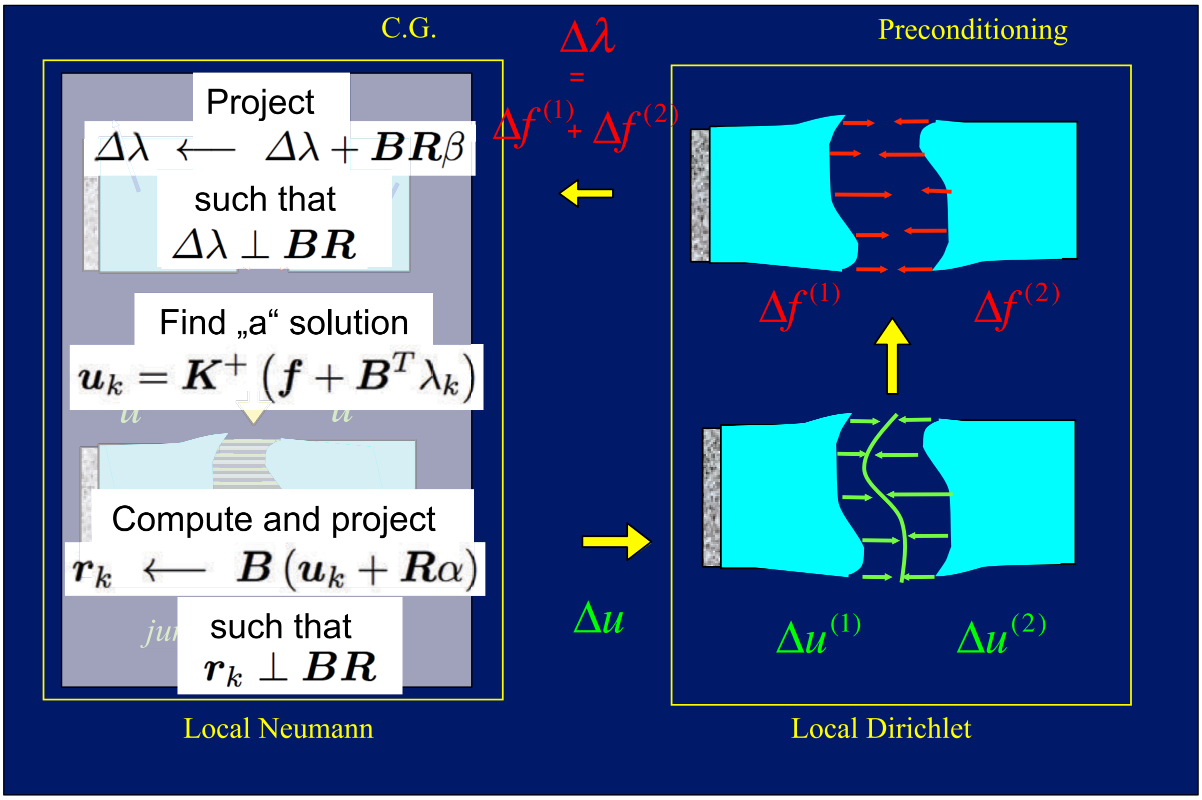
→ ensures that convergence does not deteriorate when nbr. of subdomains increases

FETI-1 *[Farhat-Roux 91]*

FETI: iteration with floating domains



FETI: iteration with floating domains



Content

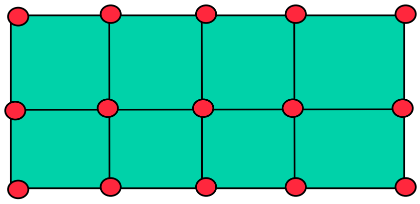
Part 1

1. Non-overlapping DD: Primal and Dual approaches
2. The FETI saga: evolution of the algorithm in the realm of engineering
   - FETI-1 (natural coarse grid – lumped preconditioner)
   - FETI-2 and FETI-DP (auxiliary coarse grids)
3. FETI for heterogeneous problems
   - Scaled preconditioners
   - FETI-Geneo (bad modes)
   - FETI-Simultaneous
4. FETI for concurrent multiscale

from FETI-1 to FETI-2

Sometimes, more compatibility constraints need to be satisfied exactly at every iteration to ensure good convergence on interface

(e.g. corners for bi-harmonic problems: plates, shells)



$$\begin{bmatrix} K & B^T(BR) & B^T \\ (R^TB^T)B & 0 & 0 \\ B & 0 & 0 \end{bmatrix} \begin{bmatrix} u \\ \beta \\ \lambda \end{bmatrix} = \begin{bmatrix} f \\ \mathbf{o} \\ \mathbf{o} \end{bmatrix}$$

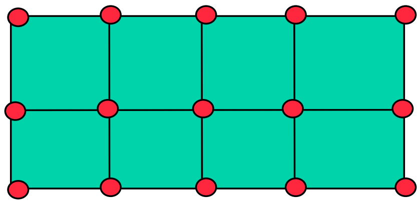
$$\begin{bmatrix} K & B^T(BR) & B^TC & B^T \\ (R^TB^T)B & 0 & 0 & 0 \\ C^TB & 0 & 0 & 0 \\ B & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u \\ \beta \\ \mu \\ \lambda \end{bmatrix} = \begin{bmatrix} f \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

C is a Boolean matrix „picking out“ the compatibility conditions at corners

μ : interface forces at corners

from FETI-1 to FETI-2

$$\begin{bmatrix} K & B^T(BR) & B^TC & B^T \\ (R^TB^T)B & 0 & 0 & 0 \\ C^TB & 0 & 0 & 0 \\ B & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u \\ \beta \\ \mu \\ \lambda \end{bmatrix} = \begin{bmatrix} f \\ 0 \\ 0 \\ 0 \end{bmatrix}$$



It defines an auxilary („non-natural“) coarse grid : *Deflation*

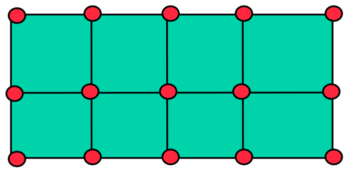
The local problems with weak compatibility can be seen as an inner FETI problem

***FETI-2** (two-level FETI)*

[Farhat-Mandel 98]

from FETI-1 to FETI-2 ... to FETI-DP

$$\begin{bmatrix} \boldsymbol{K} & \boldsymbol{B}^T(\boldsymbol{B}\boldsymbol{R}) & \boldsymbol{B}^T\boldsymbol{C} & \boldsymbol{B}^T \\ (\boldsymbol{R}^T\boldsymbol{B}^T)\boldsymbol{B} & 0 & 0 & 0 \\ \boldsymbol{C}^T\boldsymbol{B} & 0 & 0 & 0 \\ \boldsymbol{B} & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{u} \\ \beta \\ \mu \\ \lambda \end{bmatrix} = \begin{bmatrix} \boldsymbol{f} \\ 0 \\ 0 \\ 0 \end{bmatrix}$$



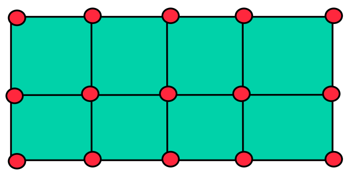
FETI-2

If there are enough corner links to fix the subdomains
the "average" compatibility is not required for the regularity of the inner problem:

$$\begin{bmatrix} \boldsymbol{K} & \boldsymbol{B}^T\boldsymbol{C} & \boldsymbol{B}^T \\ \boldsymbol{C}^T\boldsymbol{B} & 0 & 0 \\ \boldsymbol{B} & 0 & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{u} \\ \mu \\ \lambda \end{bmatrix} = \begin{bmatrix} \boldsymbol{f} \\ 0 \\ 0 \end{bmatrix}$$

from FETI-1 to FETI-2 ... to FETI-DP

$$\begin{bmatrix} \boldsymbol{K} & \boldsymbol{B}^T\boldsymbol{C} & \boldsymbol{B}^T \\ \boldsymbol{C}^T\boldsymbol{B} & 0 & 0 \\ \boldsymbol{B} & 0 & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{u} \\ \mu \\ \lambda \end{bmatrix} = \begin{bmatrix} \boldsymbol{f} \\ 0 \\ 0 \end{bmatrix}$$

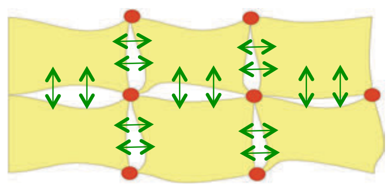


This partial compatibility can be enforced by *assembly* on the corners,
and iterating only for the interface forces for the remaining interface nodes:

$$\begin{bmatrix} \boldsymbol{L}_c^T\boldsymbol{K}\boldsymbol{L}_c & \boldsymbol{B}_r^T \\ \boldsymbol{B}_r & 0 \end{bmatrix} \begin{bmatrix} \bar{\boldsymbol{u}} \\ \lambda_r \end{bmatrix} = \begin{bmatrix} \bar{\boldsymbol{f}} \\ 0 \end{bmatrix}$$

partially assembled and regular

interface forces for non-assembled interface



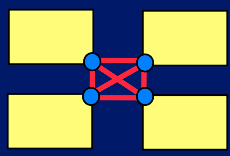
FETI-DP

[*Farhat et al.* 00]
[Farhat *et. al* 01]

