Shifted Laplace and related preconditioning for finite element approximations of the Helmholtz equation

Ivan Graham and Euan Spence (Bath, UK)

Collaborations with:

Paul Childs (Schlumberger Gould Research),
Martin Gander (Geneva)
Douglas Shanks (Bath)
Eero Vainikko (Tartu, Estonia)

Woudschoten
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Outline of talk:

- Seismic inversion, HF Helmholtz equation
- (conventional) FE discretization, preconditioned GMRES solvers
- sharp analysis of preconditioners based on absorption
- analytic wavenumber- and absorption-explicit PDE bounds
- a class of (scalable) DD preconditioners, with coarse grids
- a new convergence theory for DD for Helmholtz
- some open theoretical questions

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Chandler-Wilde, IGG, Langdon, Spence:
Numerical-asymptotic boundary integral methods in high-frequency acoustic scattering
Acta Numerica 2012

Motivation

Marine seismic

Seismic inversion

Inverse problem: reconstruct material properties of rock under sea bed (characterised by wave speed $c(x)$) from observed echos.

Regularised iterative method: repeated solution of the (forward problem): the wave equation

$$-\Delta u + \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = f \quad \text{or its elastic variant}$$

Frequency domain:

$$-\Delta u - \left(\frac{\omega}{c}\right)^2 u = f, \qquad \omega = \quad \text{frequency}$$

solve for u with approximate c.

Seismic inversion

Inverse problem: reconstruct material properties of subsurface (wave speed $c(x)$) from observed echos.

Regularised iterative method: repeated solution of the (forward problem): the wave equation

$$-\Delta u + \frac{\partial^2 u}{\partial t^2} = f \quad \text{or its elastic variant}$$

Frequency domain:

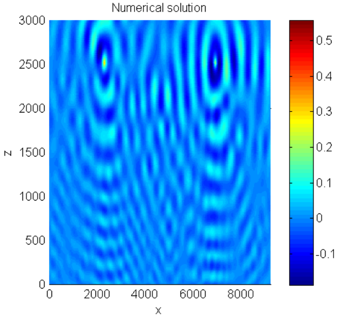
$$-\Delta u - \left(\frac{\omega L}{c}\right)^2 u = f, \qquad \omega = \quad \text{frequency}$$

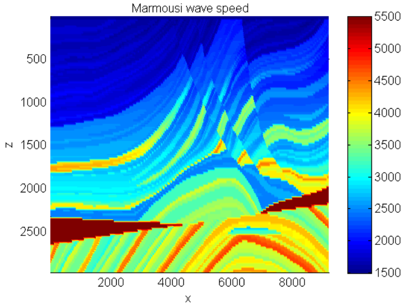
solve for u with approximate c.

Large domain of characteristic length L.

effectively high frequency

Marmousi Model Problem



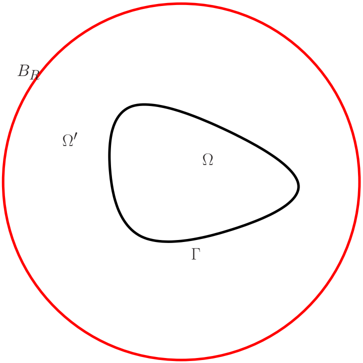


- [P. Childs, Schlumberger (2007)]: Solver of choice based on principle of limited absorption (Erlangga, Osterlee, Vuik, 2004)...
- This work: Analysis of this approach and use it to build better methods .....

Model interior impedance problem

$$\begin{aligned} -\Delta u - k^2 u &= f \quad \text{in} \quad \text{bounded domain} \quad \Omega \\ \frac{\partial u}{\partial n} - iku &= g \quad \text{on} \quad \Gamma := \partial\Omega \end{aligned}$$

....Also truncated sound-soft scattering problems in Ω'



Linear algebra problem

- weak form ,

$$a\ (u,v) \ := \ \int_\Omega (\nabla u.\nabla \overline{v} - \quad k^2\ u\overline{v}) - \mathrm{i}k\int_\Gamma u\overline{v}$$

$$= \ \int_\Omega f\overline{v} + \int_\Gamma g\overline{v}$$

- (Fixed order) finite element discretization

$$\mathbf{A}\ \ \mathbf{u} \ := \ (\mathbf{S} - \quad k^2\ \mathbf{M}^\Omega - \mathrm{i}k\mathbf{M}^\Gamma)\mathbf{u} = \mathbf{f}$$

Often: $h \sim k^{-1}$ **but pollution effect:**
for quasioptimality need $h \sim k^{-2}$?? , $h \sim k^{-3/2}$??

