High Frequency Solution Behaviour in Wave Scattering Problems

Simon Chandler-Wilde

University of Reading, UK

www.reading.ac.uk/~sms03snc

With: **Steve Langdon**, **Andrea Moiola** (Reading), **Ivan Graham**, **Euan Spence** (Bath), **Dave Hewett** (Oxford), **Valery Smyshlyaev**, **Timo Betcke** (UCL), **Marko Lindner** (TU HH), **Peter Monk** (Delaware)

PhDs **Andrew Gibbs**, **Sam Groth**, Charlotta Howarth, Ashley Twigger

Funding:
- EPSRC project(s) across Bath/Reading/UCL with **BAE Systems**, **Institute of Cancer Research**, **Met Office**, **Schlumberger Cambridge Research** as project partners.
- 4 NERC/EPSRC CASE Studentships at Bath & Reading
- EPSRC/Swiss NSF Fellowships for **Andrea, Euan, Timo**

Woudschoten, October 2014

Context

My talks apply (particularly) to **acoustic waves** — my background is in **outdoor noise propagation** and **noise barriers**.

My talks concern new **numerical-asymptotic methods** for **high frequency** wave scattering, that combine **numerical analysis ideas and tools** with **high frequency asymptotics**, see

C-W, Graham, Langdon, Spence *Acta Numerica* 21 (2012), 89–305.

This talk is the **HF asymptotics** – **numerical methods** are talk two!

Context

My talks apply (particularly) to **acoustic waves** — my background is in **outdoor noise propagation** and **noise barriers**.

My talks concern new **numerical-asymptotic methods** for **high frequency** wave scattering, that combine **numerical analysis ideas and tools** with **high frequency asymptotics**, see

Motivation. Want to factor unknown oscillatory functions into sums of products of **known oscillatory functions** and **unknown non-oscillatory functions**.

Overview

1. **Wave Equation and Helmholtz Equation**
2. **Basic Concept** of high frequency asymptotic approximations of **GO** and **GTD**
3. **Reflection** - canonical problems and high frequency GO approximations
4. **Diffraction** - canonical problems and high frequency GTD approximations
5. **The HF Kirchhoff Approximation**
6. **Preparing for NA: Quantifying Non-Oscillatorariness!**

1. WAVE EQUATION AND HELMHOLTZ EQUATION

The Wave Equation and Helmholtz Equation

$$\Delta U = \frac{1}{c^2}\frac{\partial^2 U}{\partial t^2} \quad \left(\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}\right).$$

If time-dependence is **time harmonic**, i.e., where $x = (x_1, x_2, x_3)$,

$$U(x,t) = A(x)\cos(\phi(x) - \omega t),$$

for some $\omega = 2\pi f > 0$, with $f =$ **frequency**, then

$$U(x,t) = \Re\left(u(x)\mathrm{e}^{-\mathrm{i}\omega t}\right)$$

where $u(x) = A(x)\exp(\mathrm{i}\phi(x))$ satisfies the **Helmholtz equation**

$$\Delta u + k^2 u = 0,$$

with $k = \omega/c$ the **wavenumber**.

If time-dependence is **time harmonic** then

$$U(x,t) = \Re\left(u(x)\mathrm{e}^{-\mathrm{i}\omega t}\right)$$

for some $\omega = 2\pi f > 0$, with f = **frequency**, where u satisfies the **Helmholtz equation**

$$\Delta u + k^2 u = 0,$$

with $k = \omega/c$ the **wavenumber**. E.g. if

$$u(x) = \mathrm{e}^{\mathrm{i}kx\cdot d},$$

for some **unit vector** d, then

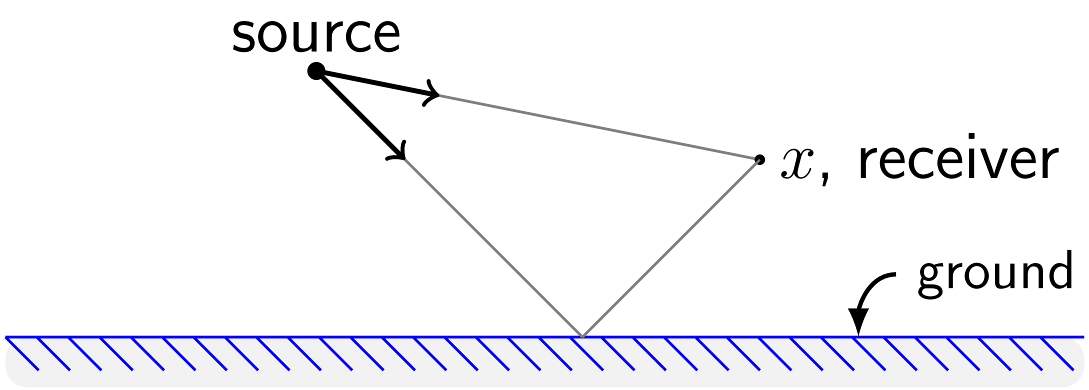
$$U(x,t) = \Re\left(u(x)\mathrm{e}^{-\mathrm{i}\omega t}\right) = \cos(kx\cdot d - \omega t)$$

is a **plane wave** travelling in direction d with **wavelength**

$$\lambda = \frac{2\pi}{k} = \frac{c}{f}.$$

2. GO AND GTD: THE BASIC CONCEPT

Geometrical Optics/Geometrical Theory of Diffraction (GTD)