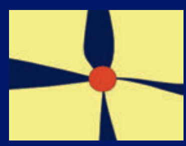
Advantage of FETI-DP vs. FETI-2

+

- No need to find local nullspace
- Less connecting variables at corners



FETI-2



FETI-DP

- “average” compatibility not enforced

smaller coarse grid

Smaller cost per iteration

Slower convergence

FETI-DP scalable in 2D
NOT scalable in 3D

-

- only point-wise compatibility enforced in inner poblem

“non-smooth” coarse grid

1

Episode 4: yet another coarse grid to FETI-DP

$$\begin{bmatrix} L_c^T K L_c & B_r^T \\ B_r & 0 \end{bmatrix} \begin{bmatrix} \bar{u} \\ \lambda_r \end{bmatrix} = \begin{bmatrix} \bar{f} \\ 0 \end{bmatrix}$$

FETI-DP ***not scalable*** in 3D

add an auxiliary coarse grid

$$\begin{bmatrix} L_c^T K L_c & B_r^T G & B_r^T \\ G^T B_r & 0 & 0 \\ B_r & 0 & 0 \end{bmatrix} \begin{bmatrix} \bar{u} \\ \gamma \\ \lambda_r \end{bmatrix} = \begin{bmatrix} \bar{f} \\ 0 \\ 0 \end{bmatrix}$$

⇨ scalable for several “smooth” choices of G

(see e.g. [Farhat *et. al* 01]
[Klawonn-Widlund 2006]
[Klawonn-Rheinbach 2007])

- If $G=BR$, FETI-DP mathematically equivalent to FETI-2
- avoids having to deal with floating domains
  - \+ no numerical issue when detecting singularity
  - \- no profit from a „natural“ coarse grid

The FETI family saga - Summary

FETI-1 :
. Dual assembly,
. CG on interface forces,
. natural coarse grid
of rigid body modes

$$\begin{bmatrix} \boldsymbol{K} & \boldsymbol{B}^T(\boldsymbol{B}\boldsymbol{R}) & \boldsymbol{B}^T \\ (\boldsymbol{R}^T\boldsymbol{B}^T)\,\boldsymbol{B} & 0 & 0 \\ \boldsymbol{B} & 0 & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{u} \\ \beta \\ \lambda \end{bmatrix} = \begin{bmatrix} \boldsymbol{f} \\ \mathbf{o} \\ \mathbf{o} \end{bmatrix}$$

FETI-2 :
. additional auxiliary
. coarse grid (Deflation)

$$\begin{bmatrix} \boldsymbol{K} & \boldsymbol{B}^T(\boldsymbol{B}\boldsymbol{R}) & \boldsymbol{B}^T\boldsymbol{C} & \boldsymbol{B}^T \\ (\boldsymbol{R}^T\boldsymbol{B}^T)\,\boldsymbol{B} & 0 & 0 & 0 \\ \boldsymbol{C}^T\boldsymbol{B} & 0 & 0 & 0 \\ \boldsymbol{B} & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{u} \\ \beta \\ \mu \\ \lambda \end{bmatrix} = \begin{bmatrix} \boldsymbol{f} \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

FETI-DP :
. enough compatibility enforced in a primal way so that local problems are regular
. additional smooth coarse

$$\begin{bmatrix} \boldsymbol{L}_c^T\boldsymbol{K}\boldsymbol{L}_c & \boldsymbol{B}_r^T\boldsymbol{G} & \boldsymbol{B}_r^T \\ \boldsymbol{G}^T\boldsymbol{B}_r & 0 & 0 \\ \boldsymbol{B}_r & 0 & 0 \end{bmatrix} \begin{bmatrix} \bar{\boldsymbol{u}} \\ \gamma \\ \lambda_r \end{bmatrix} = \begin{bmatrix} \bar{\boldsymbol{f}} \\ 0 \\ 0 \end{bmatrix}$$

Content

From previous slides,

FETI-1
solves iteratively for the interface forces
while satisfying a weak "natural" compatibility at each iteration

$$\begin{bmatrix} \boldsymbol{K} & \boldsymbol{B}^T(\boldsymbol{B}\boldsymbol{R}) & \boldsymbol{B}^T \\ (\boldsymbol{R}^T\boldsymbol{B}^T)\boldsymbol{B} & 0 & 0 \\ \boldsymbol{B} & 0 & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{u} \\ \beta \\ \lambda \end{bmatrix} = \begin{bmatrix} \boldsymbol{f} \\ \mathbf{o} \\ \mathbf{o} \end{bmatrix}$$

Bloc diagonal of local non-assembled operators ($\boldsymbol{K}$)

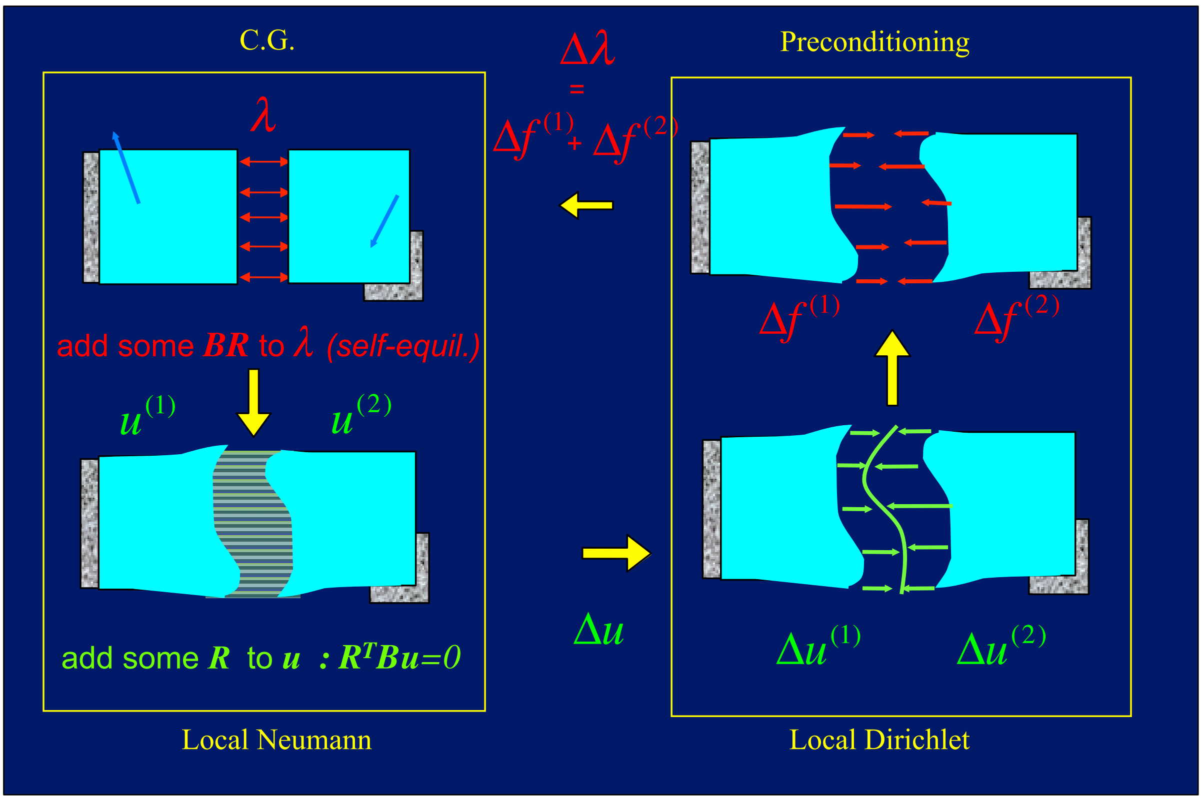
inner problem with weak campatibility

rigid body modes of floating domains (nullspace) ($\boldsymbol{R}$)

signed Boolean (interface compatibility) ($\boldsymbol{B}$)

interface forces (Lagrange multipliers) (λ)

Dual Schur (FETI): iteration



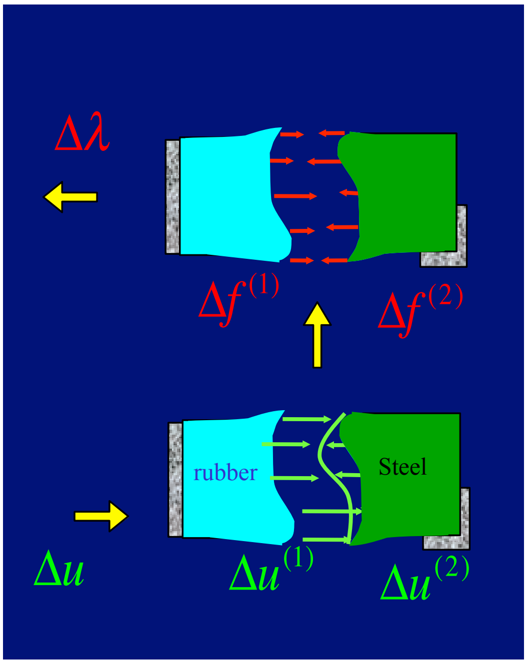
Preconditioning for heterogeneous Problems

When the problem is heterogeneous, modify the preconditioner:

assume interface force on the stiff side is closer to the exact interface forces

$$\begin{bmatrix} \cdots & \boldsymbol{W}^{(s)}\boldsymbol{B}^{(s)} & \cdots \end{bmatrix} \begin{bmatrix} \ddots & & 0 \\ & \boldsymbol{S}^{(s)} & \\ 0 & & \ddots \end{bmatrix} \begin{bmatrix} \vdots \\ \boldsymbol{B}^{(s)^T}\boldsymbol{W}^{(s)} \\ \vdots \end{bmatrix}$$

assume the exact displacement is closer to the stiff part

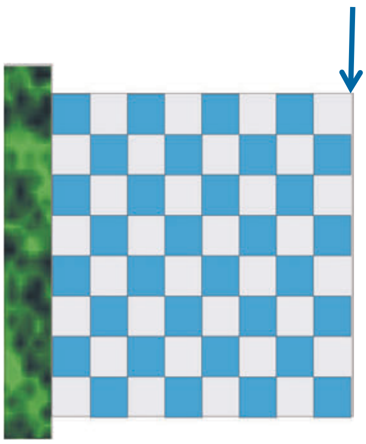


"lumped-scaling" scaling according to local stiffness

[*Rixen-Farhat* 99] [*Klawon at al.* 02]

FETI works for heterogeneous problems ...

