Hybrid Numerical-Asymptotic Methods for High Frequency Scattering Problems

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Context

My talks apply (particularly) to **acoustic waves**.

My talks concern new **numerical-asymptotic methods** for **high frequency** wave scattering based on **boundary integral equations**, that combine **numerical analysis** with **high frequency asymptotics**, see

C-W, Graham, Langdon, Spence *Acta Numerica* 21 (2012), 89–305.

The first talk was largely **HF asymptotics** – in this talk we come to **numerical methods** and their analysis!

Overview

1. Green's Representation Theorem and **boundary integral equations** (for Helmholtz)

2. A Case study for **numerical-asymptotic** methods: **the thin screen**

Step A. Represent the unknown as sum of products of **known oscillatory** and **unknown non-oscillatory** functions using **GO/GTD**.

Step B. Decide on the approximation space - combine **HF asymptotics** with hp-**approximation theory**

Step C. Implement it and see that (we hope) the cost is O(1) as $k \to \infty$!

Step D. Try to prove this by theorems about the k-dependence of everything!

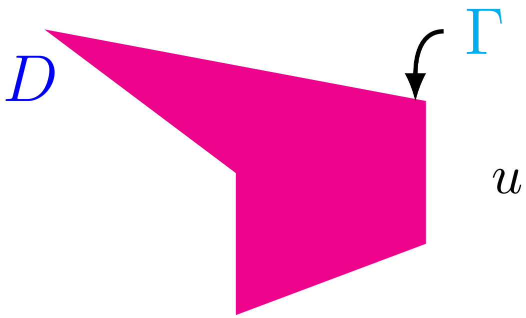
3. **Other geometries and 3D**

1. GREEN'S REPRESENTATION THEOREM AND INTEGRAL EQUATIONS FOR HELMHOLTZ

Green's Representation Theorem: slide from Talk 1

 u^{inc}

$$\Delta u + k^2 u = 0$$



$u - u^{\text{inc}}$ satisfies S.R.C.

Theorem

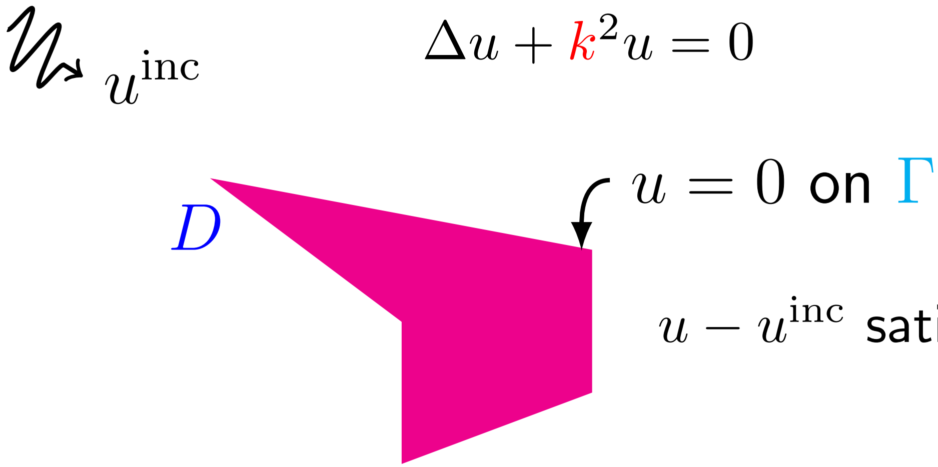
$$u(x) = u^{\text{inc}}(x) + \int_\Gamma \left(\frac{\partial u}{\partial n}(y)\Phi(x,y) - u(y)\frac{\partial \Phi(x,y)}{\partial n(y)} \right) ds(y), \quad x \in D,$$

where

$$\Phi(x,y) = \begin{cases} \frac{\mathrm{i}}{4} H_0^{(1)}(k|x-y|) & \text{(2D)}, \\ \dfrac{1}{4\pi}\dfrac{\mathrm{e}^{\mathrm{i}k|x-y|}}{|x-y|}, & \text{(3D)}. \end{cases}$$

Green's Representation Theorem

$$\Delta u + k^2 u = 0$$



$u - u^{\mathrm{inc}}$ satisfies S.R.C.

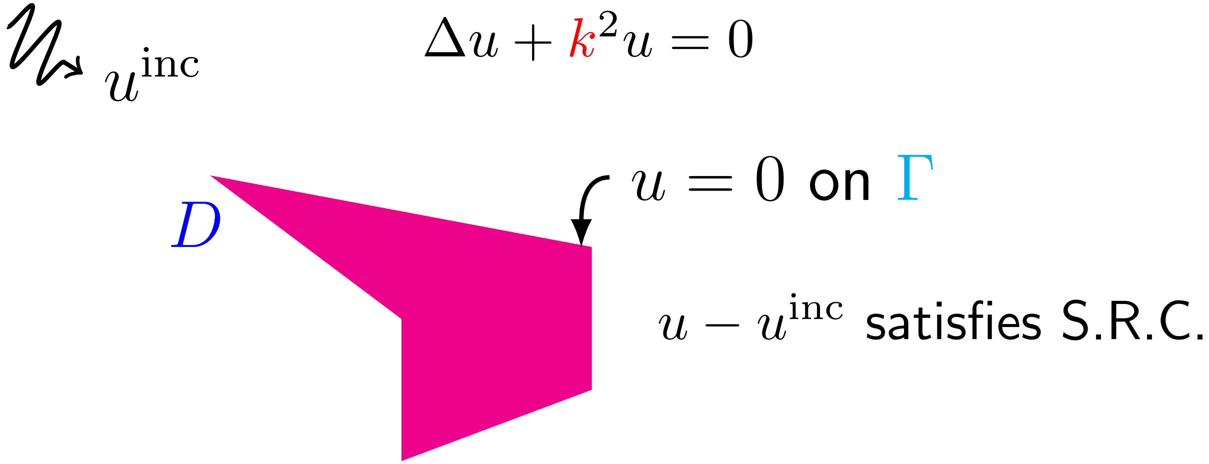
Theorem

$$u(x) = u^{\mathrm{inc}}(x) + \int_{\Gamma} \left(\frac{\partial u}{\partial n}(y)\Phi(x,y) - u(y)\frac{\partial \Phi(x,y)}{\partial n(y)} \right) ds(y), \quad x \in D.$$

.

Green's Representation Theorem

$$\Delta u + k^2 u = 0$$



Theorem

$$u(x) = u^{\mathrm{inc}}(x) + \int_{\Gamma} \frac{\partial u}{\partial n}(y)\Phi(x,y)\, ds(y), \quad x \in D.$$

.

Green's Representation Theorem for a Thin Screen

u^{inc}

$$\Delta u + k^2 u = 0$$

D

$u = 0$ on Γ

$u - u^{\mathrm{inc}}$ satisfies S.R.C.

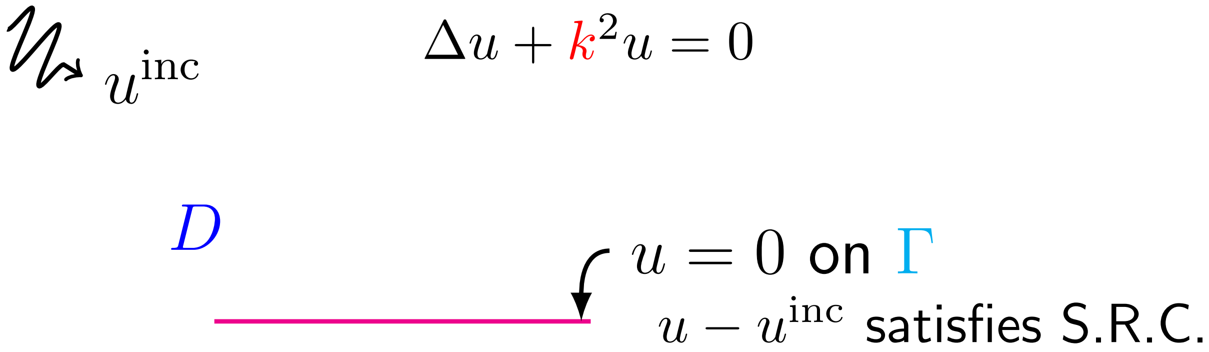
Theorem

$$u(x) = u^{\mathrm{inc}}(x) + \int_\Gamma \underbrace{\left[\frac{\partial u}{\partial n}\right]}_{\textbf{jump}}(y)\Phi(x,y)\,ds(y), \quad x \in D.$$

.

Green's Representation Theorem and BIE for a Thin Screen

$$\Delta u + k^2 u = 0$$



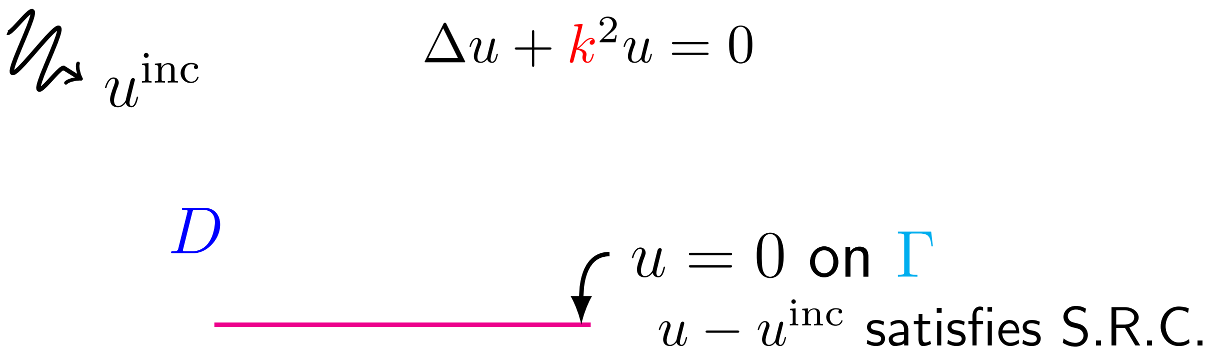
Theorem

$$u(x) = u^{\mathrm{inc}}(x) + \int_\Gamma \left[\frac{\partial u}{\partial n}\right](y)\Phi(x,y)\,ds(y), \quad x \in D.$$

Further (letting $x \to \Gamma$),

$$0 = u^{\mathrm{inc}}(x) + \int_\Gamma \left[\frac{\partial u}{\partial n}\right](y)\Phi(x,y)\,ds(y), \quad x \in \Gamma.$$

Green's Representation Theorem and BIE for a Thin Screen



Theorem

$$u(x) = u^{\text{inc}}(x) + \int_{\Gamma} \left[\frac{\partial u}{\partial n}\right](y)\Phi(x,y)\,ds(y), \quad x \in D,$$

and

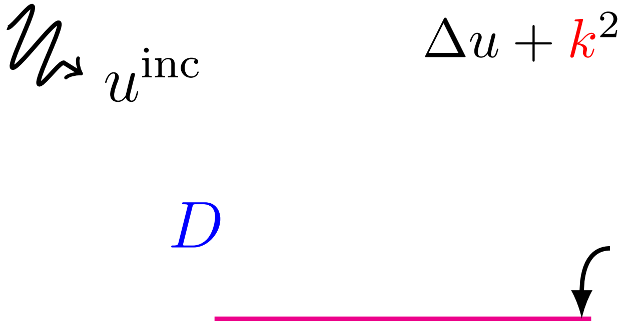
$$0 = u^{\text{inc}}(x) + \int_{\Gamma} \left[\frac{\partial u}{\partial n}\right](y)\Phi(x,y)\,ds(y), \quad x \in \Gamma,$$

in operator notation

$$S_k\left[\frac{\partial u}{\partial n}\right] = -u^{\text{inc}}|_{\Gamma} \quad \textbf{(BIE)}$$

2. A CASE STUDY FOR NUMERICAL-ASYMPTOTIC METHODS: THE THIN SCREEN

A Case Study for Numerical-Asymptotic Methods



$$\Delta u + k^2 u = 0$$

$u = 0$ on Γ

$u - u^{\text{inc}}$ satisfies S.R.C.

