Domain Decomposition solvers (FETI)

a random walk in history
and some current trends

Daniel J. Rixen
Technische Universität München
Institute of Applied Mechanics

www.amm.mw.tum.de
rixen@tum.de

8-10 October 2014
39th Woudschoten Conference,
organised by the Werkgemeenschap Scientific Computing (WSC)
When splitting the problem in parts and asking different cpu's (or threads) to take care of subproblems, will the problem be solved faster?

**FETI, Primal Schur (Balancing) method**

- around 1990 ……… basic methods, mesh decomposer technology
- 1990-2001 ……… improvements
  - preconditioners, coarse grids
  - application to Helmholtz, dynamics, non-linear ...

Here the concepts are outlined using some mechanical interpretation.
For mathematical details, see lecture of Axel Klawonn.

“...

*Pure mathematicians sometimes are satisfied with showing that the non-existence of a solution implies a logical contradiction, while engineers might consider a numerical result as the only reasonable goal.*

*Such one sided views seem to reflect human limitations rather than objective values.*

*In itself mathematics is an indivisible organism uniting theoretical contemplation and active application.*

…”

R. Courant

in *Variational Methods for the solution of problems of equilibrium and vibrations*

Here the concepts are outlined using some mechanical interpretation.
For mathematical details, see lecture of Axel Klawonn.
Content

1. Non-overlapping DD: Primal and Dual approaches
2. The FETI saga
   • FETI-1 (natural coarse grid – lumped preconditioner)
   • FETI-2 and FETI-DP (auxiliary coarse grids)
3. FETI for heterogeneous problems
   • Scaled preconditioners
   • FETI-Geneo (bad modes)
   • FETI-Simultaneous
4. FETI for concurrent multiscale

Domain Decomposition: Primal / Dual Assembly

\[
\begin{bmatrix}
K^{(1)} & 0 & \cdots & 0 \\
0 & \ddots & & \vdots \\
\vdots & & \ddots & \vdots \\
0 & \cdots & 0 & K^{(N_z)}
\end{bmatrix}
\begin{bmatrix}
u^{(1)} \\
\vdots \\
u^{(N_z)}
\end{bmatrix}
= 
\begin{bmatrix}
f^{(1)} \\
\vdots \\
f^{(N_z)}
\end{bmatrix}
\]

\[
\begin{bmatrix}
K^{(1)} & B^{(1)} & \cdots & B^{(1)} \\
0 & \ddots & & \vdots \\
\vdots & & \ddots & \vdots \\
B^{(N_z)} & \cdots & 0 & K^{(N_z)}
\end{bmatrix}
\begin{bmatrix}
u^{(1)} \\
\vdots \\
u^{(N_z)}
\end{bmatrix}
= 
\begin{bmatrix}
f^{(1)} \\
\vdots \\
f^{(N_z)}
\end{bmatrix}
\]

signed Boolean matrices
Domain Decomposition: Primal / Dual Assembly

Block-diagonal notation

\[
\begin{bmatrix}
K^{(1)} & 0 \\
0 & K^{(N_d)}
\end{bmatrix}
\begin{bmatrix}
\mathbf{u}^{(1)} \\
\vdots \\
\mathbf{u}^{(N_d)}
\end{bmatrix}
= 
\begin{bmatrix}
\mathbf{f}^{(1)} \\
\vdots \\
\mathbf{f}^{(N_d)}
\end{bmatrix}
\]

Iterative solution of interface problem

Iterate on interface dofs \( \mathbf{u}_b \)

solve for internal dofs \( \mathbf{u}_i \)

end \( \text{when interface in equilibrium} \)

Primal Schur and BDD
[Letallec et al., 91]

Iterate on interface forces \( \mathbf{\lambda} \)

solve for domain dofs \( \mathbf{u} \)

end \( \text{when interface compatible} \)

FETI
(Finite Element Tearing and Interconnecting)
[Farhat-Roux, 91]
Primal Schur: iteration

Iterate on interface dof $u_b$

Solve for internal dof $u_i$

end (when interface in equilibrium)
Dual Schur (FETI): iteration

Iterate on interface forces $\lambda$

solve for domain dofs $u$

end (when interface compatible)
The primal and dual Schur complement approaches are very similar. Most of the “tricks” used in the one method can be applied for the other. So which method to use is nearly a matter of religion …

There exists some subtle differences such as:
- treatment of cross-points (interface nodes on more than 2 domains)
- determination of an initial estimate

[ Klawonn 02, Gosselet et al. 03, Gosselet-Rey 06]

Here we outline only the dual approach (FETI) but one finds many publications on similar developments for the primal approach.