Du and Wu 2013
Melenk and Sauter 2011 (hp)

Linear algebra problem

- weak form **with absorption** $k^2 \to k^2 + \mathrm{i}\varepsilon$,

$$a_\varepsilon(u,v) := \int_\Omega (\nabla u.\nabla\overline{v} - (k^2 + i\varepsilon)u\overline{v}) - \mathrm{i}k\int_\Gamma u\overline{v}$$
$$= \int_\Omega f\overline{v} + \int_\Gamma g\overline{v} \quad \text{“Shifted Laplacian''}$$

[Equivalently $k^2 + \mathrm{i}\varepsilon \longleftrightarrow (k + \mathrm{i}\rho)^2$]

- Finite element discretization

$$\mathbf{A}_\varepsilon \mathbf{u} := (\mathbf{S} - (k^2 + i\varepsilon)\mathbf{M}^\Omega - \mathrm{i}k\mathbf{M}^\Gamma)\mathbf{u} = \mathbf{f}$$

Preconditioning with approximations of A_ε^{-1} - A very short history

Bayliss, Goldstein & Turkel 83 , Laird & Giles 02.....

Erlangga, Vuik & Oosterlee '04 :

$$B_\varepsilon^{-1} = \quad \text{Multigrid approx for} \quad \mathbf{A}_\epsilon^{-1}$$

$\epsilon \sim k^2$ (analysis via Fourier eigenvalue analysis)

Kimn & Sarkis '13 used $\varepsilon \sim k^2$ to enhance domain decomposition methods

Engquist and Ying, '11 Used $\varepsilon \sim k$ to stabilise their sweeping preconditioner

more on shifted Laplace

Y. A. Erlangga, 2008.

Y. A. Erlangga, C. W. Oosterlee, and C. Vuik, 2006.

Eigenvalues of $(\Delta + k^2(a+ib))^{-1}(\Delta + k^2)$ well clustered when $a \sim 1 \sim b$.

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M. B. Van Gijzen, Y. A. Erlangga, and C. Vuik, 2007.

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A. H. Sheikh, D. Lahaye, and C. Vuik, 2013.

Deflation - $\varepsilon \sim k^2$ Fourier analysis on model Dirichlet problems

S. Cools and W. Vanroose , 2013.

O. Ernst and M. Gander, 2012, $\varepsilon \sim k^2$ for multigrid convergence

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Preconditioning with A_ε^{-1} and its approximations

$$\mathbf{A}_\varepsilon^{-1}\mathbf{A}\mathbf{u} = \mathbf{A}_\varepsilon^{-1}\mathbf{f}.$$

"Elman theory" for GMRES requires:

$\|\mathbf{A}_\varepsilon^{-1}\mathbf{A}\| \lesssim 1,$ and $\mathrm{dist}(0, \mathbf{fov}(\mathbf{A}_\varepsilon^{-1}\mathbf{A})) \gtrsim 1$ **any norm**

Sufficient condition: $\|\mathbf{I} - \mathbf{A}_\varepsilon^{-1}\mathbf{A}\|_2 \lesssim C < 1$.

Preconditioning with $\mathrm{A}_\varepsilon^{-1}$ and its approximations

$$\mathbf{A}_\varepsilon^{-1}\mathbf{A}\mathbf{u} = \mathbf{A}_\varepsilon^{-1}\mathbf{f}.$$

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Sufficient condition: $\|\mathbf{I} - \mathbf{A}_\varepsilon^{-1}\mathbf{A}\|_2 \lesssim C < 1$.

In practice use

$$\mathbf{B}_\varepsilon^{-1}\mathbf{A}\mathbf{u} = \mathbf{B}_\varepsilon^{-1}\mathbf{f}, \quad \text{where} \quad \mathbf{B}_\varepsilon^{-1} \approx \mathbf{A}_\varepsilon^{-1}.$$ Cheaper

Writing

$$\mathbf{I} - \mathbf{B}_\varepsilon^{-1}\mathbf{A} = \mathbf{I} - \mathbf{B}_\varepsilon^{-1}\mathbf{A}_\varepsilon + \mathbf{B}_\varepsilon^{-1}\mathbf{A}_\varepsilon(\mathbf{I} - \mathbf{A}_\varepsilon^{-1}\mathbf{A}),$$

a sufficient condition is:

$$\|\mathbf{I} - \mathbf{A}_\varepsilon^{-1}\mathbf{A}\|_2 \text{ and } \|\mathbf{I} - \mathbf{B}_\varepsilon^{-1}\mathbf{A}_\varepsilon\|_2 \text{ small},$$

i.e. $\mathbf{A}_\varepsilon^{-1}$ to be a good preconditioner for $\mathbf{A}$
and $\mathbf{B}_\varepsilon^{-1}$ to be a good preconditioner for $\mathbf{A}_\varepsilon$.

Preconditioning with $\mathbf{A}_\varepsilon^{-1}$ and its approximations

$$\mathbf{A}_\varepsilon^{-1}\mathbf{A}\mathbf{u} = \mathbf{A}_\varepsilon^{-1}\mathbf{f}.$$

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Sufficient condition: $\|\mathbf{I} - \mathbf{A}_\varepsilon^{-1}\mathbf{A}\|_2 \lesssim C < 1$.

