From the invention of the Schwarz method to the Best Current Methods for Oscillatory Problems: Part 1

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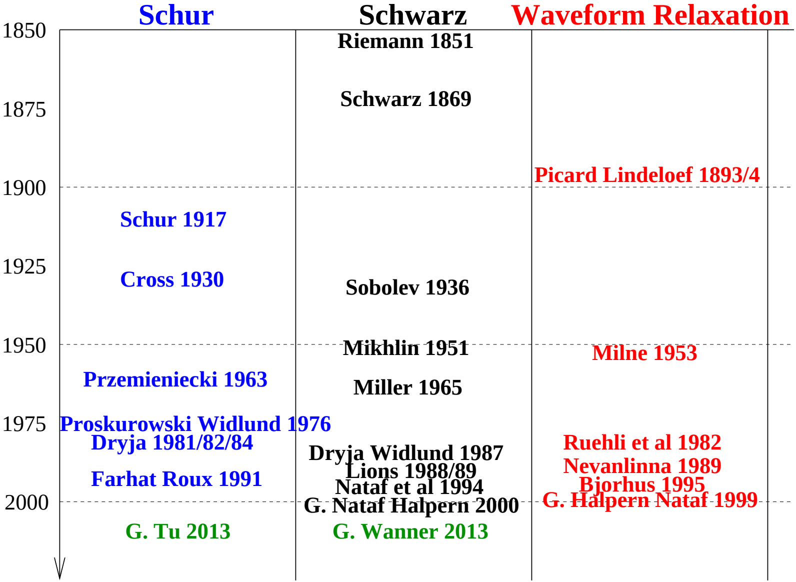
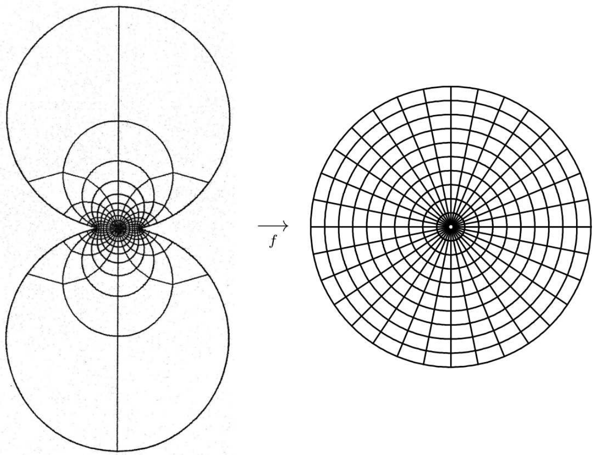
Woudschoten, October 2014

Milestones in Domain Decomposition



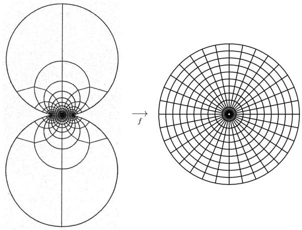
Riemann Mapping Theorem (PhD thesis 1851)

"Zwei gegebene einfach zusammenhängende Flächen können stets so aufeinander bezogen werden, dass jedem Punkte der einen ein mit ihm stetig fortrückender Punkt entspricht...;"



$$f(z) = u(x, y) + iv(x, y) \text{ analytic, } \Delta u = 0,\ \Delta v = 0.$$

Idea of Riemann's Proof



Find f which maps Ω to the unit disk and z_0 to 0: set

$$f(z) = (z - z_0)e^{g(z)}, \; g = u + iv \implies z_0 \text{ only zero}$$

In order to arrive on the boundary of the disk

$$|f(z)| = 1, \; z \in \partial\Omega \implies u(z) = -log|z - z0|, \; z \in \partial\Omega.$$

Once harmonic u with this boundary condition is found, construct v with the Cauchy-Riemann equations.

Question: Does such a u exist ???

Riemann's Audacious "Proof"

Riemann 1857, *Werke* **p. 97:**

> *"Hierzu kann in vielen Fällen ... ein Princip dienen, welches Dirichlet zur Lösung dieser Aufgabe für eine der Laplace'schen Differentialgleichung genügende Function ... in seinen Vorlesungen ... seit einer Reihe von Jahren zu geben pflegt."*

Idea: For all functions defined on a given domain Ω with the prescribed boundary values, the integral

$$J(u) = \iint_{\Omega} \frac{1}{2}\left(u_x^2 + u_y^2\right)\, dx\, dy \quad \text{is always} > 0.$$

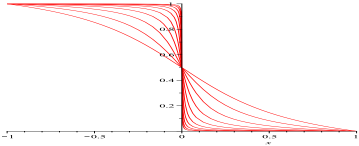
Choose among these functions the one for which this integral is minimal !
(see citation; from here originates the name "Dirichlet Principle" and "Dirichlet boundary conditions").

But is the Dirichlet Principle Correct?

Weierstrass's Critique (1869, Werke 2, p. 49):

$$\int_{-1}^{1} (x \cdot y')^2 \, dx \to \min \quad y(-1) = a, \;\; y(1) = b.$$

$$\Longrightarrow y = \frac{a+b}{2} + \frac{b-a}{2} \frac{\arctan \frac{x}{\epsilon}}{\arctan \frac{1}{\epsilon}}$$



"Die Dirichlet'sche Schlussweise führt also in dem betrachteten Falle offenbar zu einem falschen Resultat."

Riemann's Answer to Weierstrass: "... meine Existenztheoreme sind trotzdem richtig". (see F. Klein)

Helmholtz: "Für uns Physiker bleibt das Dirichletsche Prinzip ein Beweis"

International Challenge

Find harmonic functions $\Delta u = 0$ on any domain Ω with prescribed boundary conditions $u = g$ for $(x, y) \in \partial\Omega$.

Solution easy for circular domain (Poisson 1815, Poisson integration formula)

$$u(r, \phi) = \frac{1}{2\pi} \int_0^{2\pi} \frac{1 - r^2}{1 - 2r\cos(\phi - \psi) + r^2} f(\psi)\, d\psi$$

Solution also easy for rectangular domains (Fourier 1807, Fourier series).

But existence of solutions of Laplace equation on arbitrary domains appears hopeless !

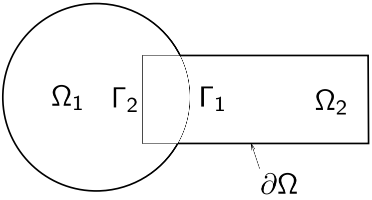
Proof of the Dirichlet Principle

H.A. Schwarz (1870, Crelle 74, 1872) Über einen Grenzübergang durch alternierendes Verfahren

"Die unter dem Namen Dirichletsches Princip bekannte Schlussweise, welche in gewissem Sinne als das Fundament des von Riemann entwickelten Zweiges der Theorie der analytischen Functionen angesehen werden muss, unterliegt, wie jetzt wohl allgemein zugestanden wird, hinsichtlich der Strenge sehr begründeten Einwendungen, deren vollständige Entfernung meines Wissens den Anstrengungen der Mathematiker bisher nicht gelungen ist".

Classical Alternating Schwarz Method

Schwarz invents a method to proof that the infimum is attained: for a general domain $\Omega := \Omega_1 \cup \Omega_2$:



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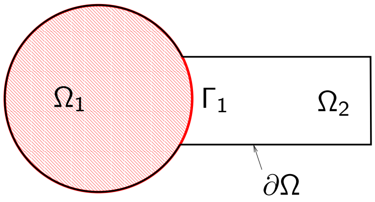

$$\begin{aligned} \Delta u_1^1 &= 0 && \text{in } \Omega_1 \\ u_1^1 &= g && \text{on } \partial\Omega \cap \overline{\Omega}_1 \\ u_1^1 &= u_2^0 && \text{on } \Gamma_1 \end{aligned}$$

solve on the disk $u_2^0 = 0$

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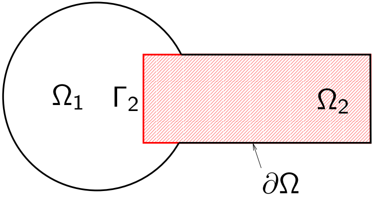

$$
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\end{aligned}
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solve on the rectangle

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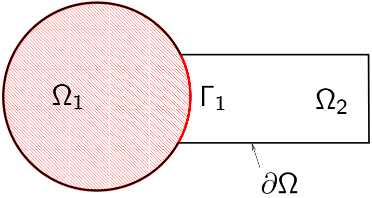

$$\begin{aligned} \Delta u_1^2 &= 0 && \text{in } \Omega_1 \\ u_1^2 &= g && \text{on } \partial\Omega \cap \overline{\Omega}_1 \\ u_1^2 &= u_2^1 && \text{on } \Gamma_1 \end{aligned}$$