Content

1. Non-overlapping DD: Primal and Dual approaches
2. The FETI saga
   - FETI-1 (natural coarse grid – lumped preconditioner)
   - FETI-2 and FETI-DP (auxiliary coarse grids)
3. FETI for heterogeneous problems
   - Scaled preconditioners
   - FETI-Geneo (bad modes)
   - FETI-Simultaneous
4. FETI for concurrent multiscale
Dual Schur (FETI): iteration

C.G.

\[ \Delta \lambda = \Delta f^{(1)} + \Delta f^{(2)} \]

Preconditioning

\[ \Delta \lambda_k = B \Delta f_k \]

\[ \Delta f_k = K_{bb} B^T r_k + K_{bi} \Delta u_i \]

\[ = (K_{bb}^{-1} - K_{bi} K_{ii}^{-1} K_{ib}) B^T r_k \]

\[ K_{ii} \Delta u_i = -K_{ib} (B^T r_k) \]

Local Neumann

Local Dirichlet

Iterate on

\[ (BK^{-1}B^T) \lambda = BK^{-1}f \]

or equivalently, considering \( S \) Schur complement on interface

\[ (BS^{-1}B^T) \lambda = BK^{-1}f \]

\[ F_I \] interface flexibility

\[ \approx (BSB^T)^{-1} \lambda \]

\[ \text{approximate the sum of the inverse by the inverse of the sum} \]

Jump on interface

\[ \Delta u \]

\[ \Delta u^{(1)} \]

\[ \Delta u^{(2)} \]
Dual Schur (FETI): lumped preconditioner

- **C.G.**
  - \( \lambda \)
  - \( u^{(1)} \) → \( u^{(2)} \)
  - *jump on interface*

- **Preconditioning**
  - \( \Delta \lambda \)
  - \( \Delta f^{(1)} \) & \( \Delta f^{(2)} \)
  - \( \Delta u^{(1)} \) & \( \Delta u^{(2)} \)

- **Local Neumann** & **Local lumped Dirichlet**

---

Dual Schur (FETI): lumped preconditioner

- **C.G.**
  - \( \lambda \)
  - \( u^{(1)} \) → \( u^{(2)} \)
  - *jump on interface*

- **Preconditioning**
  - \( \Delta \lambda \)
  - Compute \( \Delta \lambda_k = B \Delta f_k \)
  - Compute \( K_{ii} \Delta u_i = K_{ib} (B^T r_k) \)
  - Solve

- **Local Neumann** & **Local lumped Dirichlet**
The basic FETI and its natural coarse grid

\[
\begin{bmatrix}
  K & B^T \\
  B & 0
\end{bmatrix}
\begin{bmatrix}
  u \\
  \lambda
\end{bmatrix}
= \begin{bmatrix}
  f \\
  0
\end{bmatrix}
\]

Not enough constraints
Badly defined local problems
Singular K

Force the inner problem to have a bit of compatibility to make it regular: at every iteration enforce a weak compatibility

\[
\begin{bmatrix}
  K & B^T T & B^T \\
  T^T B & 0 & 0 \\
  B & 0 & 0
\end{bmatrix}
\begin{bmatrix}
  u \\
  \beta \\
  \lambda
\end{bmatrix}
= \begin{bmatrix}
  f \\
  0 \\
  0
\end{bmatrix}
\]

\(T\) such that local problems are well posed
\(\rightarrow\) weak coupling of domains
\(\rightarrow\) small coarse grid

This interpretation of FETI explained in [Rixen et al. 01]
The basic FETI and its natural coarse grid

If $T=0$, the problem is singular: nullspace = Rigid Body Modes of the floating domains

So if there exists a nullspace for the inner problem, it must have the form: $u_{null} = R\gamma$

How to choose $T$ such that $T^TBu_{null} = T^TR\gamma = 0$ implies $u_{null} = 0$?