$$u(x) = \sum_j u_j(x)$$

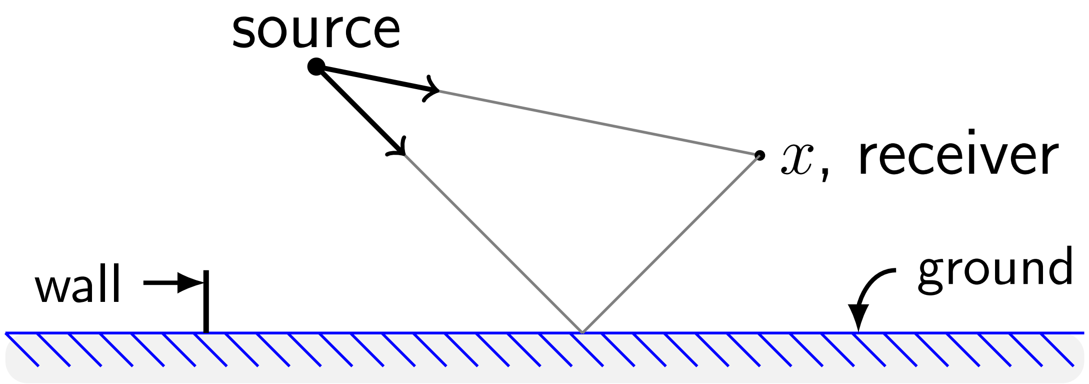
where sum is over **rays** passing through x, with

$$\arg u_j(x) \quad = \quad \textbf{optical length} \text{ of ray path } = k s_j$$

$$|u_j(x)| \quad = \quad \textbf{amplitude} \text{ determined by energy conservation,}$$

but with multiplication of $u_j(x)$ by **coefficients** accounting for **reflection**, **refraction**, and **diffraction** events.

Geometrical Optics/Geometrical Theory of Diffraction (GTD)



$$u(x) = \sum_j u_j(x)$$

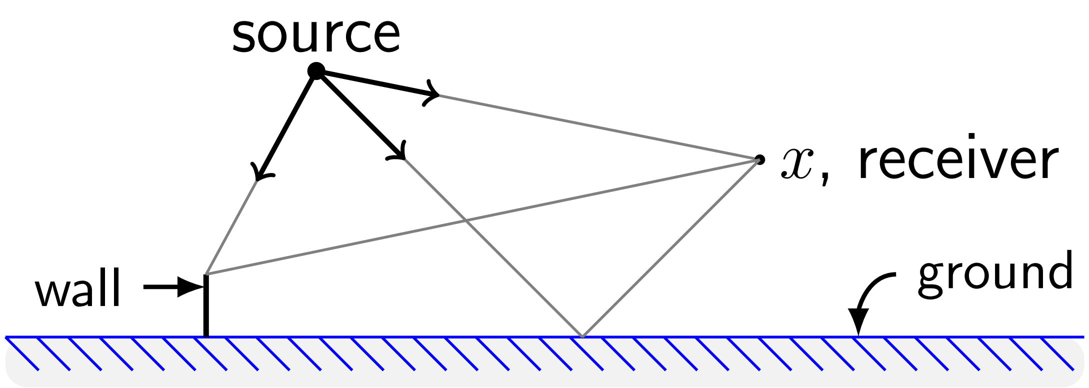
where sum is over **rays** passing through x, with

$$\arg u_j(x) \quad = \quad \textbf{optical length} \text{ of ray path } = ks_j$$

$$|u_j(x)| \quad = \quad \textbf{amplitude} \text{ determined by energy conservation,}$$

but with multiplication of $u_j(x)$ by **coefficients** accounting for **reflection**, **refraction**, and **diffraction** events.

Geometrical Optics/Geometrical Theory of Diffraction (GTD)



$$u(x) = \sum_j u_j(x)$$

where sum is over **rays** passing through x, with

$\arg u_j(x)$ = **optical length** of ray path $= ks_j$

$|u_j(x)|$ = **amplitude** determined by energy conservation,

but with multiplication of $u_j(x)$ by **coefficients** accounting for **reflection**, **refraction**, and **diffraction** events.

Geometrical Optics/Geometrical Theory of Diffraction (GTD)

$$u(x) = \sum_j u_j(x)$$

where sum is over **rays** passing through x, with

$$\arg u_j(x) = \textbf{optical length of ray path} = k s_j$$

$$|u_j(x)| = \textbf{amplitude determined by energy conservation},$$

but with multiplication of $u_j(x)$ by **coefficients** accounting for **reflection**, **refraction**, and **diffraction** events.

The concept/theory/idea of GO/GTD is that at **high frequency** these interaction events are **local** and so these coefficients can be computed by solving **canonical problems**, with simple geometries and incident fields.

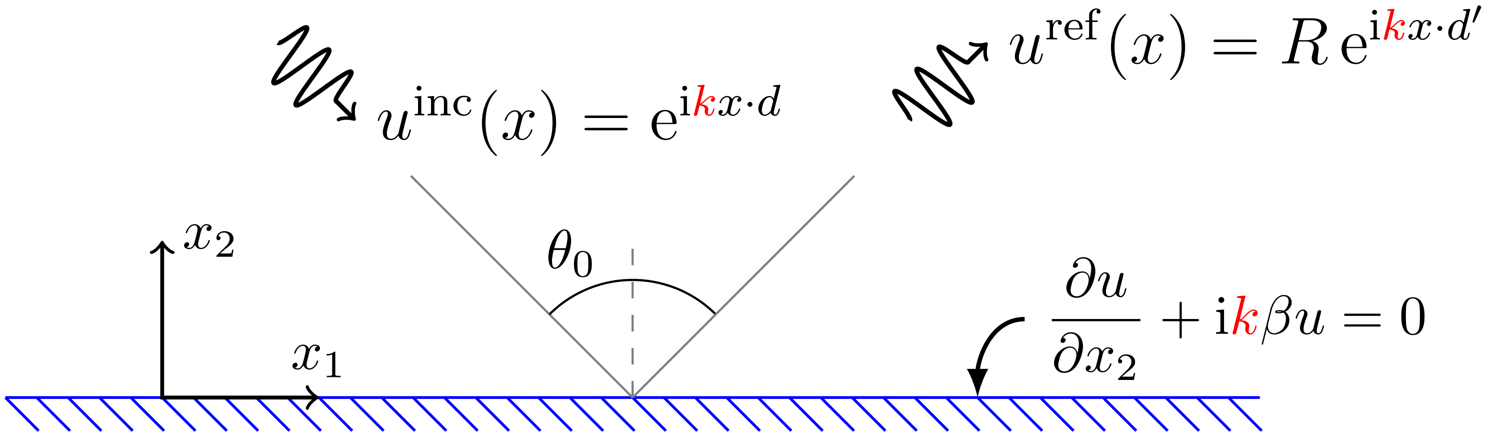
3. REFLECTION: CANONICAL PROBLEMS AND HIGH FREQUENCY GO APPROXIMATIONS[a]