In practice use

$$\mathbf{B}_\varepsilon^{-1}\mathbf{A}\mathbf{u} = \mathbf{B}_\varepsilon^{-1}\mathbf{f},$$

$\mathbf{B}_\varepsilon^{-1}$ easily computed approximation of $\mathbf{A}_\varepsilon^{-1}$. Writing

$$\mathbf{I} - \mathbf{B}_\varepsilon^{-1}\mathbf{A} \; = \; \mathbf{I} - \mathbf{B}_\varepsilon^{-1}\mathbf{A}_\varepsilon \; + \; \mathbf{B}_\varepsilon^{-1}\mathbf{A}_\varepsilon(\mathbf{I} - \mathbf{A}_\varepsilon^{-1}\mathbf{A}),$$

so we require

$$\|\mathbf{I} - \mathbf{A}_\varepsilon^{-1}\mathbf{A}\|_2 \; \text{and} \; \|\mathbf{I} - \mathbf{B}_\varepsilon^{-1}\mathbf{A}_\varepsilon\|_2 \quad \text{small},$$

i.e. **$\mathbf{A}_\varepsilon^{-1}$ to be a good preconditioner for $\mathbf{A}$**
and $\mathbf{B}_\varepsilon^{-1}$ to be a good preconditioner for $\mathbf{A}_\varepsilon$. **Part 1**

Preconditioning with A_ε^{-1} and its approximations

$$\mathbf{A}_\varepsilon^{-1}\mathbf{A}\mathbf{u} = \mathbf{A}_\varepsilon^{-1}\mathbf{f}.$$

"Elman theory" for GMRES requires:

$$\|\mathbf{A}_\varepsilon^{-1}\mathbf{A}\| \lesssim 1, \quad \text{and} \quad \mathrm{dist}(0, \mathbf{fov}(\mathbf{A}_\varepsilon^{-1}\mathbf{A})) \gtrsim 1$$

Sufficient condition: $\|\mathbf{I} - \mathbf{A}_\varepsilon^{-1}\mathbf{A}\|_2 \lesssim C < 1$.

In practice use

$$\mathbf{B}_\varepsilon^{-1}\mathbf{A}\mathbf{u} = \mathbf{B}_\varepsilon^{-1}\mathbf{f},$$

B_ε^{-1} easily computed approximation of A_ε^{-1}. Writing

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so we require

$$\|\mathbf{I} - \mathbf{A}_\varepsilon^{-1}\mathbf{A}\|_2 \text{ and } \|\mathbf{I} - \mathbf{B}_\varepsilon^{-1}\mathbf{A}_\varepsilon\|_2 \quad \text{small ,}$$

i.e. A_ε^{-1} **to be a good preconditioner for** A
and B_ε^{-1} to be a good preconditioner for A_ε . Part 2

Theorem 1 (with Martin Gander and Euan Spence)
For Lipschitz star-shaped domains
Quasiuniform meshes:

$$\|\mathbf{I} - \mathbf{A}_\epsilon^{-1}\mathbf{A}\| \quad \lesssim \quad \frac{\epsilon}{k} \ .$$

Shape regular meshes:

$$\|\mathbf{I} - \mathbf{D}^{1/2}\mathbf{A}_\epsilon^{-1}\mathbf{A}\mathbf{D}^{-1/2}\| \quad \lesssim \quad \frac{\epsilon}{k} \ .$$

$\mathbf{D} = \mathrm{diag}(\mathbf{M}^\Omega)$.

So ϵ/k sufficiently small $\implies$ $k-$independent GMRES convergence.

Shifted Laplacian preconditioner $\varepsilon = k$

Solving $\mathbf{A}_\varepsilon^{-1}\mathbf{A}\mathbf{x} = \mathbf{A}_\varepsilon^{-1}\mathbf{1}$ on unit square

	k	# GMRES
	10	**6**
$h \sim k^{-3/2}$	20	**6**
	40	**6**
	80	**6**

Shifted Laplacian preconditioner $\varepsilon = k^{3/2}$

Solving $\mathbf{A}_\varepsilon^{-1}\mathbf{A}\mathbf{x} = \mathbf{A}_\varepsilon^{-1}\mathbf{1}$ on unit square

	k	# GMRES
	10	**8**
$h \sim k^{-3/2}$	20	**11**
	40	**14**
	80	**16**

Shifted Laplacian preconditioner $\varepsilon = k^2$

Solving $\mathbf{A}_\varepsilon^{-1}\mathbf{A}\mathbf{x} = \mathbf{A}_\varepsilon^{-1}\mathbf{1}$ on unit square

	k	# GMRES
	10	**13**
$h \sim k^{-3/2}$	20	**24**
	40	**48**
	80	**86**

Proof of Theorem 1: via continuous problem

$$a_\epsilon(u,v) = \int_\Omega f\overline{v} + \int_\Gamma g\overline{v}\ , \quad v \in H^1(\Omega) \qquad (*)$$

Theorem (Stability) Assume Ω is Lipschitz and star-shaped. Then, if ϵ/k sufficiently small,

$$\underbrace{\|\nabla u\|^2_{L^2(\Omega)} + k^2\|u\|^2_{L^2(\Omega)}}_{=:\|u\|^2_{1,k}} \lesssim \|f\|^2_{L^2(\Omega)} + \|g\|^2_{L^2(\Gamma)}\ , \quad k \to \infty$$

"$\lesssim$" indept of k and ϵ cf. Melenk 95, Cummings & Feng 06

More absorption: $k \lesssim \epsilon \lesssim k^2$ general Lipschitz domain OK.