$T^TB$ must be full column rank

A „natural“ choice for a minimum weak compatibility is $T = BR$

Note: $BR$ is full column rank (otherwise singular global problem) [Rixen 98]
The FETI and its natural coarse grid

One requires at each iteration on the interface forces $\lambda$ that the compatibility is satisfied on average (trace of the rigid-body modes) – "Natural coarse grid"

The average compatibility is enforced by determining the interface forces in $\text{Image}(BR)$ such that the interface forces are orhtogonal to $\text{null}(K)$ – "self-equilibrated"

enforced by projecting the iterates such that $R^T (B^T \lambda_k - f) = 0$

The compatibility must be satisfied in the subspace $BR$ at every iteration → "coarse grid”
→ ensures that convergence does not deteriorate when nbr. of subdomains increases

FETI-1 [Farhat-Roux 91]

FETI: iteration with floating domains

C.G. Preconditioning

$\Delta \lambda = \Delta f^{(1)} + \Delta f^{(2)}$

$\Delta u = \Delta u^{(1)} + \Delta u^{(2)}$

Local Neumann Local Dirichlet

add some $BR$ to $\lambda$ (self-equil.)

add some $R$ to $u : R^T Bu = 0$
FETI: iteration with floating domains

C.G.

\[ \Delta \lambda \leftarrow \Delta \lambda + BR \beta \]

such that

\[ \Delta \lambda \perp BR \]

Find a solution

\[ u_k = K^+ (f + B^T \lambda_k) \]

Compute and project

\[ r_k \leftarrow B (u_k + R \alpha) \]

such that

\[ r_k \perp BR \]

Preconditioning

\[ \Delta \lambda = \Delta f^{(1)} + \Delta f^{(2)} \]

\[ \Delta u^{(1)} \]

\[ \Delta u^{(2)} \]

Content

1. Non-overlapping DD: Primal and Dual approaches
2. The FETI saga: evolution of the algorithm in the realm of engineering
   • FETI-1 (natural coarse grid – lumped preconditioner)
   • FETI-2 and FETI-DP (auxiliary coarse grids)
3. FETI for heterogeneous problems
   • Scaled preconditioners
   • FETI-Geneo (bad modes)
   • FETI-Simultaneous
4. FETI for concurrent multiscale
Sometimes, more compatibility constraints need to be satisfied exactly at every iteration to ensure good convergence on interface (e.g. corners for bi-harmonic problems: plates, shells)

\[ \begin{bmatrix} K & B^T(BR) & B^T \\ (R^TB^T) B & 0 & 0 \\ B & 0 & 0 \end{bmatrix} \begin{bmatrix} u \\ \beta \\ \lambda \end{bmatrix} = \begin{bmatrix} f \\ 0 \\ 0 \end{bmatrix} \]

\[ \begin{bmatrix} K & B^T(BR) & B^T C & B^T \\ (R^TB^T) B & 0 & 0 & 0 \\ C^TB & 0 & 0 & 0 \\ B & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u \\ \beta \\ \mu \\ \lambda \end{bmatrix} = \begin{bmatrix} f \\ 0 \\ 0 \\ 0 \end{bmatrix} \]

\( C \) is a Boolean matrix “picking out” the compatibility conditions at corners

\( \mu \) : interface forces at corners

It defines an auxiliary („non-natural“) coarse grid: Deflation

The local problems with weak compatibility can be seen as an inner FETI problem

**FETI-2** (two-level FETI)

[Farhat-Mandel 98]
If there are enough corner links to fix the subdomains the “average” compatibility is not required for the regularity of the inner problem:

\[
\begin{bmatrix}
K & B^T C & B^T \\
C^T B & 0 & 0 \\
B & 0 & 0
\end{bmatrix}
\begin{bmatrix}
u \\ \mu \\ \lambda
\end{bmatrix}
= \begin{bmatrix} f \\ 0 \\ 0 \end{bmatrix}
\]

This partial compatibility can be enforced by assembly on the corners, and iterating only for the interface forces for the remaining interface nodes:

\[
\begin{bmatrix}
L_c^T K L_c & B_r^T \\
B_r & 0
\end{bmatrix}
\begin{bmatrix}
\tilde{u} \\ \lambda_r
\end{bmatrix}
= \begin{bmatrix} \tilde{f} \\ 0 \end{bmatrix}
\]

\[\text{FETI-DP}\]