---

[a] "Uber die Ausbreitung der Wellen in der drahtlosen Telegraphie", A. Sonmmerfeld, *Ann. Phys.* (1909)

A Canonical Reflection Problem

$$\Delta u + k^2 u = 0$$



$d = (d_1, d_2)$ is direction of **incident** plane wave,

$d' = (d_1, -d_2)$ is direction of **reflected** plane wave,

β is the **admittance** in the **impedance boundary condition**,

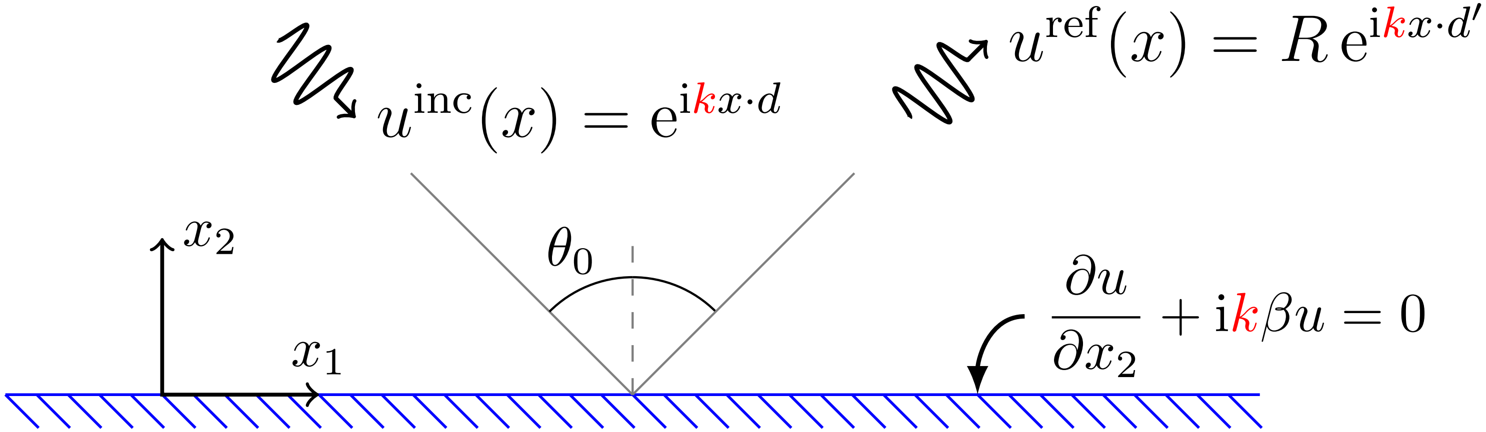
θ_0 is the **angle of incidence**,

R is the **reflection coefficient**.

.

A Canonical Reflection Problem

$$\Delta u + k^2 u = 0$$



The **total field** is

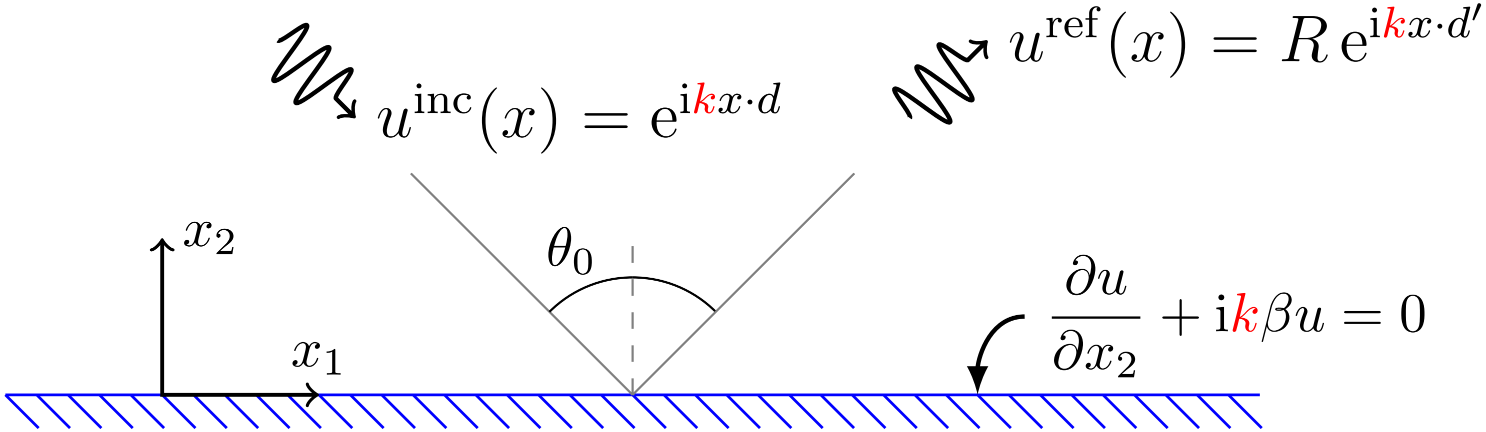
$$u(x) = u^{\mathrm{inc}}(x) + u^{\mathrm{ref}}(x) = \mathrm{e}^{\mathrm{i}kx\cdot d} + R\,\mathrm{e}^{\mathrm{i}kx\cdot d'}$$

where

$$R = \textbf{ reflection coefficient } = \frac{\cos\theta_0 - \beta}{\cos\theta_0 + \beta}.$$