Theorem

$$u(x) = u^{\text{inc}}(x) + \int_\Gamma \left[\frac{\partial u}{\partial n}\right](y)\Phi(x,y)\,ds(y), \quad x \in D,$$

and

$$0 = u^{\text{inc}}(x) + \int_\Gamma \left[\frac{\partial u}{\partial n}\right](y)\Phi(x,y)\,ds(y), \quad x \in \Gamma,$$

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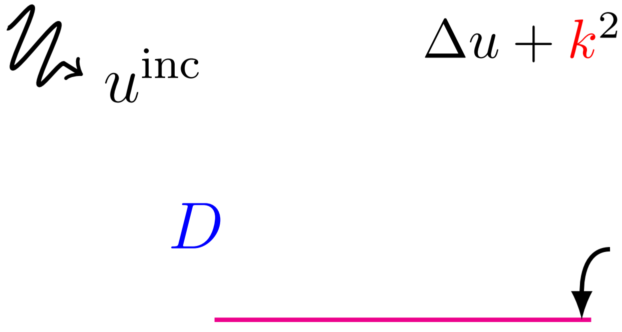
$$S_k \left[\frac{\partial u}{\partial n}\right] = -u^{\text{inc}}|_\Gamma \quad \textbf{(BIE)}$$

A Case Study for Numerical-Asymptotic Methods



$$\Delta u + k^2 u = 0$$

$u = 0$ on Γ

$u - u^{\mathrm{inc}}$ satisfies S.R.C.

We will solve

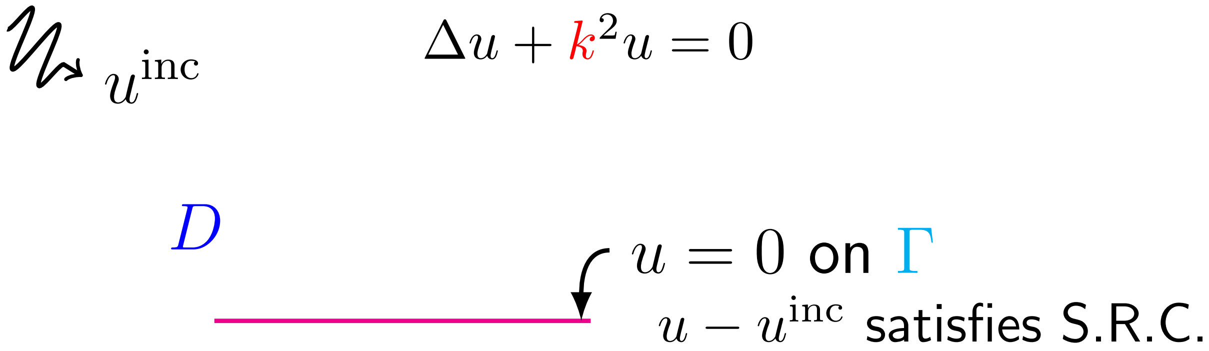
$$0 = u^{\mathrm{inc}}(x) + \int_{\Gamma} \left[\frac{\partial u}{\partial n}\right](y)\Phi(x,y)\, ds(y), \quad x \in \Gamma,$$

in operator notation

$$S_k \left[\frac{\partial u}{\partial n}\right] = -u^{\mathrm{inc}}|_{\Gamma}, \quad \textbf{(BIE)}$$

by a **Galerkin BEM** representing $\left[\frac{\partial u}{\partial n}\right]$ as a sum of products of **known oscillatory functions** and **unknown non-oscillatory functions**.

A Case Study for Numerical-Asymptotic Methods



We will solve the BIE by a Galerkin BEM representing $\left[\frac{\partial u}{\partial n}\right]$ as a sum of products of **known oscillatory functions** and **unknown non-oscillatory functions**.

The steps are:

A. Work out what this representation for $\left[\frac{\partial u}{\partial n}\right]$ is - use **GTD**.

B. Decide on the approximation space - use **HF asymptotics** and hp-**approximation** theory

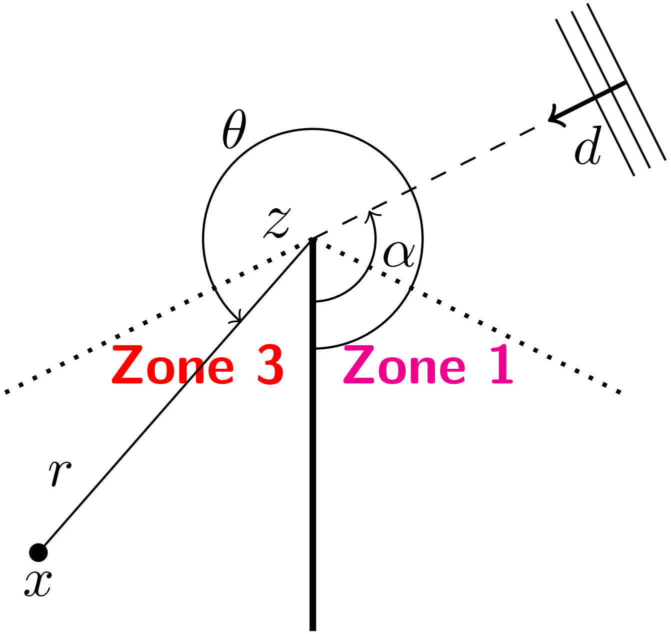
C. Implement it and see that the cost is $O(1)$ as $k \to \infty$!

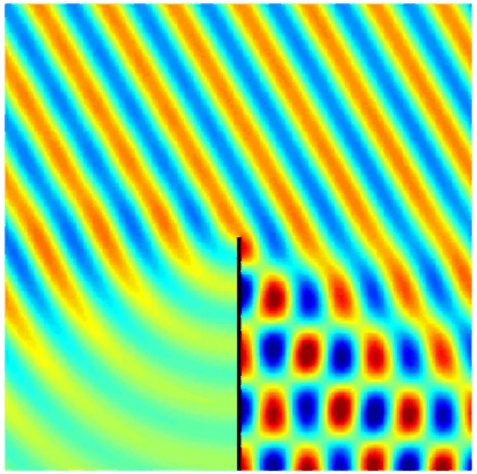
D. Try to prove this!

STEP A. REPRESENT THE UNKNOWN AS SUM OF PRODUCTS OF KNOWN OSCILLATORY AND UNKNOWN NON-OSCILLATORY FUNCTIONS USING GO/GTD.

The GTD Approach to Diffraction: Recap from Talk 1

$$u^{\mathrm{inc}}(x) = \mathrm{e}^{\mathrm{i}kx\cdot d}$$

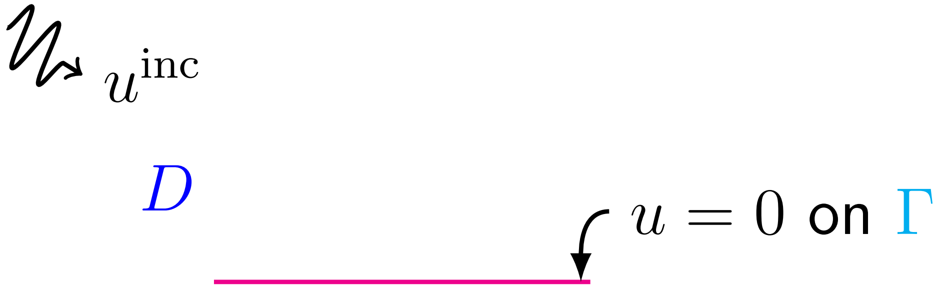




$\Re\, u(x)$

$$u(x) \approx \begin{cases} u^{\mathrm{inc}}(x) + u^{\mathrm{ref}}(x) + u^{\mathrm{d}}(x), & x \text{ in Zone 1} \\ u^{\mathrm{d}}(x), & x \text{ in Zone 3} \end{cases}$$

where $u^{\mathrm{d}}(x) = u^{\mathrm{inc}}(z)\,\mathcal{D}(\theta,\alpha)\,\dfrac{\mathrm{e}^{\mathrm{i}kr}}{\sqrt{kr}}$, $\mathcal{D}$ the **diffraction coefficient**.



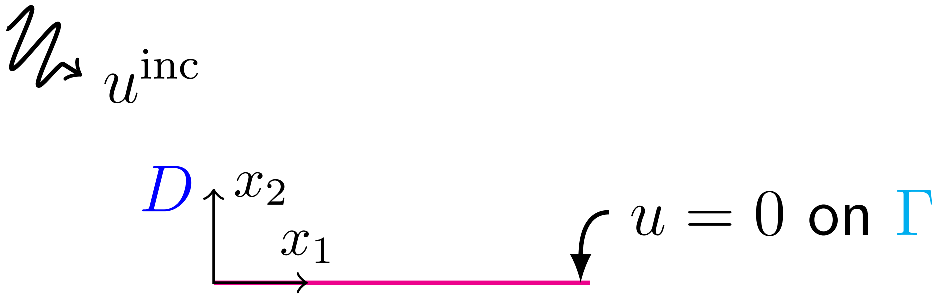
We will solve the BIE by a Galerkin BEM representing $\left[\frac{\partial u}{\partial n}\right]$ as a sum of products of **known oscillatory functions** and **unknown non-oscillatory functions**.

Step A. Work out what this representation for $\left[\frac{\partial u}{\partial n}\right]$ is - use **GTD**.

Above the screen (see the GO example yesterday)

$$\frac{\partial u}{\partial n} \approx 2\frac{\partial u^{\mathrm{inc}}}{\partial n}$$

.



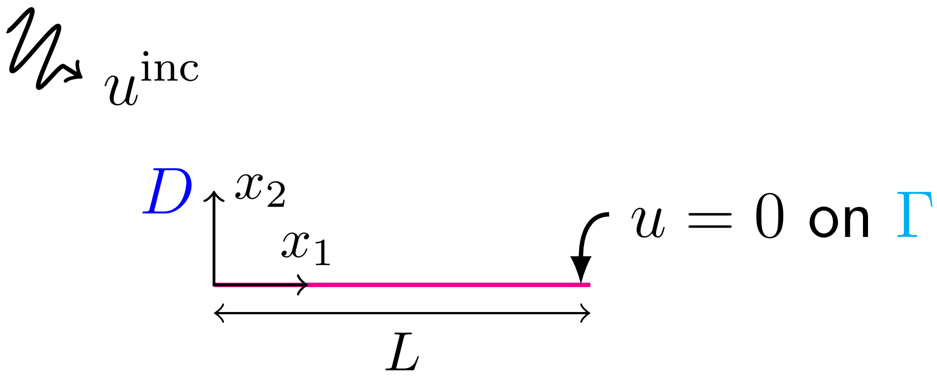
We will solve the BIE by a Galerkin BEM representing $\left[\frac{\partial u}{\partial n}\right]$ as a sum of products of **known oscillatory functions** and **unknown non-oscillatory functions**.

Step A. Work out what this representation for $\left[\frac{\partial u}{\partial n}\right]$ is - use **GTD**.

Above the screen ... and adding in the GTD terms ...

$$\frac{\partial u}{\partial n}(x) \approx 2\frac{\partial u^{\text{inc}}}{\partial n}(x) + c_+ \frac{\mathrm{e}^{\mathrm{i}kx_1}}{\sqrt{kx_1}} + \ldots$$

.



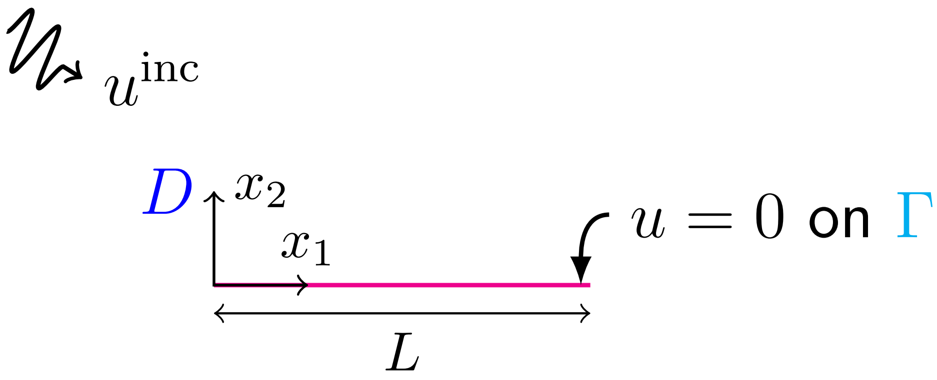
We will solve the BIE by a Galerkin BEM representing $\left[\frac{\partial u}{\partial n}\right]$ as a sum of products of **known oscillatory functions** and **unknown non-oscillatory functions**.