Key technique in proof (star-shaped case)

Rellich/Morawetz Identity

$$\mathcal{M}u = \mathbf{x}.\nabla u + \alpha u, \qquad \alpha = (d-1)/2$$
$$\mathcal{L}u = \Delta u + k^2 u$$

$$\|\nabla u\|^2_{L^2(\Omega)} + k^2\|u\|^2_{L^2(\Omega)} = -2\,\mathrm{Re}\int_\Omega (\overline{\mathcal{M}u}\mathcal{L}u) + \int_\Gamma \left[2\,\mathrm{Re}(\overline{\mathcal{M}u}\frac{\partial u}{\partial n}) + (k^2|u|^2 - |\nabla u|^2)(\mathbf{x}.\mathbf{n})\right]$$

Key technique in proof (star-shaped case)

Rellich/Morawetz Identity

$$\mathcal{M}u = \mathbf{x}.\nabla u + \alpha u, \qquad \alpha = (d-1)/2$$
$$\mathcal{L}u = \Delta u + k^2 u$$

$$\begin{aligned}\|\nabla u\|^2_{L^2(\Omega)} + k^2\|u\|^2_{L^2(\Omega)} \quad &= \quad -2\,\mathrm{Re}\int_\Omega (\overline{\mathcal{M}u}\mathcal{L}u) \\ &+ \int_\Gamma \left[2\,\mathrm{Re}(\overline{\mathcal{M}u}\frac{\partial u}{\partial n}) \; + \; (k^2|u|^2 - |\nabla u|^2)(\mathbf{x}.\mathbf{n})\right]\end{aligned}$$

cf. “Green’s identity”

$$\|\nabla u\|^2_{L^2(\Omega)} - k^2\|u\|^2_{L^2(\Omega)} \; = \; -\int_\Omega (\overline{u}\mathcal{L}u) \; + \; \int_\Gamma \overline{u}\frac{\partial u}{\partial n}$$

Bound for $\|A_\epsilon^{-1}\|_2$

Fix $\mathbf{f} \in \mathbb{C}^N$, and consider the solution of $\mathbf{A}_\varepsilon \mathbf{u} = \mathbf{f}$.

Then $u_h := \sum_j u_j \phi_j$ is FE solution of problem

$$a_\epsilon(u, v) = (f_h, v)$$

with $$\|f_h\|_{L_2(\Omega)} \sim h^{-d/2} \|\mathbf{f}\|_2.$$

Then

$$\begin{aligned} k\, h^{d/2}\, \|\mathbf{u}\|_2 &\sim k\|u_h\|_{L_2(\Omega)} \\ &\leq \|u_h\|_{1,k} \\ &\leq \|u - u_h\|_{1,k} + \|u\|_{1,k} \\ &\leq 2\|u\|_{1,k} \quad \text{quasioptimality} \\ &\lesssim \|f_h\|_{L_2(\Omega)} \quad \text{stability} \end{aligned}$$

and so

$$\|\mathbf{A}_\epsilon^{-1}\| \lesssim h^{-d} k^{-1}, \quad \text{for all} \quad \varepsilon \lesssim k^2$$

PDE Theory to bound the matrix $\mathbf{A}_\epsilon^{-1}$

Fix $\mathbf{f} \in \mathbb{C}^N$, and consider the solution of $\mathbf{A}_\varepsilon \mathbf{u} = \mathbf{f}$.

Then $u_h := \sum_j u_j \phi_j$ is FE solution of problem

$$a_\epsilon(u, v) = (f_h, v)$$

with $$\|f_h\|_{L_2(\Omega)} \sim h^{-d/2}\|\mathbf{f}\|_2.$$

Then

$$\begin{aligned} k\ h^{d/2}\ \|\mathbf{u}\|_2 &\sim k\|u_h\|_{L_2(\Omega)} \\ &\leq \|u_h\|_{1,k} && \text{(A)} \\ &\leq \|u - u_h\|_{1,k} + \|u\|_{1,k} \\ &\lesssim 2\|u\|_{1,k} && \text{quasioptimality} \\ &\lesssim \|f_h\|_{L_2(\Omega)} && \text{stability} \quad \text{(B)} \end{aligned}$$

and so

$$\|\mathbf{A}_\epsilon^{-1}\| \lesssim h^{-d}k^{-1}, \quad \text{for all} \quad \varepsilon \lesssim k^2$$

By H.Wu (2013) (A) $\lesssim$ (B) when $hk^{3/2} \lesssim 1$. (without ε)

Corollary

$$\begin{aligned}\|\mathbf{I}-\mathbf{A}_{\epsilon}^{-1}\mathbf{A}\| &\leq \|\mathbf{A}_{\varepsilon}^{-1}\|\|\mathbf{A}_{\varepsilon}-\mathbf{A}\| \\ &\leq h^{-d}k^{-1}\,\|i\epsilon\mathbf{M}\| \\ &\lesssim \frac{\epsilon}{k}\,.\end{aligned}$$