Checkerboard problem



8 X 8 subdomains

2 materials:
$E_1 / E_2 = 4096$

80 X 80 plane stress elements

Convergence:
10^{-8} on primal residual

20 iterations if *k-scaling*

FETI works for heterogeneous problems ...

1 000 000 d.o.f
Highly heterogeneous

FETI with scaling:

250 CPU: 370 sec

500 CPU: 160 sec

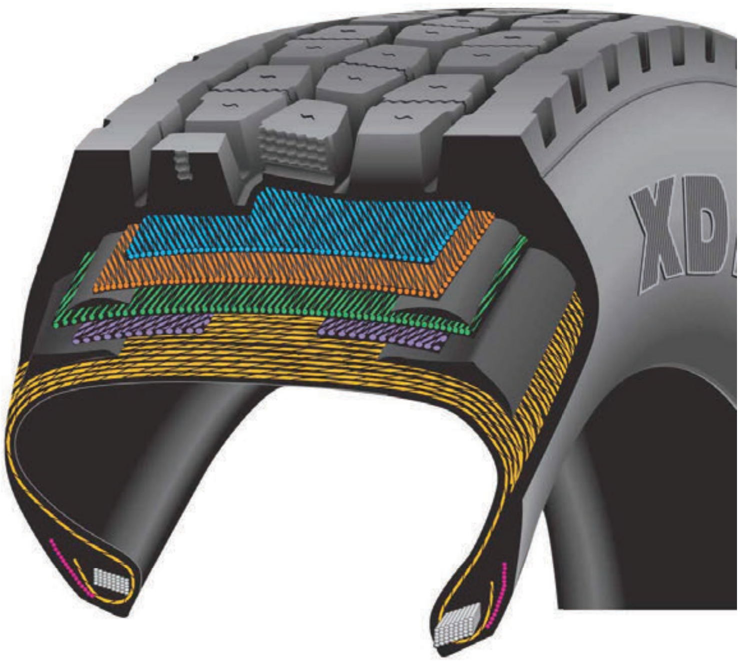
Needed to partition according to the materials !

Reentry vehicle (SANDIA)

[Bhardwaj, Day, Farhat, Lesoinne, Pierson, Rixen], 2000

But then .... the really hard problems:

www.michelintruck.com



Steel cables, rubber, thin structures .....

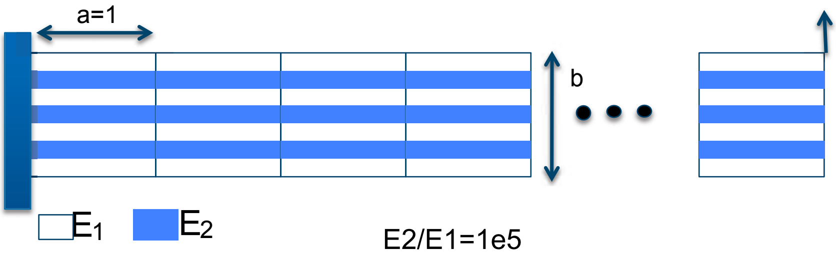
Collaboration with U.Paris VI / Michelin : N. Spilane, F. Nataf, V. Dolean, P. Hauret

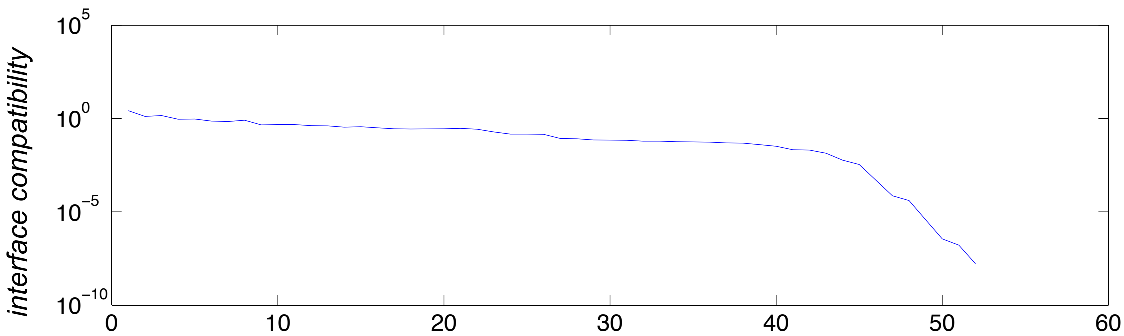
But then .... the really hard problems:

When decomposed into slices,
we have the classical „Schwarz-Wälder kirsch“ problem

High heterogeneties ALONG the interface ! (scaling does not help)

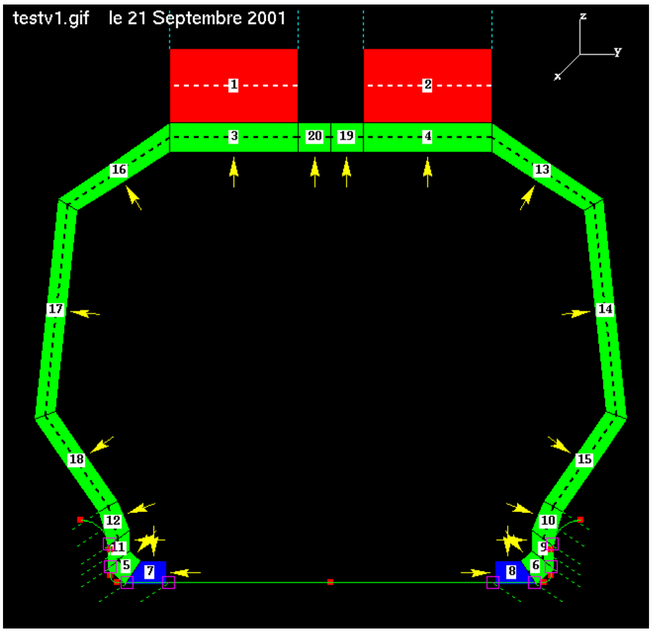
Typical converge of FETI on heterogeneous interface ALONG the interface

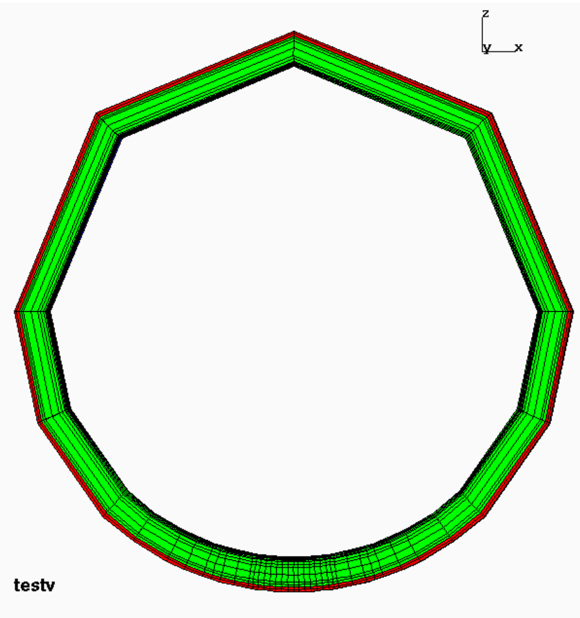




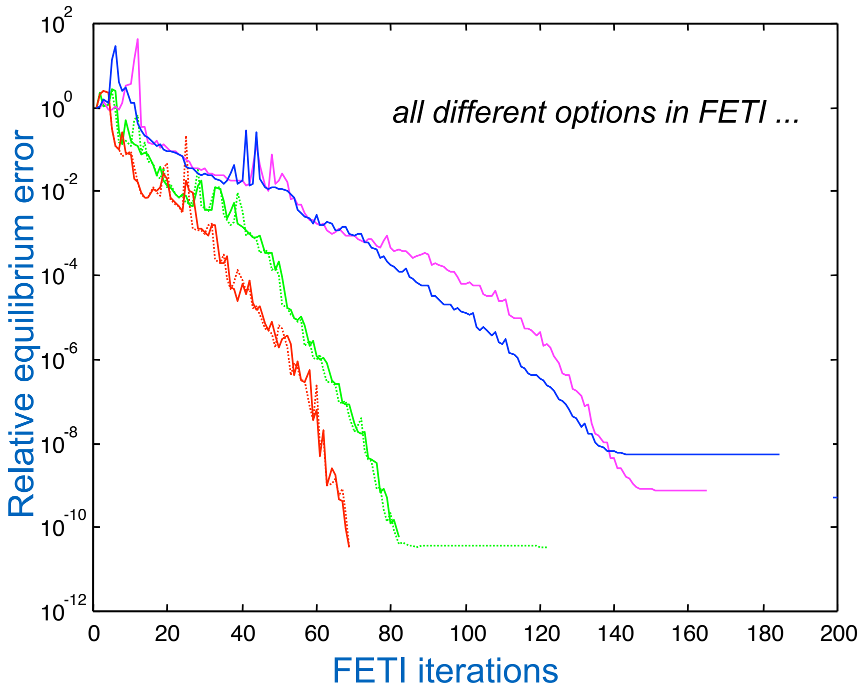
BAD !!!

