From the invention of the Schwarz method to the Best Current Methods for Oscillatory Problems: Part 2

Martin J. Gander
martin.gander@unige.ch

University of Geneva

Woudschoten, October 2014

Are There Two Helmholtz Equations ?

$(\Delta + k^2)u = f$ **or** $(\Delta - \eta)u = f,\ \eta > 0$?

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XVI.

Theorie der Luftschwingungen in Röhren mit offenen Enden.

Journal für reine und angewandte Mathematik. Bd. 57 S. 1—72. (1859.)

Die mathematische Theorie der Orgelpfeifen ist von den bedeutendsten mathematischen Physikern vielfältig behandelt worden, aber seit den ersten Schritten, welche D. Bernoulli und Euler gethan haben, und durch welche die Hauptzüge der Erscheinung eine annähernde Erklärung fanden, um keinen wesentlichen Schritt vorgerückt. Der Grund davon hat haupt-

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und Schallschwingungen erregen. In allen anderen Theilen der Luftmasse ist $q = 0$, und dort sind daher die Functionen Ψ der Bedingung unterworfen:

$$0 = k^2 \Psi + \frac{d^2 \Psi}{dx^2} + \frac{d^2 \Psi}{dy^2} + \frac{d^2 \Psi}{dz^2}. \tag{3b}$$

Ich werde im Folgenden den immer wiederkehrenden Ausdruck:

$$\frac{d^2\Phi}{dx^2} + \frac{d^2\Phi}{dy^2} + \frac{d^2\Phi}{dz^2}$$

nach dem Vorgang von Green mit $\nabla_x \Phi$, oder wo es unzweideutig ist, mit $\nabla \Phi$ bezeichnen.

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welche in allen Theilen des Raumes S der Gleichung genügt:

$$\triangle \Psi + k^2 \Psi = 0,$$

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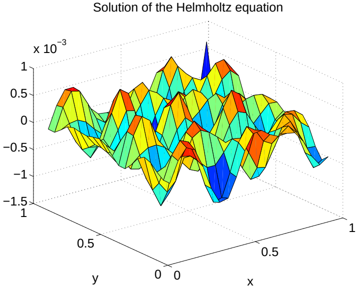

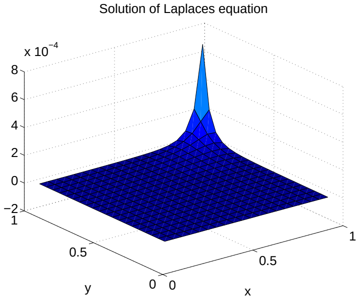

Preconditioning with ILU, Multigrid or Schwarz

Example: Open cavity with point source at the center

	QMR		ILU('0')		ILU(1e-2)	
k	it	Mflops	it	Mflops	it	Mflops
5	197	120.1	60	60.4	22	28.3
10	737	1858.2	370	1489.3	80	421.4
20	$>$ 2000	$>$ 20335.1	—	—	$>$ 2000	$>$ 42320.1

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k	Smoothing steps	2.5π	5π	10π	20π
Iterative	$\nu = 2$	7	div	div	div
Preconditioner	$\nu = 2$	6	12	41	127
Iterative	$\nu = 5$	7	stag	div	div
Preconditioner	$\nu = 5$	5	13	41	223

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k	Overlap	10π	20π	40π	80π	160π
Iterative	h	div	div	div	div	div
Preconditioner	h	20	33	45	69	110
Iterative	fixed	div	div	div	div	div
Preconditioner	fixed	16	23	43	86	155

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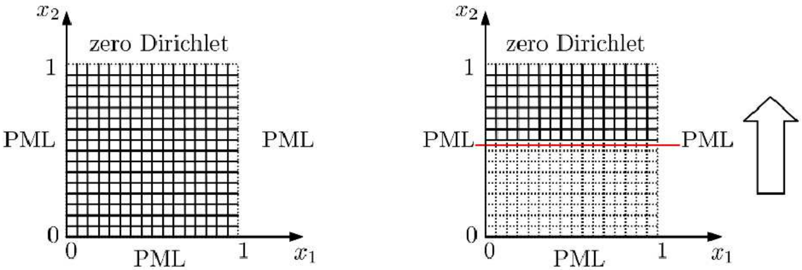
"Why it is Difficult to Solve Helmholtz Problems with Classical Iterative Methods", Ernst and MJG 2012

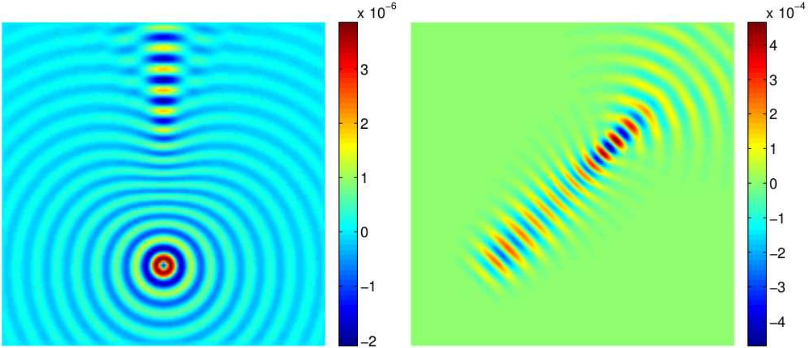
New Method 1: Sweeping Preconditioner

Enquist, Ying 2010: *Sweeping Preconditioner for the Helmholtz Equation*

“The paper introduces the sweeping preconditioner, which is highly efficient for iterative solutions of the variable coefficient Helmholtz equation including very high frequency problems. The first central idea of this novel approach is to construct an approximate factorization of the discretized Helmholtz equation by sweeping the domain layer by layer, starting from an absorbing layer or boundary condition.”