A Canonical Reflection Problem

$$\Delta u + k^2 u = 0$$



$R =$ **reflection coefficient** $= \dfrac{\cos\theta_0 - \beta}{\cos\theta_0 + \beta}$,

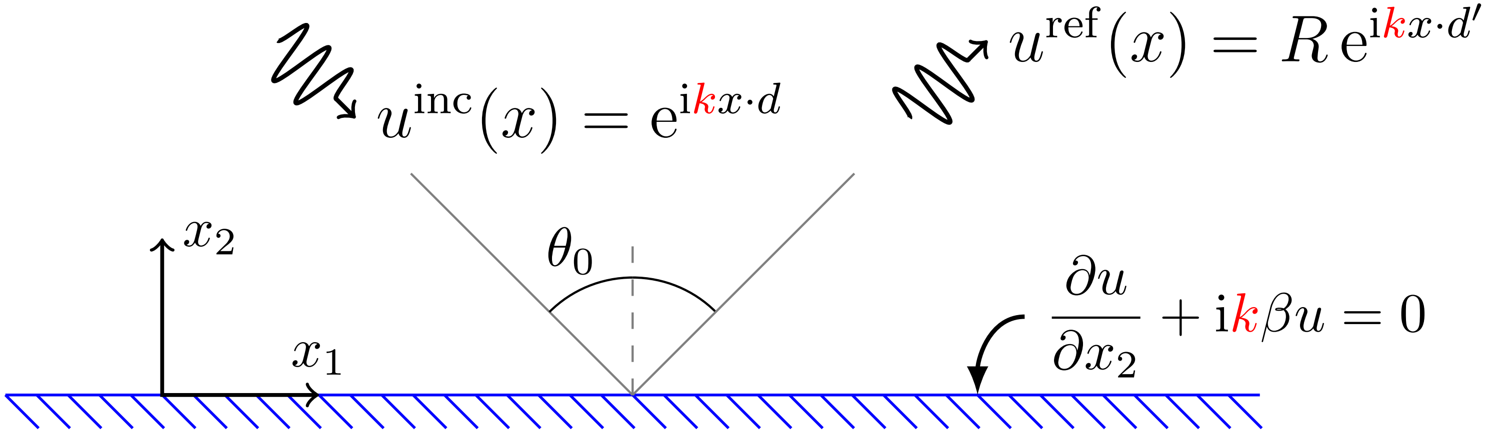
in particular

$R = 1$ if $\beta = 0$ (**sound hard/Neumann**),

$R = -1$ if $\beta \to \infty$ (**sound soft/Dirichlet**).

A Canonical Reflection Problem

$$\Delta u + k^2 u = 0$$



$$R = \textbf{ reflection coefficient } = \frac{\cos\theta_0 - \beta}{\cos\theta_0 + \beta}.$$

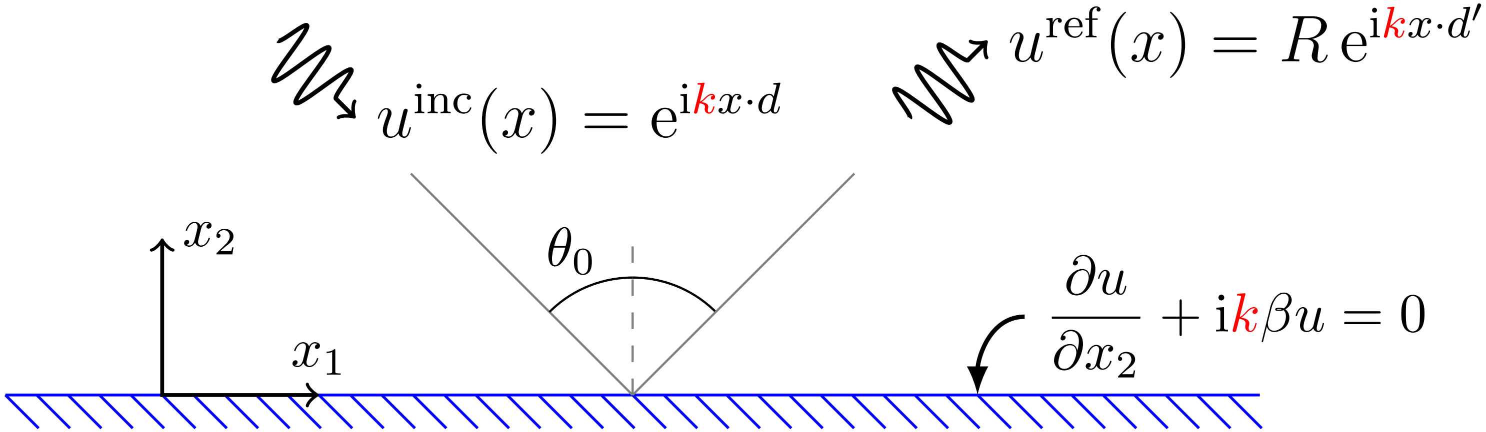
and note that

$$|R| \le 1 \quad \Leftrightarrow \quad \Re\,\beta \ge 0.$$

.