Tire simple test case



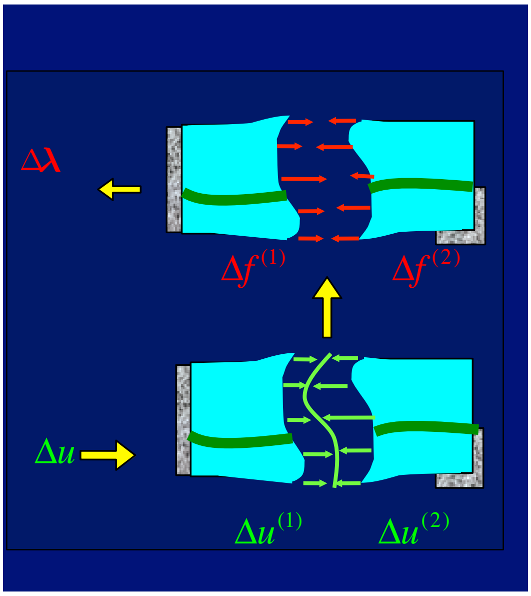


All the known tools of FETI are of no help ....



bad !!

Why does the scaled Dirichlet preconditioner not work for heterogeneities ALONG the interfaces ?

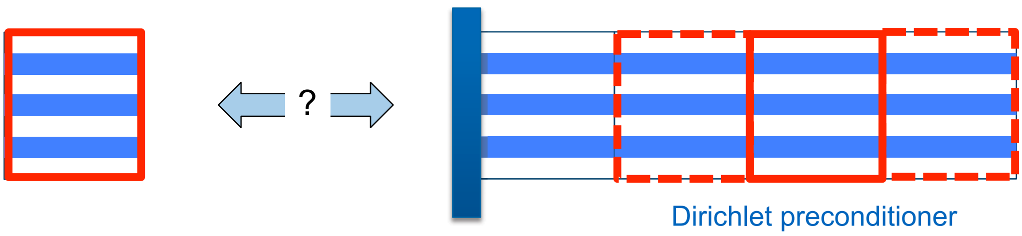


Scaling does not help since it looks only at heterogeneities ACROSS the interface

The Dirichlet preconditioner, by construction,
does not know anything about the ASSEMBLED interface !!

Finding the bad modes on the interface

„bad modes“
=
behavior of the assembly
that is important for the solution
but cannot be seen by an isolated subdomain



Dirichlet preconditioner
~„assembled“ interface stiffness

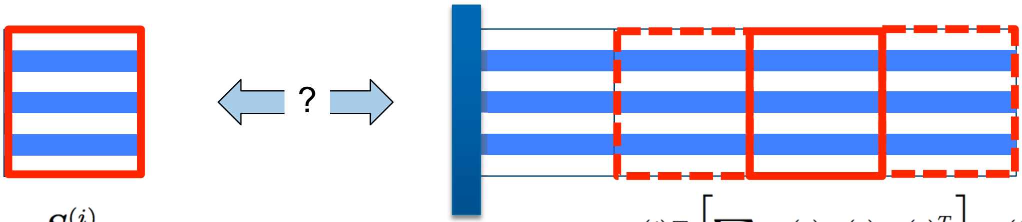
$$\boldsymbol{S}^{(i)}$$

interface stiffness for
an isolated subdomain

$$\boldsymbol{B}^{(i)T}\left[\sum_s \boldsymbol{B}^{(s)}\boldsymbol{S}^{(s)}\boldsymbol{B}^{(s)^T}\right]\boldsymbol{B}^{(i)}$$

~ assembled interface stiffness
seen by one subdomain

Finding the bad modes on the interface



$\boldsymbol{S}^{(i)}$

$$\boldsymbol{B}^{(i)T}\left[\sum_s \boldsymbol{B}^{(s)}\boldsymbol{S}^{(s)}\boldsymbol{B}^{(s)^T}\right]\boldsymbol{B}^{(i)}$$

Build the GenEO eigenvalue problem
(called Geneo because the idea originates from the *Generalized Eigenvalues in the Overlap*
developped for Schwarz [*Nataf et al. 11*], [*Spillane et al. 13*], [*Dolean et. al 12*])

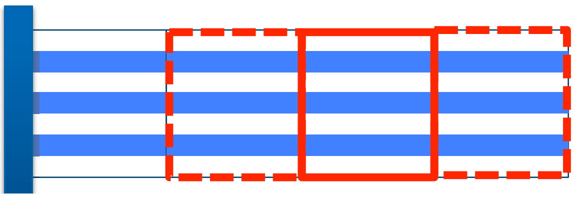
$$\boldsymbol{S}^{(i)}\boldsymbol{p}_k^{(i)} = \mu_k \left[\sum_{s\ neighbor\ of\ i} \boldsymbol{B}^{(i)^T}\boldsymbol{B}^{(s)}\boldsymbol{S}^{(s)}\boldsymbol{B}^{(s)^T}\boldsymbol{B}^{(i)}\right]\boldsymbol{p}_k^{(i)}$$

The bad modes are those with low eigenvalue. [*Spillane-Rixen* 13]

The related interface force $\boldsymbol{S}^{(i)}\boldsymbol{p}_k^{(i)}$ are treated in auxiliary coarse grid (FETI-2)

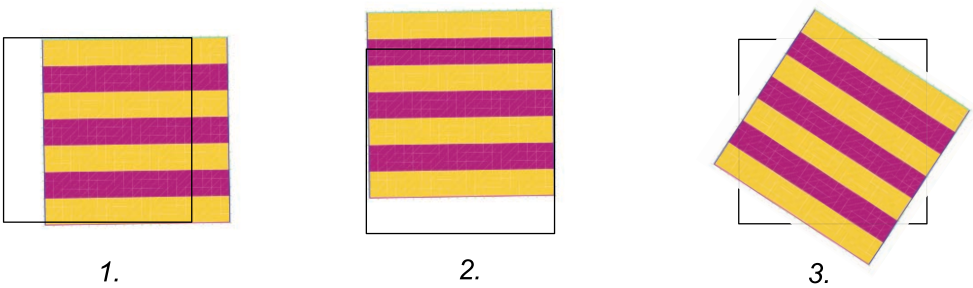
FETI-GenEO

Bad modes: example



$$\boldsymbol{S}^{(i)}\boldsymbol{p}_k^{(i)} = \mu_k \left[\sum_{s\ neighbor\ of\ i} \boldsymbol{B}^{(i)^T}\boldsymbol{B}^{(s)}\boldsymbol{S}^{(s)}\boldsymbol{B}^{(s)^T}\boldsymbol{B}^{(i)}\right]\boldsymbol{p}_k^{(i)}$$

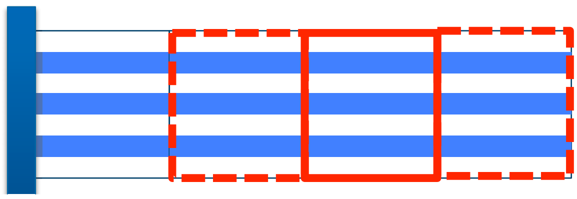
First modes :



1. *2.* *3.*

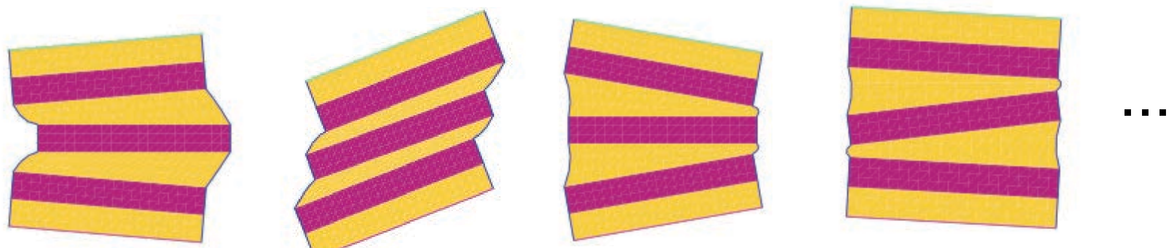
3 rigid body modes (already in coarse grid of FETI, so not in Geneo coarse grid)

Bad modes: example



$$S^{(i)} p_k^{(i)} = \mu_k \left[\sum_{s \ neighbor \ of \ i} B^{(i)^T} B^{(s)} S^{(s)} B^{(s)^T} B^{(i)} \right] p_k^{(i)}$$

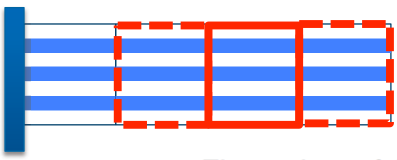
First modes :