Step A. Work out what this representation for $\left[\frac{\partial u}{\partial n}\right]$ is - use **GTD**.

Above the screen ... and adding in the GTD terms ...

$$\frac{\partial u}{\partial n}(x) \approx 2\frac{\partial u^{\mathrm{inc}}}{\partial n}(x) + c_+ \frac{\mathrm{e}^{\mathrm{i}kx_1}}{\sqrt{kx_1}} + c_- \frac{\mathrm{e}^{-\mathrm{i}kx_1}}{\sqrt{k(L-x_1)}}$$

.



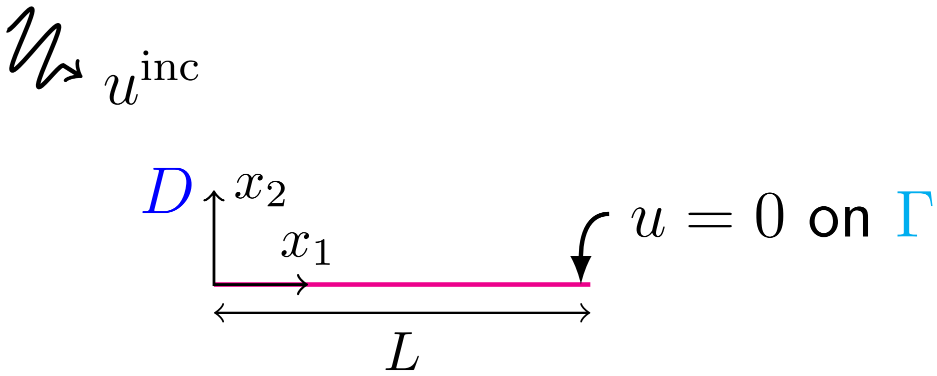
We will solve the BIE by a Galerkin BEM representing $\left[\frac{\partial u}{\partial n}\right]$ as a sum of products of **known oscillatory functions** and **unknown non-oscillatory functions**.

Step A. Work out what this representation for $\left[\frac{\partial u}{\partial n}\right]$ is - use **GTD**.

While below the screen

$$\frac{\partial u}{\partial n}(x) \approx c'_+ \frac{\mathrm{e}^{\mathrm{i}kx_1}}{\sqrt{kx_1}} + c'_- \frac{\mathrm{e}^{-\mathrm{i}kx_1}}{\sqrt{k(L-x_1)}}$$

.



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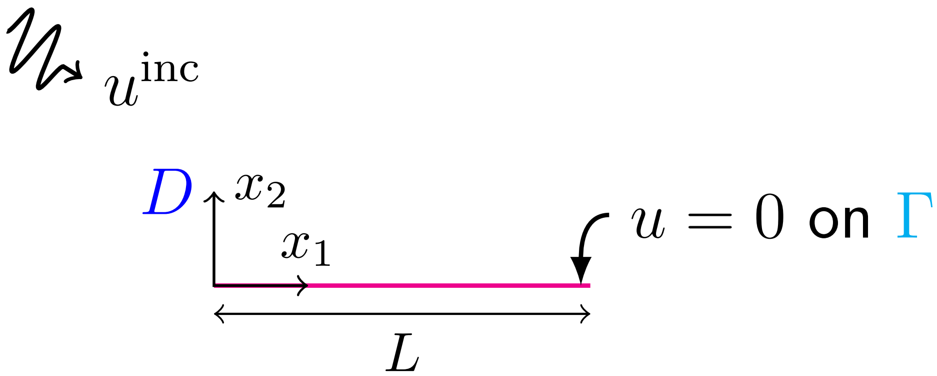
Step A. Work out what this representation for $\left[\frac{\partial u}{\partial n}\right]$ is - use **GTD**.

So

$$\left[\frac{\partial u}{\partial n}\right](x) \approx 2\frac{\partial u^{\mathrm{inc}}}{\partial n}(x) + C_+ \frac{\mathrm{e}^{\mathrm{i}kx_1}}{\sqrt{kx_1}} + C_- \frac{\mathrm{e}^{-\mathrm{i}kx_1}}{\sqrt{k(L-x_1)}}.$$

.



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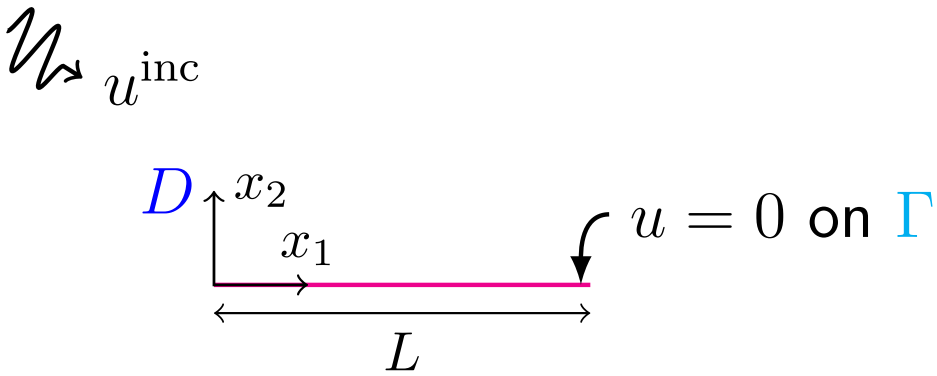
Step A. Work out what this representation for $\left[\frac{\partial u}{\partial n}\right]$ is - use **GTD**.

... and our representation is ...

$$\left[\frac{\partial u}{\partial n}\right](x) = 2\frac{\partial u^{\mathrm{inc}}}{\partial n}(x) + \mathrm{e}^{\mathrm{i}kx_1} \underbrace{F_+(x_1)}_{\text{unknown}} + \mathrm{e}^{-\mathrm{i}kx_1} \underbrace{F_-(L - x_1)}_{\text{unknown}}$$

.



We will represent $\left[\frac{\partial u}{\partial n}\right]$ as a sum of products of **known oscillatory functions** and **unknown non-oscillatory functions**.

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Step B. Show that $F_\pm$ are **non-oscillatory** and choose **piecewise polynomial** approximation spaces for them.

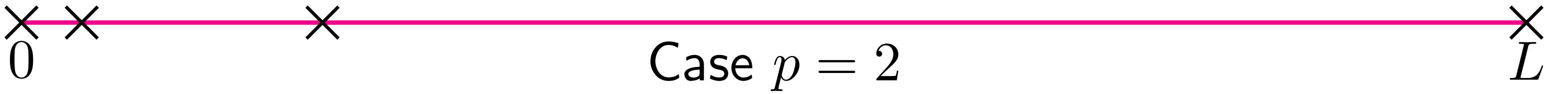
STEP B. RECAP – A PIECEWISE POLYNOMIAL QUIZ

How best to approximate $F(t) = t^{1/2}$ on $[0, L]$ with smallest $L^\infty(0, L)$ error using piecewise polynomials of degree p?

0 $\qquad$ L

.

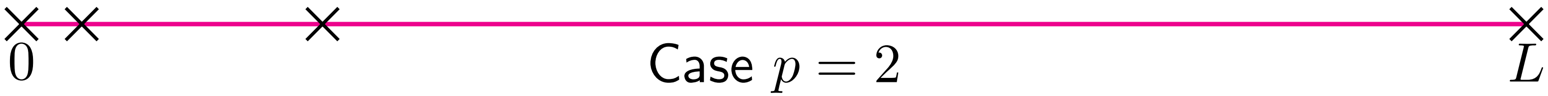
How best to approximate $F(t) = t^{1/2}$ on $[0, L]$ with smallest $L^{\infty}(0, L)$ error using piecewise polynomials of degree p?



ANSWER! Grid-points 0 and $L\alpha^j$, $j = 0, ..., p$, with $\alpha \approx 0.2$, and polynomial of degree p on each subinterval.

.

How best to approximate $F(t) = t^{1/2}$ on $[0, L]$ with smallest $L^\infty(0, L)$ error using piecewise polynomials of degree p?



ANSWER! Grid-points 0 and $L\alpha^j$, $j = 0, ..., p$, with $\alpha \approx 0.2$, and polynomial of degree p on each subinterval.

This is standard hp-**approximation on a geometrically graded mesh**.

.

How best to approximate $F(t) = t^{-1/2}$ on $[0, L]$ with smallest $L^q(0, L)$ error ($1 \le q < 2$) using piecewise polynomials of degree p?

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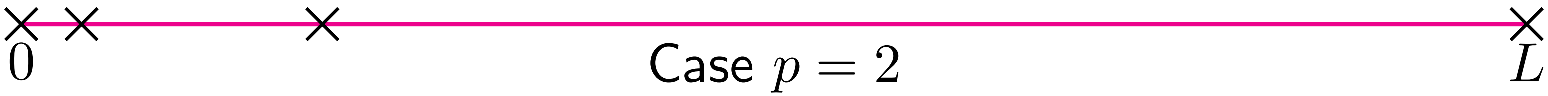
How best to approximate $F(t) = t^{-1/2}$ on $[0, L]$ with smallest $L^q(0, L)$ error ($1 \leq q < 2$) using piecewise polynomials of degree p?



ANSWER! Grid-points 0 and $L\alpha^j$, $j = 0, ..., p$, with $\alpha \approx 0.2$, and polynomial of degree p on each subinterval.

This is standard hp-**approximation on a geometrically graded mesh** and

minimum error $\leq C\exp(-cp) = C\exp(-c\sqrt{N})$**,**

where $N = D.O.F.$

.

Suppose that $F(z)$ is analytic in $\Re z > 0$ and

$$|F(z)| \leq |z|^{-1/2}, \quad \Re z > 0.$$

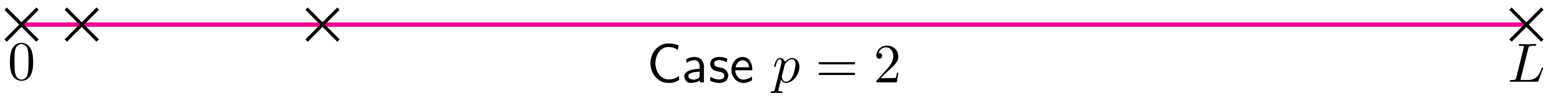
How best to approximate F on $[0, L]$ with smallest $L^q(0, L)$ error ($1 \leq q < 2$) using piecewise polynomials of degree p?

0 L

Suppose that $F(z)$ is analytic in $\Re z > 0$ and

$$|F(z)| \leq |z|^{-1/2}, \quad \Re z > 0.$$

How best to approximate F on $[0, L]$ with smallest $L^q(0, L)$ error ($1 \leq q < 2$) using piecewise polynomials of degree p?



ANSWER! Grid-points 0 and $L\alpha^j$, $j = 0, ..., p$, with $\alpha \approx 0.2$, and polynomial of degree p on each subinterval.

This is standard hp-**approximation on a geometrically graded mesh** and

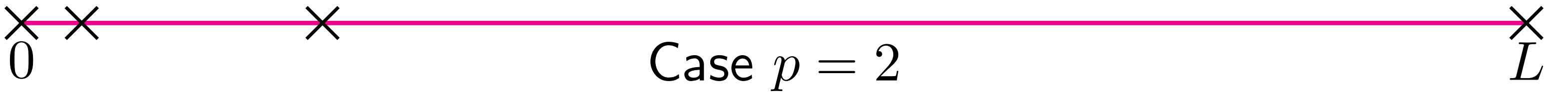
minimum error $\leq C \exp(-cp) = C \exp(-c\sqrt{N})$**,**

where $N = D.O.F.$**.**

Suppose that $F(z)$ is analytic in $\Re z > 0$ and

$$|F(z)| \leq |z|^{-1/2}, \quad \Re z > 0.$$

How best to approximate F on $[0, L]$ with smallest $\widetilde{H}^{-1/2}(\Gamma)$ error using piecewise polynomials of degree p?