Corollary

$$\begin{aligned} \|\mathbf{I} - \mathbf{A}_\epsilon^{-1}\mathbf{A}\| &\leq \|\mathbf{A}_\varepsilon^{-1}\| \|\mathbf{A}_\varepsilon - \mathbf{A}\| \\ &\leq h^{-d}k^{-1} \, \|i\epsilon\mathbf{M}\| \\ &\lesssim \frac{\epsilon}{k} \, . \end{aligned}$$

Locally refined meshes:

$$\|\mathbf{I} - \mathbf{D}^{1/2}\mathbf{A}_\epsilon^{-1}\mathbf{A}\mathbf{D}^{-1/2}\| \quad \lesssim \quad \frac{\epsilon}{k} \, .$$

Same for right preconditioning

Exterior scattering problem with refinement

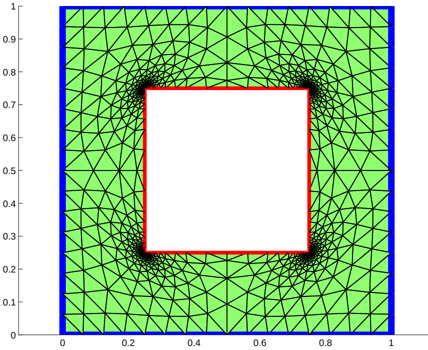
$h \sim k^{-1}$,

Solving $\mathbf{A}_\varepsilon^{-1}\mathbf{A}\mathbf{x} = \mathbf{A}_\varepsilon^{-1}\mathbf{1}$ on unit square

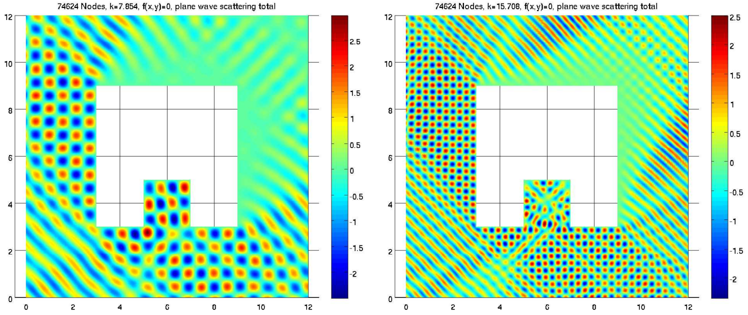
$\#$ **GMRES**

with diagonal scaling

k	$\varepsilon = k$	$\varepsilon = k^{3/2}$
20	**5**	**8**
40	**5**	**11**
80	**5**	**13**
160	**5**	**16**



A trapping domain



k	$\varepsilon = k$	$\varepsilon = k^{3/2}$
$10\pi/8$	**18**	**29**
$20\pi/8$	**19**	**41**
$40\pi/8$	**21**	**60**
$80\pi/8$	**22**	**89**

Stability result fails when ε grows slower than k "quasimodes"

Betcke, Chandler-Wilde, IGG, Langdon, Lindner, 2010

Part 2: How to approximate A_ε^{-1}?

Engquist & Ying (2012):
"Since the shifted Laplacian operator is elliptic, standard algorithms such as multigrid can be used for its inversion"

Domain Decomposition:

Many non-overlapping methods ($\varepsilon = 0$)

Benamou & Després 1997.....Gander, Magoules, Nataf, Halpern, Dolean........

General issue: coarse grids, scalability?

Conjecture If ε large enough, classical overlapping DD methods with coarse grids will work (giving scalable solvers).

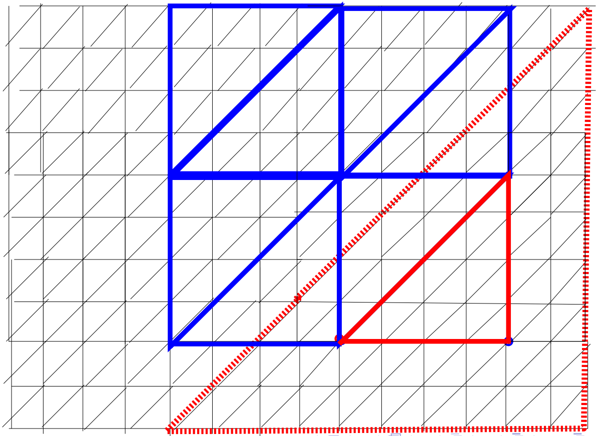
However Classical analysis for $\varepsilon = 0$ (Cai & Widlund, 1992) leads to coarse grid size $H \sim k^{-2}$

Classical additive Schwarz

To solve a problem on a fine grid FE space $\mathcal{S}_h$

- **Coarse space** $\mathcal{S}_H$ (here linear FE) **on a coarse grid**
- **Subdomain spaces** $\mathcal{S}_i$ **on subdomains** Ω_i, overlap δ

$H_{sub} \sim H$ in this case



Classical additive Schwarz p/c for matrix C

Approximation of C^{-1}:

$$\sum_i \mathbf{R}_i^T \mathbf{C}_i^{-1} \mathbf{R}_i \; + \; \mathbf{R}_H^T \mathbf{C}_H^{-1} \mathbf{R}_H$$