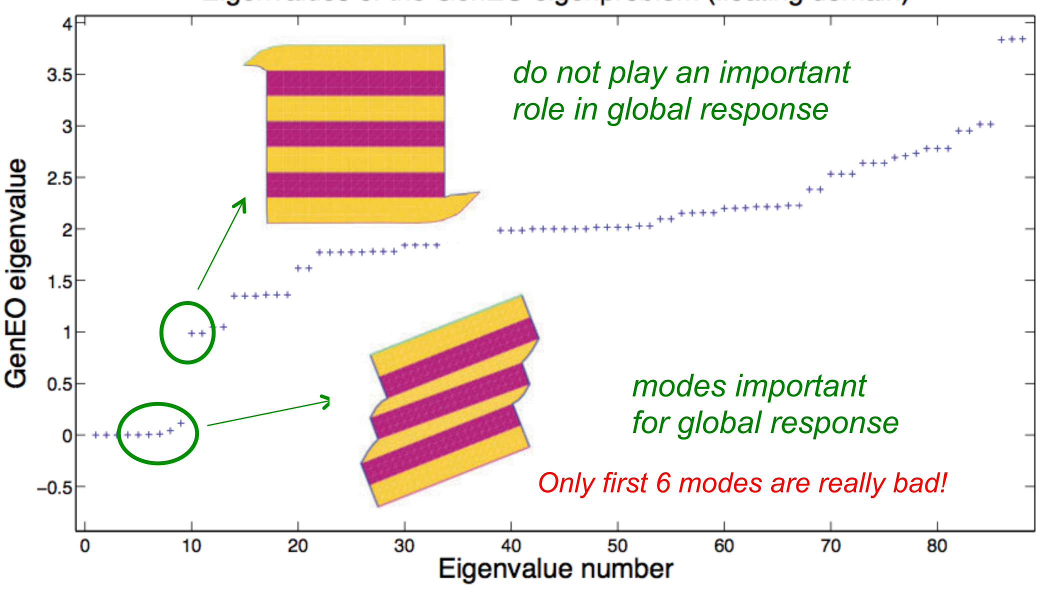
4. *5.* *6.* *7.*

... represent quasi-rigid motion of hard layers → coarse grid is a model of total „skeleton“

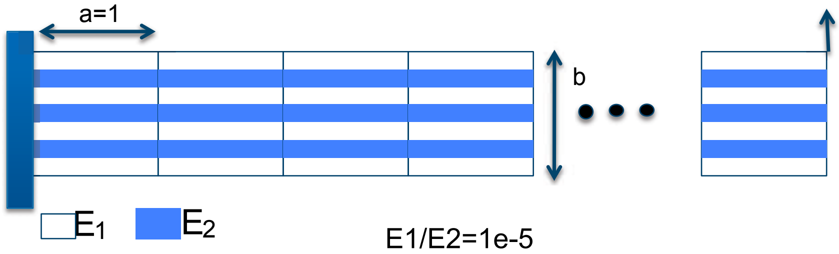
Bad modes: example



Eigenvalues of the GenEO eigenproblem (floating domain)



FETI-Geneo: results



Condition # of prec.deflat. operator

# of modes in coarse grid

N subdomains	FETI-GenEO κ	FETI-GenEO $\#U_0$	FETI-GenEO it	FETI-1 κ	FETI-1 it
4	3	14	5	$1.4 \cdot 10^3$	20
8	1.34	38	5	$1.9 \cdot 10^3$	39
16	1.34	86	4	$2.1 \cdot 10^3$	75
32	1.35	182	4	$2.2 \cdot 10^3$	137
64	1.35	374	4	$2.2 \cdot 10^3$	190

The Geneo coarse grid guarantees robustness !

More results and mathematical analysis in [*Spillane-Rixen* 13]

FETI-Geneo

- The Geneo coarse grid allows robust convergence for hard problems:
  - heterogeneity along the interface
  - bad aspect ratios
  - jagged interface decompositions (observed)
- Not easy to know a priori what the optimum size of the Geneo coarse grid is
- Computing the bad modes (e.g. by a Krylov-based method) requires solving many Dirichlet problems
- The bad modes can be computed in parallel (one e.v.p per domain)

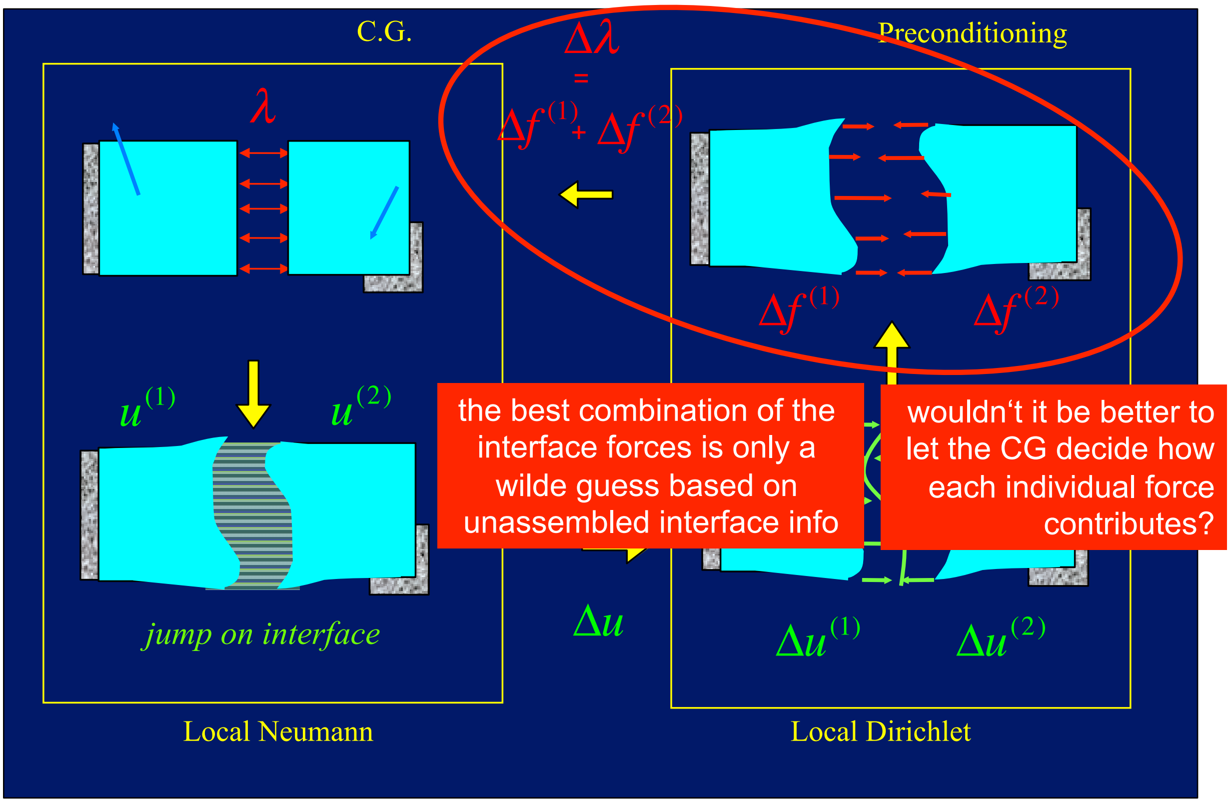
Content

1. Non-overlapping DD: Primal and Dual approaches
2. The FETI saga: evolution of the algorithm in the realm of engineering
   - FETI-1 (natural coarse grid – lumped preconditioner)
   - FETI-2 and FETI-DP (auxiliary coarse grids)

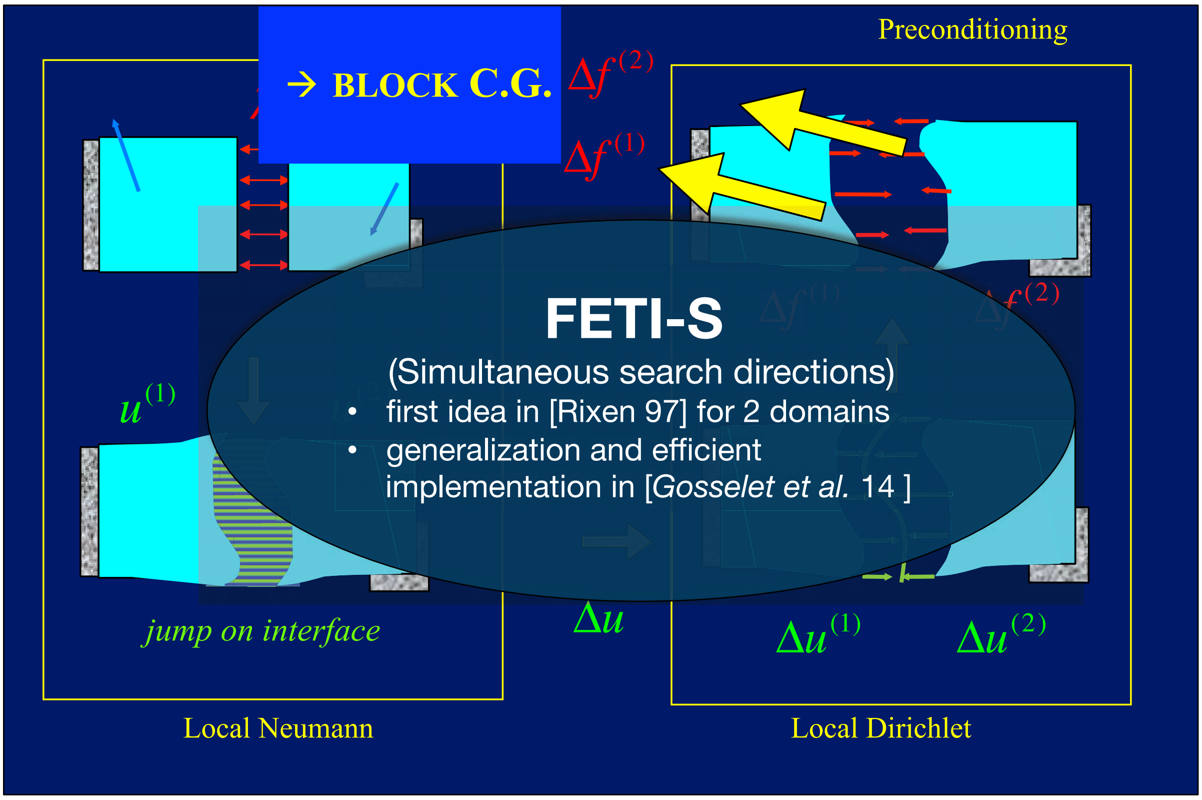
Part 2

3. FETI for heterogeneous problems
   - Scaled preconditioners
   - FETI-Geneo (bad modes)
   - FETI-Simultaneous
4. FETI for concurrent multiscale