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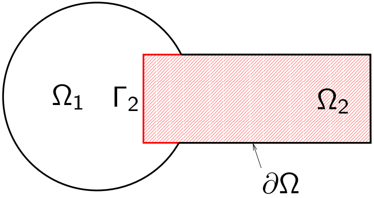

$$\begin{aligned} \Delta u_2^2 &= 0 && \text{in } \Omega_2 \\ u_2^2 &= g && \text{on } \partial\Omega \cap \overline{\Omega}_2 \\ u_2^2 &= u_1^2 && \text{on } \Gamma_2 \end{aligned}$$

solve on the rectangle

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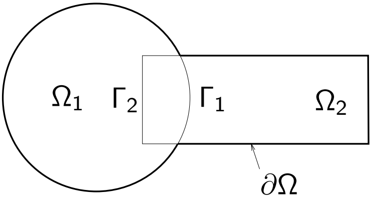

$$\begin{aligned} \Delta u_1^n &= 0 && \text{in } \Omega_1 \\ u_1^n &= g && \text{on } \partial\Omega \cap \overline{\Omega}_1 \\ u_1^n &= u_2^{n-1} && \text{on } \Gamma_1 \end{aligned} \qquad \begin{aligned} \Delta u_2^n &= 0 && \text{in } \Omega_2 \\ u_2^n &= g && \text{on } \partial\Omega \cap \overline{\Omega}_2 \\ u_2^n &= u_1^n && \text{on } \Gamma_2 \end{aligned}$$

solve on the disk | solve on the rectangle

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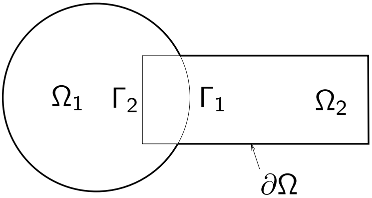

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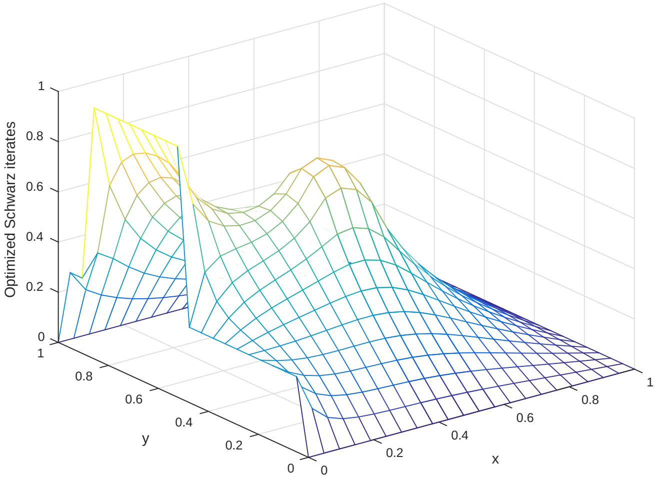
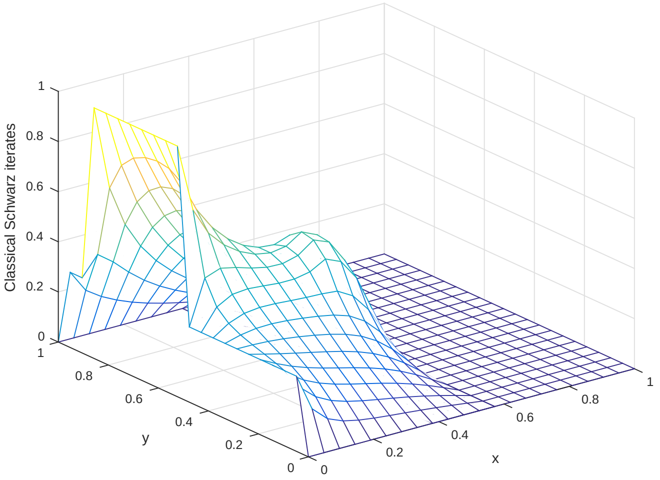
solve on the disk　　　　solve on the rectangle

- Schwarz proved convergence in 1869 using the maximum principle.

