Error analysis and fast solvers for high-frequency scattering problems

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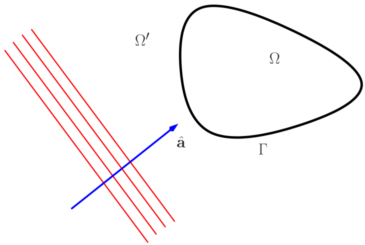
High freq. problem for the Helmholtz equation

Given an object $\Omega \subset \mathbb{R}^d$, with boundary Γ and exterior Ω',
Incident plane wave, e.g. : $u_I(x) = \exp(\mathrm{i}\boldsymbol{k}\mathbf{x} \cdot \widehat{\mathbf{a}})$



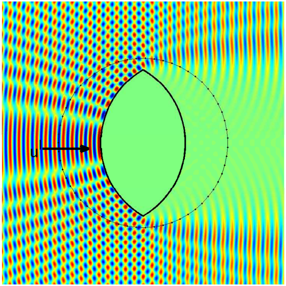
Total wave $u = u_I + u_S$, where **Scattered wave** u_S satisfies:

$$\Delta u_S + \boldsymbol{k}^2 u_S \;=\; 0 \quad \text{in } \Omega'$$

plus **boundary condition** (Mostly $u_I + u_S = 0$ on Γ) and

radiation condition: $\frac{\partial u^S}{\partial r} - i\boldsymbol{k}u^S \;=\; o(r^{-(d-1)/2})$ as $r \to \infty$

Numerical-asymptotic methods

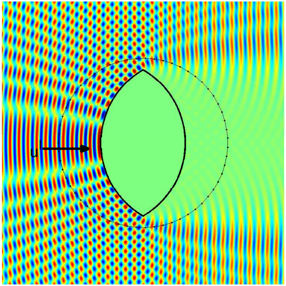


Constant wavenumber k - asymptotic information (mostly BEM)

Computing in “time independent of frequency”.

Links to Daan and Simon’s talks

Numerical-asymptotic methods



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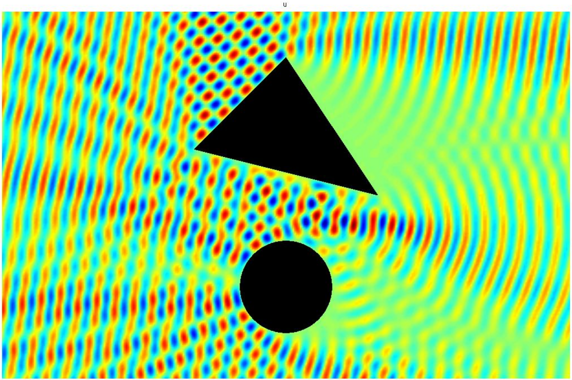
Geometry dependent methods

Numerical-asymptotic methods

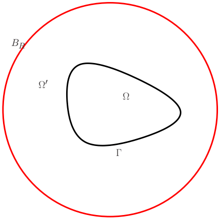


Still a role for conventional BEM

Second Talk: Truncated problems

$$
\begin{aligned}
\Delta u + k^2 u &= 0 \quad \text{in } \Omega' \cap B_R \\
u &= -u_I \quad \text{on} \quad \Gamma \\
\frac{\partial u}{\partial n} - iku &= 0 \quad \text{on} \quad B_R
\end{aligned}
$$

for large R



Model "cavity" problem

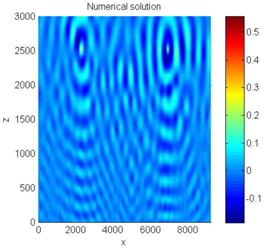
$$
\begin{aligned}
\Delta u + k^2 u &= f \quad \text{in} \quad \text{bounded domain} \quad \Omega \\
\frac{\partial u}{\partial n} - iku &= g \quad \text{on} \quad \Gamma := \partial\Omega
\end{aligned}
$$

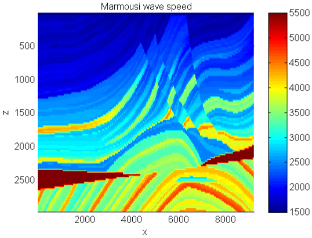
Heterogeneity

Seismic inversion problem:

$$-\Delta u - \left(\frac{\omega L}{c(x)}\right)^2 u = f, \qquad \omega = \quad \text{frequency}$$

solve for u with approximate c.