Numerical Experiment (Enquist, Ying 2010)



				Test 1		Test 2	
$\omega/(2\pi)$	q	$N=n^3$	T_{setup}	N_{iter}	T_{solve}	N_{iter}	T_{solve}
5	8	39^3	4.80e+00	11	4.53e+00	11	4.63e+00
10	8	79^3	6.37e+01	11	4.92e+01	11	4.93e+01
20	8	159^3	8.27e+02	12	5.53e+02	12	5.94e+02

New Method 2: Source Transfer DDM

Chen, Xiang 2012: *A Source Transfer Domain Decomposition Method for Helmholtz Equations in Unbounded Domain*

"The method is based on the decomposition of the domain into non-overlapping layers and the idea of source transfer which transfers the sources equivalently layer by layer so that the solution in the final layer can be solved using a PML method defined locally outside the last two layers."

ALGORITHM 2.1. (SOURCE TRANSFER FOR PML PROBLEM IN $\mathbb{R}^2$)

1° *Let* $\bar{f}_1 = f_1$ *in* $\mathbb{R}^2$;

2° *For* $i = 1, 2, \cdots, N-2$, *compute* $\bar{f}_{i+1} = f_{i+1} + \Psi_{i+1}(\bar{f}_i)$, *where*

$$\Psi_{i+1}(\bar{f}_i) = \begin{cases} J^{-1}\nabla \cdot (A\nabla(\beta_{i+1}u_i)) + k^2(\beta_{i+1}u_i) & in \quad \Omega_{i+1}, \\ 0 & in \quad \mathbb{R}^2 \backslash \bar{\Omega}_{i+1}, \end{cases}$$

and u_i *is given by*

$$u_i(x) = \int_{\Omega_i} \bar{f}_i(y)\tilde{G}(x,y)dy.$$

Numerical Experiment (Chen, Xiang 2012)

$k/2\pi$	q	DOF	N_{iter}	T_{solve}
30	10	300^2	5	2.43
60	10	600^2	5	9.77
120	10	1200^2	6	44.58
240	10	2400^2	7	225.15
480	10	4800^2	8	1122.37
960	10	9600^2	12	8047.68

TABLE 4.1

Numerical results for different wave numbers k when $q = 10$, where N_{tier} is the number of iterations of the preconditioned GMRES method and T_{solve} is the overall solution time in seconds.

$k/2\pi$	q	NOF	N_{iter}	T_{solve}
30	20	600^2	3	8.11
60	20	1200^2	3	26.58
120	20	2400^2	4	127.94
240	20	4800^2	5	676.45

TABLE 4.2

Numerical results for different wave numbers k when $q = 20$, where N_{tier} is the number of iterations of the preconditioned GMRES method and T_{solve} is the overall solution time in seconds.

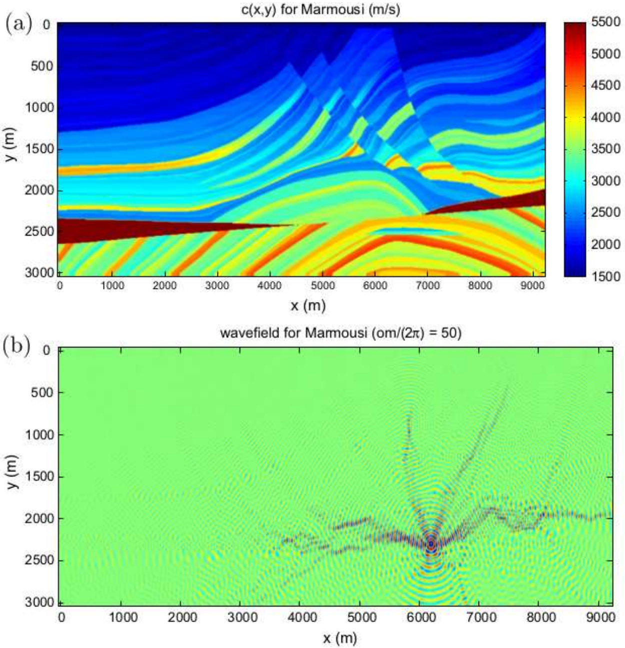
New Method 3: PML DDM

Stolk 2013: *A rapidly converging domain decomposition method for the Helmholtz equation*

"A new domain decomposition method is introduced for the heterogeneous 2-D and 3-D Helmholtz equations. Transmission conditions based on the perfectly matched layer (PML) are derived that avoid artificial reflections and match incoming and outgoing waves at the subdomain interfaces."

"Our most remarkable finding concerns the situation where the domain is split into many thin layers along one of the axes, say J subdomains numbered from 1 to J. Following [3] we will also call these quasi 2-D subdomains. Generally, an increase in the number of subdomains leads to an increase in the number of iterations required for convergence. Here we propose and study a method where the number of iterations is essentially independent of the number of subdomains."

Numerical Experiments (Stolk 2013)



Numerical Experiments (Stolk 2013)

Table 2
Convergence results for Example 1. Displayed is the number of iterations for reduction of the residual by 10^{-6} as a function of the size of the domain and the number of subdomains.

$N_x \times N_y$	h (m)	$\frac{\omega}{2\pi}$ (Hz)	Number of x-subdomains				
			3	10	30	100	300
600×212	16	12.5	4	5	6		
1175×400	8	25	5	6	7		
2325×775	4	50	6	6	7	9	
4625×1525	2	100	6	6	7	8	
9225×3025	1	200		7	8	9	13 (8) (*)

(*) 13 was obtained for $w_{\text{pml}} = 5$, 8 for $w_{\text{pml}} = 6$.

Table 5
Comparison of convergence between Robin and PML-based transmission conditions for a constant medium.

$N_x \times N_y$	h	$\frac{\omega}{2\pi}$	J	PML	Robin
100×100	0.01	10	10	3	9
200×200	0.005	20	20	4	13
400×400	0.0025	40	40	4	20
800×800	0.00125	80	80	5	42
1600×1600	0.000625	160	160	7	103

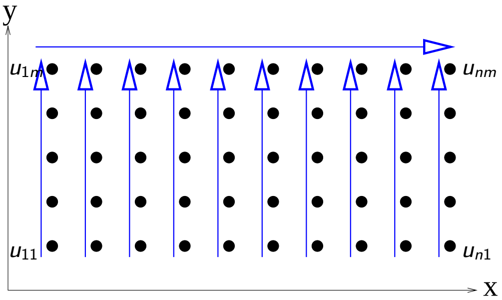
Table 6
Comparison of convergence between Robin and PML-based transmission conditions for the random medium displayed in Fig. 3.