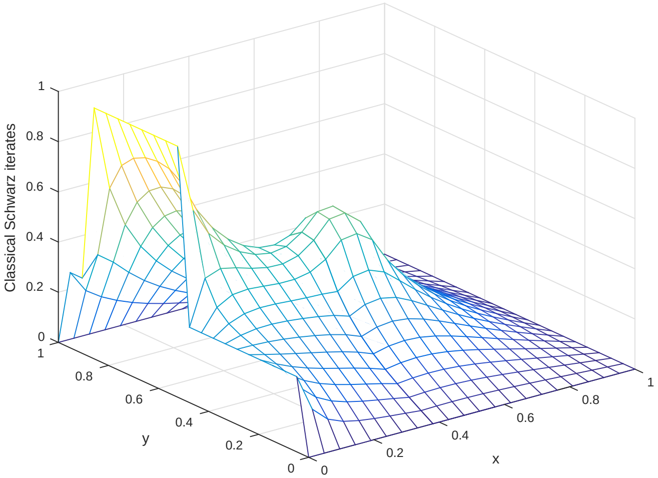
Example: Heating a Room



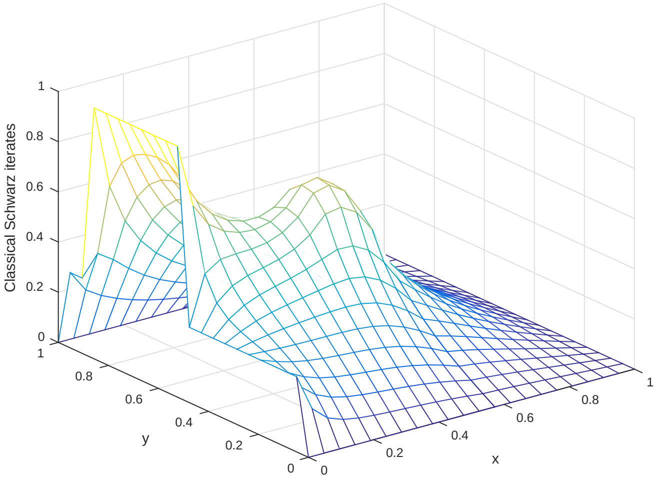
Iteration 1 Left



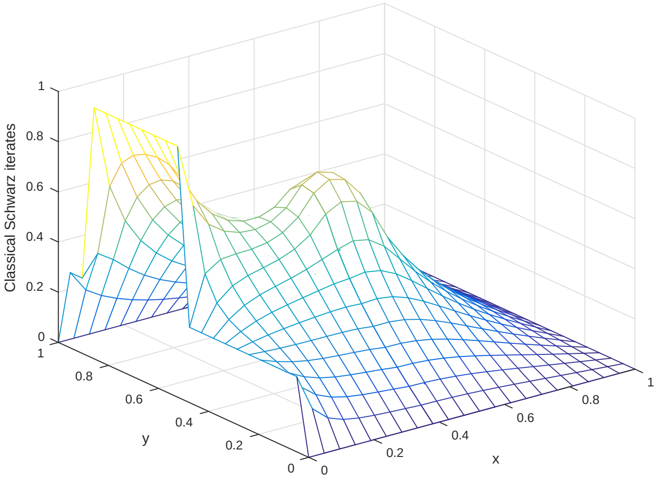
Iteration 1 Right



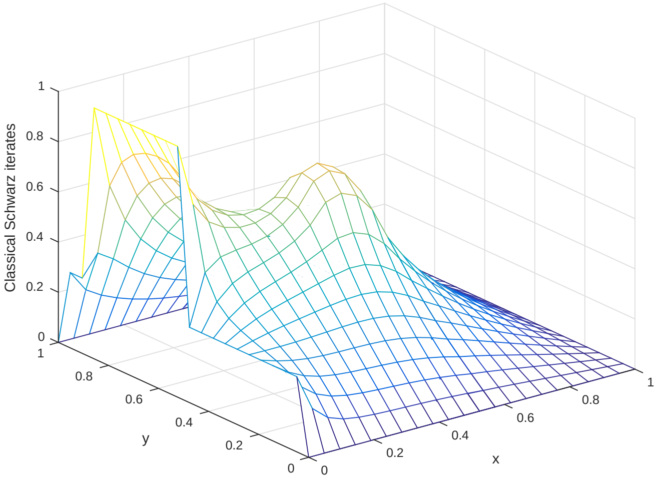
Iteration 2 Left



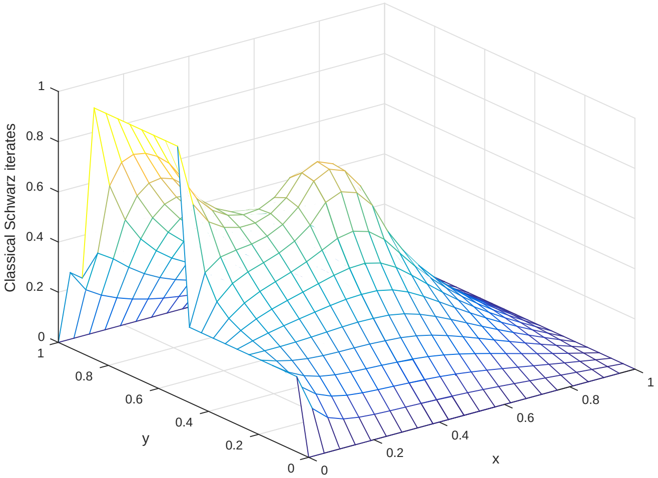
Iteration 2 Right



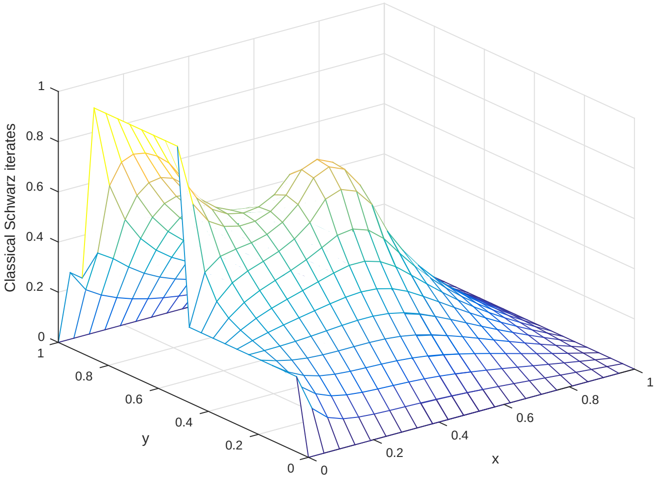
Iteration 3 Left



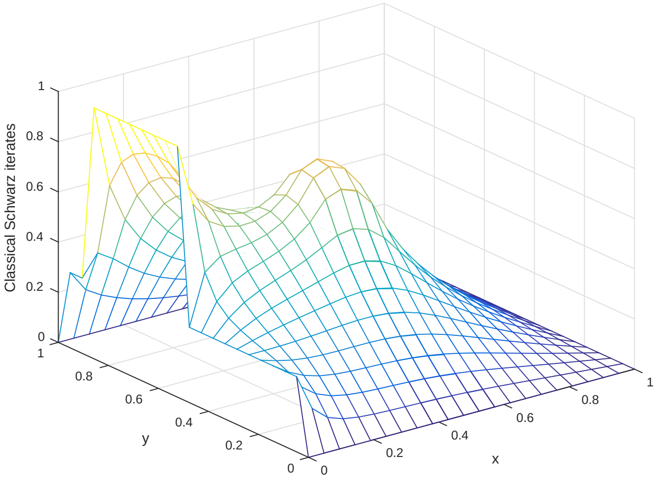
Iteration 3 Right



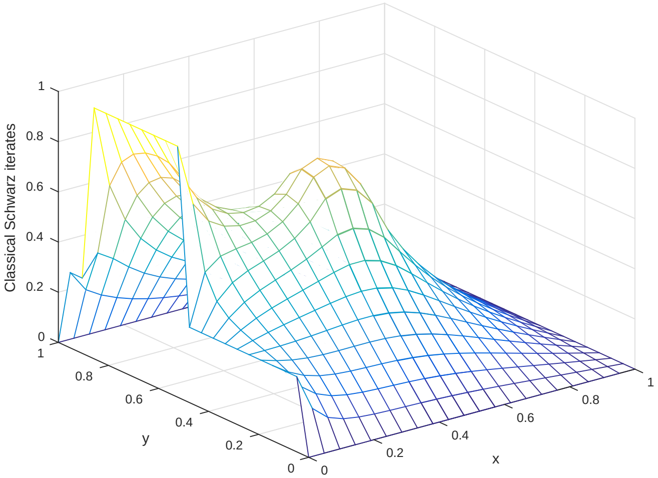
Iteration 4 Left



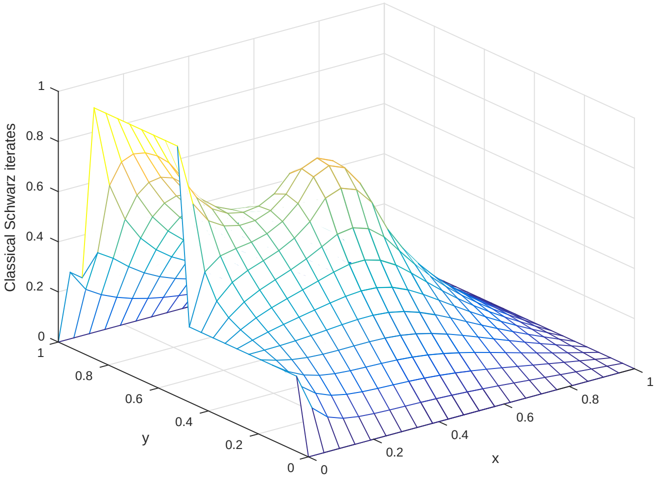
Iteration 4 Right



Iteration 5 Left



Iteration 5 Right



Early Theoretical and Numerical Results

Sobolev (1936): L'algorithme de Schwarz dans la Théorie de l'Elasticitée, Doklady

Mikhlin (1951): On the Schwarz algorithm, Doklady

Miller (1965): Numerical Analogs to the Schwarz alternating procedure, Numer. Math.
"Schwarz's method presents some intriguing possibilities for numerical methods. Firstly, quite simple explicit solutions by classical methods are often known for simple regions such as rectangles or circles."

Lions (1988): On the Schwarz Alternating Method II
"Let us observe, by the way, that the Schwarz alternating method seems to be the only domain decomposition method converging for two entirely different reasons: variational characterization of the Schwarz sequence and maximum principle."

Convergence Theorems

Theorem (Drjya and Widlund (1989))

Condition number of the additive Schwarz preconditioned system with coarse grid is bounded by

$$\kappa(M_{AS}A) \leq C\left(1 + \frac{H}{\delta}\right)$$

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The above estimate can not be improved.

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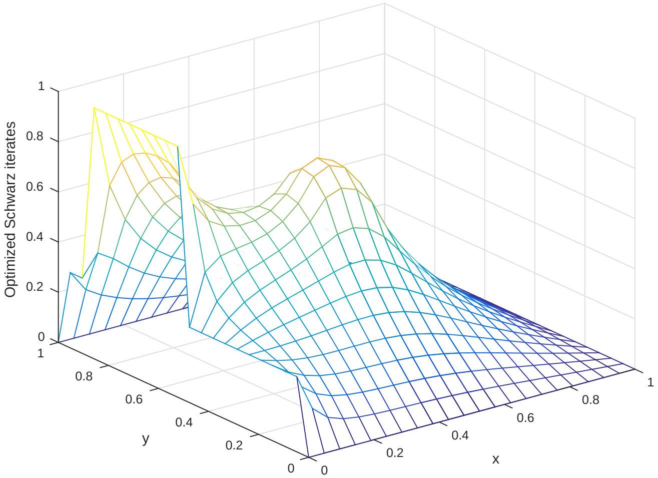
The above estimate can not be improved.

Theorem (Dubois, G, Loisel, St-Cyr, Szyld (2011))

Contraction factor of a zeroth order **optimized Schwarz method (2000)** *with coarse grid is*

$$\rho = 1 - O\left(\left(\frac{\delta}{H}\right)^{\frac{1}{3}}\right).$$

Example: Heating a Room



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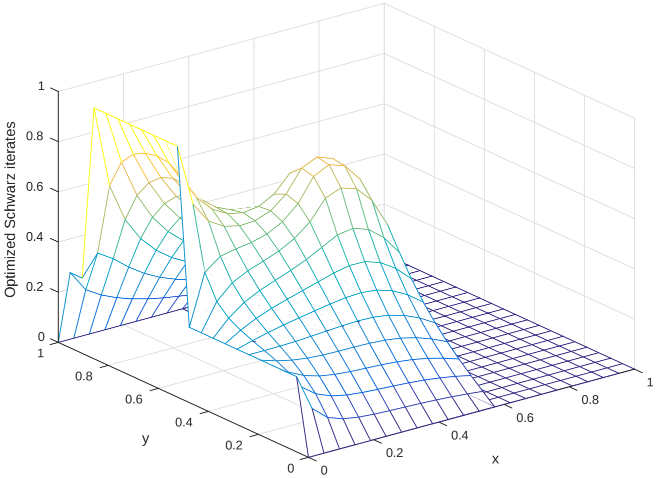