Second talk: Conventional discretisation and fast solvers

Key reference: Erlangga, Osterlee, Vuik, 2004...

Link to Martin's talks and Domain Decomposition

Outline of my talks:

Two problems on **conventional methods**.

1. When is the error in the $h-$ version BEM bounded independently of k?

2. Give an analysis of preconditioning methods for standard $h-$ version FEM

Both have solutions which use high-frequency analysis.

References for the talks:

- I.G. Graham, M. Loehndorf, J.M. Melenk and E.A. Spence, When is the error in the h-BEM for solving the Helmholtz equation bounded independently of k? To appear in BIT, 2014.

- M. J. Gander, I. G. Graham and E. A. Spence, How should one choose the shift for the shifted Laplacian to be a good preconditioner for the Helmholtz equation?, submitted 2014.

- I.G. Graham, E.A. Spence and E. Vainikko, Convergence of additive Schwarz methods for the Helmholtz equation with and without absorption, in preparation.

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• I.G. Graham, E.A. Spence and E. Vainikko, Convergence of additive Schwarz methods for the Helmholtz equation with and without absorption, in preparation.

• S.N. Chandler-Wilde, I.G. Graham, S. Langdon and E.A. Spence, Boundary Integral Equation Methods for High Frequency Scattering Problems Acta Numerica 2012.

First problem

When is the error in the $h-$ version boundary element method bounded independently of k?

Fundamental solution for the Helmholtz equation

$$\Delta u + k^2 u \;=\; 0$$

$$G_k(x,y) = \begin{cases} \frac{i}{4} H_0^{(1)}(k|x-y|) & \text{2D} \\[2ex] \dfrac{\exp(ik|x-y|)}{4\pi|x-y|} & \text{3D} \end{cases}$$

Phase: $k|x-y|$

single layer potential : $(\mathcal{S}_k\phi)(x) \;=\; \int_\Gamma G_k(x,y)\phi(y)dS(y),$

double layer: $(\mathcal{D}_k\phi)(x) \;=\; \int_\Gamma [\partial_{n(y)} G_k(x,y)]\phi(y)dS(y),$

adjoint double layer: $\mathcal{D}'_k$ (switch roles of x and y).

Combined potential boundary integral formulations

Exterior scattering problem with incident field u_I:

Green's identity for u_S in Ω':

$$\mathcal{S}_k(\partial_n u_S \qquad) - \mathcal{D}_k(u_S \qquad) = (-u_S \qquad) \quad \text{in} \quad \Omega' \qquad (1)$$

Combined potential boundary integral formulations

Exterior scattering problem with incident field u_I:

Green's identity for u_I in Ω:

$$\mathcal{S}_k(\partial_n u_S + \partial_n u_I) - \mathcal{D}_k(u_S + u_I) \; = \; (-u_S + 0) \quad \text{in} \quad \Omega' \qquad (1)$$

Combined potential boundary integral formulations

Exterior scattering problem with incident field u_I:

Green's identity for u_I in Ω:

$$\mathcal{S}_k(\underbrace{\partial_n u_S + \partial_n u_I}_{\partial_n u}) - \mathcal{D}_k(\underbrace{u_S + u_I}_{=0}) \; = \; (\underbrace{-u_S + 0}_{u_I}) \quad \text{in} \quad \Omega' \qquad (1)$$

Combined potential boundary integral formulations

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Limit to boundary Γ: Equation for unknown $v := \partial_n u$
but with spurious frequencies.

k

Combined potential boundary integral formulations

Exterior scattering problem with incident field u_I:

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Limit to boundary Γ: Equation for unknown $v := \partial_n u$ but with spurious frequencies.