$N_x \times N_y$	h	$\frac{\omega}{2\pi}$	J	PML	Robin
100×100	0.01	7.14	10	7	11
200×200	0.005	14.29	20	6	14
400×400	0.0025	28.57	40	6	20
800×800	0.00125	57.14	80	7	34
1600×1600	0.000625	114.3	160	8	74

Underlying Fundamental Algorithms 1

An optimal discrete algorithm based on matrix factorization

$$(\Delta + k^2)u = f \qquad \text{in } \Omega = (0,1)\times(0,\pi)$$



$$A\mathbf{u} = \begin{bmatrix} D_1 & L_{1,2} & & \\ L_{2,1} & D_2 & \ddots & \\ & \ddots & \ddots & L_{n-1,n} \\ & & L_{n,n-1} & D_n \end{bmatrix} \begin{pmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \\ \vdots \\ \mathbf{u}_n \end{pmatrix} = \begin{pmatrix} \mathbf{f}_1 \\ \mathbf{f}_2 \\ \vdots \\ \mathbf{f}_n \end{pmatrix}$$

Block Factorization

The block LU factorization of the block tridiagonal matrix is

$$A = \begin{bmatrix} D_1 & L_{1,2} & & \\ L_{2,1} & D_2 & L_{2,3} & \\ & \ddots & \ddots & \ddots \\ & & L_{n,n-1} & D_n \end{bmatrix} = LU$$

$$= \begin{bmatrix} T_1 & & & \\ L_{2,1} & T_2 & & \\ & \ddots & \ddots & \\ & & L_{n,n-1} & T_n \end{bmatrix} \begin{bmatrix} I & T_1^{-1}L_{1,2} & & \\ & I & T_2^{-1}L_{2,3} & \\ & & \ddots & \ddots \\ & & & I \end{bmatrix}$$

where the diagonal blocks T_i satisfy the recurrence

$$T_i = \begin{cases} D_1 & i = 1, \\ D_i - L_{i,i-1}T_{i-1}^{-1}L_{i-1,i} & 1 < i \leq n. \end{cases}$$

Forward and Backward Substitution

$$Au = LUu = f \quad \Longleftrightarrow \quad Lv = f, \quad Uu = v$$

Forward substitution:

$$\begin{bmatrix} T_1 & & & \\ L_{2,1} & T_2 & & \\ & \ddots & \ddots & \\ & & L_{n,n-1} & T_n \end{bmatrix} \begin{pmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \\ \vdots \\ \mathbf{v}_n \end{pmatrix} = \begin{pmatrix} \mathbf{f}_1 \\ \mathbf{f}_2 \\ \vdots \\ \mathbf{f}_n \end{pmatrix}$$

$$\begin{aligned}
\mathbf{v}_1 &= T_1^{-1}\mathbf{f}_1 \\
\mathbf{v}_2 &= T_2^{-1}(\mathbf{f}_2 - L_{2,1}\mathbf{v}_1) = T_2^{-1}(\mathbf{f}_2 - L_{2,1}T_1^{-1}\mathbf{f}_1) = T_2^{-1}\bar{\mathbf{f}}_2 \\
\mathbf{v}_3 &= T_3^{-1}(\mathbf{f}_3 - L_{3,2}\mathbf{v}_2) = T_3^{-1}(\mathbf{f}_3 - L_{3,2}T_2^{-1}\bar{\mathbf{f}}_2) = T_3^{-1}\bar{\mathbf{f}}_3 \\
&\vdots \qquad\qquad\qquad\qquad \vdots \qquad\qquad\qquad\qquad \vdots
\end{aligned}$$

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$\Longrightarrow$ Source transfer: $\bar{\mathbf{f}}_j = \mathbf{f}_j - L_{j,j-1}T_{j-1}^{-1}\bar{\mathbf{f}}_{j-1}$ and $\mathbf{v}_n = \mathbf{u}_n$

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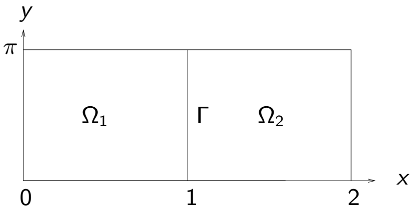
$\Longrightarrow$ Sweeping preconditioner: forward and backward sweep

Underlying Fundamental Algorithms 2

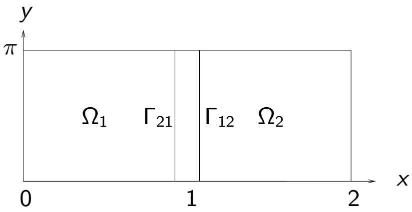


Nataf et al (1994): Optimal Schwarz algorithm

$$
\begin{aligned}
(\Delta + k^2)u_1^n &= f & \text{in } \Omega_1\\
\partial_x u_1^n + \mathrm{DtN}_2(u_1^n) &= \partial_x u_2^{n-1} + \mathrm{DtN}_2(u_2^{n-1}) & \text{on } \Gamma\\
(\Delta + k^2)u_2^n &= f & \text{in } \Omega_2\\
\partial_x u_2^n - \mathrm{DtN}_1(u_2^n) &= \partial_x u_1^{n-1} - \mathrm{DtN}_1(u_1^{n-1}) & \text{on } \Gamma
\end{aligned}
$$

This algorithm converges in two iterations,

Underlying Fundamental Algorithms 2



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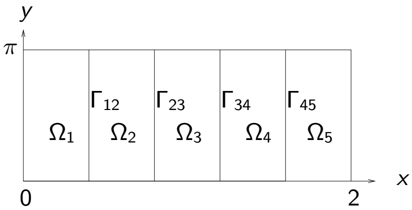
(\Delta + k^2)u_2^n & = & f & \text{in } \Omega_2 \\
\partial_x u_2^n - \mathrm{DtN}_1(u_2^n) & = & \partial_x u_1^{n-1} - \mathrm{DtN}_1(u_1^{n-1}) & \text{on } \Gamma_{21}
\end{array}
$$

This algorithm converges in two iterations, independantly of the overlap!