A Canonical Reflection Problem

$$\Delta u + k^2 u = 0$$



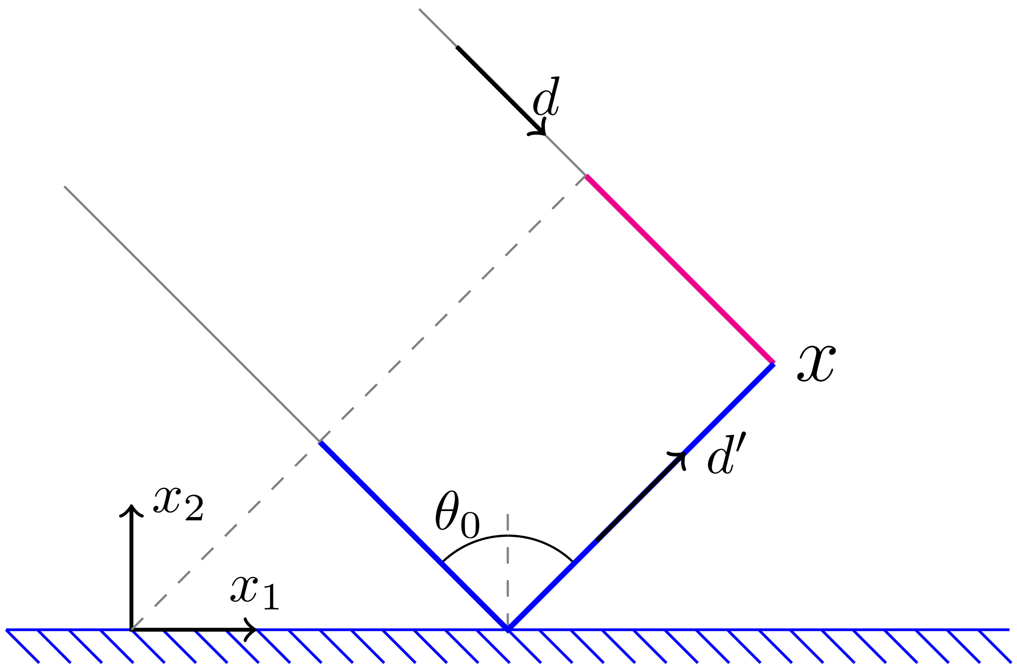
$$R = \textbf{ reflection coefficient } = \frac{\cos\theta_0 - \beta}{\cos\theta_0 + \beta},$$

and

$$\frac{\partial u}{\partial x_2} = (1-R)\frac{\partial u^{\mathrm{inc}}}{\partial x_2} = 2\frac{\partial u^{\mathrm{inc}}}{\partial x_2} \text{ if } \beta \to \infty \textbf{ (sound soft/Dirichlet)}.$$

.

The Ray Perspective

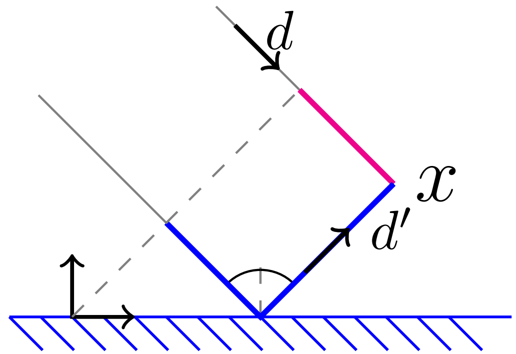


$$u(x) \;=\; e^{ik x\cdot d} + R\, e^{ik x\cdot d'} = \exp(ik\, s) + R\, \exp(ik\, s'),$$

s = distance along direct **magenta** ray,

s' = distance along reflected **blue** ray.

Geometrical Optics (and the Geometrical Theory of Diffraction)



$$u(x) = \sum_j u_j(x)$$

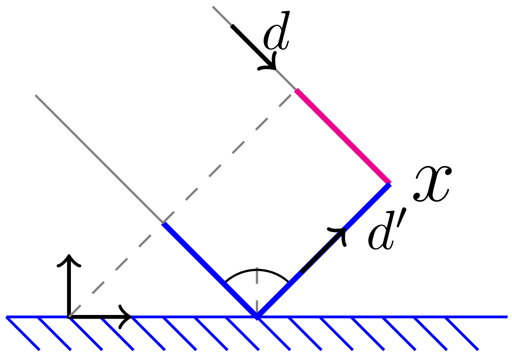
where sum is over rays passing through x, with

$$\arg u_j(x) \quad = \quad \textbf{optical length} \text{ of ray path } = ks_j$$

$$|u_j(x)| \quad = \quad \textbf{amplitude} \text{ determined by energy conservation,}$$

but with multiplication of $u_j(x)$ by **coefficients** accounting for **reflection**, **refraction**, and **diffraction** events, these events depending **only on the local geometry**.

Geometrical Optics (and the Geometrical Theory of Diffraction)



$$u(x) = \sum_j u_j(x) = \exp(\mathrm{i}k\, s) + R\,\exp(\mathrm{i}k\, s'),$$

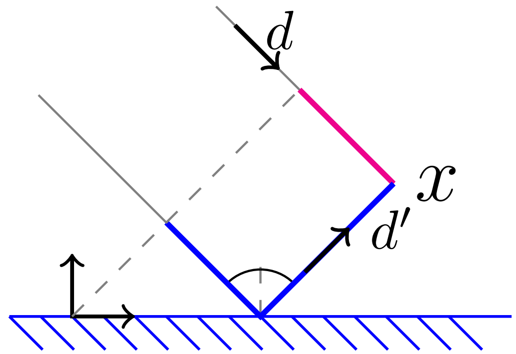
where sum is over rays passing through x, with

$$\arg u_j(x) \quad = \quad \textbf{optical length} \text{ of ray path } = ks_j$$

$$|u_j(x)| \quad = \quad \textbf{amplitude} \text{ determined by energy conservation,}$$

but with multiplication of $u_j(x)$ by **coefficients** accounting for **reflection**, **refraction**, and **diffraction** events, these events depending **only on the local geometry**.