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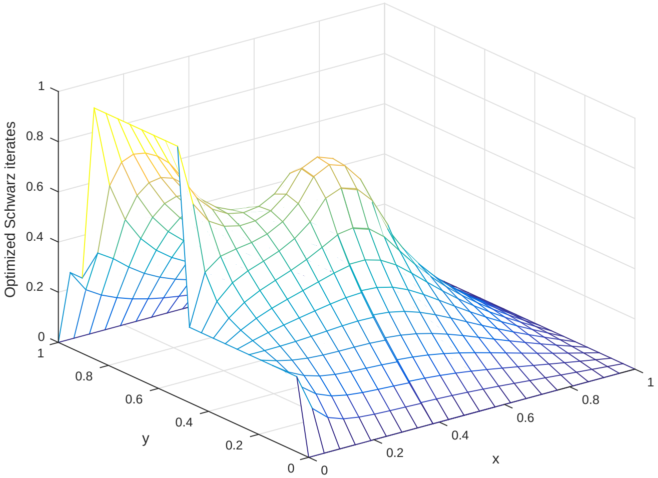

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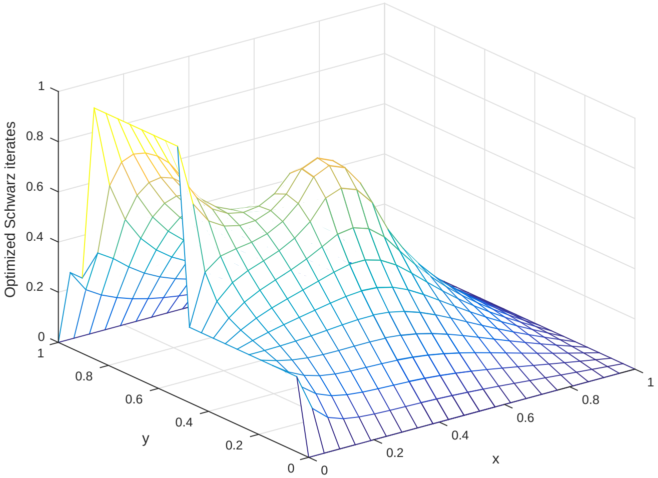

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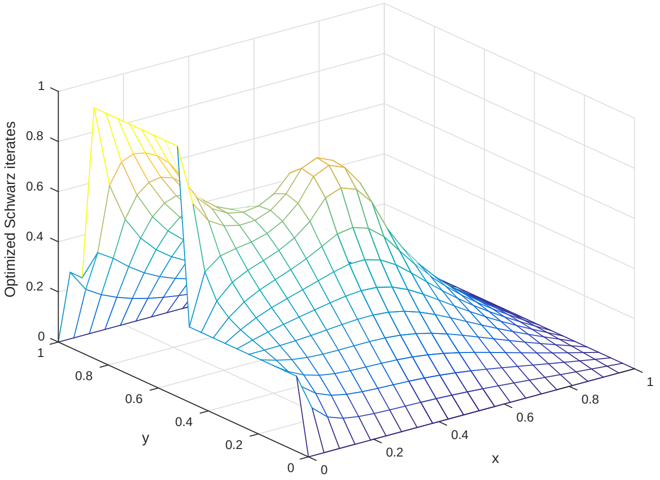
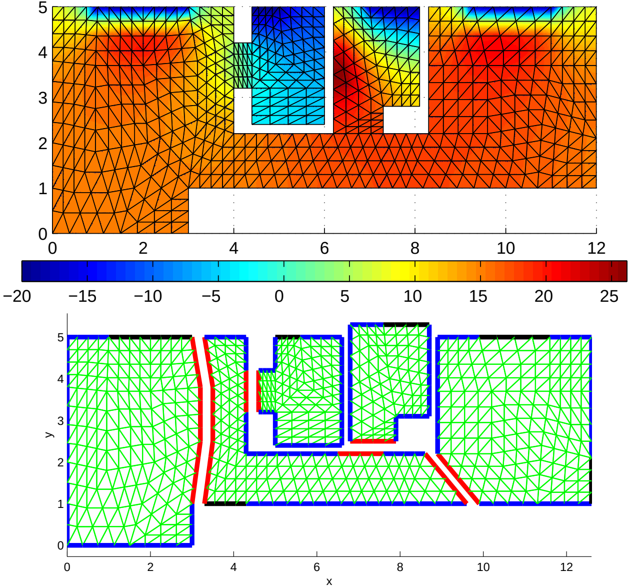
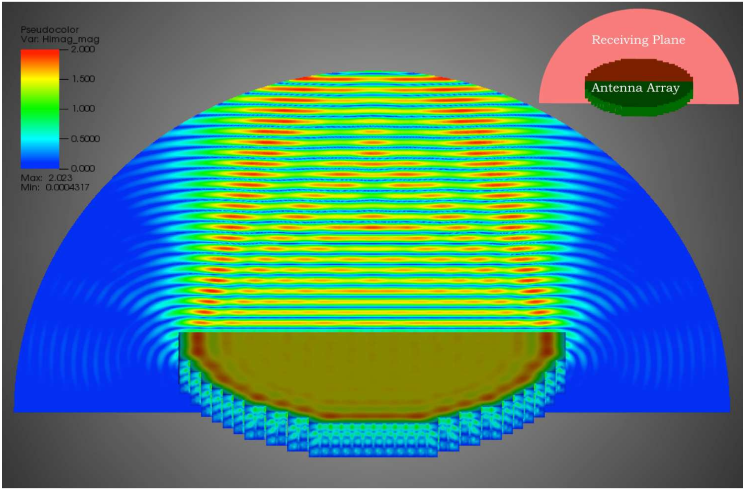


Apartment in Montreal

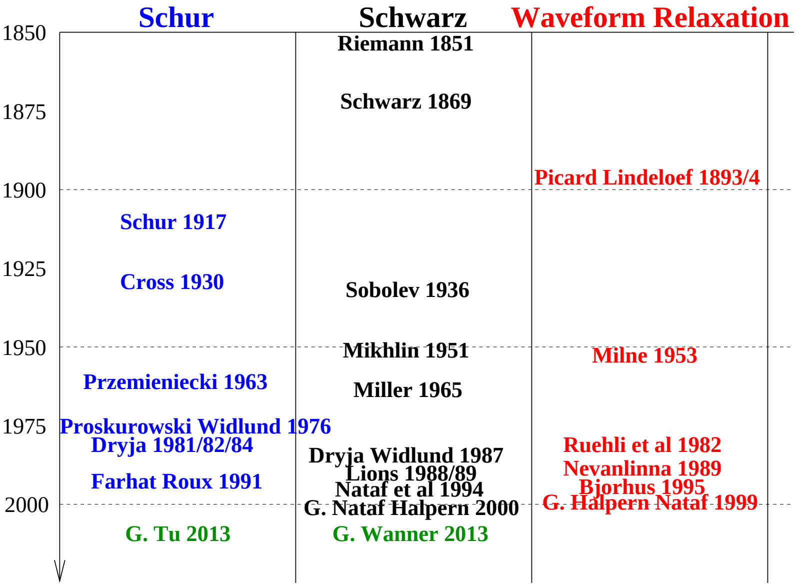


Vivaldi Antenna Array



864 antenna elements (computations by Zhen Peng)

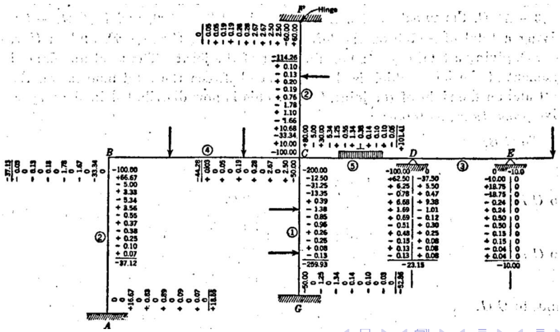
Schur or Substructuring Methods



Origin of Schur Methods

Cross (1930): Analysis of continuous frames by distributing fixed-end moments

> *"The reactions in beams, bents, and arches which are immovably fixed at their ends have been extensively discussed. They can be found comparatively readily by methods which are more or less standard. The method of analysis herein presented enables one to derive from these the moments, shears, and thrusts required in the design of complicated continuous frames."*



Aircraft Industry at Boeing

Przemieniecki 1963: Matrix structural analysis of substructures

"In the present method each substructure is first analyzed separately, assuming that all common boundaries with adjacent substructures are completely fixed: these boundaries are then relaxed simultaneously and the actual boundary displacements are determined from the equations of equilibrium of forces at the boundary joints."