$\mathbf{R}_i$ = restriction to $\mathcal{S}_i$,
$\mathbf{C}_i = \mathbf{R}_i \mathbf{C} \mathbf{R}_i^T$
Dirichlet BCs

$\mathbf{R}_H$ = restriction to $\mathcal{S}_H$
$\mathbf{C}_H = \mathbf{R}_H \mathbf{C} \mathbf{R}_H^T$

Apply to A_ε to get $\mathrm{B}_\varepsilon^{-1}$

Non-standard DD theory - applied to A_ε

Coercivity Lemma There exisits $|\Theta| = 1$, with

$$\mathrm{Im}\left[\Theta a_\varepsilon(v,v)\right] \ \gtrsim \ \frac{\varepsilon}{k^2}\|v\|_{1,k}^2. \qquad (\star)$$

Projections onto subpaces:

$$a_\varepsilon(Q_i v_h, w_i) = a_\varepsilon(v_h, w_i), \quad v_h \in \mathcal{S}_h, \quad w_i \in \mathcal{S}_i \ .$$

Non-standard DD theory - applied to A_ε

Coercivity Lemma There exisits $|\Theta| = 1$, with

$$\mathrm{Im}\,[\Theta a_\varepsilon(v,v)] \gtrsim \frac{\varepsilon}{k^2} \underbrace{\|v\|_{1,k}^2}_{\|\nabla u\|_\Omega^2 + k^2\|u\|_\Omega^2} . \qquad (\star)$$

Projections onto subpaces:

$$a_\varepsilon(Q_H v_h, w_H) = a_\varepsilon(v_h, w_H), \quad v_h \in \mathcal{S}_h, \quad w_H \in \mathcal{S}_H .$$

Guaranteed well-defined by $(\star)$.

Analysis of $\mathbf{B}_\varepsilon^{-1}\mathbf{A}_\varepsilon$ equivalent to analysing

$$Q := \sum_i Q_i + Q_H \quad \text{operator in FE space} \quad \mathcal{S}_h .$$

Convergence results

Assume $\varepsilon \sim k^2$ and overlap $\delta \sim H$.

Theorem IGG, Spence, Vainikko, 2014

For all coarse grid sizes H,

$$\|Q\|_{1,k} \lesssim 1 .$$

Theorem IGG, Spence, Vainikko, 2014

There exists $C > 0$ so that

$$\mathrm{dist}(0, \mathsf{fov}(Q)) \gtrsim 1,$$

provided $kH < C$ (no pollution!).

Hence $k-$independent GMRES convergence.

Convergence results

Assume $\varepsilon \sim k^2$. and overlap δ.

Theorem IGG, E. Spence, E. Vainikko, 2014

For all coarse grid sizes H,

$$\|Q\|_{1,k} \lesssim 1 \, .$$

Theorem IGG, E. Spence, E. Vainikko, 2014

There exists $C > 0$ so that

$$\mathrm{dist}(0, \mathsf{fov}(Q)) \gtrsim \left(1 + \frac{H}{\delta}\right)^{-2},$$

provided $kH < C$ (no pollution!).

Numerical experiments: unit square

$$\varepsilon = k^2 \qquad h \sim k^{-3/2}, \qquad H \sim k^{-1} \qquad \delta \sim H$$

Classical additive Schwarz - see later for better!

k	#**GMRES**
20	**14**
40	**15**
60	**15**
80	**17**

Some steps in proof $\varepsilon \sim k^2$

$$(v_h, Qv_h)_{1,k} = \sum_j (v_h, Q_j v_h)_{1,k} + (v_h, Q_H v_h)_{1,k}$$

Some steps in proof $\varepsilon \sim k^2$

$$(v_h, Qv_h)_{1,k} = \sum_j (v_h, Q_j v_h)_{1,k} + (v_h, Q_H v_h)_{1,k}$$

$$(v_h, Q_H v_h)_{1,k} = \|Q_H v_h\|_{1,k}^2 + ((I - Q_H)v_h, Q_H v_h)_{1,k}$$

Some steps in proof $\varepsilon \sim k^2$

$$(v_h, Qv_h)_{1,k} \;=\; \sum_j (v_h, Q_j v_h)_{1,k} + (v_h, Q_H v_h)_{1,k}$$

$$(v_h, Q_H v_h)_{1,k} \;=\; \|Q_H v_h\|_{1,k}^2 + ((I - Q_H)v_h, Q_H v_h)_{1,k}$$

$$((I - Q_H)v_h, Q_H v_h)_{1,k} = \underbrace{a_\varepsilon((I - Q_H)v_h, Q_H v_h)}_{=0,\quad \text{Galerkin orthogonality}} + L_2 \quad \text{terms}$$

Bound L^2 terms using duality, regularity $\Longrightarrow$ condition on kH

Some steps in proof $\varepsilon \sim k^2$

$$(v_h, Qv_h)_{1,k} = \sum_j (v_h, Q_j v_h)_{1,k} + (v_h, Q_H v_h)_{1,k}$$

$$(v_h, Q_H v_h)_{1,k} = \|Q_H v_h\|_{1,k}^2 + ((I - Q_H)v_h, Q_H v_h)_{1,k}$$

$$((I - Q_H)v_h, Q_H v_h)_{1,k} = \underbrace{a_\varepsilon((I - Q_H)v_h, Q_H v_h)}_{=0, \quad \text{Galerkin orthogonality}} + L_2 \quad \text{terms}$$