Underlying Fundamental Algorithms 2



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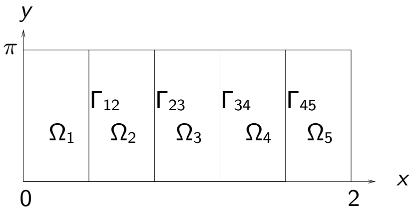
With N subdomains, it converges in N iterations,

Underlying Fundamental Algorithms 2



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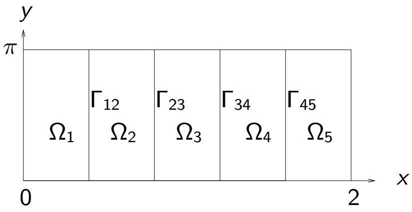
\partial_x u_i^n - \mathrm{DtN}_{i,i-1}(u_i^n) &= \partial_x u_{i-1}^{n-1} - \mathrm{DtN}_{i,i-1}(u_{i-1}^{n-1}) && \text{on } \Gamma_{i-1,i}
\end{aligned}
$$

With N subdomains, it converges in N iterations,
or in two when sweeping back and forth once, independently of N $\Longrightarrow$ Stolk DDM.

Underlying Fundamental Algorithms 2



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(\Delta + k^2)u_i^n &= f & \text{in } \Omega_i\\
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\partial_x u_i^n - \mathrm{DtN}_{i,i-1}(u_i^n) &= \partial_x u_{i-1}^{n-1} - \mathrm{DtN}_{i,i-1}(u_{i-1}^{n-1}) & \text{on } \Gamma_{i-1,i}
\end{aligned}
$$

With N subdomains, it converges in N iterations, or in two when sweeping back and forth once, independently of N $\Longrightarrow$ Stolk DDM.

It is the continuous formulation of block LU !

Practical Algorithms 1

DtN operator is non local. Fourier symbol is

$$\widehat{\mathrm{DtN}}(u) = \sqrt{k^2 - \xi^2}$$

Use approximations for practical algorithms:

- absorbing boundary conditions (ABCs)

$$\sqrt{k^2 - \xi^2} \approx p + q\xi^2.$$

- and perfectly matched layers (PMLs)

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Examples: OO0 and OO2 (Japhet, Nataf 1999), OSM (many in 1999-2014, MJG, Kwok 2010), Source Transfer (Chen, Xiang 2012), Helmholtz DDM (Stolk 2013)

Practical Algorithms 2

Block factorization: need to approximate dense matrices T_i

$$T_i = \begin{cases} D_1 & i = 1, \\ D_i - L_{i,i-1} T_{i-1}^{-1} L_{i-1,i} & 1 < i \leq n. \end{cases}$$

$\Longrightarrow$ Use algebraic (compute and approximate the limit by a sparse matrix) or continuous techniques (as in OSM)

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- based on grid layers instead of subdomains
- sweep forward and backward

Examples: Frequency Filtering (Wittum 1992), AILU (G, Nataf 2000), Sweeping preconditioner (Enquist, Ying 2010)

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Disadvantage: can not use adaptive mesh refinement !

MJG, Zhang 2014: Iterative Solvers for the Helmholtz Equation: Factorizations, Sweeping Preconditioners and Schwarz Methods

Maxwell's Equations

James C. Maxwell, in "Faradays Lines of Force" (1856):

"All the mathematical sciences are founded on relations between physical laws and laws of numbers, so that the aim of exact science is to reduce the problems of nature to the determination of quantities by operations with numbers."

The original Maxwell's equations (20 equations in 20 unknowns) are nowadays written in the form

$$\varepsilon\frac{\partial \boldsymbol{\mathcal{E}}}{\partial t} - \operatorname{curl} \boldsymbol{\mathcal{H}} + \sigma\boldsymbol{\mathcal{E}} = -\mathbf{J}, \qquad \mu\frac{\partial \boldsymbol{\mathcal{H}}}{\partial t} + \operatorname{curl} \boldsymbol{\mathcal{E}} = 0$$

Related form using only the electric field:

$$\varepsilon\frac{\partial^2 \boldsymbol{\mathcal{E}}}{\partial t^2} + \sigma\frac{\partial \boldsymbol{\mathcal{E}}}{\partial t} + \operatorname{curl} \frac{1}{\mu}\operatorname{curl} \boldsymbol{\mathcal{E}} = -\frac{\partial \boldsymbol{\mathcal{J}}}{\partial t}$$

Time Harmonic and Time Discretized Variants

Time harmonic Maxwell's equations:

$$\begin{aligned}\mathcal{E}(\mathbf{x},t) &= \mathcal{R}e(\mathbf{E}(\mathbf{x})\exp(i\omega t))\\ \mathcal{H}(\mathbf{x},t) &= \mathcal{R}e(\mathbf{H}(\mathbf{x})\exp(i\omega t))\end{aligned}$$

$$\Longrightarrow \boxed{(\sigma + i\omega\varepsilon)\mathbf{E} - \operatorname{curl}\mathbf{H} = -\mathbf{J}, \quad i\omega\mu\mathbf{H} + \operatorname{curl}\mathbf{E} = \mathbf{0}.}$$

Time discretized Maxwell's equations:

$$\begin{cases} \varepsilon\frac{\mathbf{E}^{n+1}-\mathbf{E}^n}{\Delta t} - \operatorname{curl}\left(\frac{\mathbf{H}^{n+1}+\mathbf{H}^n}{2}\right) + \sigma\left(\frac{\mathbf{E}^{n+1}+\mathbf{E}^n}{2}\right) = -\mathbf{J},\\ \mu\frac{\mathbf{H}^{n+1}-\mathbf{H}^n}{\Delta t} + \operatorname{curl}\left(\frac{\mathbf{E}^{n+1}+\mathbf{E}^n}{2}\right) = \mathbf{0}.\end{cases}$$

$$\Longrightarrow \boxed{(\sigma + \varepsilon\sqrt{\eta})\mathbf{E} - \operatorname{curl}\mathbf{H} = -\tilde{\mathbf{J}}, \quad \mu\sqrt{\eta}\mathbf{H} + \operatorname{curl}\mathbf{E} = \mathbf{g},}$$

where $(\mathbf{E},\mathbf{H}) := (\mathbf{E}^{n+1},\mathbf{H}^{n+1})$, $\sqrt{\eta} := \frac{2}{\Delta t}$,
$\widetilde{\mathbf{J}} := \mathbf{J} - \sqrt{\eta}\varepsilon\mathbf{E}^n + 2\sigma\mathbf{E}^n - \operatorname{curl}\mathbf{H}^n$, $\mathbf{g} = \sqrt{\eta}\mu\mathbf{H}^n - \operatorname{curl}\mathbf{E}^n$.