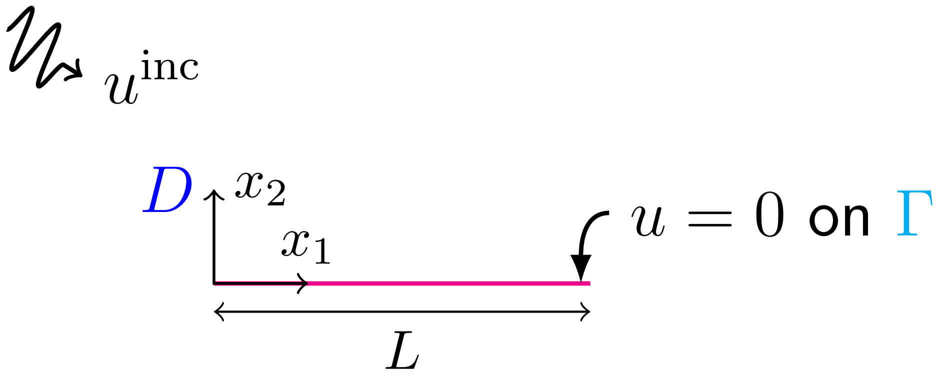


ANSWER! Grid-points 0 and $L\alpha^j$, $j = 0, ..., p$, with $\alpha \approx 0.2$, and polynomial of degree p on each subinterval.

This follows since $\widetilde{H}^{-1/2}(\Gamma)$ is continuously embedded in $L^q(\Gamma) = L^q(0, L)$ for $q > 1$, and hence

minimum error $\leq C \exp(-cp) = C \exp(-c\sqrt{N})$,

where $N = D.O.F.$



Non-Oscillatorariness Theorem. For some $F_\pm(z)$, analytic in $\Re z > 0$ with

$$|F_\pm(z)| \leq Ck^{3/2}|z|^{-1/2}, \quad \Re z > 0,$$

it holds that

$$\left[\frac{\partial u}{\partial n}\right](x) = 2\frac{\partial u^{\text{inc}}}{\partial n}(x) + \mathrm{e}^{\mathrm{i}kx_1}F_+(x_1) + \mathrm{e}^{-\mathrm{i}kx_1}F_-(L - x_1), \quad x \in \Gamma.$$

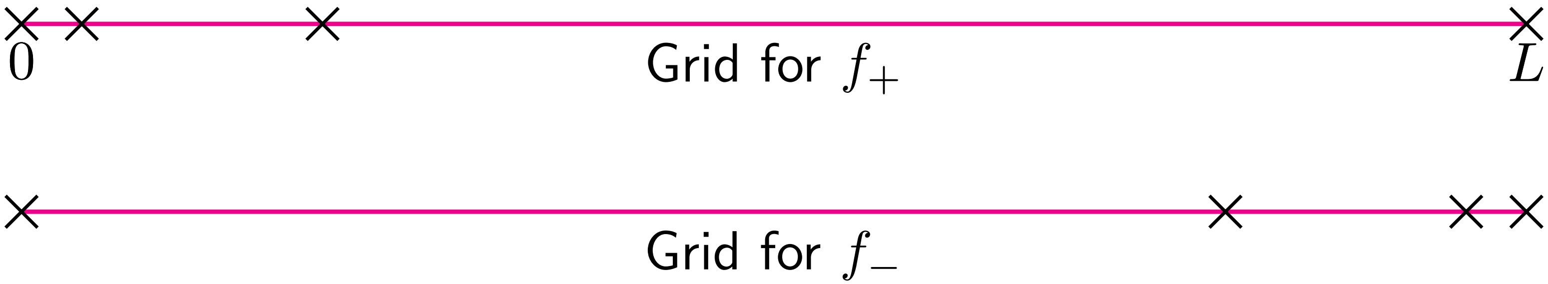
Proof. Hewett, Langdon, C-W, to appear *IMA J. Numer. Anal.*.

For $x \in \Gamma$,

$$\left[\frac{\partial u}{\partial n}\right](x) = 2\frac{\partial u^{\mathrm{inc}}}{\partial n}(x) + \mathrm{e}^{\mathrm{i}kx_1}F_+(x_1) + \mathrm{e}^{-\mathrm{i}kx_1}F_-(L - x_1)$$

$$\approx 2\frac{\partial u^{\mathrm{inc}}}{\partial n}(x) + \mathrm{e}^{\mathrm{i}kx_1}f_+(x_1) + \mathrm{e}^{-\mathrm{i}kx_1}f_-(x_1),$$

where f_+ and f_- are piecewise polynomials of degree p on geometrically graded meshes, each with p intervals: i.e., hp-approximation.



This is our Galerkin approximation space.

A Case Study for Numerical-Asymptotic Methods

u^{inc}

$$\Delta u + k^2 u = 0$$

D

$u = 0$ on Γ

$u - u^{\mathrm{inc}}$ satisfies S.R.C.

We will solve

$$0 = u^{\mathrm{inc}}(x) + \int_{\Gamma} \left[\frac{\partial u}{\partial n}\right](y)\Phi(x,y)\,ds(y), \quad x \in \Gamma,$$

in operator notation

$$S_k \left[\frac{\partial u}{\partial n}\right] = -u^{\mathrm{inc}}|_{\Gamma}, \quad \textbf{(BIE)}$$

by a **Galerkin BEM using this ansatz**.

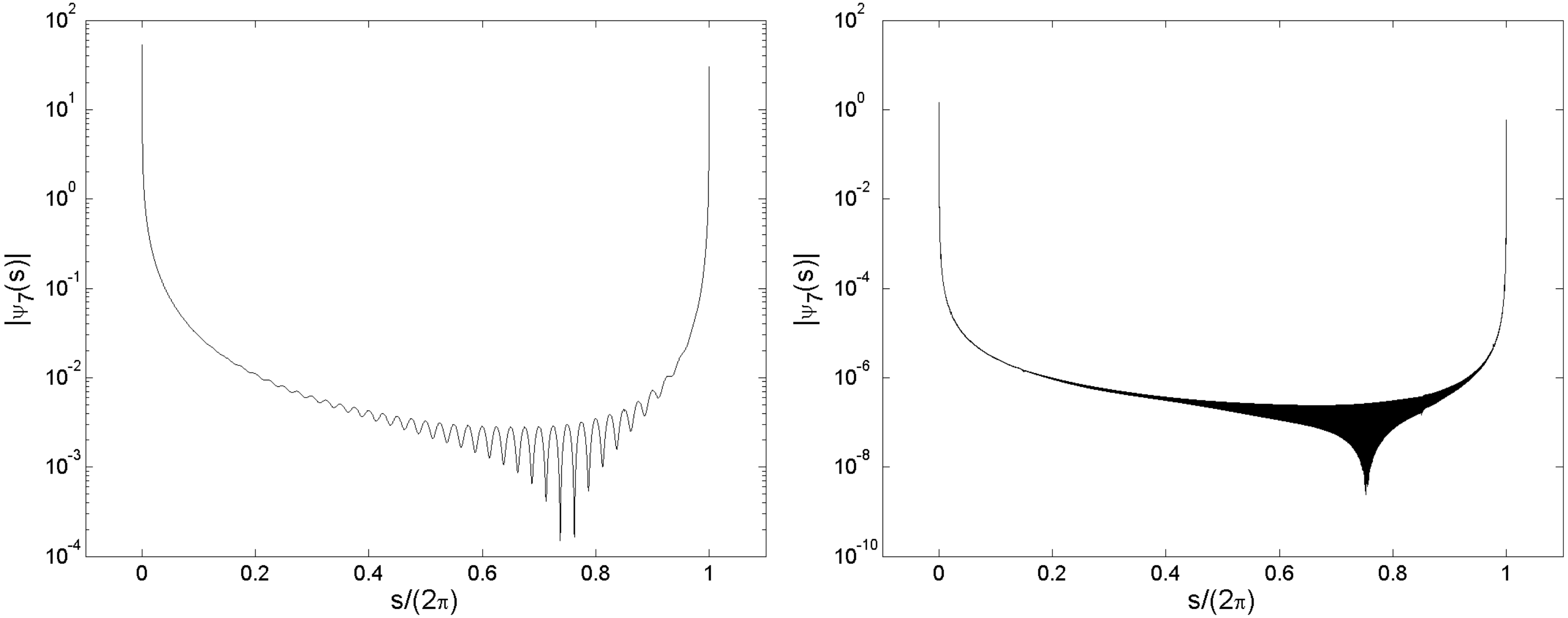
STEP C. IMPLEMENT IT AND SEE THAT THE COST IS $O(1)$ AS $k \to \infty$!

Difficulty (and main cost) is assembly of the matrix $[a_{mn}]$ which requires **2D highly oscillatory integrals**:

$$a_{mn} = \int_{\Gamma_m} \int_{\Gamma_n} H_0^{(1)}(k|x_1 - y_1|) \exp(\pm \mathrm{i} k x_1 \pm \mathrm{i} k y_1) p_m(x_1) p_n(y_1) \, dx_1 \, dy_1,$$

where p_m and p_n are polynomials supported on elements Γ_m and Γ_n.

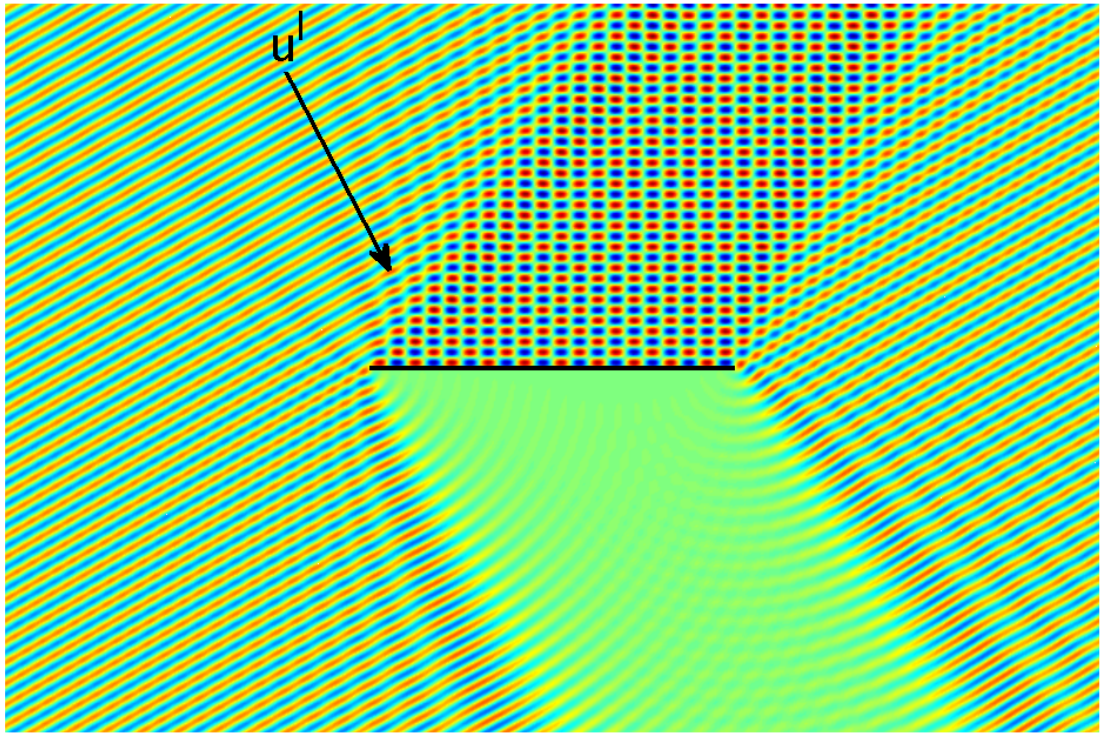
For details of our Filon quadrature see Hewett, Langdon, C-W (2014).