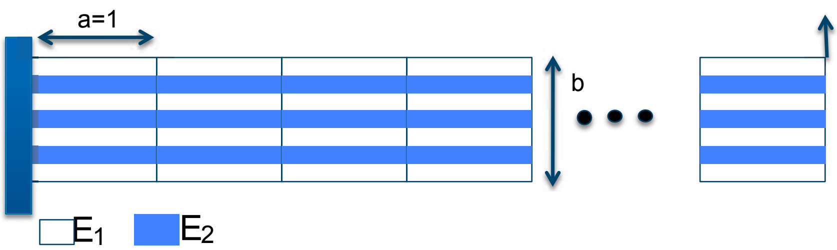
Heterogeneities along interface: What is wrong with FETI ??



IDEA: Recycle the maximum information computed in the preconditioner

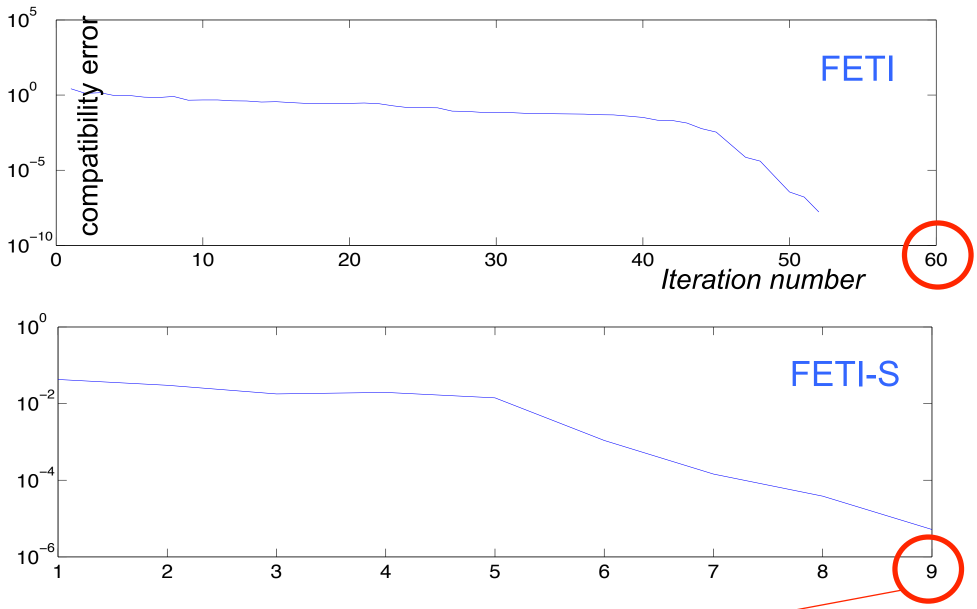


The „Schwarz – wälder kirsch“ problem



N domains = 8,
E1/E2 = 1e-5
FETI
FETI-S
Preconditioner: Dirichlet

Convergence of FETI and FETI-S



9 X 8 = 72 directions of descent

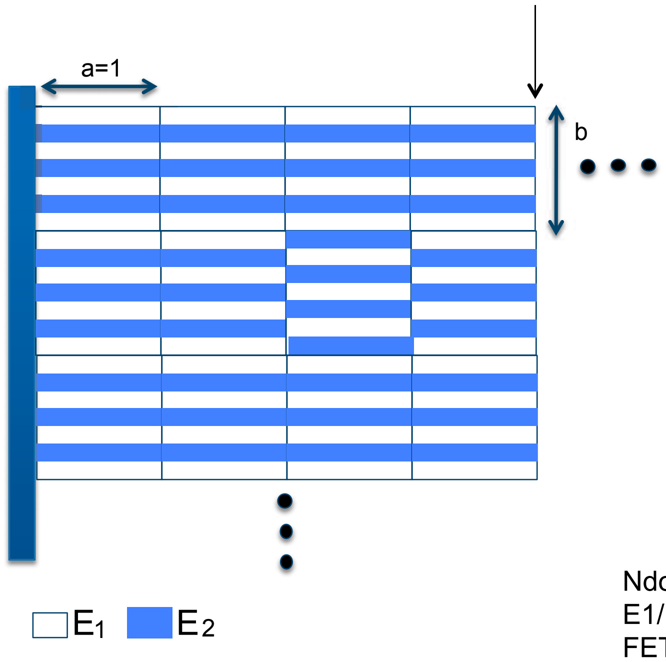
Magic? but local direction of descent

→ only ~ 3X9 Neumann Problem /domain (Block solves)

→ only 9 Dirichlet Problems / domain

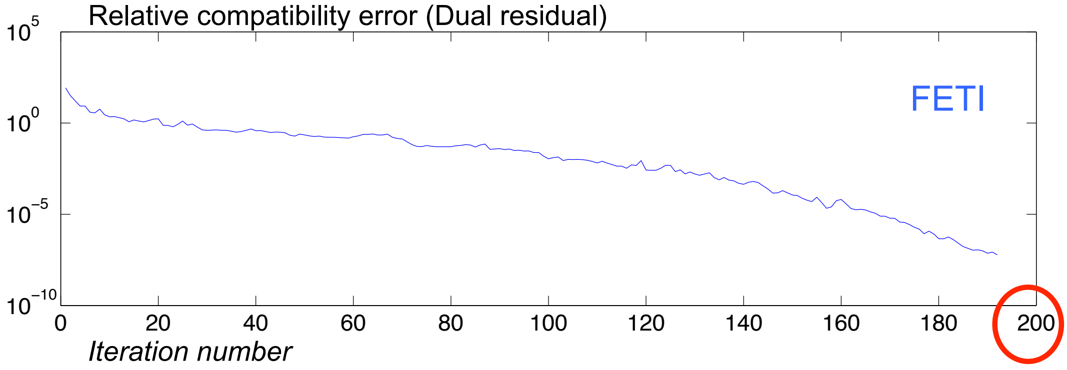
Magic !

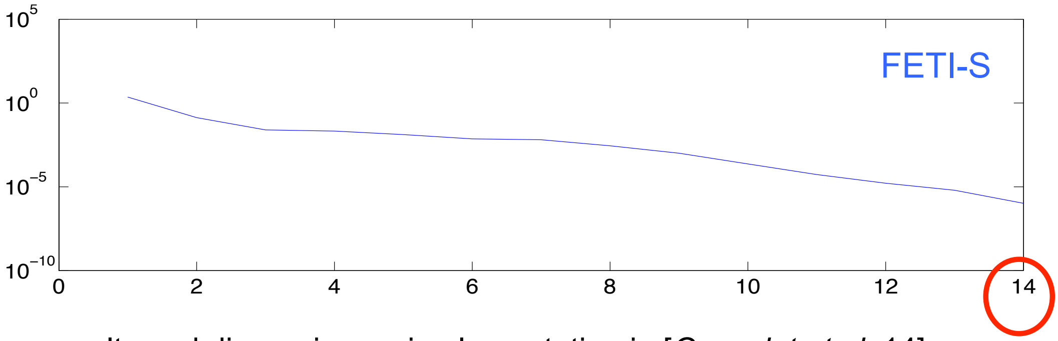
The „Schwarz – wälder kirsch“ problem
No piece of cake ....