Take normal derivative in (1) and combine with $-\mathrm{i}k\times$ (1):
"direct" combined potential formulation

$$\mathcal{R}'_k v := \left(\frac{1}{2}I + \mathcal{D}'_k\right) v - \mathrm{i}k\mathcal{S}_k v = \partial_n u_I - \mathrm{i}k u_I \,, \quad \text{or } k \to \eta$$

Combined potential boundary integral formulations

Exterior scattering problem with incident field u_I:

Green's identity for u_I in Ω:

$$\mathcal{S}_k(\underbrace{\partial_n u_S + \partial_n u_I}_{\partial_n u}) - \mathcal{D}_k(\underbrace{u_S + u_I}_{=0}) \;=\; (\underbrace{-u_S + 0}_{u_I}) \quad \text{in} \quad \Omega' \qquad (1)$$

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Alternative **"indirect"** method:

$$\mathcal{R}_k\phi \;:=\; \left(\frac{1}{2}I + \mathcal{D}_k\right)\phi - \mathrm{i}k\mathcal{S}_k\phi \;=\; u_I\,,$$

BEM analysis - Classical setting

"Fredholm integral equations of the Second kind"

$$\begin{aligned} \mathcal{R}'_k v &= (\lambda I + \mathcal{L}'_k) v &= f_k \\ \mathcal{R}_k \phi &= (\lambda I + \mathcal{L}_k)\phi &= g_k \qquad (\lambda = 1/2) \end{aligned}$$

Galerkin method in approximating space $\mathcal{V}_N$ (or $\mathcal{V}_h$).

e.g. piecewise polynomials of fixed degree p.

Solution v_N or ϕ_N, e.g.

$$(\lambda I + \mathcal{P}_N \mathcal{L}'_k) v_N = \mathcal{P}_N f_k$$

BEM analysis - Classical setting

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e.g. piecewise polynomials of fixed degree p.

Solution v_N or ϕ_N, e.g.

$$(\lambda I + \mathcal{P}_N \mathcal{L}'_k) v_N = \mathcal{P}_N f_k$$

$$v - v_N = \lambda \underbrace{(\lambda I - \mathcal{P}_N \mathcal{L}'_k)^{-1}}_{\text{stability}} \underbrace{(v - \mathcal{P}_N v)}_{\text{best approx}}$$

Question 1 (best approximation error)

When are

$$\frac{\inf_{w_N \in \mathcal{V}_N} \|v - w_N\|_{L^2(\Gamma)}}{\|v\|_{L^2(\Gamma)}}$$

and

$$\frac{\inf_{w_N \in \mathcal{V}_N} \|\phi - w_N\|_{L^2(\Gamma)}}{\|\phi\|_{L^2(\Gamma)}}$$

bounded independently of k?

Question 2 (quasioptimality)

When are

$$\frac{\|v - v_N\|_{L^2(\Gamma)}}{\inf_{w_N \in \mathcal{V}_N} \|v - w_N\|_{L^2(\Gamma)}}$$

and

$$\frac{\|\phi - \phi_N\|_{L^2(\Gamma)}}{\inf_{w_N \in \mathcal{V}_N} \|\phi - w_N\|_{L^2(\Gamma)}}$$

bounded independently of k?

If both hold...(bound on relative errors)

$$\frac{\|v - v_N\|_{L^2(\Gamma)}}{\|v\|_{L^2(\Gamma)}}$$

and

$$\frac{\|\phi - \phi_N\|_{L^2(\Gamma)}}{\|\phi\|_{L^2(\Gamma)}}$$

bounded indpendently of k.

Answers: Question 1 ("direct" version $v = \partial_n u$)

When is

$$\frac{\inf_{w_N \in \mathcal{V}_N} \|v - w_N\|_{L^2(\Gamma)}}{\|v\|_{L^2(\Gamma)}}$$

bounded independently of k?

Theorem If Ω is C^∞ and convex then for $h-$BEM,

$$\inf_{w_h \in \mathcal{V}_h} \|v - w_h\|_{L^2(\Gamma)} \lesssim (hk)^p \|v\|_{L^2(\Gamma)}$$

so $hk \lesssim 1$ is sufficient for Question 1.