Bound L^2 terms using duality, regularity $\implies$ condition on kH

Some steps in proof $\varepsilon \sim k^2$

$$(v_h, Qv_h)_{1,k} = \sum_j (v_h, Q_j v_h)_{1,k} + (v_h, Q_H v_h)_{1,k}$$

$$(v_h, Q_H v_h)_{1,k} = \|Q_H v_h\|_{1,k}^2 + ((I - Q_H)v_h, Q_H v_h)_{1,k}$$

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Bound L^2 terms using duality, regularity $\Longrightarrow$ condition on kH

$$|(v_h, Qv_h)_{1,k}| \gtrsim \sum_j \|Q_j v_h\|_{1,k}^2 + \|Q_H v_h\|_{1,k}^2$$

$$\gtrsim \|v_h\|_{1,k}^2$$

Some steps in proof $\varepsilon \ll k^2$

$$(v_h, Qv_h)_{1,k} = \sum_j (v_h, Q_j v_h)_{1,k} + (v_h, Q_H v_h)_{1,k}$$

$$(v_h, Q_H v_h)_{1,k} = \|Q_H v_h\|_{1,k}^2 + ((I - Q_H)v_h, Q_H v_h)_{1,k}$$

$$((I - Q_H)v_h, Q_H v_h)_{1,k} = \underbrace{a_\varepsilon((I - Q_H)v_h, Q_H v_h)}_{=0} + L_2 \quad \text{terms}$$

Galerkin Orthogonality, duality, regularity $\implies$ condition on kH

$$\begin{aligned} \mathrm{Re}(v_h, Qv_h)_{1,k} &\gtrsim \sum_j \|Q_j v_h\|_{1,k}^2 + \|Q_H v_h\|_{1,k}^2 \\ &\gtrsim \left(\frac{\varepsilon}{k^2}\right)^2 \|v_h\|_{1,k}^2 \quad \text{when} \quad Hk \lesssim (\varepsilon/k^2)^2 \end{aligned}$$

Useful Variants

- **Hybrid**: Multiplicative between coarse and local solves

Mandel and Brezina: 1994,96

- **RAS**: only add up once on regions of overlap

Cai & Sarkis, 1999, Kimn & Sarkis 2010

$\mathrm{B}_\varepsilon^{-1}$ as preconditioner for A_ε $\quad \varepsilon = k^2$

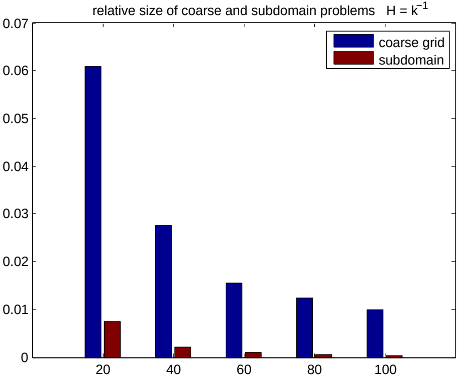
$h \sim k^{-3/2}, \quad n \sim k^3,$ Hybrid RAS,

$Hk \sim 1$

Relative **Coarse** and **subdomain** problem size

Scale $= 0.07$

k	**#GMRES**
20	**8**
40	**8**
60	**8**
80	**8**
100	**8**



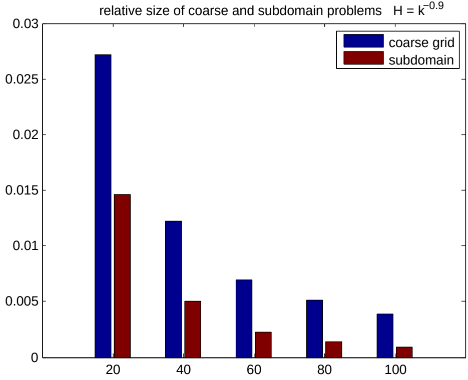
B_ε^{-1} as preconditioner for A_ε $\quad \varepsilon = k^2$

$h \sim k^{-3/2}, \quad n \sim k^3,$ Hybrid RAS,

$Hk \sim k^{0.1}$

Scale $= 0.03$

k	#GMRES
20	9
40	10
60	10
80	10
100	10



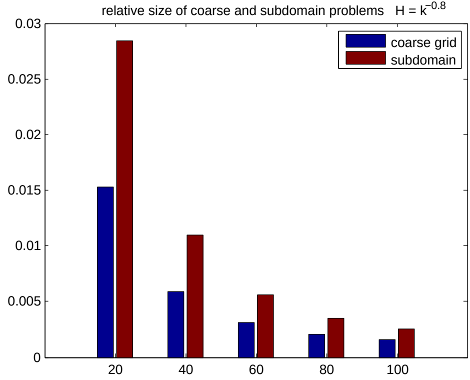
B_ε^{-1} as preconditioner for A_ε $\quad \varepsilon = k^2$