Are There Two Maxwell's Equations ?

Eliminating **H**, we obtain in the time harmonic case

$$(-\omega^2\varepsilon + i\omega\sigma)\mathbf{E} + \mathrm{curl}\,\frac{1}{\mu}\mathrm{curl}\,\mathbf{E} = -i\omega\mathbf{J}.$$

and in the time discretized case

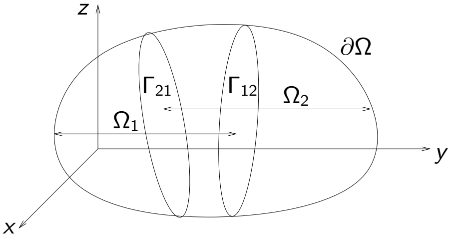
$$(\varepsilon\sqrt{\eta} + \sigma)\mathbf{E} + \mathrm{curl}\,\frac{1}{\mu\sqrt{\eta}}\mathrm{curl}\,\mathbf{E} = -\tilde{\mathbf{J}} - \mathrm{curl}\,\frac{1}{\mu\sqrt{\eta}}\mathbf{g}.$$

Many solvers for the time discretized case:

- Domain decomposition: Toselli 1997, Toselli, Widlund, Wohlmuth 1998, Zou 2011, ...
- Multigrid: Hiptmair 1999, Hiptmair, Xu 2007,...

Very few solvers for the time harmonic case with large wave number!

Schwarz Methods for Maxwell's Equation



$$
\begin{aligned}
-i\omega\varepsilon \mathbf{E}^{1,n} + \text{curl } \mathbf{H}^{1,n} - \sigma \mathbf{E}^{1,n} &= \mathbf{J} && \text{in } \Omega_1 \\
i\omega\mu \mathbf{H}^{1,n} + \text{curl } \mathbf{E}^{1,n} &= \mathbf{0} && \text{in } \Omega_1 \\
(\mathcal{B}_{\mathbf{n}_1} + \mathcal{S}_1 \mathcal{B}_{\mathbf{n}_2})(\mathbf{E}^{1,n}, \mathbf{H}^{1,n}) &= (\mathcal{B}_{\mathbf{n}_1} + \mathcal{S}_1 \mathcal{B}_{\mathbf{n}_2})(\mathbf{E}^{2,n-1}, \mathbf{H}^{2,n-1}) && \text{on } \Gamma_{12} \\
-i\omega\varepsilon \mathbf{E}^{2,n} + \text{curl } \mathbf{H}^{2,n} - \sigma \mathbf{E}^{2,n} &= \mathbf{J} && \text{in } \Omega_2 \\
i\omega\mu \mathbf{H}^{2,n} + \text{curl } \mathbf{E}^{2,n} &= \mathbf{0} && \text{in } \Omega_2 \\
(\mathcal{B}_{\mathbf{n}_2} + \mathcal{S}_2 \mathcal{B}_{\mathbf{n}_1})(\mathbf{E}^{2,n}, \mathbf{H}^{2,n}) &= (\mathcal{B}_{\mathbf{n}_2} + \mathcal{S}_2 \mathcal{B}_{\mathbf{n}_1})(\mathbf{E}^{1,n-1}, \mathbf{H}^{1,n-1}) && \text{on } \Gamma_{21}
\end{aligned}
$$

where $\mathcal{B}_{\mathbf{n}}$ is the impedance operator

$$\mathcal{B}_{\mathbf{n}}(\mathbf{E}, \mathbf{H}) := \mathbf{n} \times \frac{\mathbf{E}}{Z} + \mathbf{n} \times (\mathbf{H} \times \mathbf{n}) = \mathbf{s}.$$

Result for One New Variant

Theorem (Dolean, G, Gerardo-Giorda (2009))

For $\sigma = 0$, if $\mathcal{S}_l$ have the Fourier symbol

$$\sigma_l := \mathcal{F}(\mathcal{S}_l) = \gamma_l \left[\begin{array}{cc} k_y^2 - k_z^2 & -2k_y k_z \\ -2k_y k_z & k_z^2 - k_y^2 \end{array} \right], \ \gamma_l \in \mathbb{C}(k_z, k_y),$$

then the convergence factor of the new algorithm is

$$\rho = \left| \frac{(\sqrt{|\mathbf{k}|^2 - \tilde{\omega}^2} - i\tilde{\omega})^2}{(\sqrt{|\mathbf{k}|^2 - \tilde{\omega}^2} + i\tilde{\omega})^2} \frac{1 - \gamma_1(\sqrt{|\mathbf{k}|^2 - \tilde{\omega}^2} + i\tilde{\omega})^2}{1 - \gamma_1(\sqrt{|\mathbf{k}|^2 - \tilde{\omega}^2} - i\tilde{\omega})^2} \frac{1 - \gamma_2(\sqrt{|\mathbf{k}|^2 - \tilde{\omega}^2} + i\tilde{\omega})^2}{1 - \gamma_2(\sqrt{|\mathbf{k}|^2 - \tilde{\omega}^2} - i\tilde{\omega})^2} e^{-2\sqrt{|\mathbf{k}|^2 - \tilde{\omega}^2} L} \right|^{\frac{1}{2}}$$

Theorem (Optimal Schwarz Method (2009))

The choice $\gamma_l = 1/(\sqrt{|\mathbf{k}|^2 - \tilde{\omega}^2} + i\tilde{\omega})^2$ is optimal, since then $\rho \equiv 0$, for all Fourier modes $\mathbf{k}$; the method converges in two iterations.