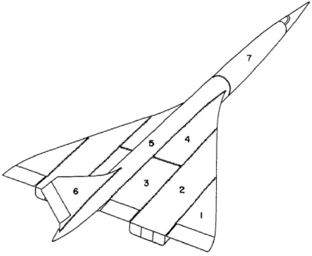


FIG. 3. Typical substructure arrangement for delta aircraft.

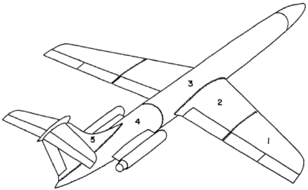


FIG. 4. Typical substructure arrangement for conventional aircraft.

Historical Example of Przemieniecki

Let P be the exterior forces, K the stiffness matrix, and U the displacement vector satisfying

$$KU = P.$$

Partition U into U_i interior in each substructure, and U_b on the boundaries between substructures:

$$\begin{bmatrix} K_{bb} & K_{bi} \\ K_{ib} & K_{ii} \end{bmatrix} \begin{bmatrix} U_b \\ U_i \end{bmatrix} = \begin{bmatrix} P_b \\ P_i \end{bmatrix}.$$

Eliminating interior unknowns Przemieniecki obtains

$$(K_{bb} - K_{bi}K_{ii}^{-1}K_{ib})U_b = P_b - K_{bi}K_{ii}^{-1}P_i$$

Issai Schur (1917): Über Potenzreihen, die im Innern des Einheitskreises beschränkt sind, Crelle.

Ist nämlich die Determinante von P nicht Null, so wird, wenn E die Einheitsmatrix des Grades n bedeutet,

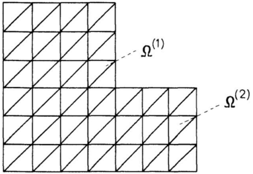
$$\begin{pmatrix} P^{-1}, & 0 \\ -RP^{-1}, & E \end{pmatrix} \begin{pmatrix} P, & Q \\ R, & S \end{pmatrix} = \begin{pmatrix} E, & P^{-1}Q \\ 0, & S - RP^{-1}Q \end{pmatrix}.$$

Direct Versus Iterative Solution

Proskurowski and Widlund (1976): On the numerical solution of Helmholtz's equation by the capacitance matrix method

> *"This new formulation leads to well-conditioned capacitance matrix equations which can be solved quite efficiently by the conjugate gradient method."*

Dryja (1981/82/84):



$$\gamma_0 (K^{1/2} y, y)_{R^p} \leqq (C y, y)_{R^p} \leqq \gamma_1 (K^{1/2} y, y)_{R^p}$$

γ_0 *and* γ_1 *are the following constants*

$$\gamma_0 = r/(1+(1+r^2)^{1/2}), \qquad \gamma_1 = (1+2r(1+2\sigma))\, b_2/(4r\, b_1)$$
$$r = (32)^{1/2}(a_2 - a_1)/b_1.$$

> *"The system is solved by generalized conjugate gradient method with $K^{1/2}$ as the preconditioning."*

Primal and Dual Schur Complement Methods

The Schur complement system Przemieniecki obtained

$$(K_{bb} - K_{bi}K_{ii}^{-1}K_{ib})U_b = P_b - K_{bi}K_{ii}^{-1}P_i$$

is in modern notation

$$\mathcal{S}_P u_\Gamma = f_P$$

Farhat, Roux (1991): assume derivatives u'_Γ are fixed

$$\mathcal{S}_D u'_\Gamma = f_D$$

Theorem

$\mathcal{S}_P$ and $\mathcal{S}_D$ have a condition number $O(\frac{1}{h})$. However $\mathcal{S}_P\mathcal{S}_D$ and $\mathcal{S}_D\mathcal{S}_P$ have a condition number of $O(1)$.

J	A	Schur Primal	Schur Dual	Dual-Primal	Primal-Dual
10	48.37	6.55	7.28	1.11	1.11
20	178.06	13.04	14.31	1.10	1.10
40	680.62	25.91	28.26	1.09	1.09

FETI and Balancing Neumann-Neumann

FETI is $\mathcal{S}_P\mathcal{S}_D$ and Balancing Neumann-Neumann is $\mathcal{S}_D\mathcal{S}_P$

Theorem

The condition number of FETI (with natural coarse grid) or balancing Neumann-Neumann is bounded by

$$C(1+\ln(\frac{H}{h}))^2$$

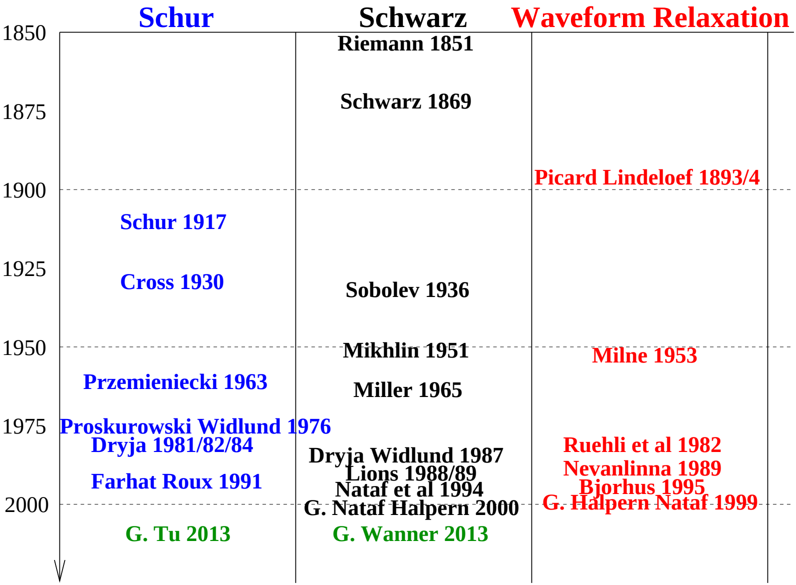
where C is a constant independent of H and h.

Proofs:

- For Neumann-Neumann see Drjya and Widlund (1995) and Mandel and Brezina (1996)
- For FETI, see Mandel and Tezaur (1996)

More recent variants: FETI-H, FETI-2LM, FETI-DP, . . .

Waveform Relaxation Methods



Martin J. Gander

Picard 1893 and Lindelöf 1894

Émile Picard (1893): Sur l'application des méthodes d'approximations successives à l'étude de certaines équations différentielles ordinaires

$$v' = f(v) \quad \Longrightarrow \quad v^n(t) = v(0) + \int_0^t f(v^{n-1}(\tau))d\tau$$

Ernest Lindelöf (1894): Sur l'application des méthodes d'approximations successives à l'étude des intégrales réelles des équations différentielles ordinaires

Theorem (Superlinear Convergence)

On bounded time intervals $t \in [0, T]$, the iterates satisfy the superlinear error bound

$$||\mathbf{v} - \mathbf{v}^n|| \leq \frac{(CT)^n}{n!}||\mathbf{v} - \mathbf{v}^0||$$

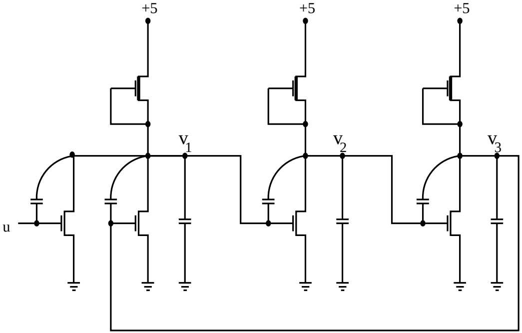
Milne (1953): "Actually this method of continuing the computation is highly inefficient and is not recommended"

Lelarasmee, Ruehli and Sangiovanni-Vincentelli

The Waveform Relaxation Method for Time-Domain Analysis of Large Scale Integrated Circuits. IEEE Trans. on Computer-Aided Design of Int. Circ. a. Sys. 1982

> *"The spectacular growth in the scale of integrated circuits being designed in the VLSI era has generated the need for new methods of circuit simulation. "Standard" circuit simulators, such as SPICE and ASTAP, simply take too much CPU time and too much storage to analyze a VLSI circuit".*

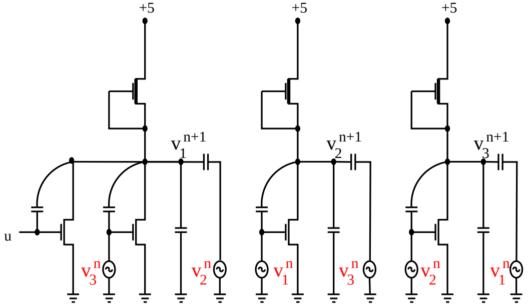
MOS ring oscillator from 1982



Using Kirchhoff's and Ohm's laws gives system of ODEs:

$$\begin{aligned} \frac{\partial \mathbf{v}}{\partial t} &= f(\mathbf{v}), \qquad 0 < t < T \\ \mathbf{v}(0) &= \mathbf{g} \end{aligned}$$

Waveform Relaxation Decomposition



Iteration using sub-circuit solutions only:

$$
\begin{aligned}
\partial_t v_1^{n+1} &= f_1(v_1^{n+1}, v_2^n, v_3^n) \\
\partial_t v_2^{n+1} &= f_2(v_1^n, v_2^{n+1}, v_3^n) \\
\partial_t v_3^{n+1} &= f_3(v_1^n, v_2^n, v_3^{n+1})
\end{aligned}
$$

Signals along wires are called 'waveforms', which gave the algorithm its name: **Waveform Relaxation**.