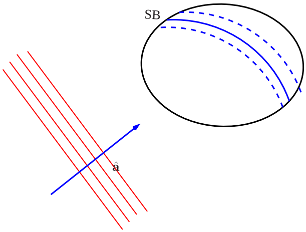
Proof uses

$$v(\mathbf{x}) \;:=\; \partial u/\partial n(\mathbf{x}) \;=\; kV(\mathbf{x},k)\exp(ik\mathbf{x}\cdot\hat{\mathbf{a}})\;, \quad x\in\Gamma,$$

Theorem Dominguez, IGG, Smyshlyaev, 2007

$$|D^n V(x,k)| \le \begin{cases} C_n, & n=0,1,\\ C_n\, k^{-1}\,(k^{-1/3}+\mathrm{dist}(x,SB))^{-(n+2)} & n\ge 2,\end{cases}$$

where $SB=\{\mathbf{x}\in\Gamma : \mathbf{n}(\mathbf{x}).\hat{\mathbf{a}}=0\}$ shadow boundary.



Proves, e.g. $\|v\|_{H^1(\Gamma)} \lesssim k\|v\|_{L^2(\Gamma)}$

Answers: Question 1 (“direct” version $v = \partial_n u$)

When is

$$\frac{\inf_{w_N \in \mathcal{V}_N} \|v - w_N\|_{L^2(\Gamma)}}{\|v\|_{L^2(\Gamma)}}$$

bounded independently of k?

Theorem If Ω is a convex polygon then there is a mesh with $\mathcal{O}(N)$ points so that ,

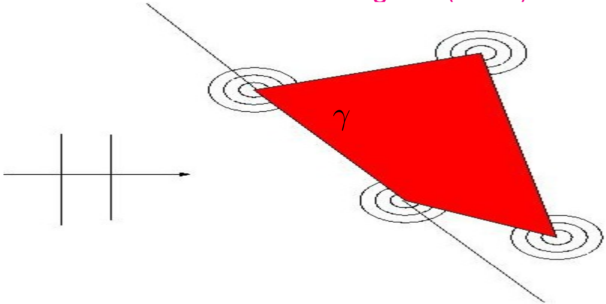
$$\inf_{w_h \in \mathcal{V}_h} \|v - w_h\|_{L^2(\Gamma)} \lesssim \frac{k}{N}\|v\|_{L^2(\Gamma)}$$

so $k/N \lesssim 1$ is sufficient for Question 1.

(Requires $\sup_{\mathbf{x}\in\Omega'} |u(\mathbf{x})|$.)

Proof uses:

Theorem Chandler-Wilde and Langdon (2007)



$$\frac{\partial u}{\partial n}(s) \quad = \quad 2\frac{\partial u^I}{\partial n}(s) + \mathrm{e}^{\mathrm{i}ks}v_+(s) + \mathrm{e}^{-\mathrm{i}ks}v_-(s)$$

where s is distance along γ, and

$$k^{-n}|v_+^{(n)}(s)| \le \begin{cases} C_n(ks)^{-1/2-n}, & ks \ge 1, \\ C_n(ks)^{-\alpha-n}, & 0 < ks \le 1, \end{cases}$$

where $\alpha < 1/2$ depends on the corner angle.

Answers: Question 1: Indirect method

$$\lambda\phi \;=\; \mathcal{L}_k\phi \;=\; \mathrm{i}k\mathcal{S}_k\phi + \mathcal{D}_k\phi$$

To estimate the derivatives of ϕ:

$$\|\mathcal{S}_k\|_{H^1\leftarrow L_2} \;\lesssim\; k^{(d-1)/2} \quad (\Gamma \;\text{ Lipschitz})$$

$$\|\mathcal{D}_k\|_{H^1\leftarrow L_2} \;\lesssim\; k^{(d+1)/2} \quad (\Gamma \;\text{ smooth enough})$$

These imply $\|\phi\|_{H^1(\Gamma)} \;\lesssim\; k^{(d+1)/2}\|\phi\|_{L^2(\Gamma)}$

And so $hk^{(d+1)/2} \lesssim 1$ is sufficient for Question 1.

Answers: Question 2 (classical approach)

$$\begin{aligned} \mathcal{R}'_k v := \quad (\lambda I + \mathcal{L}'_k)v \quad &= f_k \quad \text{compact perturbation} \\ (\lambda I + \mathcal{P}_h \mathcal{L}'_k)v_h \quad &= \mathcal{P}_h f_k \quad \text{Galerkin method} \end{aligned}$$

Lemma [Atkinson, Anselone, 1960's ....]