Similar results for two other variants.

Optimized Schwarz Methods

1. taking $\gamma_1 = \gamma_2 = 0$: classical algorithm
2. taking $\gamma_1 = \gamma_2 = \frac{1}{|\mathbf{k}|^2} \frac{s - i\omega\sqrt{\varepsilon\mu}}{s + i\omega\sqrt{\varepsilon\mu}}$ with $s \in \mathbb{C}$:

$$\rho_2(\omega, \varepsilon, \mu, L, |\mathbf{k}|, s) = \left| \left(\frac{\sqrt{|\mathbf{k}|^2 - \omega^2\varepsilon\mu} - s}{\sqrt{|\mathbf{k}|^2 - \omega^2\varepsilon\mu} + s} \right)^2 e^{-2\sqrt{|\mathbf{k}|^2 - \omega^2\varepsilon\mu} L} \right|^{\frac{1}{2}}$$

3. taking $\gamma_1 = \gamma_2 = \frac{1}{|\mathbf{k}|^2 - 2\omega^2\varepsilon\mu + 2i\omega\sqrt{\varepsilon\mu}s}$ with $s \in \mathbb{C}$:

$$\rho_3(\omega, \varepsilon, \mu, L, |\mathbf{k}|, s) = \left| \frac{\sqrt{|\mathbf{k}|^2 - \omega^2\varepsilon\mu} - i\omega\sqrt{\varepsilon\mu}}{\sqrt{|\mathbf{k}|^2 - \omega^2\varepsilon\mu} + i\omega\sqrt{\varepsilon\mu}} \right| \rho_2(\omega, \varepsilon, \mu, L, |\mathbf{k}|, s)$$

4. taking $\gamma_l = \frac{1}{|\mathbf{k}|^2} \frac{s_l - i\omega\sqrt{\varepsilon\mu}}{s_l + i\omega\sqrt{\varepsilon\mu}}$, $l = 1, 2,\ s_1 \neq s_2$.
5. taking $\gamma_l = \frac{1}{|\mathbf{k}|^2 - 2\omega^2\varepsilon\mu + 2i\omega\sqrt{\varepsilon\mu}s_l}$, $l = 1, 2,\ s_1 \neq s_2$.

Optimal Choice Based on Asymptotic Analysis

Based on equioscillation, we find for the choice $s_l = p_l(1+i)$ the optimized results:

	with overlap, $L = h$		without overlap, $L = 0$	
Case	ρ	parameters	ρ	parameters
1	$1 - \sqrt{k_+ - \omega^2 h}$	none	1	none
2	$1 - 2C_\omega^{\frac{1}{6}} h^{\frac{1}{3}}$	$p = \frac{C_\omega^{\frac{1}{3}}}{2\cdot h^{\frac{1}{3}}}$	$1 - \frac{\sqrt{2}C_\omega^{\frac{1}{4}}}{\sqrt{C}}\sqrt{h}$	$p = \frac{\sqrt{C}C_\omega^{\frac{1}{4}}}{\sqrt{2}\sqrt{h}}$
3	$1 - 2(k_+^2 - \omega^2)^{\frac{1}{6}} h^{\frac{1}{3}}$	$p = \frac{(k_+^2-\omega^2)^{\frac{1}{3}}}{2\cdot h^{\frac{1}{3}}}$	$1 - \frac{\sqrt{2}(k_+^2-\omega^2)^{\frac{1}{4}}}{\sqrt{C}}\sqrt{h}$	$p = \frac{\sqrt{C}(k_+^2-\omega^2)^{\frac{1}{4}}}{\sqrt{2}\sqrt{h}}$
4	$1 - 2^{\frac{2}{5}} C_\omega^{\frac{1}{10}} h^{\frac{1}{5}}$	$\begin{cases} p_1 = \frac{C_\omega^{\frac{2}{5}}}{2^{\frac{7}{5}}\cdot h^{\frac{1}{5}}} \\ p_2 = \frac{C_\omega^{\frac{1}{5}}}{2^{\frac{6}{5}}\cdot h^{\frac{3}{5}}} \end{cases}$	$1 - \frac{C_\omega^{\frac{1}{8}}}{C^{\frac{1}{4}}} h^{\frac{1}{4}}$	$\begin{cases} p_1 = \frac{C_\omega^{\frac{3}{8}}\cdot C^{\frac{1}{4}}}{2\cdot h^{\frac{1}{4}}} \\ p_2 = \frac{C_\omega^{\frac{1}{8}}\cdot C^{\frac{3}{4}}}{h^{\frac{3}{4}}} \end{cases}$
5	$1-2^{\frac{2}{5}}(k_+^2-\omega^2)^{\frac{1}{10}} h^{\frac{1}{5}}$	$\begin{cases} p_1 = \frac{(k_+^2-\omega^2)^{\frac{2}{5}}}{2^{\frac{7}{5}}\cdot h^{\frac{1}{5}}} \\ p_2 = \frac{(k_+^2-\omega^2)^{\frac{1}{5}}}{2^{\frac{6}{5}}\cdot h^{\frac{3}{5}}} \end{cases}$	$1 - \frac{(k_+^2-\omega^2)^{\frac{1}{8}}}{C^{\frac{1}{4}}} h^{\frac{1}{4}}$	$\begin{cases} p_1 = \frac{(k_+^2-\omega^2)^{\frac{3}{8}} C^{\frac{1}{4}}}{2\cdot h^{\frac{1}{4}}} \\ p_2 = \frac{(k_+^2-\omega^2)^{\frac{1}{8}} C^{\frac{3}{4}}}{h^{\frac{3}{4}}} \end{cases}$

where $\tilde{\omega} = \omega\sqrt{\varepsilon\mu}$ and $C_{\tilde{\omega}} = \min\left(k_+^2 - \tilde{\omega}^2, \tilde{\omega}^2 - k_-^2\right)$.