Geometrical Optics (and the Geometrical Theory of Diffraction)



$$u(x) = \sum_j u_j(x) = \exp(\mathrm{i}k\, s) + R\, \exp(\mathrm{i}k\, s'),$$

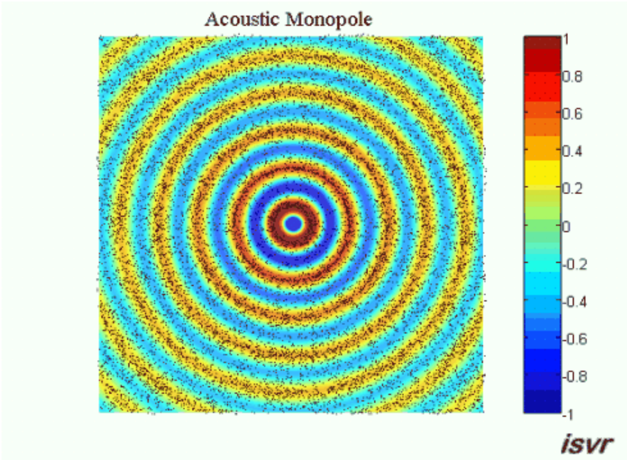
where sum is over rays passing through x, with

$\arg u_j(x)$ = **optical length** of ray path $= k s_j$

$|u_j(x)|$ = **amplitude** determined by energy conservation,

but with multiplication of $u_j(x)$ by **coefficients** accounting for **reflection**, **refraction**, and **diffraction** events, these events depending **only on the local geometry**. **In general only valid for k large.**

A First High Frequency (HF) Geometrical Optics Approximation



$\Re\, u^{\text{inc}}(x)$

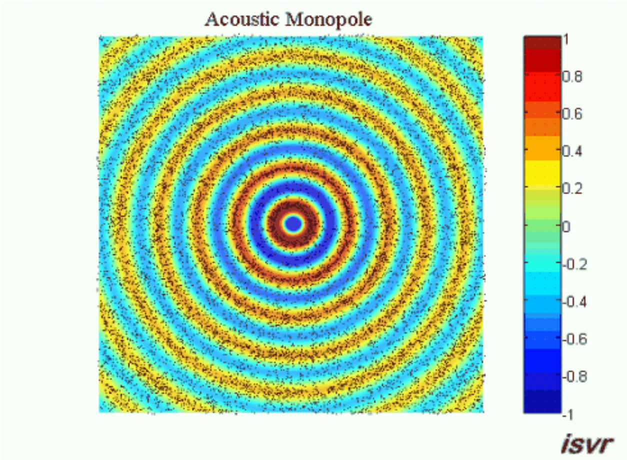
In this example the (2D) incident field is a cylindrical wave generated by a **monopole point source** at $y = (y_1, y_2)$,

$$u^{\text{inc}}(x) = \Phi(x, y) := \frac{\mathrm{i}}{4} H_0^{(1)}(k\mathcal{R})$$

where $\mathcal{R} = |x - y|$ and $H_0^{(1)}$ is **Hankel function** of 1st kind of order 0.

.

A First High Frequency (HF) Geometrical Optics Approximation



$\Re\, u^{\mathrm{inc}}(x)$

In this example the (2D) incident field is a cylindrical wave generated by a **monopole point source** at $y = (y_1, y_2)$,

$$u^{\mathrm{inc}}(x) = \Phi(x, y) := \frac{\mathrm{i}}{4} H_0^{(1)}(k\mathcal{R}).$$

$H_0^{(1)}(z)$ is **analytic** except for **log singularity** at 0 and

$$H_0^{(1)}(z) \sim \mathrm{const}\, \mathrm{e}^{\mathrm{i}z} z^{-1/2} \quad \text{as} \quad z \to \infty,$$