If $\quad \|(I - \mathcal{P}_h)\mathcal{L}'_k\| \|(\lambda I + \mathcal{L}'_k)^{-1}\| << 1,$

then $\quad \|v - v_h\| \lesssim \|(\lambda I + \mathcal{L}'_k)^{-1}\| \inf_{w_h \in \mathcal{V}_h} \|v - w_h\|$

Application:

$$\|(I - \mathcal{P}_h)\mathcal{L}'_k\| \lesssim h\|\mathcal{L}'_k\|_{L^2 \to H^1} \lesssim hk^{(d+1)/2}$$

and in addition:

$$\|(\lambda I + \mathcal{L}'_k)^{-1}\| \lesssim 1$$ [Chandler-Wilde & Monk, 2008]

Lipschitz star-shaped

Theorem Hence quasioptimality if $hk^{(d+1)/2} \leq C$

Tools

We used in this talk

• $k-$ explicit bounds on norms of $\mathcal{L}_k$, $\mathcal{L}'_k$
(where $\mathcal{R}'_k = \frac{1}{2}I + \mathcal{L}'_k$), etc.
needed smooth enough domains

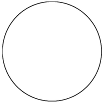
• $k-$ explicit bounds on inverses $(\mathcal{R}_k)^{-1}, (\mathcal{R}'_k)^{-1}$
needed Lipschitz star-shaped

The Subtlety of Behaviour of $\|\mathcal{L}_k\|$ and $\|\mathcal{R}_k^{-1}\|$
Equivalently $\|\mathcal{L}'_k\|$ and $\|(\mathcal{R}'_k)^{-1}\|$

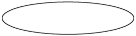
$\|\mathcal{L}_k\|, \quad \|\mathcal{R}_k^{-1}\|$

$\sim k^{1/3}, \quad \sim 1$

Circle

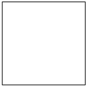


Ellipse



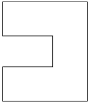
$\sim k^{1/2}, \quad \sim 1$

Square

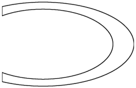


$\sim k_m^{1/2}, \quad \sim k_m^{7/5}$

Rectangular cavity



Elliptic Cavity



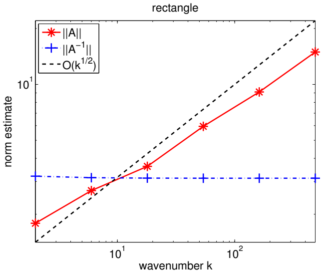
$\sim k_m^{1/2}, \quad \sim \mathrm{e}^{\gamma k_m}$

Chandler-Wilde, IGG et al (2009),
Betcke, Chandler-Wilde, IGG et al (2011).

Numerical Experiments: domain $[0, 0.5] \times [0, 5]$

$$\sqrt{1+\gamma_p^2} \approx \frac{\|v - v_N\|_{L^2(\Gamma)}}{\inf_{w_N \in \mathcal{V}_N} \|v - w_N\|_{L^2(\Gamma)}}$$

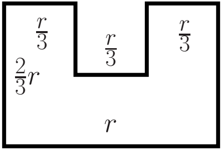
degree $p = 0, 1$

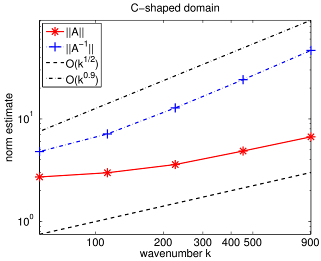


N	k	γ_0	γ_1
22	2	0.368234	0.136623
66	6	0.334368	0.121106
198	18	0.337487	0.120028
594	54	0.335113	0.120023
1782	162	0.333687	0.12
5346	486	0.333559	0.119998

$hk \sim 1$

Numerical Experiments: trapping domain





m	k	N	γ_0	γ_1
3	56.5	120	0.480033	0.174585
6	113.1	240	0.487655	0.174454
12	226.2	480	0.51861	0.174301
24	452.4	960	0.527743	0.174264
48	904.8	1920	0.549879	0.174278

Tools

Open question: Prove $hk \lesssim 1$ sufficient for quasioptimality

Tools

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• $k-$ explicit bounds on norms of $\mathcal{L}_k$, $\mathcal{L}'_k$
(where $\mathcal{R}'_k = \frac{1}{2}I + \mathcal{L}'_k$), etc.
needed smooth enough domains

• $k-$ explicit bounds on inverses $(\mathcal{R}_k)^{-1}, (\mathcal{R}'_k)^{-1}$
needed Lipschitz star-shaped

We will need in the next talk

• Bound on the solution operator for the Helmholtz BVP PDE itself.