[Farhat et al. 00]
[Farhat et. al 01]
**Advantage of FETI-DP vs. FETI-2**

- No need to find local nullspace
- Less connecting variables at corners

FETI-2

FETI-DP

• “average” compatibility not enforced

Smaller cost per iteration

Slower convergence

FETI-DP scalable in 2D

NOT scalable in 3D

• only point-wise compatibility enforced in inner problem

“non-smooth” coarse grid

**Episode 4: yet another coarse grid to FETI-DP**

\[
\begin{bmatrix}
L_c^T \mathbf{K} L_c & B_r^T \\
B_r & 0
\end{bmatrix}
\begin{bmatrix}
\mathbf{u} \\
\lambda_r
\end{bmatrix} =
\begin{bmatrix}
f \\
0
\end{bmatrix}
\]

FETI-DP not scalable in 3D

add an auxiliary coarse grid

\[
\begin{bmatrix}
L_c^T \mathbf{K} L_c & B_r^T \mathbf{G} & B_r^T \\
\mathbf{G}^T B_r & 0 & 0 \\
B_r & 0 & 0
\end{bmatrix}
\begin{bmatrix}
\mathbf{u} \\
\gamma \\
\lambda_r
\end{bmatrix} =
\begin{bmatrix}
f \\
0 \\
0
\end{bmatrix}
\]

scalable for several “smooth” choices of \(\mathbf{G}\)

(see e.g. [Farhat et. al 01] [Klawonn-Widlund 2006] [Klawonn-Rheinbach 2007])

- If \(\mathbf{G} = \mathbf{B} \mathbf{R}\), FETI-DP mathematically equivalent to FETI-2
- avoids having to deal with floating domains
  + no numerical issue when detecting singularity
  - no profit from a „natural” coarse grid
The FETI family saga - Summary

FETI-1:
- Dual assembly,
- CG on interface forces,
- natural coarse grid
  of rigid body modes

FETI-2:
- additional auxiliary
- coarse grid (Deflation)

FETI-DP:
- enough compatibility enforced
  in a primal way so that local
  problems are regular
- additional smooth coarse

Content

1. Non-overlapping DD: Primal and Dual approaches
2. The FETI saga: evolution of the algorithm in the realm of engineering
   - FETI-1 (natural coarse grid – lumped preconditioner)
   - FETI-2 and FETI-DP (auxiliary coarse grids)
3. FETI for heterogeneous problems
   - Scaled preconditioners
   - FETI-Geneo (bad modes)
   - FETI-Simultaneous
4. FETI for concurrent multiscale
From previous slides,

FETI-1 solves iteratively for the interface forces while satisfying a weak “natural” compatibility at each iteration.

Dual Schur (FETI): iteration

- **C.G.**
  - $\lambda$
  - $u^{(1)}$ and $u^{(2)}$
  - Add some $BR$ to $\lambda$ (self-equil.)

- **Preconditioning**
  - $\Delta \lambda = \Delta f^{(1)} + \Delta f^{(2)}$
  - $\Delta u$
  - $\Delta u^{(1)}$ and $\Delta u^{(2)}$

- **Local Neumann**
  - Add some $R$ to $u : R^T Bu = 0$

- **Local Dirichlet**
  - $K$ and $B^T (BR)$
  - Bloc diagonal of local non-assembled operators
  - Signed Boolean (interface compatibility)
  - Interface forces (Lagrange multipliers)
  - Rigid body modes of floating domains (nullspace)
Preconditioning for heterogeneous Problems

When the problem is heterogeneous, modify the preconditioner:

- Assume the exact displacement is closer to the stiff part.
- Assume the interface force on the stiff side is closer to the exact interface forces.

"lumped-scaling" scaling according to local stiffness

[Rixen-Farhat 99] [Klawon at al. 02]

FETI works for heterogeneous problems ...

Checkerboard problem

8 X 8 subdomains
2 materials: $E_1 / E_2 = 4096$
80 X 80 plane stress elements
Convergence: $10^{-8}$ on primal residual

20 iterations if $k$-scaling
FETI works for heterogeneous problems ...