Ndomains = 4 X 8 = 32,
E1/E2 = 1e-5
FETI
FETI-S
Prec: Dirichlet

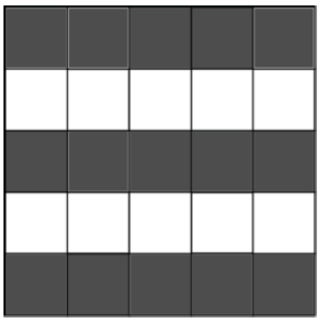
Convergence of interface problem



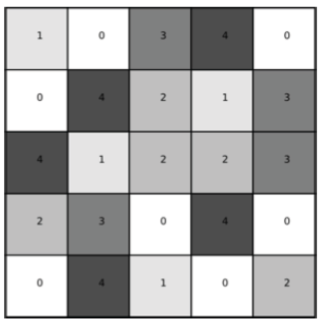


More results and discussion on implementation in [*Gosselet et al.* 14]

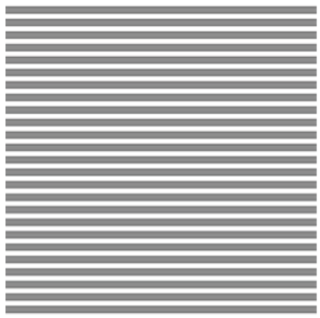
Convergence of interface problem



1.(25 Domains, 2e6 dofs)



2.(25 Domains, 2e6 dofs)



3.(50 Domains, 10e6 dofs)

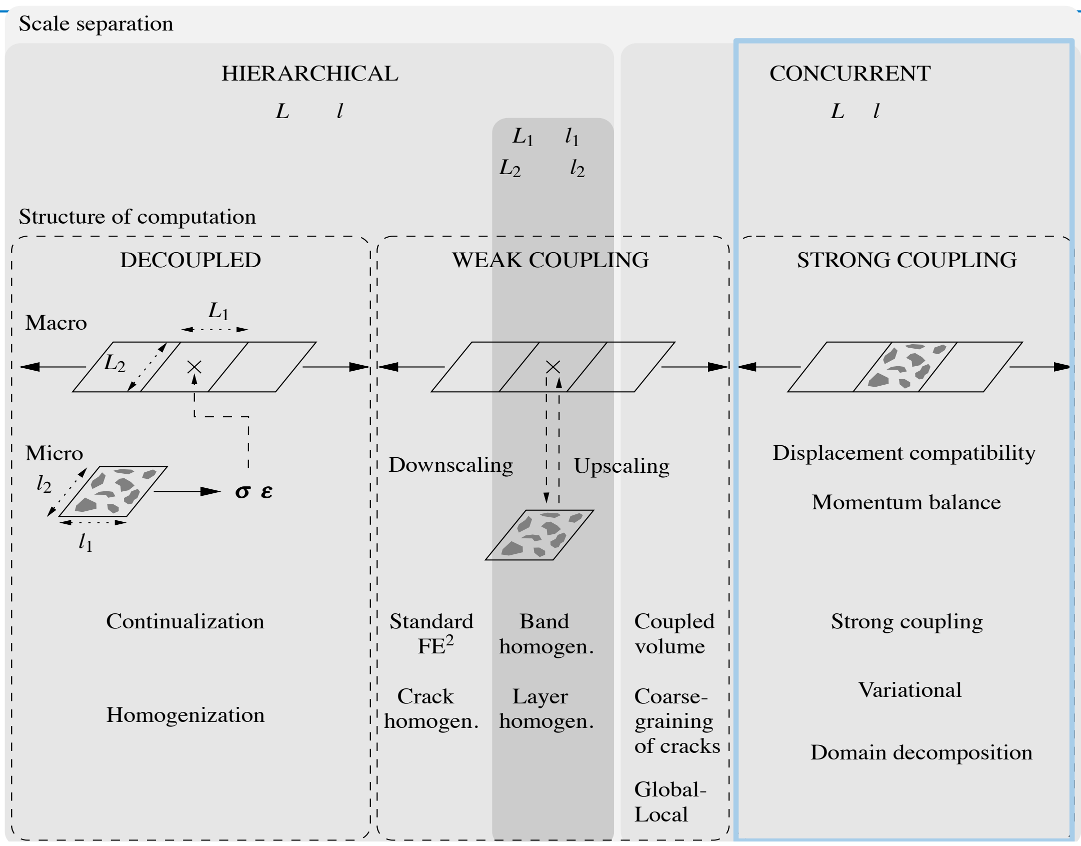
	Decomp.	Hete.	Solver	#iterations	#search directions	Max #local resolutions	Time (s)
1.	Checker	Layers	FETI S-FETI	105 39	105 975	210 273	24 24
2.	Checker	Random	FETI S-FETI	386 102	386 2550	772 714	112 68
3.	Slices	Layers	FETI S-FETI	107 2	107 100	214 10	70 3

Table VI. CPU performance of FETI and S-FETI for a 2D elasticity problem

More results and discussion on implementation in [*Gosselet et al.* 14]

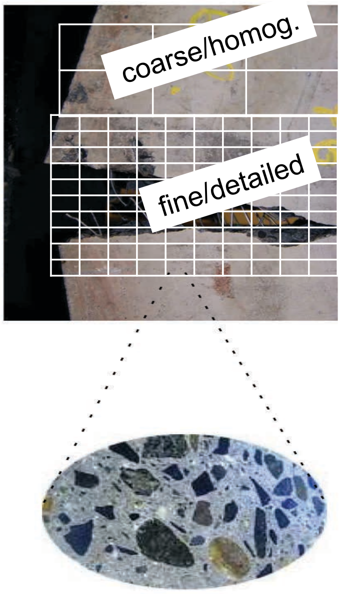
Content

Overview of multliscale approaches



[Lloberas-Valls 12]

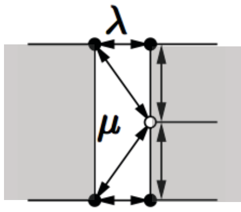
FETI in multiscale computation



Advantage of FETI over BDD:
can handle non-matching meshes

$$\begin{bmatrix} \mathbf{K} & \mathbf{B}^T \\ \mathbf{B} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ \boldsymbol{\lambda} \end{bmatrix} = \begin{bmatrix} \mathbf{f} \\ \mathbf{0} \end{bmatrix}$$

non-Boolean for non-conforming interface
(Mortar, collocation ...)

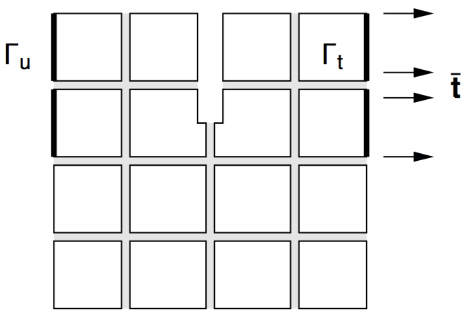


Work from ***Oriol Lloberas-Valls*** (now Cimne, UPC, Barcelona)
supervised by L.Sluijs, A.Simone (TU Deflt) & D. Rixen (TU München)
[Lloberas-Valls *et al.*, 11-12-12]

FETI in multiscale computation

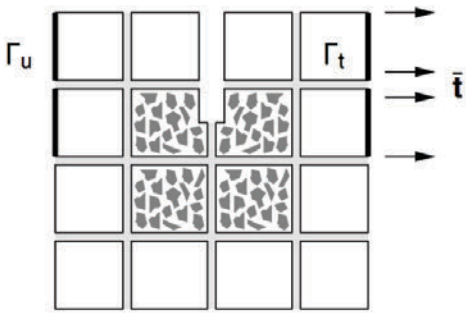
Adaptive multiscale strategy for increasing load

When stresses are low
(no damage likely to occur)



Coarse resolution with effective
(homogenized) elastic properties

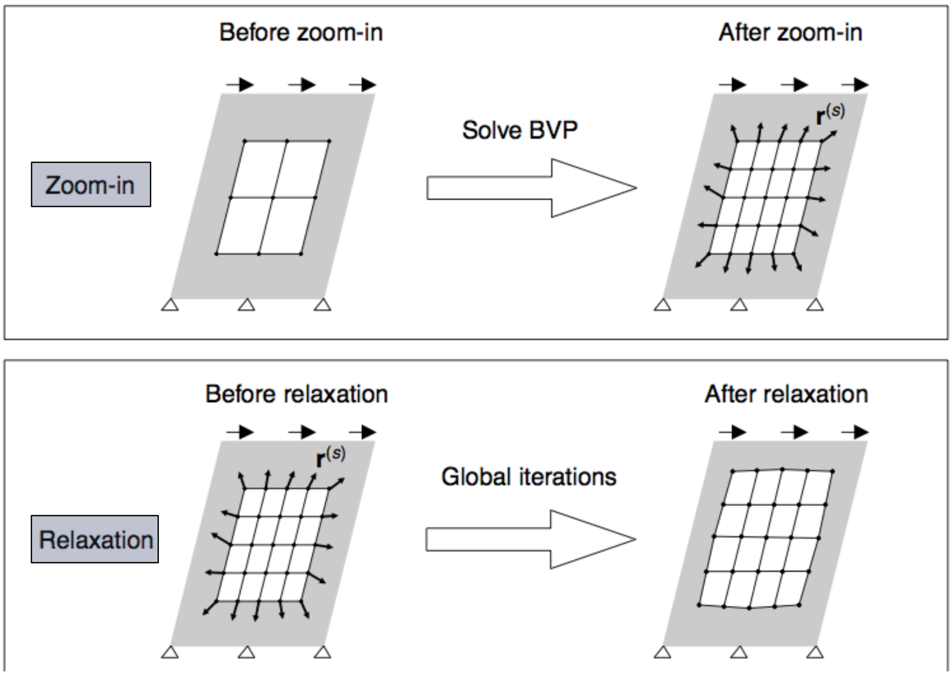
When the stress level indicates
that damage could occur, refine!