Transverse Electric Waves

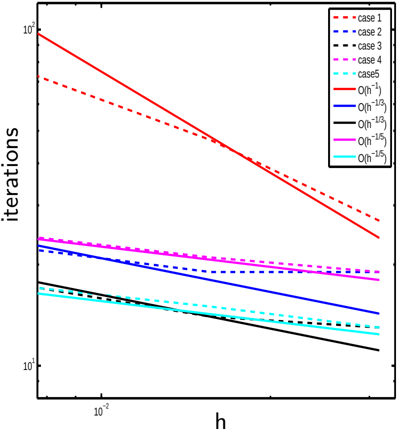
Joint work with Dolean and Gerardo-Giorda (2009)

We consider the transverse electric waves problem (TE) in $\Omega = (0,1)^2$:

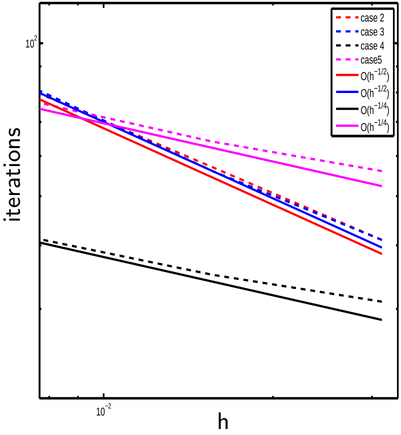
- Subdomains $\Omega_1 = (0,\beta) \times (0,1)$ and $\Omega_2 = (\alpha,1) \times (0,1)$
- Overlap $L = \beta - \alpha$
- pulsation $\tilde{\omega} = 2\pi$
- relative residual reduction of 10^{-6}

	with overlap, $L = h$				without overlap, $L = 0$			
h	1/32	1/64	1/128	1/256	1/32	1/64	1/128	1/256
Case 1	27(21)	47(27)	72(33)	118(45)	-(73)	-(100)	-(138)	-(181)
Case 2	19(14)	19(15)	22(17)	26(19)	41(26)	57(34)	79(40)	111(47)
Case 3	13(13)	14(14)	17(17)	21(18)	41(23)	56(25)	80(28)	115(35)
Case 4	19(14)	21(16)	24(18)	27(19)	31(24)	35(28)	41(30)	47(33)
Case 5	13(13)	15(15)	17(18)	19(19)	56(26)	64(30)	76(32)	76(35)

Comparison of the Asymptotics



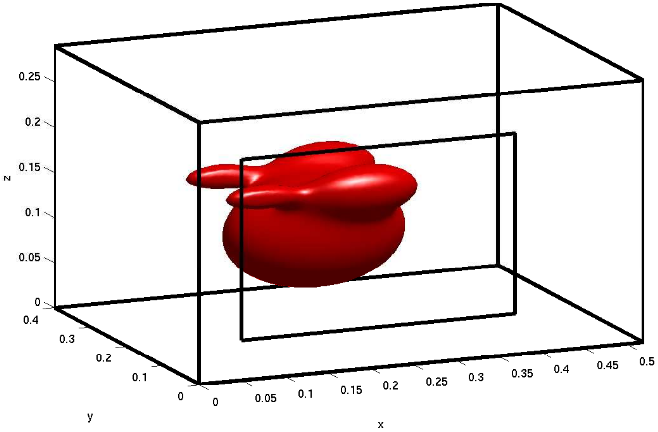
overlapping case



non-overlapping case

The Chicken Problem

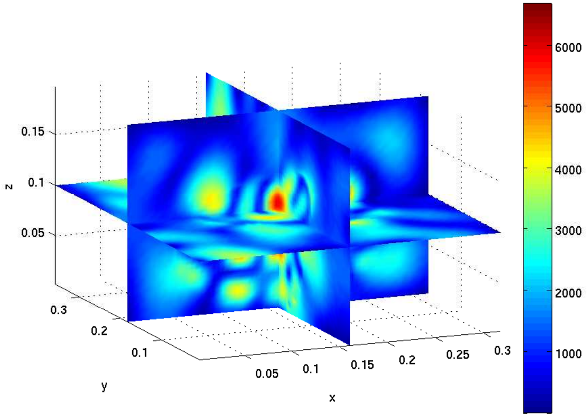
Joint work with Dolean (2009)



Heating a chicken in our Whirlpool Talent Combi 4 microwave oven

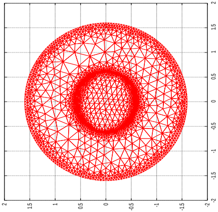
The Chicken Problem

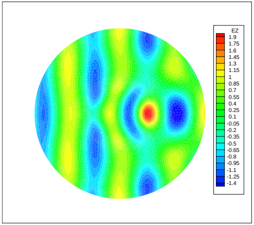
The electric field intensity in the chicken



Scattering of a plane wave by a dielectric cylinder

Joint Work with Dolean, El Bouajaji and Lanteri (2010)

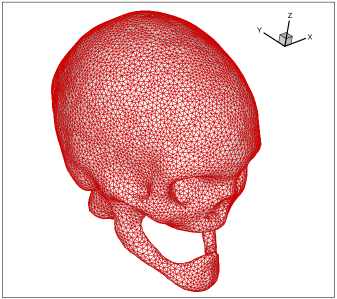


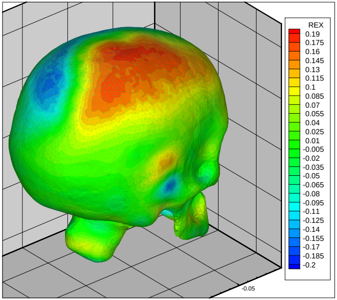


Method	L^2 err. E_z Classical	L^2 err. E_z Optimized	N_s	# iter Classical	# iter Optimized	
DGTH-$\mathbb{P}_1$	0.16400	0.16457	4	317	52	(6.1)
-	0.16400	0.16467	16	393	83	(4.7)
DGTH-$\mathbb{P}_2$	0.05701	0.05705	4	650	61	(10.7)
-	0.05701	0.05706	16	734	109	(6.7)
DGTH-$\mathbb{P}_3$	0.05519	0.05519	4	1067	71	(15.0)
-	0.05519	0.05519	16	1143	139	(8.2)
DGTH-$\mathbb{P}_4$	0.05428	0.05427	4	1619	83	(19.5)
-	0.05427	0.05527	16	1753	170	(10.3)
DGTH-$\mathbb{P}_{p_K}$	0.05487	0.05486	4	352	49	(7.2)
-	0.05487	0.05491	16	414	81	(5.1)