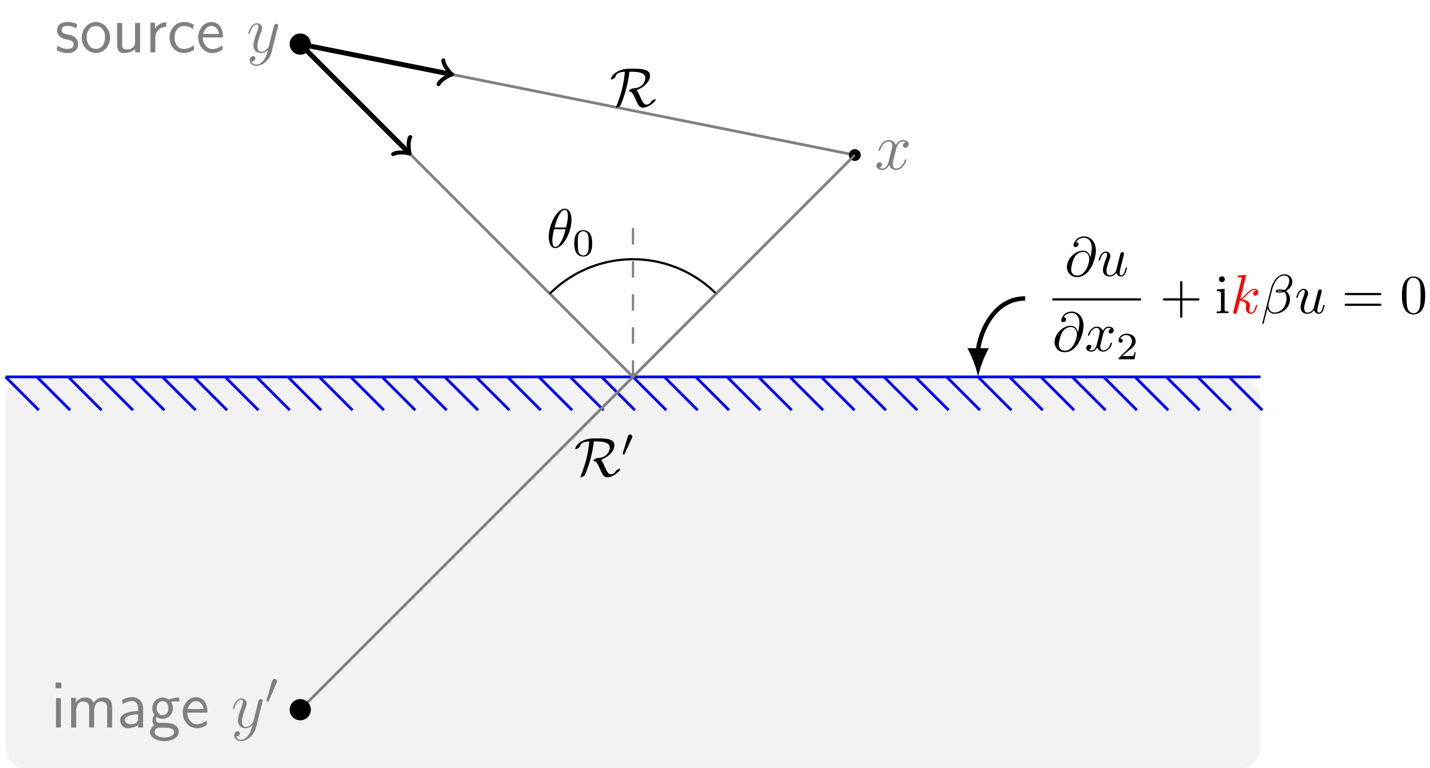
so

$$u^{\mathrm{inc}}(x) \sim \mathrm{const}\, \frac{\mathrm{e}^{\mathrm{i}k\mathcal{R}}}{\sqrt{k\mathcal{R}}} \quad \text{as} \quad |x| \to \infty$$

is **locally like a plane wave – key to GTD working**.

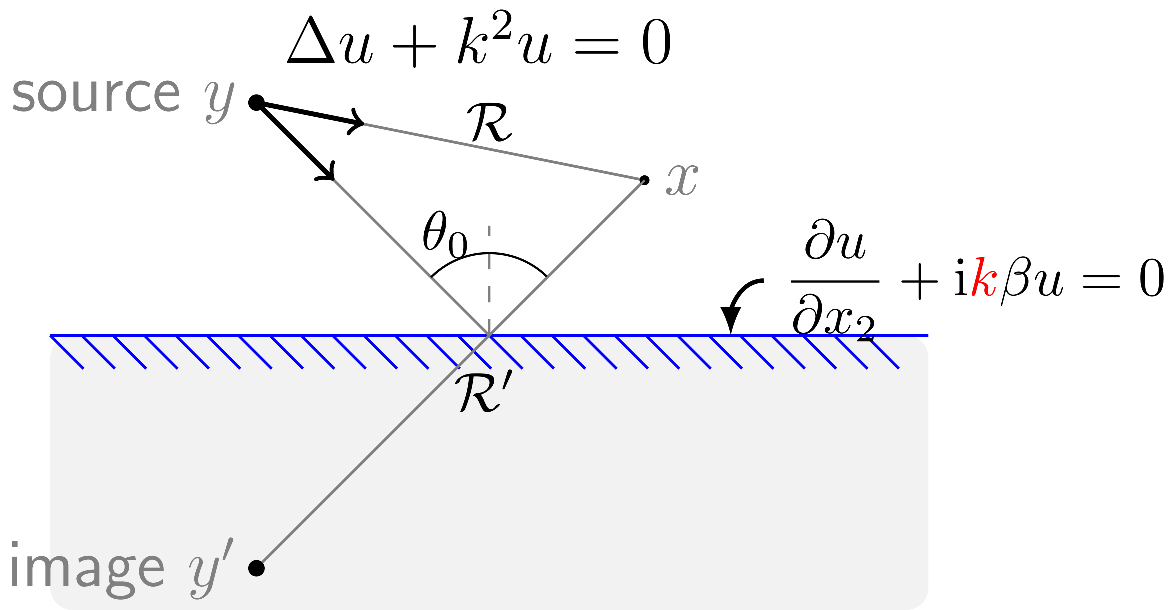
A First High Frequency (HF) Geometrical Optics Approximation

$$\Delta u + k^2 u = 0$$



$$u^{\mathrm{inc}}(x) = \Phi(x, y) := \frac{\mathrm{i}}{4} H_0^{(1)}(k\mathcal{R}).$$

A First High Frequency (HF) Geometrical Optics Approximation



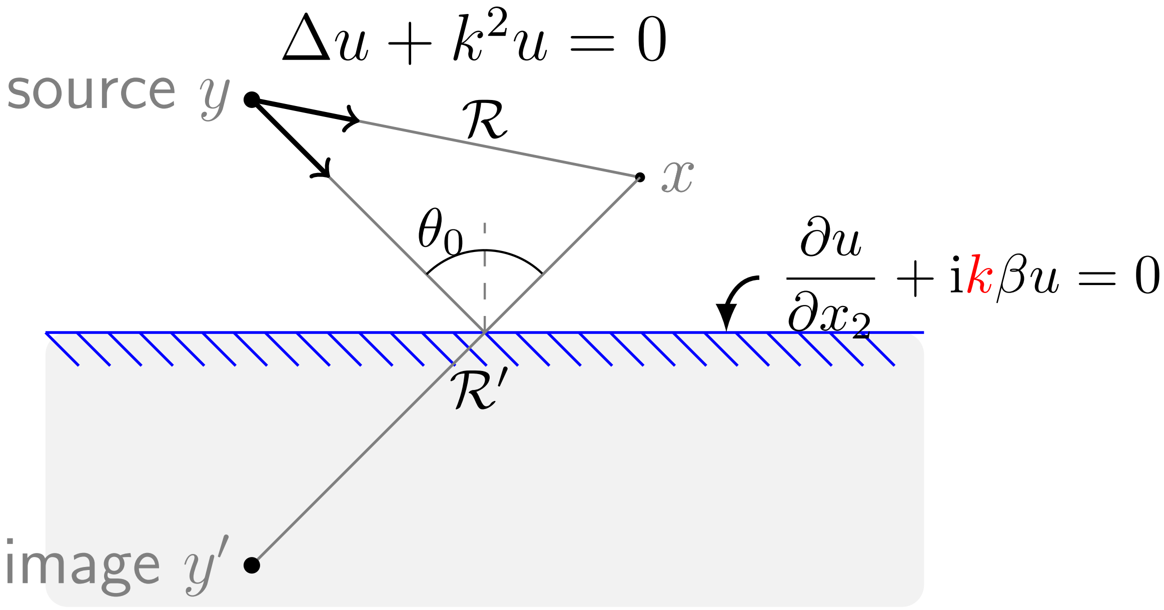
$$u(x) \approx \Phi(x,y) + R\,\Phi(x,y') = \frac{\mathrm{i}}{4} H_0^{(1)}(k\mathcal{R}) + R\,\frac{\mathrm{i}}{4} H_0^{(1)}(k\mathcal{R}'),$$

where

$$R = \textbf{ reflection coefficient } = \frac{\cos\theta_0 - \beta}{\cos\theta_0 + \beta}.$$

This is **accurate provided** $k\mathcal{R}' \gg 1$ and $k\mathcal{R}'|\beta + \cos\theta_0|^2 \gg 1$.