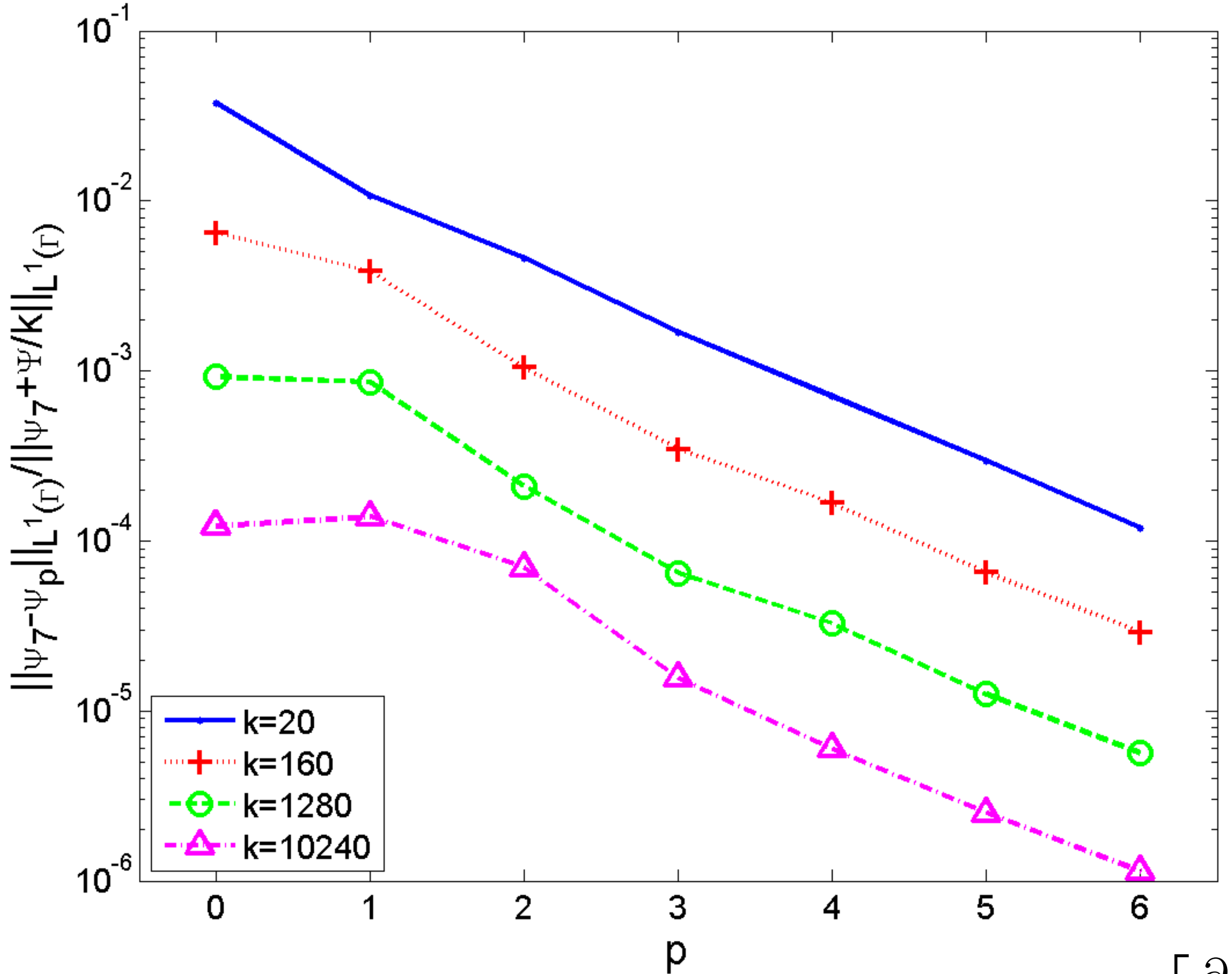
Plots of the amplitude of the diffracted component, i.e.

$$\left| \left[\frac{\partial u}{\partial n} \right](x) - 2\frac{\partial u^{\mathrm{inc}}}{\partial n}(x) \right| \quad \text{against} \quad \frac{x_1}{L},$$

for $\dfrac{L}{\lambda} = 20$ (left), $\dfrac{L}{\lambda} = 10240$ (right).



$\Re\, u$ in D for $\dfrac{L}{\lambda} = 20.$



Relative $L^1(\Gamma)$ errors in Galerkin approximation to $\left[\frac{\partial u}{\partial n}\right]$.

$\frac{L}{\lambda}$	$\lambda N/L$	$\frac{\|[\partial u/\partial n] - \phi_{64}\|_{L^1(\Gamma)}}{\|[\partial u/\partial n]\|_{L^1(\Gamma)}}$	cpu time (secs)
10	6.40×10^{0}	1.38×10^{-2}	47
40	1.60×10^{0}	1.40×10^{-2}	42
160	4.00×10^{-1}	1.40×10^{-2}	47
640	1.00×10^{-1}	1.39×10^{-2}	42
2560	2.50×10^{-2}	1.38×10^{-2}	42
10240	6.25×10^{-3}	1.37×10^{-2}	40

Relative $L^1(\Gamma)$ error in computing $[\partial u/\partial n]$:
64 degrees of freedom, grazing incidence.

STEP D. TRY TO PROVE THAT THE METHOD IS $O(1)$ BY THEOREMS ABOUT THE k-DEPENDENCE OF EVERYTHING!

Error Analysis

We are solving by a Galerkin BEM

$$0 = u^{\rm inc}(x) + \int_\Gamma \left[\frac{\partial u}{\partial n}\right](y)\Phi(x,y)\,ds(y), \quad x \in \Gamma,$$

in operator notation

$$S_k \left[\frac{\partial u}{\partial n}\right] = -u^{\rm inc}|_\Gamma.$$

By explicit Fourier analysis we can show, for a general planar screen (in 2D or 3D), k-**explicit coercivity** and **continuity** of S_k: for some $C, \alpha > 0$ independent of k and L,

$$\|S_k\| \leq C(kL)^{1/2}, \quad |\langle S_k\phi, \phi\rangle| \geq \alpha \|\phi\|^2_{\widetilde{H}^{-1/2}(\Gamma)},$$

where $\|S_k\|$ is the norm of $S_k : \widetilde{H}^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$ and $\langle \cdot, \cdot \rangle$ is the $L^2(\Gamma)$ inner product.

Error Analysis

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Error Analysis

$$S_k \left[\frac{\partial u}{\partial n}\right] = -u^{\mathrm{inc}}|_\Gamma.$$

By explicit Fourier analysis we can show, for a general planar screen (in 2D or 3D), k-**explicit coercivity** and **continuity** of S_k: for some $C, \alpha > 0$ independent of k and L,

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where $\|S_k\|$ is the norm of $S_k : \widetilde{H}^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$.

By **Céa's lemma** the Galerkin solution ϕ_N is well-defined and

$$\left\| \left[\frac{\partial u}{\partial n}\right] - \phi_N \right\|_{\widetilde{H}^{-1/2}(\Gamma)} \leq \frac{C}{\alpha}(kL)^{1/2} \inf_{\psi_N} \left\| \left[\frac{\partial u}{\partial n}\right] - \psi_N \right\|_{\widetilde{H}^{-1/2}(\Gamma)},$$

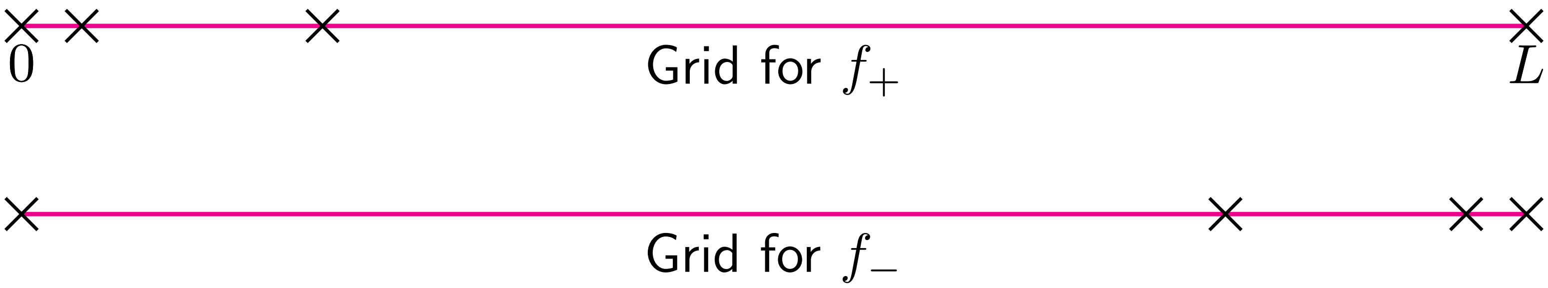
where the infimum is taken over all ψ_N in the N-dimensional Galerkin subspace.

For $x \in \Gamma$,

$$\left[\frac{\partial u}{\partial n}\right](x) = 2\frac{\partial u^{\mathrm{inc}}}{\partial n}(x) + \mathrm{e}^{\mathrm{i}kx_1} F_+(x_1) + \mathrm{e}^{-\mathrm{i}kx_1} F_-(L - x_1)$$

$$\approx 2\frac{\partial u^{\mathrm{inc}}}{\partial n}(x) + \mathrm{e}^{\mathrm{i}kx_1} f_+(x_1) + \mathrm{e}^{-\mathrm{i}kx_1} f_-(x_1),$$

where f_+ and f_- are piecewise polynomials of degree p on geometrically graded meshes, each with p intervals: i.e., hp-approximation.



This is our Galerkin approximation space.

$$\left[\frac{\partial u}{\partial n}\right](x) \quad \approx \quad 2\frac{\partial u^{\mathrm{inc}}}{\partial n}(x) + \mathrm{e}^{\mathrm{i}kx_1} f_+(x_1) + \mathrm{e}^{-\mathrm{i}kx_1} f_-(x_1),$$

where f_+ and f_- are piecewise polynomials of degree p on geometrically graded meshes, each with p intervals: i.e., hp-approximation.

Theorem If ϕ_N is the best $\widetilde{H}^{-1/2}(\Gamma)$ approximation to $\left[\frac{\partial u}{\partial n}\right]$ of this form, then

$$\left\| \left[\frac{\partial u}{\partial n}\right] - \phi_N \right\|_{\widetilde{H}^{-1/2}(\Gamma)} \leq C k^{3/2} (\log k)^{1/2} \exp(-c\sqrt{N}),$$

where C and c depend (only) on Γ, and $N \propto p^2$ is the number of D.O.F.

$$\left[\frac{\partial u}{\partial n}\right](x) \quad \approx \quad 2\frac{\partial u^{\text{inc}}}{\partial n}(x) + \mathrm{e}^{\mathrm{i}kx_1} f_+(x_1) + \mathrm{e}^{-\mathrm{i}kx_1} f_-(x_1),$$

where f_+ and f_- are piecewise polynomials of degree p on geometrically graded meshes, each with p intervals: i.e., hp-approximation.

Theorem **If ϕ_N is the Galerkin approximation** to $\left[\frac{\partial u}{\partial n}\right]$ of this form, then

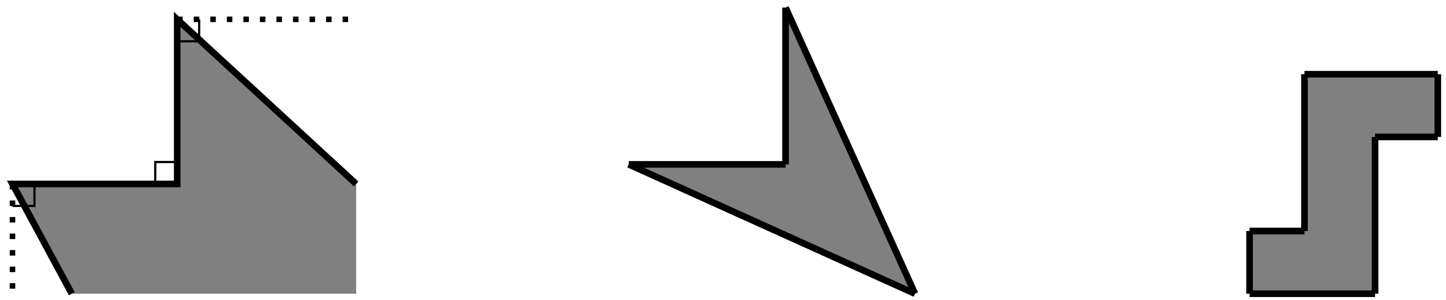
$$\left\| \left[\frac{\partial u}{\partial n}\right] - \phi_N \right\|_{\widetilde{H}^{-1/2}(\Gamma)} \leq C k^2 (\log k)^{1/2} \exp(-c\sqrt{N}),$$

where C and c depend (only) on Γ, and $N \propto p^2$ is the number of D.O.F.