1 000 000 d.o.f
Highly heterogeneous

FETI with scaling:
250 CPU: 370 sec
500 CPU: 160 sec

*Needed to partition according to the materials!*

*Reentry vehicle (SANDIA)*
[Bhardwaj, Day, Farhat, Lesoinne, Pierson, Rixen], 2000

But then .... the really hard problems:

Steel cables, rubber, thin structures ..... 

Collaboration with U.Paris VI / Michelin : N. Spilane, F. Nataf, V. Dolean, P. Hauret
But then .... the really hard problems:

When decomposed into slices, we have the classical "Schwarz-Wälder kirsch" problem

High heterogeneities ALONG the interface ! (scaling does not help)

Typical converge of FETI on heterogeneous interface ALONG the interface

![Graph showing interface compatibility and iteration number]

Iteration number

BAD !!!
Tire simple test case

All the known tools of FETI are of no help....

all different options in FETI...

bad !!
Why does the scaled Dirichlet preconditioner not work for heterogeneities ALONG the interfaces?

Scaling does not help since it looks only at heterogeneities ACROSS the interface.

The Dirichlet preconditioner, by construction, does not know anything about the ASSEMBLED interface!!

Finding the bad modes on the interface

"bad modes" = behavior of the assembly that is important for the solution but cannot be seen by an isolated subdomain

\[ S^{(i)} \]
interference stiffness for an isolated subdomain

\[ B^{(i)^T} \left[ \sum_s B^{(s)} S^{(s)} B^{(s)^T} \right] B^{(i)} \]
~ assembled interface stiffness seen by one subdomain
Finding the bad modes on the interface

Build the GenEO eigenvalue problem
(called Geneo because the idea originates from the Generalized Eigenvalues in the Overlap developed for Schwarz [Nataf et al. 11], [Spillane et al. 13], [Dolean et. al 12])

\[ S^{(i)} p_k^{(i)} = \mu_k \left[ \sum_{s \text{ neighbor of } i} B^{(i)T} B^{(s)} S^{(s)} B^{(s)T} B^{(i)} \right] p_k^{(i)} \]

The bad modes are those with low eigenvalue. [Spillane-Rixen 13]

The related interface force \( S^{(i)} p_k^{(i)} \) are treated in auxiliary coarse grid (FETI-2)

FETI-GenEO

Bad modes: example

First modes:

1. \[ \text{3 rigid body modes (already in coarse grid of FETI, so not in Geneo coarse grid)} \]
Bad modes: example

\[ S^{(i)} p_k^{(i)} = \mu_k \sum_{\text{a neighbor of } i} B^{(i)^T} B^{(s)} S^{(s)} B^{(s)^T} B^{(i)} p_k^{(i)} \]

First modes:

4. 5. 6. 7.

... represent quasi-rigid motion of hard layers → coarse grid is a model of total "skeleton"

Bad modes: example

Eigenvales of the GenEO eigenproblem (floating domain)

do not play an important role in global response

modes important for global response

Only first 6 modes are really bad!
FETI-Geneo: results

The Geneo coarse grid guarantees robustness!

More results and mathematical analysis in [Spillane-Rixen 13]

FETI-Geneo

- The Geneo coarse grid allows robust convergence for hard problems:
  - heterogeneity along the interface
  - bad aspect ratios
  - jagged interface decompositions (observed)

- Not easy to know a priori what the optimum size of the Geneo coarse grid is

- Computing the bad modes (e.g. by a Krylov-based method) requires solving many Dirichlet problems

- The bad modes can be computed in parallel (one e.v.p per domain)
Content

1. Non-overlapping DD: Primal and Dual approaches
2. The FETI saga: evolution of the algorithm in the realm of engineering
   • FETI-1 (natural coarse grid – lumped preconditioner)
   • FETI-2 and FETI-DP (auxiliary coarse grids)
3. FETI for heterogeneous problems
   • Scaled preconditioners
   • FETI-Geneo (bad modes)
   • FETI-Simultaneous
4. FETI for concurrent multiscale

Heterogeneities along interface:
What is wrong with FETI??
IDEA: Recycle the maximum information computed in the preconditioner

FETI-S
(Simultaneous search directions)
• first idea in [Rixen 97] for 2 domains
• generalization and efficient implementation in [Gosselet et al. 14]

The „Schwarz – wälder kirsch“ problem

N domains = 8,
E1/E2 = 1e-5
FETI
FETI-S
Preconditioner: Dirichlet
Convergence of FETI and FETI-S

Convergence plots for FETI and FETI-S methods. The FETI plot shows a slower convergence rate compared to the FETI-S plot, which converges much faster.