• connection between the two illustrates the role of star-shaped.

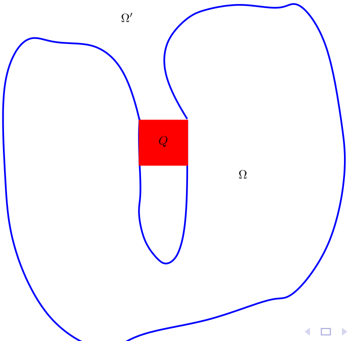
“Trapping domains”

Can things get bad in the non-star-shaped case?

Theorem

If the exterior domain Ω' contains a square Q of side length a and the boundary Γ coincides with two parallel sides of Q, then if $2ak = m\pi$ for any positive integer m,

$$\|\mathcal{R}_k^{-1}\| \gtrsim (ak)^{9/10} .$$



"Quasimodes"

(family of) sources g and solutions v of Helmholtz problem

$$\Delta v + k^2 v = g \quad \text{in} \quad \Omega' \qquad v = 0 \quad \text{on} \quad \Gamma$$

+ Sommerfeld condition, where

$$\|v\|_{L^2(\Omega')} \;\geq\; M_k \|g\|_{L^2(\Omega')}, \quad \text{with} \quad M_k \quad \text{"large"}$$

. This could contradict the bound

$$\|v\|_{L^2(\Omega')} \;\lesssim\; \frac{1}{k} \|f\|_{L^2(\Omega')}$$

which holds in star-shaped case (see next lecture).

In fact

$$\mathcal{R}_k' \left(\partial_n v\right) = \left(\partial_n v^I - \mathrm{i} k v^I\right)$$

where v^I is the Newtonian potential generated by g

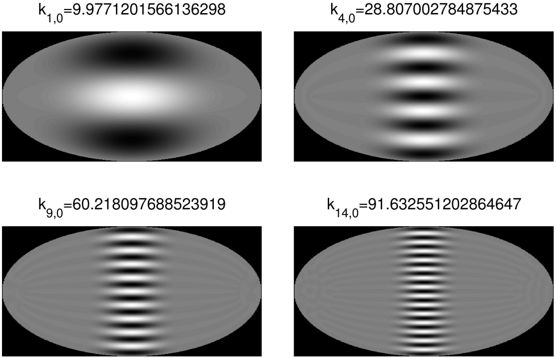
implies growth of $\|(\mathcal{R}_k')^{-1}\|$

More generally ...

$$\|(\mathcal{R}_k')^{-1}\| \gtrsim k^{-(d-2)} M_k - \mathcal{O}(k^{(d-1)/2})$$

With elliptic cavity M_k can increase exponentially.

[Betcke, Chandler-Wilde, IGG, Langdon, Lindner, 2011]



Final theorem for today...

Model interior impedance problem:

$$\begin{aligned} -\Delta u - k^2 u &= f \quad \text{in} \quad \text{bounded domain} \quad \Omega \\ \frac{\partial u}{\partial n} - iku &= g \quad \text{on} \quad \Gamma := \partial\Omega \end{aligned}$$

....Also truncated sound-soft scattering problems in Ω'

Theorem (Stability) Assume Ω is Lipschitz and star-shaped. Then,

$$\underbrace{\|\nabla u\|^2_{L^2(\Omega)} + k^2\|u\|^2_{L^2(\Omega)}}_{=:\|u\|^2_{1,k}} \lesssim \|f\|^2_{L^2(\Omega)} + \|g\|^2_{L^2(\Gamma)}\,, \qquad k \to \infty$$

[Melenk 95, Cummings & Feng 06]

Central result in next lecture