3. EXTENSIONS TO OTHER GEOMETRIES AND 3D

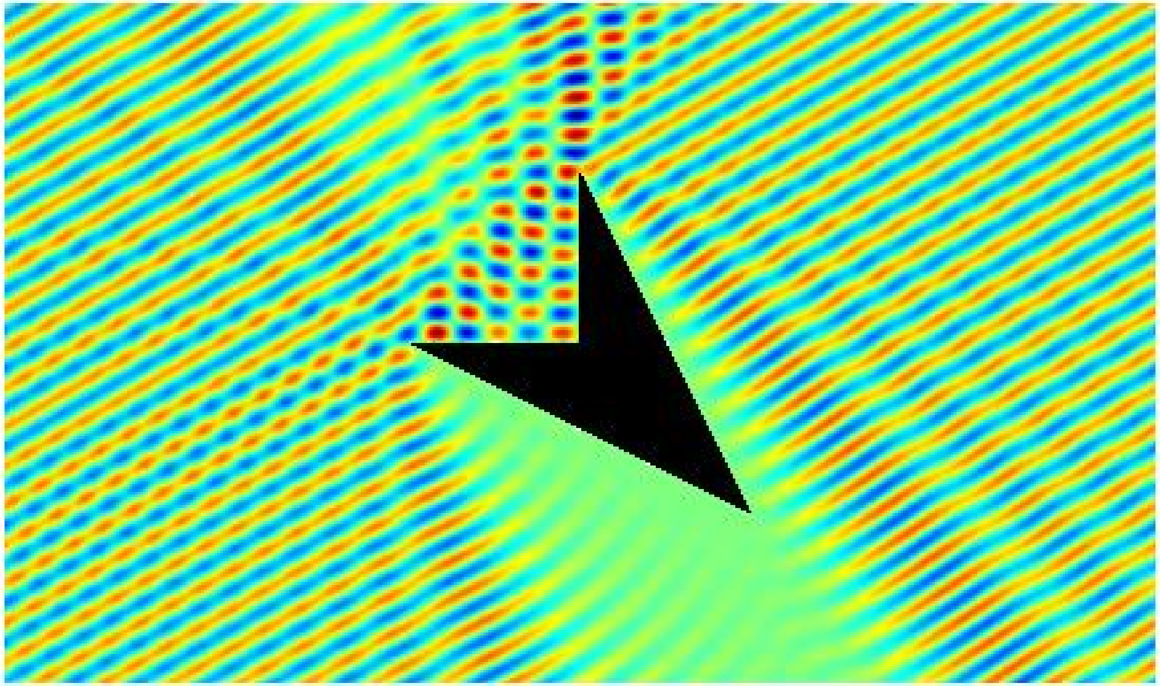
Polygons: Convex and Non-Convex

C-W, Hewett, Langdon, Twigger, *Numer Math* (2014)

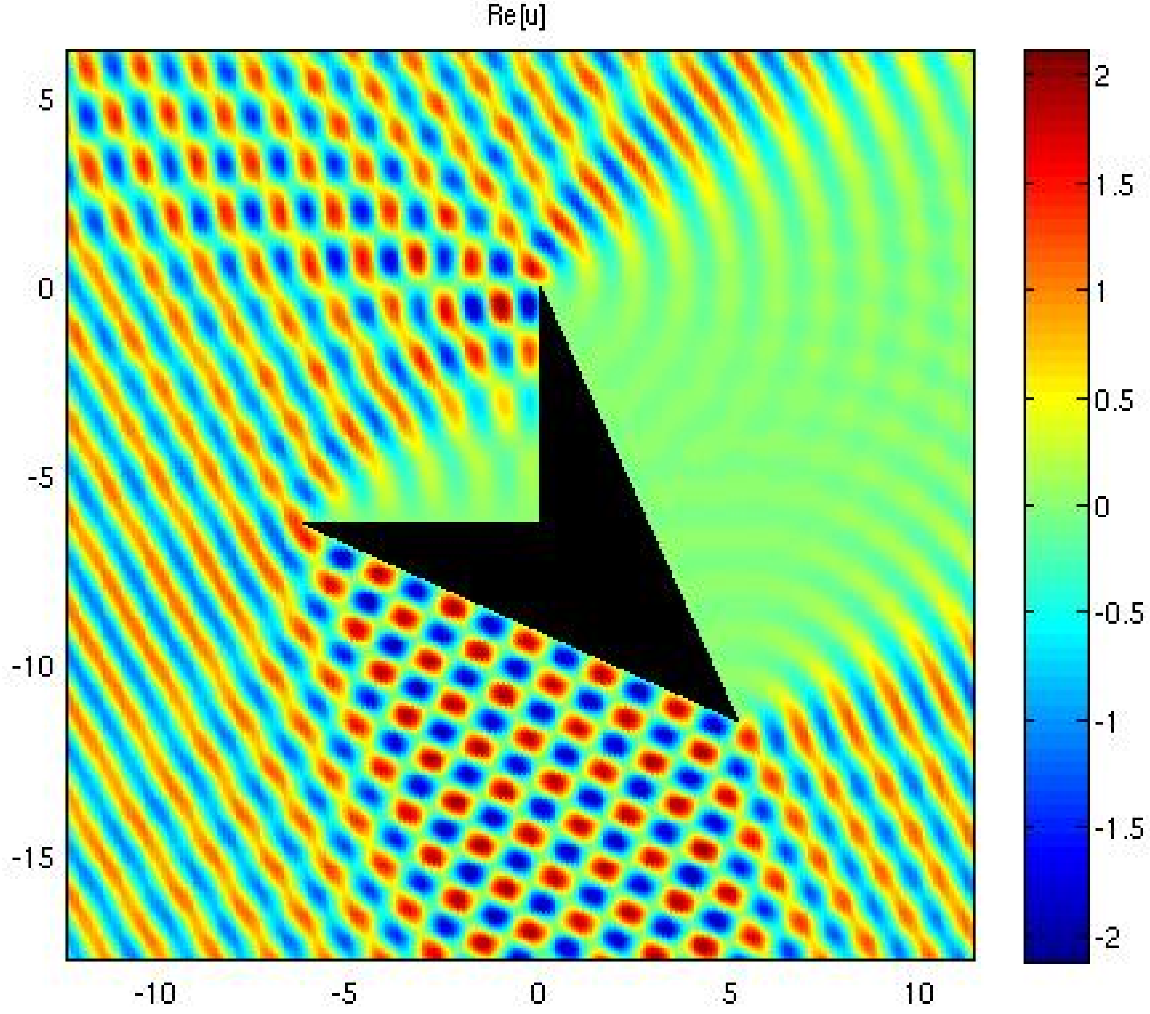


We can, using GO/GTD, design an approximation space for $\frac{\partial u}{\partial n}$ which provably needs only $O(\log^2 k)$ degrees of freedom as $k \to \infty$ and in experiments only $O(1)$.

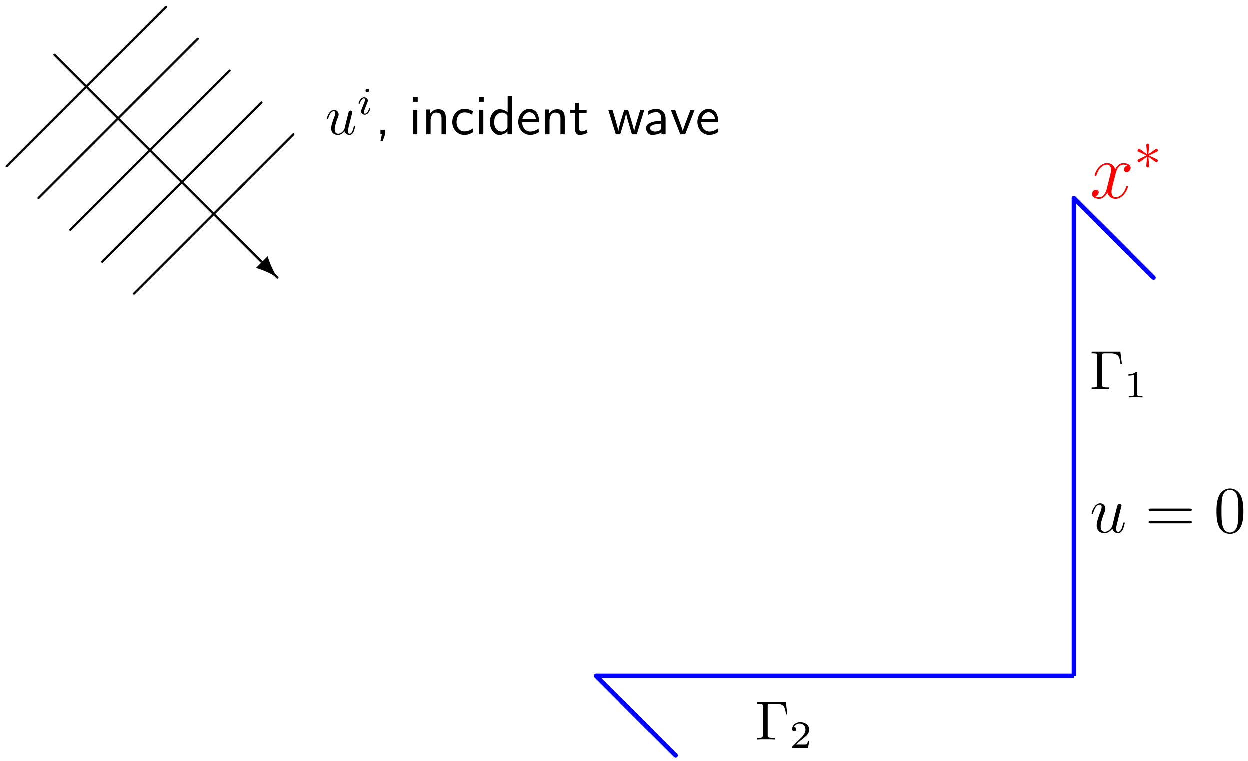
Solution Behaviour: $\Re u$



Solution Behaviour: $\Re u$



Solution Behaviour on Γ_2



On Γ_2,

$$\frac{\partial u}{\partial n} = \text{known} + \mathrm{e}^{\mathrm{i}k|x-x^*|}F(x_1) + \mathrm{e}^{\mathrm{i}kx_1}F_+(x_1) + \mathrm{e}^{-\mathrm{i}kx_1}F_-(x_1)$$

where 'known' = Fresnel integral and F is **analytic** and bounded in fixed neighbourhood of Γ_2, and again $N = O(\log^2 k)$ as $k \to \infty$ is provably enough.

hp-BEM Based on this Ansastz

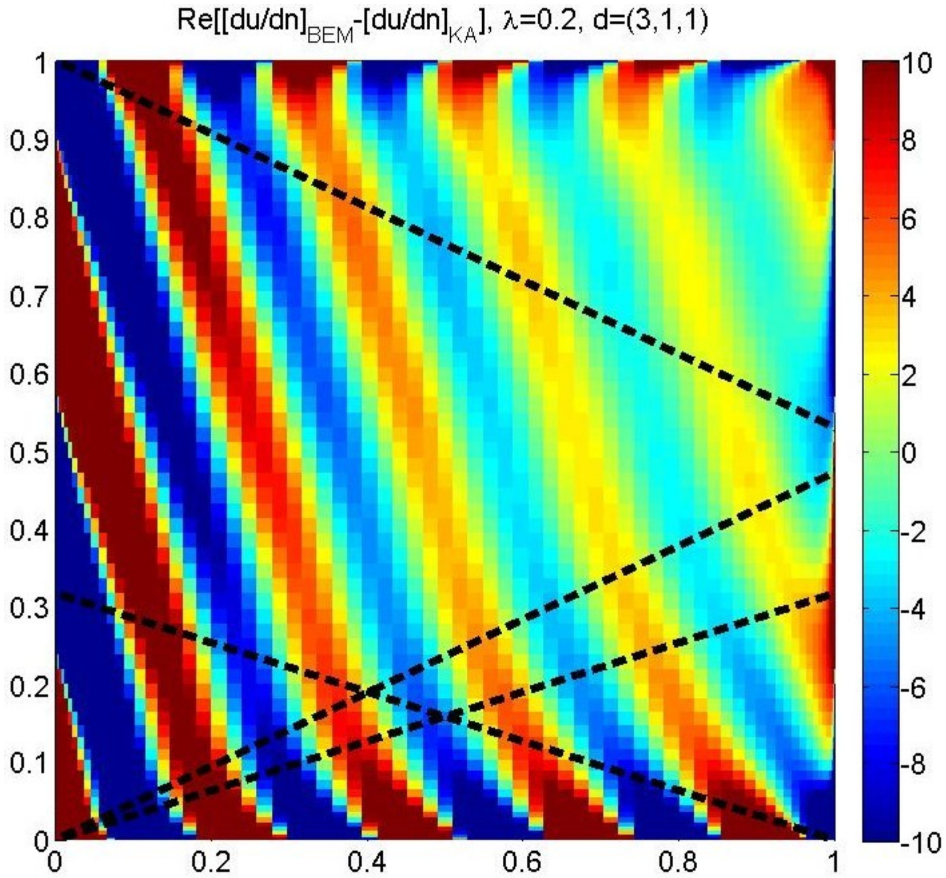
k	dof	dof per λ	L^2 error	Relative L^2 error
5	320	10.7	2.09e-2	1.51e-2
10	320	5.3	1.07e-2	1.11e-2
20	320	2.7	4.60e-3	6.91e-3
40	320	1.3	3.13e-3	6.83e-3

C-W, Langdon, Hewett, Twigger, *Numer Math* (2014).