Exposure of head tissue to plane wave 1800 MHz

Joint with Dolean, El Bouajaji, Lanteri, Perrussel (2010)





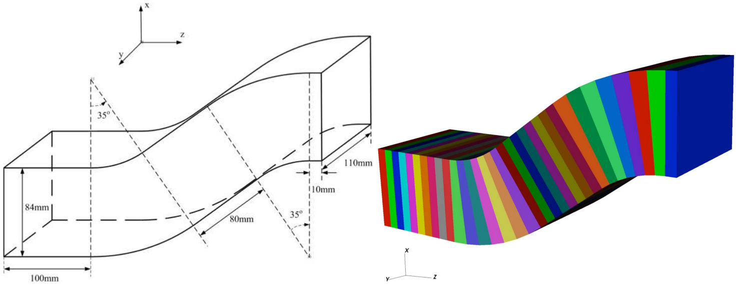
# d.o.f	N_s	RAM LU	time LU	iter	Elapsed time
26,854,848	160	2.1/3.1 GB	496 sec	30	1314 sec
-	320	0.8/1.2 GB	132 sec	36	528 sec
44,491,968	256	2.2/3.2 GB	528 sec	42	1824 sec
-	512	0.8/1.3 GB	142 sec	49	785 sec

DGTH-$\mathbb{P}_1$, single precision MUMPS LU, contour lines of E_x

Skin ($\varepsilon_r = 43.85$ and $\sigma = 1.23$ S/m), skull ($\varepsilon_r = 15.56$ and $\sigma = 0.43$ S/m), CSF (Cerebro Spinal Fluid) ($\varepsilon_r = 67.20$ and $\sigma = 2.92$ S/m), brain ($\varepsilon_r = 43.55$ and $\sigma = 1.15$ S/m)

COBRA Cavity

Joint work with Dolean, Lanteri, Lee and Peng (2014):



33 subdomains, $h = \lambda_0/4$, 21′562′026 DOFs plane wave incident opon the cavity aperture $f = 17.6GHz$

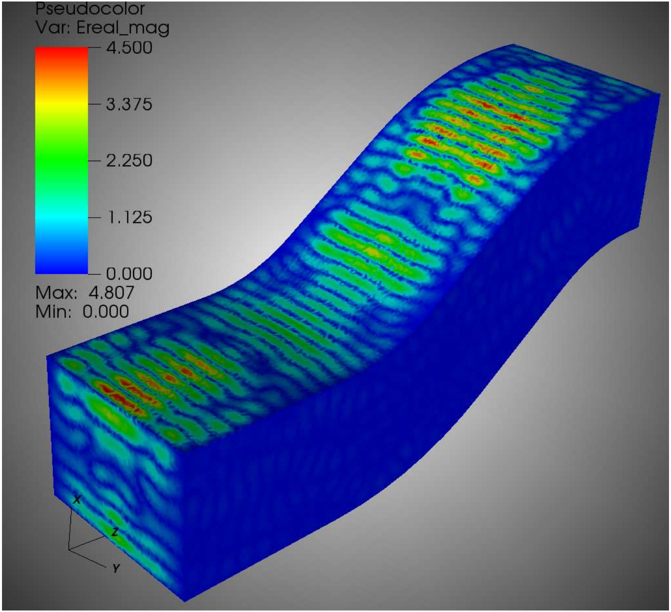
Orthogonal incidence:

18 iterations to reach 1e-3 (29 without optimization)

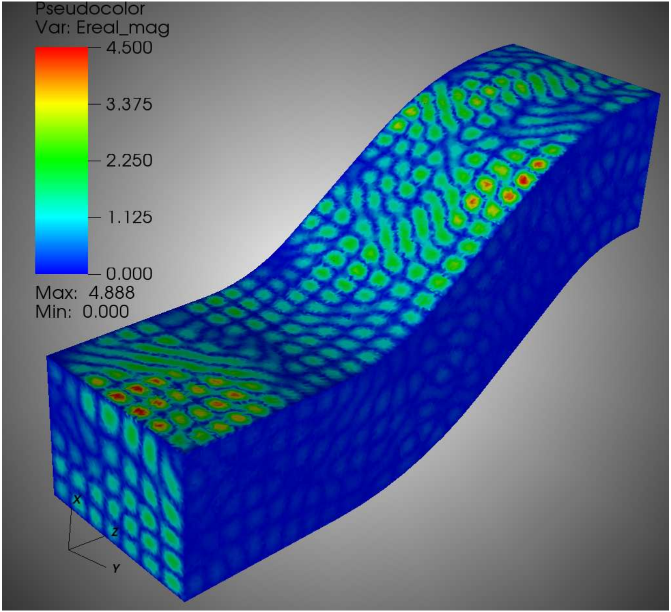
Oblique incidence:

23 iterations to reach 1e-3 (51 without optimization)

COBRA Cavity Results 1



COBRA Cavity Results 2



Conclusions

- New preconditioners based on one fundamental idea:
  1. Block factorizations
  2. Schwarz methods for strip decompositions
- Both are direct solvers, if *DtN* or Schur complements are used:

  $\Longrightarrow$ **Optimal Schwarz methods**

- Approximations lead to practical algorithms:

  $\Longrightarrow$ **Optimized Schwarz methods**

  Examples: OO0, OO2, Generalized Schwarz, Robin Schwarz, Source Transfer, Frequency filtering, AILU, Sweeping Preconditioner, and many more

Preprints are available at www.unige.ch/∼gander

MJG, Zhang 2014: Iterative Solvers for the Helmholtz Equation: Factorizations, Sweeping Preconditioners and Schwarz Methods