$h \sim k^{-3/2}, \quad n \sim k^3,$ Hybrid RAS,

$Hk \sim k^{0.2}$

Scale $= 0.03$

k	#GMRES
20	10
40	10
60	11
80	11
100	11



Solving the real problem: B_k^{-1} as preconditioner for A

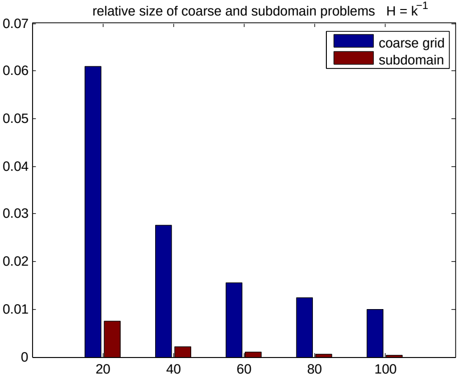
$h \sim k^{-3/2}$, $n \sim k^3$, Hybrid RAS,

$\varepsilon \sim k$ **empirically best choice** (cf. multigrid)

$Hk \sim 1$

k	# **GMRES**
20	**12**
40	**15**
60	**20**
80	**26**
100	**33**

Scale $= 0.07$



Solving the real problem: B_k^{-1} as preconditioner for A

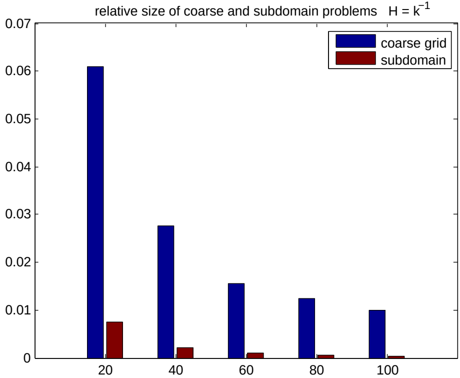
$h \sim k^{-3/2}$, $n \sim k^3$, Hybrid RAS,

$\varepsilon \sim k$ **empirically best choice** (cf. multigrid)

$Hk \sim 1$

No coarse grid **Scale** $= 0.07$

k	# **GMRES**
20	**58**
40	**181**
60	**316**
80	**434**
100	**576**



Problem becomes “less elliptic” as $\varepsilon << k^2$.

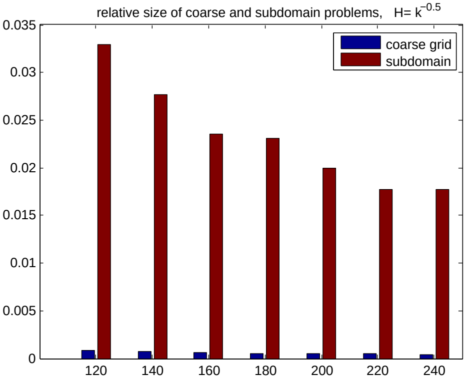
local Dirichlet $\rightarrow$ local impedance (or PML) Toselli , 1999

Solving the real problem: B_k^{-1} as preconditioner for A

20 grid points per wavelength, $h \sim k^{-1}$, $n \sim k^2$,
Impedance subdomain problems $Hk \sim k^{0.5}$

Scale $= 0.035$

k	**#GMRES**
120	**51**
140	**56**
160	**59**
180	**57**
200	**61**
220	**64**
240	**65**



 # GMRES $\sim \log k$

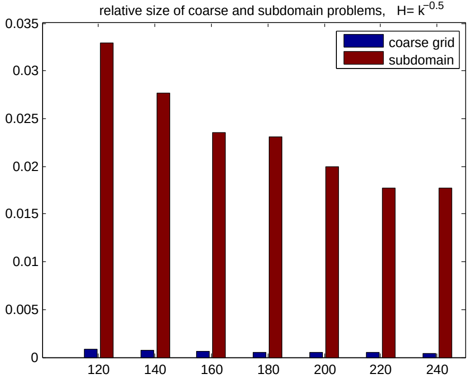
Solving the real problem: B_k^{-1} as preconditioner for A

20 grid points per wavelength, $h \sim k^{-1}$, $n \sim k^2$,

Dirichlet subdomain problems $Hk \sim k^{0.5}$

Scale $= 0.035$

k	**#GMRES**
120	**487**
140	**595**
160	$>$ **600**
180	$>$ **600**
200	
220	
240	



Summary

- k and ϵ explicit analysis allows rigorous explanation of some empirical observations and formulation of new methods.

- When $\epsilon \in [0, k]$, $\mathbf{A}_\epsilon^{-1}$ is optimal preconditioner for $\mathbf{A}$

- When $\epsilon \sim k^2$, $\mathbf{B}_\varepsilon^{-1}$ is "optimal" for $\mathbf{A}_\varepsilon$ $(H \sim k^{-1})$

- Analysis is for classical DP method - introduce more wavelike components

- When preconditioning $\mathbf{A}$ with $\mathbf{B}_\varepsilon^{-1}$, empirical best choice is $\varepsilon \sim k$

- New framework for DD analysis for larger k.

- Open questions in analysis when $\frac{\varepsilon}{k^2} \ll 1$