3D Thin Screen: Square Plate

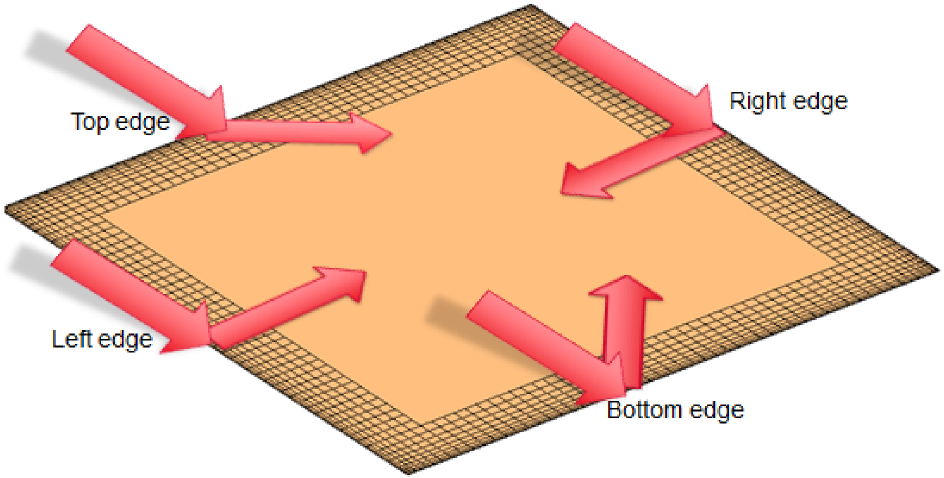
Hargreaves, Hewett, Langdon, Lam,

EPSRC project Reading/Salford

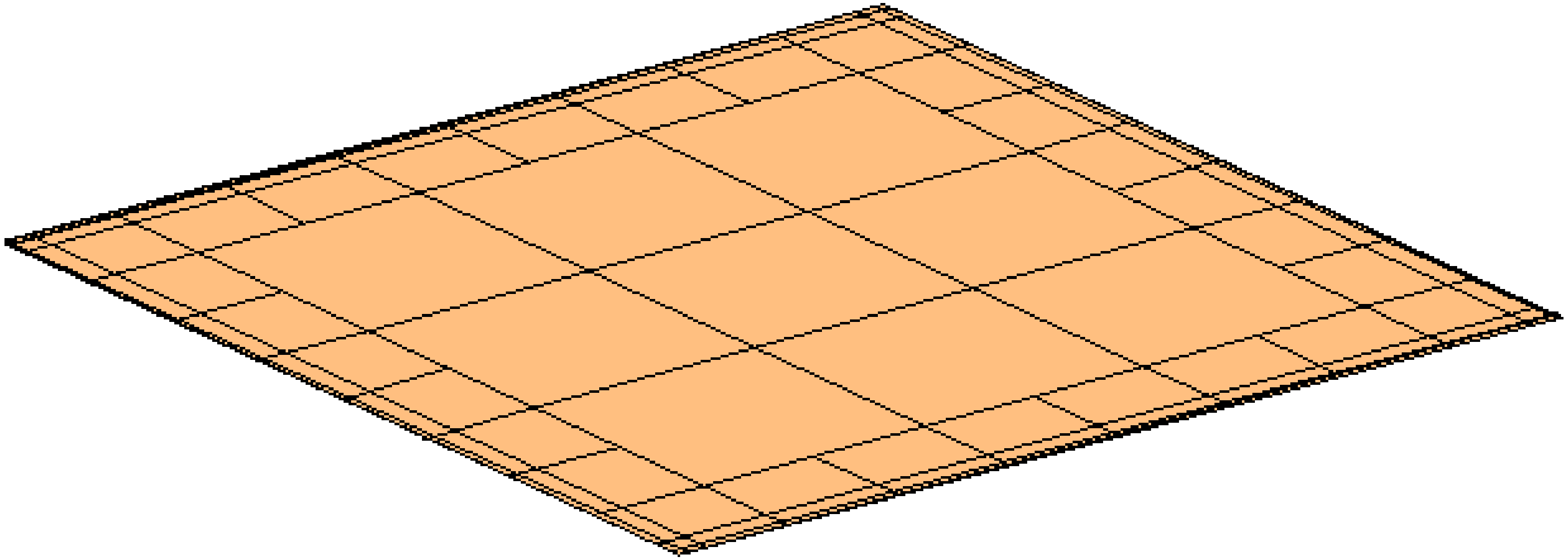


Approximation Methodology

- Subtract leading order oscillatory behaviour (incident field).
- Small conventional elements around the rim (to represent singular behaviour at edge).
- Large hybrid elements in the centre; basis functions are plane waves multiplied by polynomial basis functions (order p).
- Phase functions on hybrid elements correspond to first order diffraction directions ("edge plane waves").

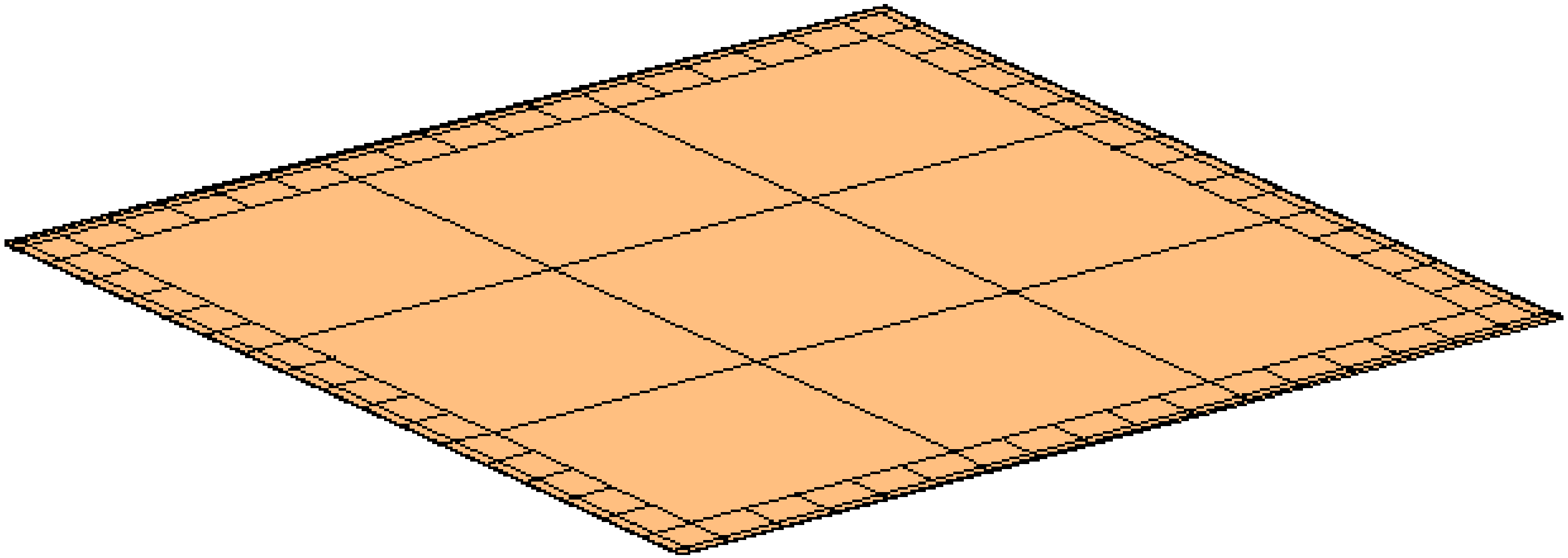


Mesh and required number of DOFS, $k = 5$



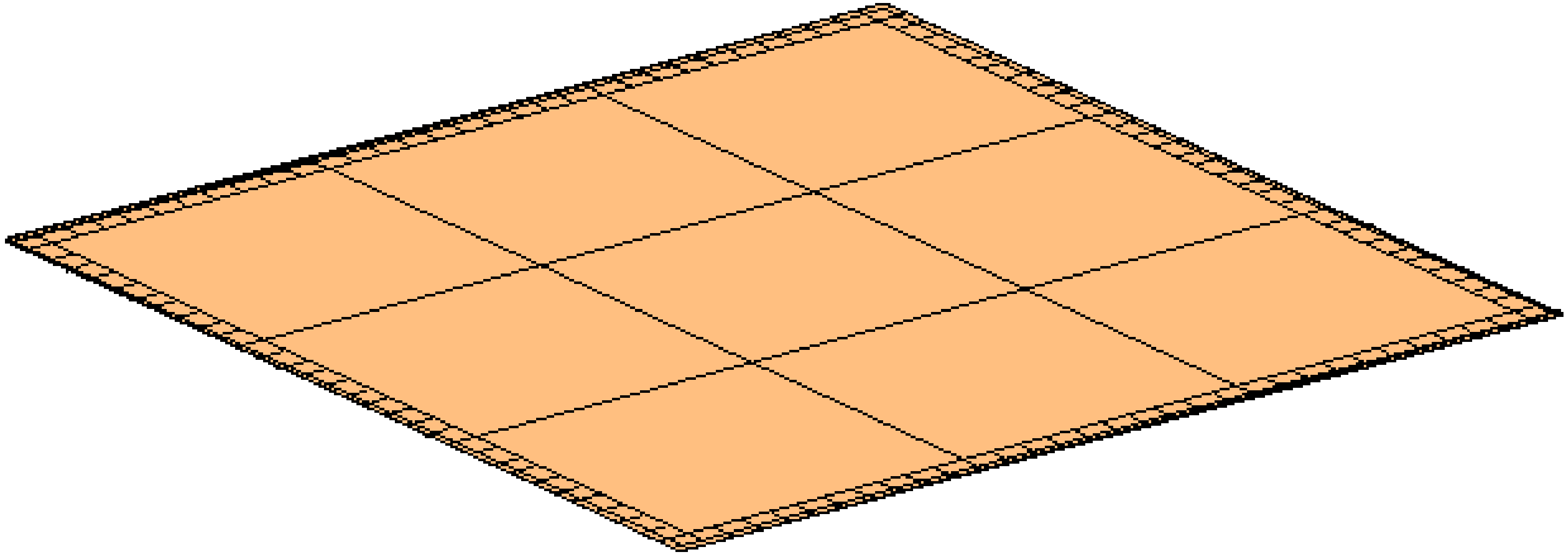
#DOF	Constant	Linear	Quadratic	Cubic
Regular	196	576	1,296	2,500
Hybrid	88	396	1,044	2,176