9 X 8 = 72 directions of descent but local direction of descent → only ~ 3X9 Neumann Problem / domain (Block solves) → only 9 Dirichlet Problems / domain

Magic!

The „Schwarz – wälder kirsch“ problem
No piece of cake ....

The Schwarz problem illustrated with a grid of domains. The grid size is 4x8, and the convergence test shows a clear improvement with the FETI-S method compared to FETI.
Convergence of interface problem

More results and discussion on implementation in [Gosselet et al. 14]

Convergence of interface problem

More results and discussion on implementation in [Gosselet et al. 14]
Content

1. Non-overlapping DD: Primal and Dual approaches
2. The FETI saga: evolution of the algorithm in the realm of engineering
   • FETI-1 (natural coarse grid – lumped preconditioner)
   • FETI-2 and FETI-DP (auxiliary coarse grids)
3. FETI for heterogeneous problems
   • Scaled preconditioners
   • FETI-Geneo (bad modes)
   • FETI-Simultaneous
4. FETI for concurrent multiscale

Overview of multiscale approaches

Part 2

Lengths $L$ and $l$ refer to macro and microscales, respectively. In the former they are completely separated ($L_l$) whilst in the latter they remain coupled ($L_l$). Additionally, the structure of computation, as suggested by Belytschko and Song [3], is considered as a secondary criterion to make a subdivision of techniques. Consequently, one can distinguish between decoupled (or sequential), weak coupling, and strong coupling multiscale techniques. The following overview is based on this secondary criteria since it allows to distinguish more accurately between up-to-date established and emerging techniques.

2.1 Decoupled (or sequential) techniques
In these approaches, information is passed in one direction from the microscopic (or mesoscopic) to the macroscopic level. This information exchange is performed as a preprocessing step before the macroscopic analysis is initiated. Since the flow of information is performed only once at the beginning of the analysis, these techniques extract a...
FETI in multiscale computation

Advantage of FETI over BDD:
can handle non-matching meshes

\[
\begin{bmatrix}
K & B^T \\
B & 0
\end{bmatrix}
\begin{bmatrix}
u \\
\lambda
\end{bmatrix}
= 
\begin{bmatrix}
f \\
0
\end{bmatrix}
\]

non-Boolean for non-conforming interface
(Mortar, collocation ...)

Work from Oriol Lloberas-Valls (now Cimne, UPC, Barcelona) supervised by L. Sluijs, A. Simone (TU Delft) & D. Rixen (TU München) [Lloberas-Valls et al., 11-12-12]

FETI in multiscale computation

Adaptive multiscale strategy for increasing load

When stresses are low
(no damage likely to occur)

When the stress level indicates
that damage could occur, refine!

Coarse resolution with effective
(homogenized) elastic properties

Refine ONLY critical domains
(become non-linear due to material)
FETI in multiscale computation

Reequilibrate after refinement

Example 1 – Fibers in matrix

- Fibers as strong inclusions:
  \[ E_f = 20 \times 10^4 \text{ Mpa}, \quad E_i = 20 \times 10^3 \text{ Mpa}, \quad E_m = 40 \times 10^3 \text{ Mpa} \]

due to interface high heterogeneity,
weak compatibility condition with special stiffness weighting

[Lloberas-Valls et al., 12]
Example 1

Example 2: concrete-like specimen
If increasing the number of domains
→ the solution is (nearly) not changing (objectivity)
→ the effectiveness increases (even less non-linear and fine domains)
FETI in multiscale computation

- Efficient concurrent multiscale method
  (non-linear only where needed, linear domains reused)

- Good accuracy compared to fully refined models

- More research needed to improve effectiveness
  - Reuse of search directions for different load steps when domains change
  - Preconditioners and coarse grids
  - ...

Selected References (1)


Selected References (2)


Selected References (3)