Schwarz Waveform Relaxation for PDEs

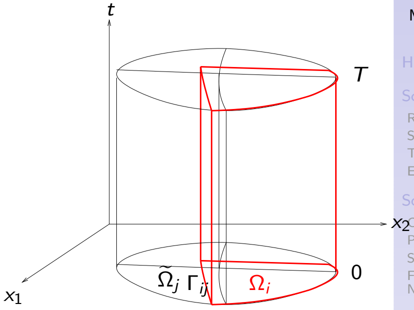
For a given evolution PDE,

$$\partial_t u = \mathcal{L}u + f, \quad \text{in } \Omega \times (0, T),$$

with initial condition

$$u(x, 0) = u_0,$$

the Schwarz waveform relaxation algorithm is:



$$\begin{aligned} \partial_t u_i^n &= \mathcal{L}u_i^n + f && \text{in } \Omega_i \times (0, T), \\ u_i^n(\cdot, \cdot, 0) &= u_0 && \text{in } \Omega_i, \\ u_i^n &= u_j^{n-1} && \text{on } \Gamma_{ij} \times (0, T) \end{aligned}$$

- Many convergence results: heat equation, wave equation, advection reaction diffusion, Maxwell

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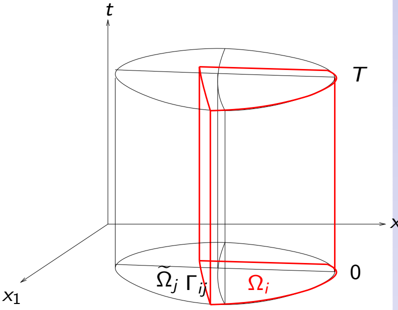

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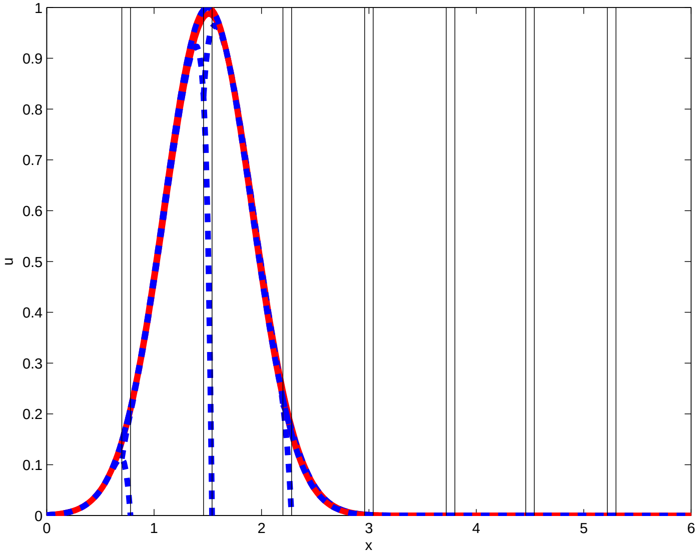
- Many convergence results: heat equation, wave equation, advection reaction diffusion, Maxwell
- Need to use optimized transmission conditions

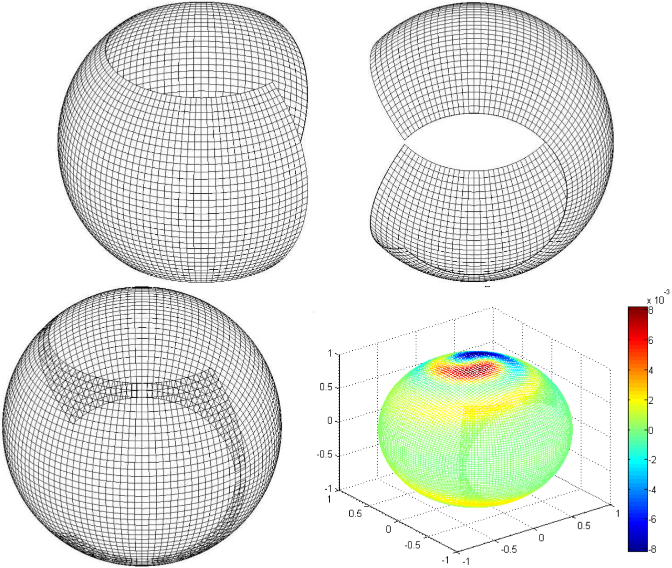
A Numerical Experiment

For an advection reaction diffusion equation in 1d:



Global Weather Simulation: Cyclogenesis Test

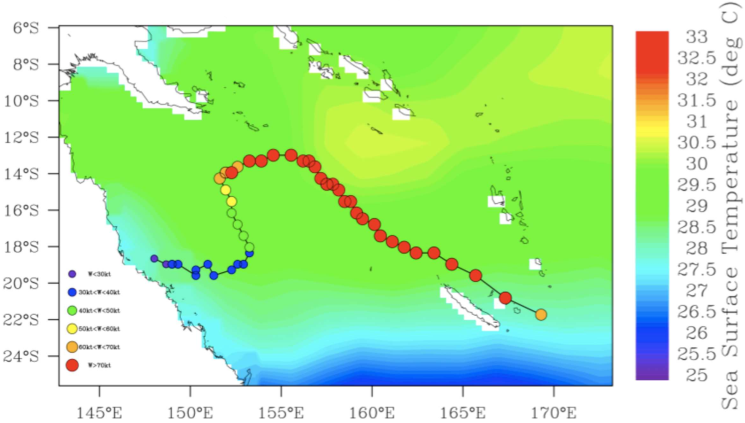
On the Yin-Yang grid (with Côté and Qaddouri 2006)



Track of Hurricane Erica (2003)

Group of E. Blayo (University of Grenoble)

- Primitive equation ocean model (ROMS 2005)
- Non hydrostatic atmospheric model (WRF 2007)



Without ocean-atmosphere Schwarz coupling

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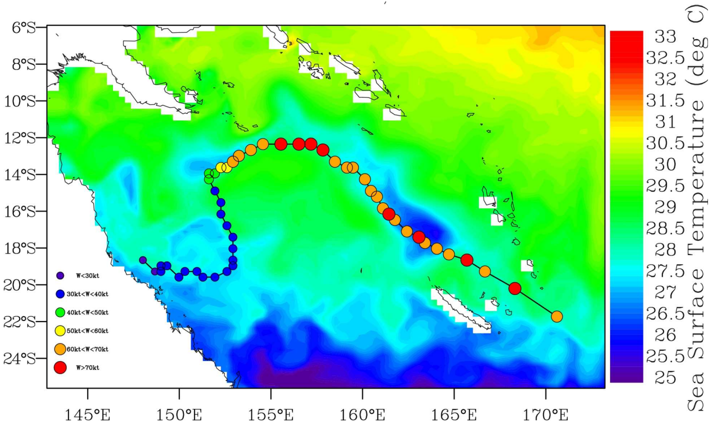


With ocean-atmosphere Schwarz coupling

The Parareal Algorithm

J-L. Lions, Y. Maday, G. Turinici (2001): A “Parareal” in Time Discretization of PDEs

The parareal algorithm for the model problem

$$u' = f(u)$$

is defined using two propagation operators:

1. $G(t_2, t_1, u_1)$ is a rough approximation to $u(t_2)$ with initial condition $u(t_1) = u_1$,
2. $F(t_2, t_1, u_1)$ is a more accurate approximation of the solution $u(t_2)$ with initial condition $u(t_1) = u_1$.