Refine ONLY critical domains
(become non-linear due to material)

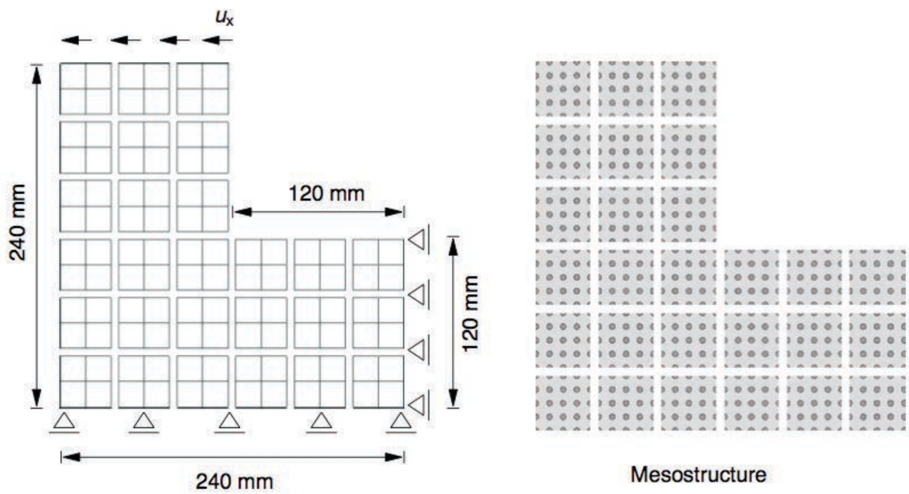
FETI in multiscale computation

Reequilibrate after refinment



FETI in multiscale computation

Example 1 – Fibers in matrix



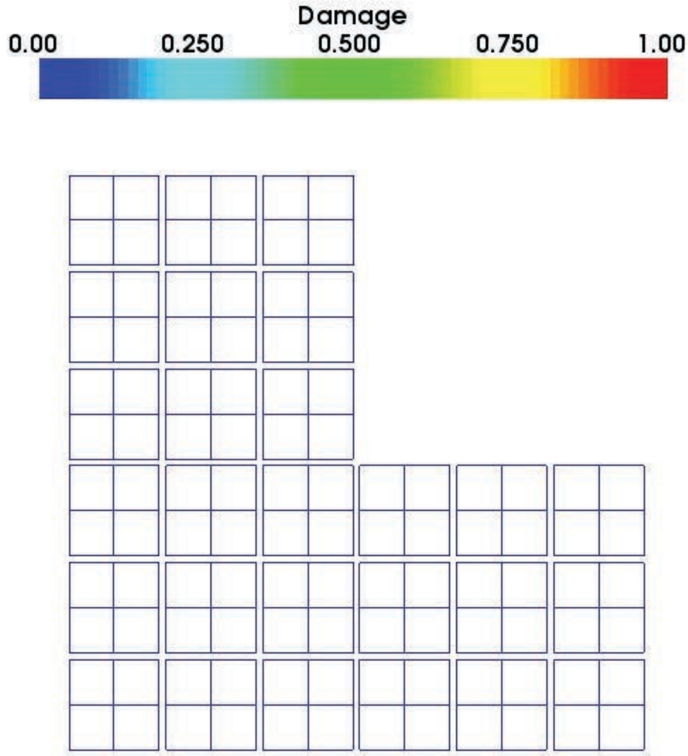
- Fibers as strong inclusions:

  $E_f = 20 \times 10^4$ Mpa, $E_I = 20 \times 10^3$ Mpa, $E_m = 40 \times 10^3$ Mpa

due to interface high heterogeneity,
weak compatibility condition with special stiffness weihgting
[Lloberas-Valls *et al.*, 12]

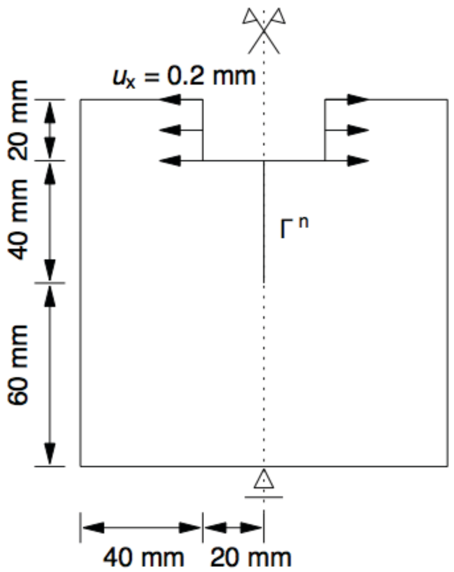
FETI in multiscale computation

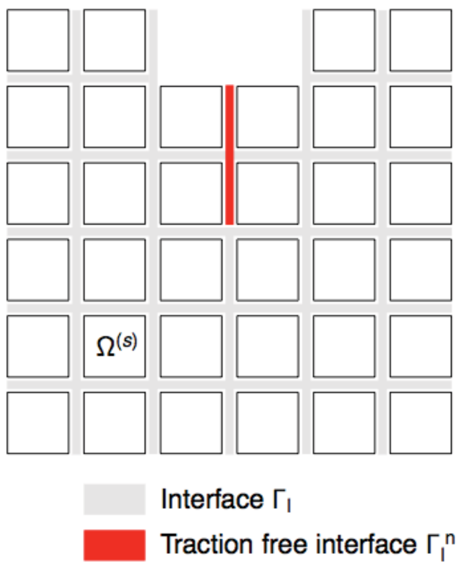
Example 1



FETI in multiscale computation

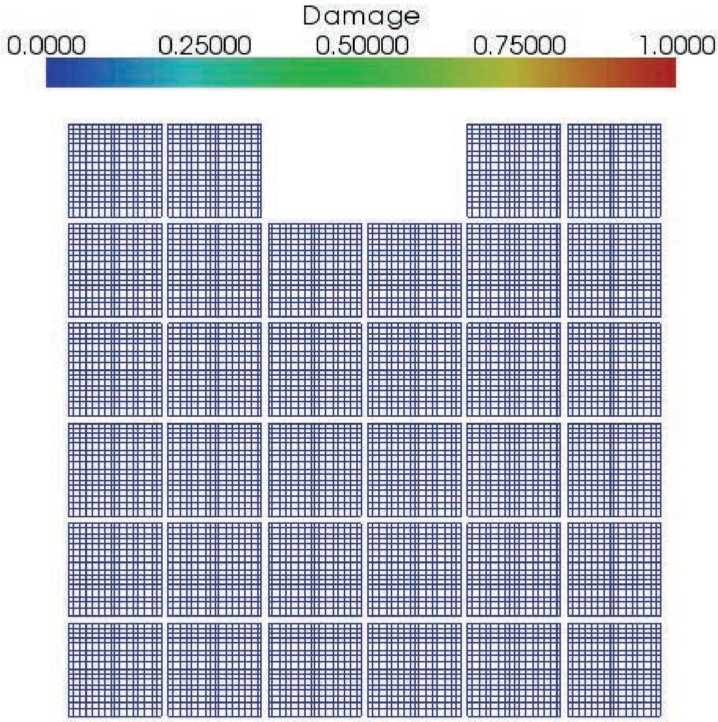
Example 2: concrete-like specimen





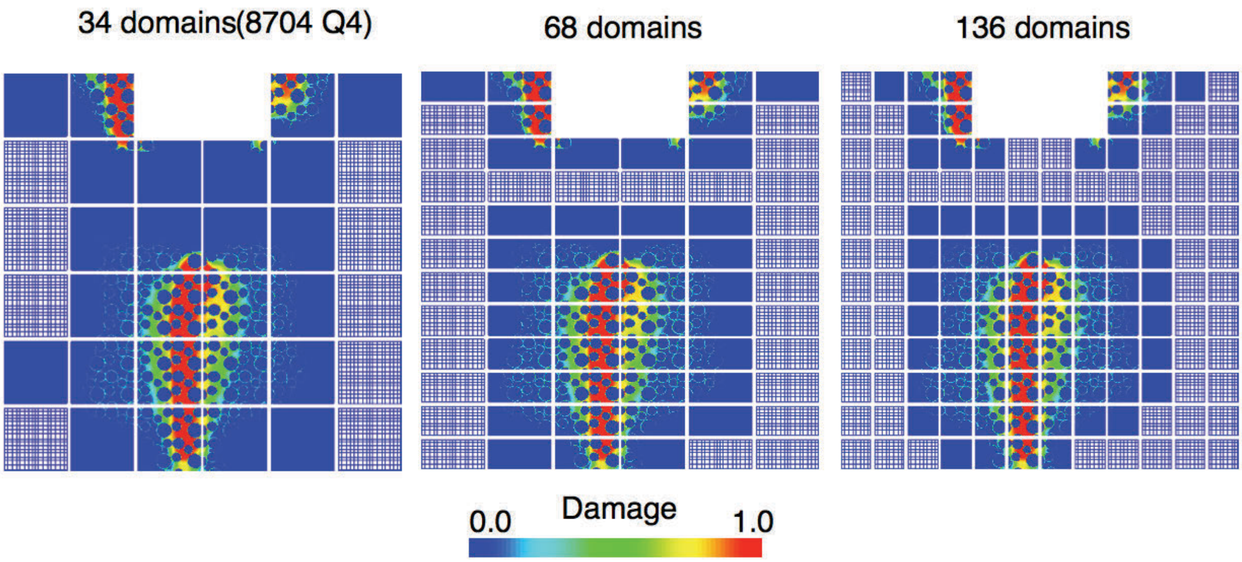
FETI in multiscale computation

Example 2



FETI in multiscale computation

Example 2



If increasing the number of domains

→ the solution is (nearly) not changing (objectivity)

→ the effectivness increases (even less non-linear and fine domains)

FETI in multiscale computation

- Efficient concurrent multiscale method
  (non-linear only where needed, linear domains reused)

- good accuracy compared to fully refined models

- more research needed to improve effectiveness
  - reuse of search directions for different load steps when domains change
  - preconditioners and coarse grids
  - ...

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