Mesh and required number of DOFS, $k = 10$



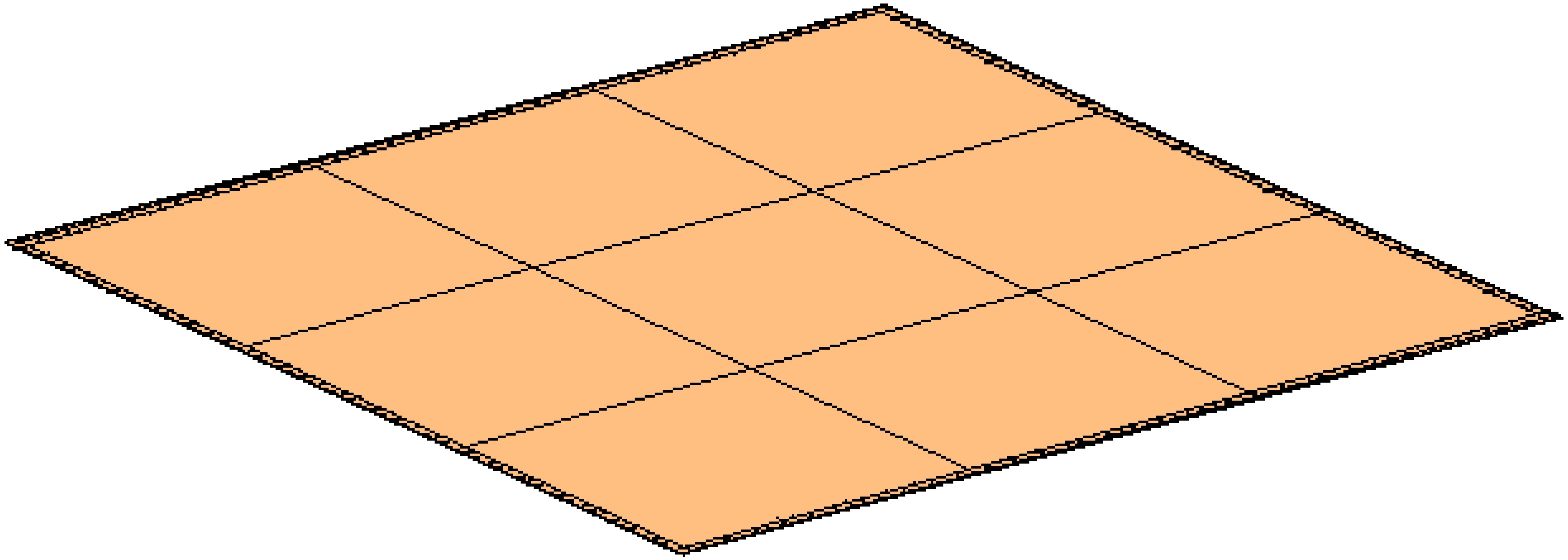
#DOF	Constant	Linear	Quadratic	Cubic
Regular	1,024	2,601	5,184	9,025
Hybrid	160	720	1,908	3,976

Mesh and required number of DOFS, $k = 20$



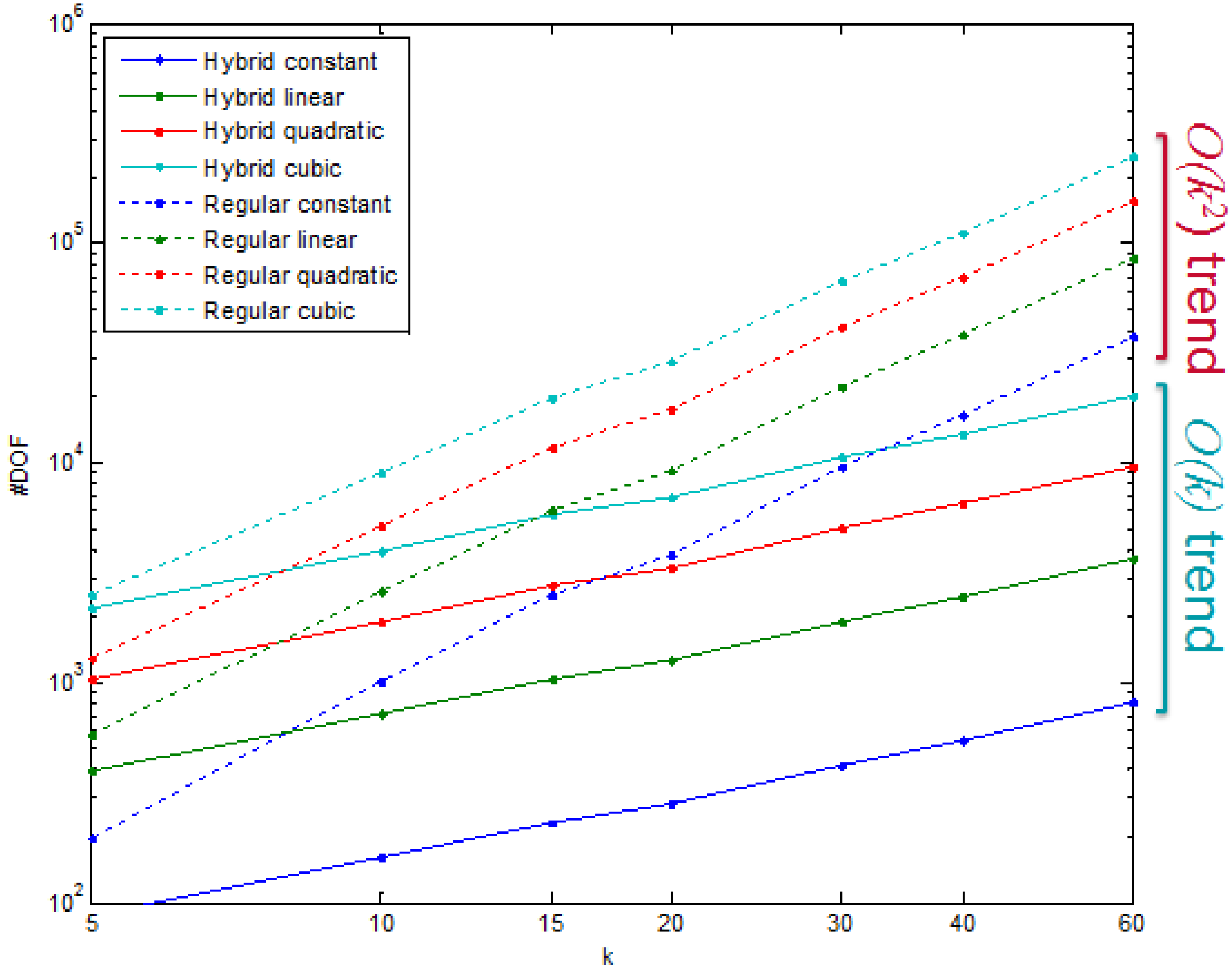
#DOF	Constant	Linear	Quadratic	Cubic
Regular	3,844	9,216	17,424	28,900
Hybrid	280	1,260	3,348	6,976

Mesh and required number of DOFS, $k = 40$

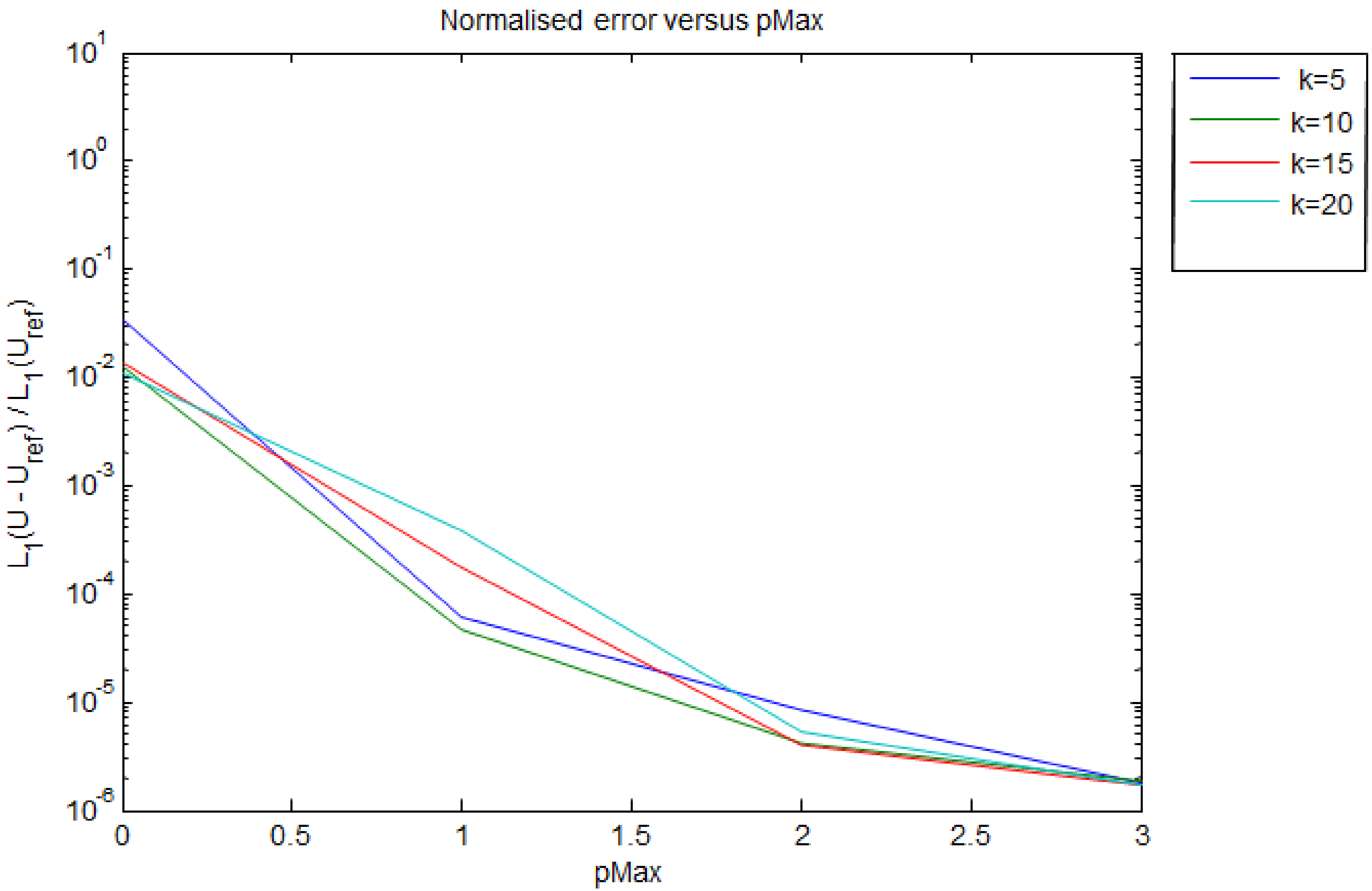


#DOF	Constant	Linear	Quadratic	Cubic
Regular	16,384	38,025	69,696	112,225
Hybrid	544	2,448	6,516	13,576

Degrees of freedom trend



Convergence of hybrid scheme



Other Geometries

- **Smooth convex obstacles**: see Bruno et al. *Phil. Trans R. Soc.* (2004), Dominguez, Graham et al *Numer. Math.* (2007), Huybrechs & Vandewalle *SISC* (2007)
- **Piecewise smooth convex polygons**: see Langdon, Mokgolele, C-W *J. Comp. Appl. Math* (2010)
- **Inhomogeneous impedance plane: outdoor noise propagation**: see C-W, Langdon *Phil. Trans R. Soc.* (2004), Langdon & C-W *SINUM* (2006)
- **Penetrable scatterers**: see Groth, Hewett, Langdon *IMA J. Appl. Math.* (2014)

Recap

1. Green's Representation Theorem and **boundary integral equations** (for Helmholtz)

2. A Case study for **numerical-asymptotic** methods: **the thin screen**

Step A. Represent the unknown as sum of products of **known oscillatory** and **unknown non-oscillatory** functions using **GO/GTD**.

Step B. Decide on the approximation space - combine **HF asymptotics** with hp-**approximation theory**

Step C. Implement it and see that (we hope) the cost is O(1) as $k \to \infty$!

Step D. Try to prove this by theorems about the k-dependence of everything!

3. **Other geometries and 3D**

References

Two review papers:

- "Numerical-asymptotic boundary integral methods in high-frequency acoustic scattering", C-W, I.G. Graham, S Langdon, & E.A. Spence, *Acta Numerica* (2012).
- **"Acoustic scattering: high frequency boundary element methods and unified transform methods"**, C-W & Langdon, to appear (preprint on Researchgate).

Note **Unified transform methods** $\approx$ **WBM from Daan's talk**.