Starting with a coarse approximation U_n^0 at the time points $t_1, t_2, \ldots, t_N$, parareal performs for $k = 0, 1, \ldots$ the correction iteration

$$U_{n+1}^{k+1} = G(t_{n+1}, t_n, U_n^{k+1}) + F(t_{n+1}, t_n, U_n^k) - G(t_{n+1}, t_n, U_n^k).$$

Precise Convergence Theorem for Parareal

For the non-linear IVP $u' = f(u), \quad u(t_0) = u_0.$

Theorem (G, Hairer 2005)

Let $F(t_{n+1}, t_n, U_n^k)$ denote the exact solution at t_{n+1} and $G(t_{n+1}, t_n, U_n^k)$ be a one step method with local truncation error bounded by $C_1 \Delta T^{p+1}$. If

$$|G(t+\Delta T, t, x) - G(t+\Delta T, t, y)| \leq (1 + C_2 \Delta T)|x - y|,$$

then

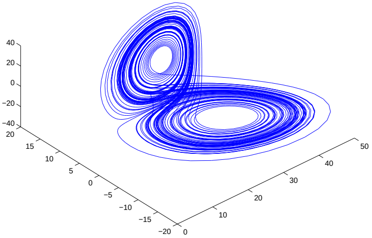
$$\max_{1\leq n\leq N} |u(t_n) - U_n^k| \leq \frac{C_1 \Delta T^{k(p+1)}}{k!}(1 + C_2 \Delta T)^{N-1-k} \prod_{j=1}^{k}(N-j) \max_{1\leq n\leq N} |u(t_n) - U_n^0|$$

$$\leq \frac{(C_1 T)^k}{k!} e^{C_2(T-(k+1)\Delta T)} \Delta T^{pk} \max_{1\leq n\leq N} |u(t_n) - U_n^0|.$$

$\Longrightarrow$ Superlinear Convergence estimate since Parareal is a Waveform Relaxation technique

Results for the Lorenz Equations

$$\begin{aligned}\dot{x} &= -\sigma x + \sigma y\\ \dot{y} &= -xz + rx - y\\ \dot{z} &= xy - bz\end{aligned}$$

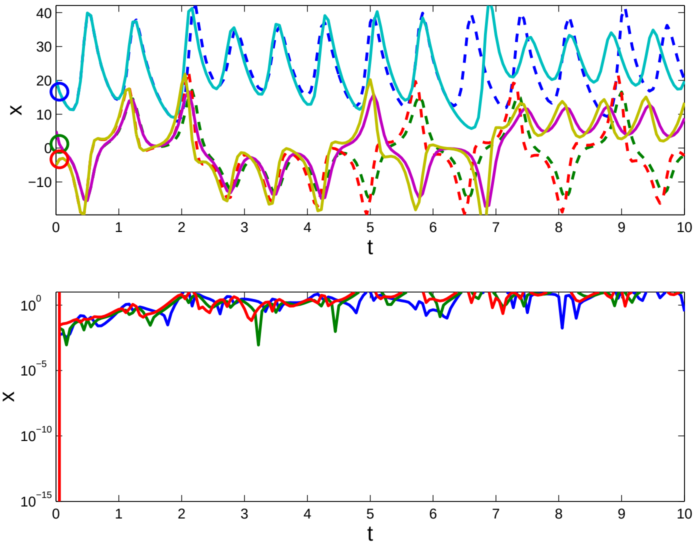


Parameters: $\sigma = 10$, $r = 28$ and $b = \frac{8}{3} \Longrightarrow$ chaotic regime.

Initial conditions: $(x, y, z)(0) = (20, 5, -5)$

Simulation time: $t \in [0, T = 10]$

Discretization: Fourth order Runge Kutta, $\Delta T = \frac{T}{180}$, $\Delta t = \frac{T}{1800}$.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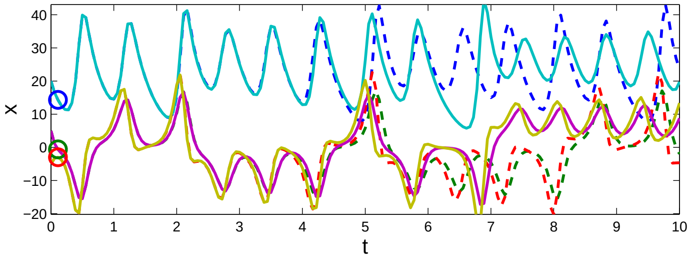

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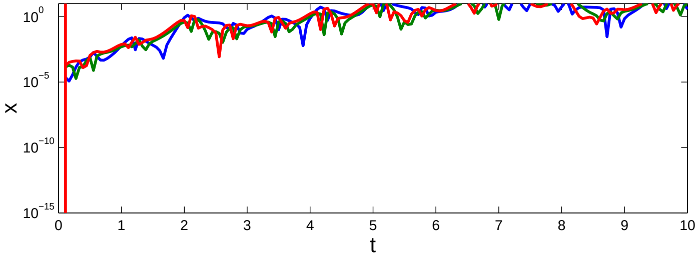

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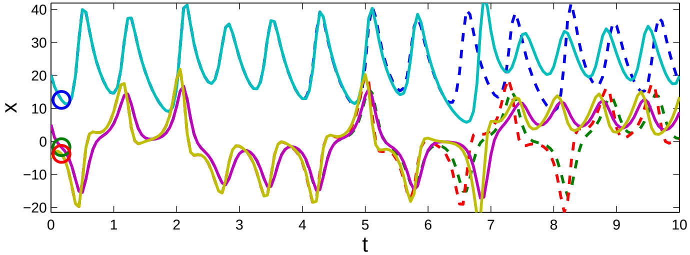
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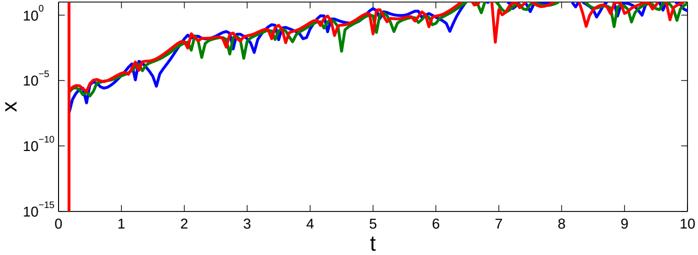
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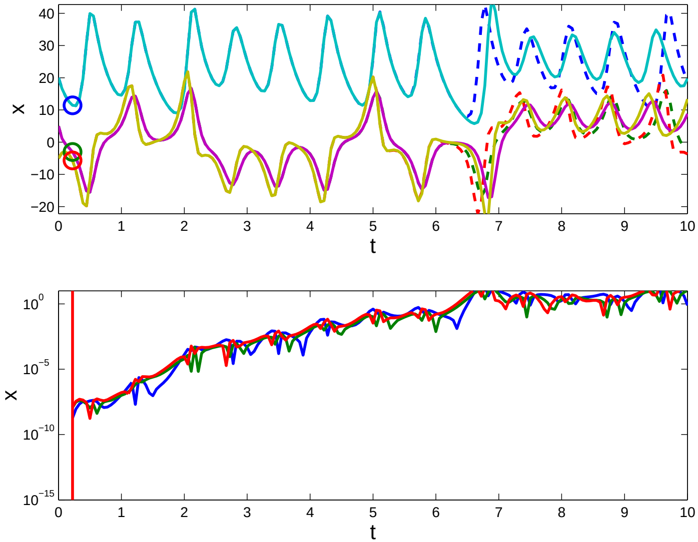
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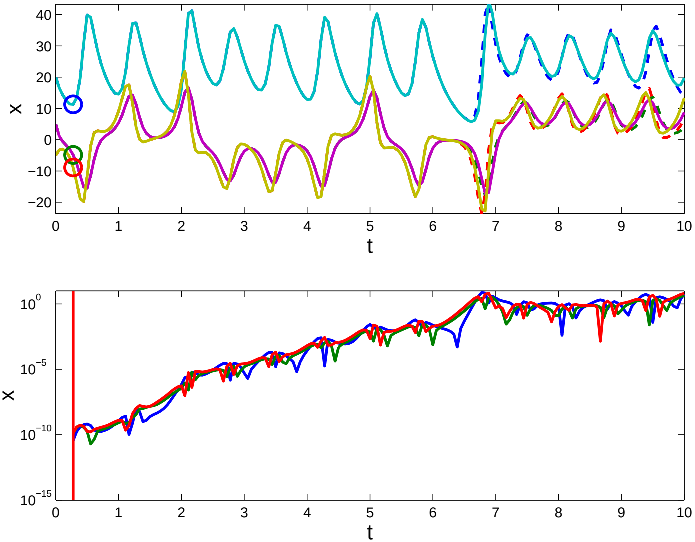
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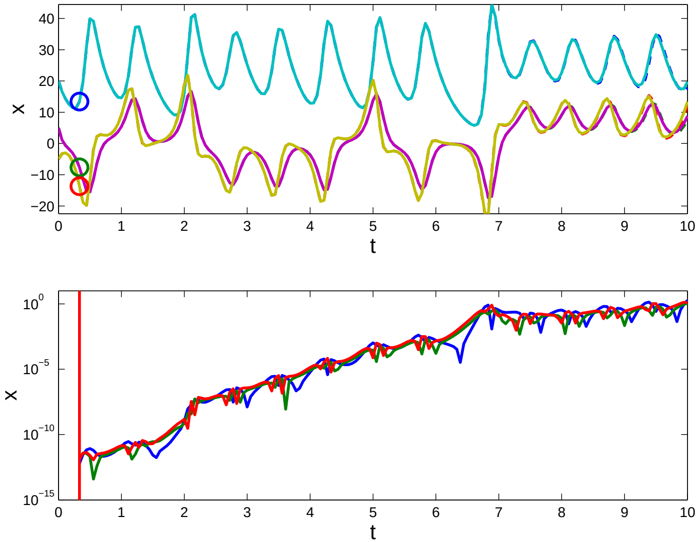

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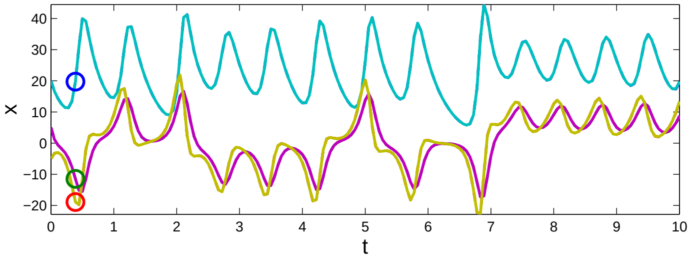

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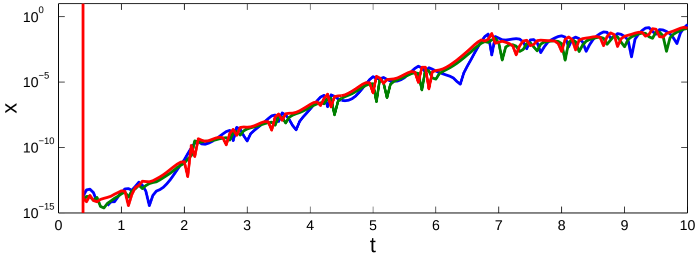

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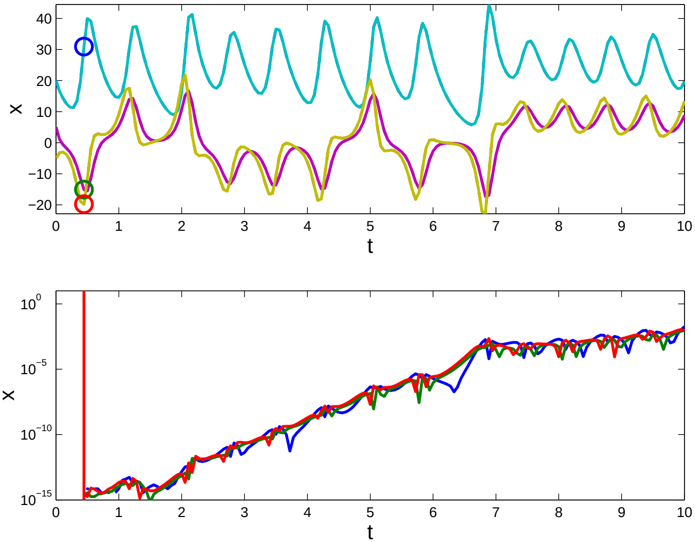

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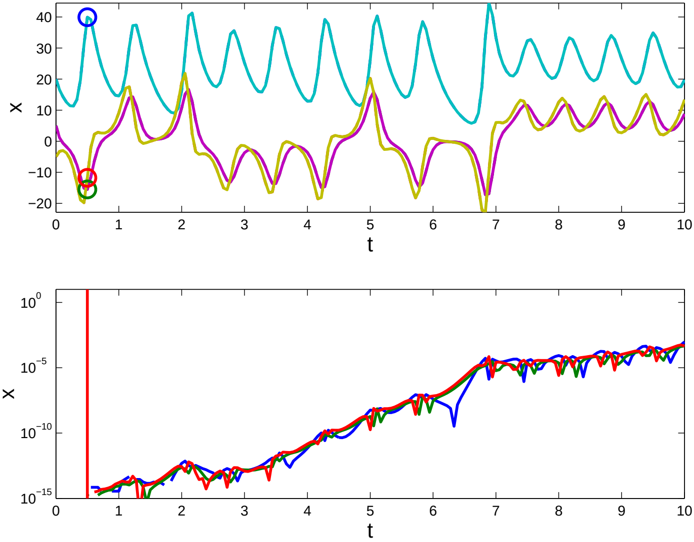
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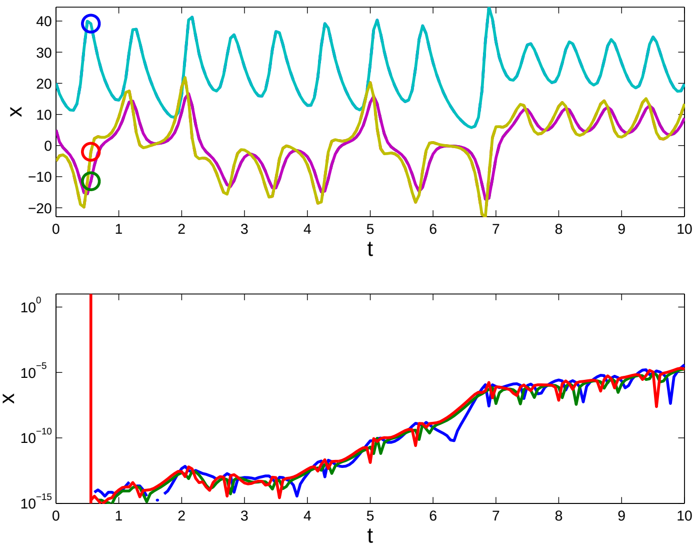
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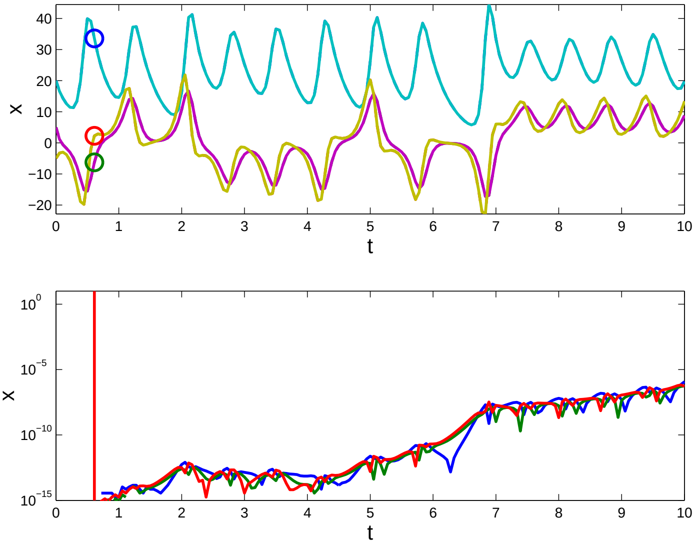

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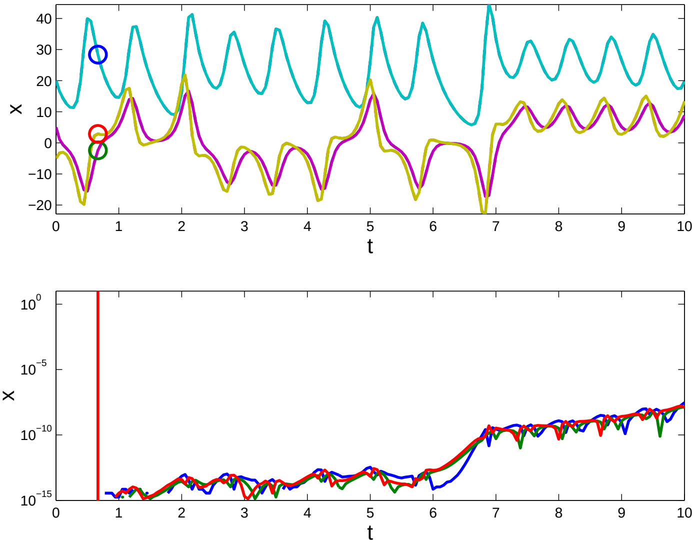

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References

Schwarz Methods Over the Course of Time, M.J. Gander, ETNA, Vol. 31, pp. 228–255, 2008.

Méthodes de décomposition de domaines, M.J. Gander and L. Halpern, Techniques de l'ingénieur, 2012.

From Euler, Ritz and Galerkin to Modern Computing, M.J. Gander and G. Wanner, SIAM Review, 2013.

The Origins of the Alternating Schwarz Method, M.J. Gander and G. Wanner, Domain Decomposition Methods in Science and Engineering XXI, 2014.

On the Origins of Iterative Substructuring Methods, M.J. Gander and X.Tu, Domain Decomposition Methods in Science and Engineering XXI, 2014.