A First High Frequency (HF) Geometrical Optics Approximation

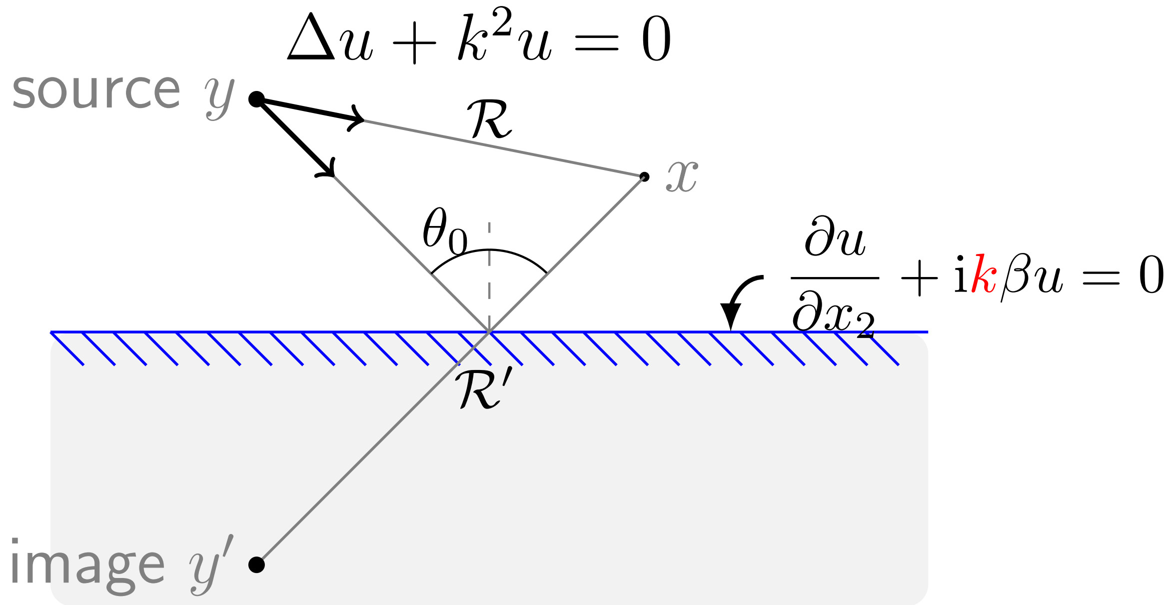


This HF Geometrical Optics approximation is **accurate provided** $k\mathcal{R}' \gg 1$ and $k\mathcal{R}'|\beta + \cos\theta_0|^2 \gg 1$.

Unfortunately in outdoor sound propagation often $|\beta| \ll 1$ and $\cos_0\theta \ll 1$ so this approximation poor.

.

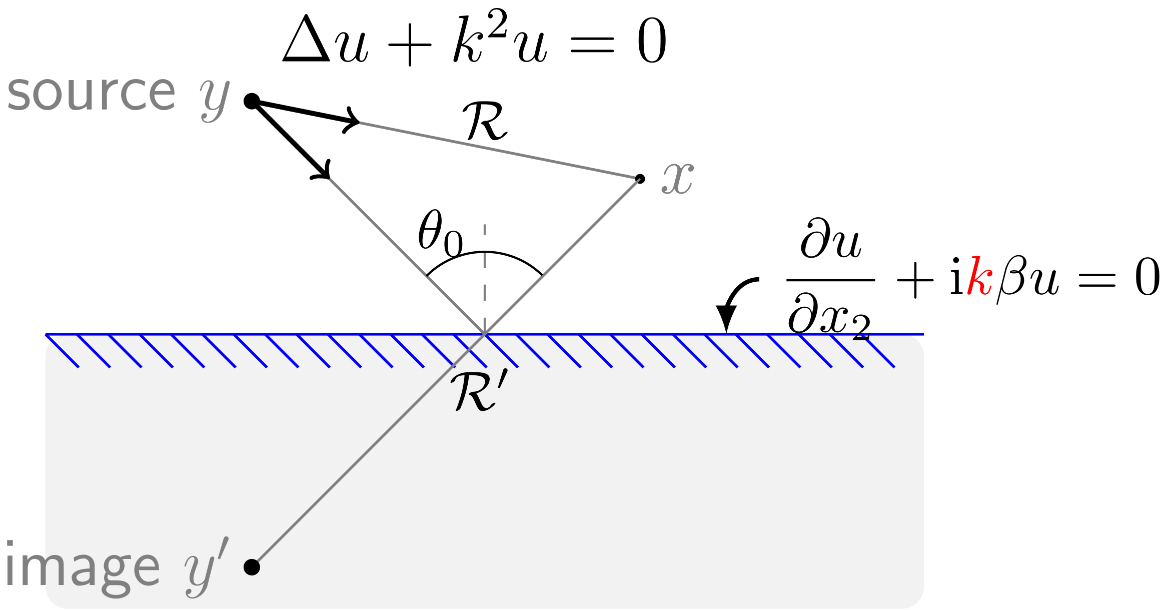
A First High Frequency (HF) Geometrical Optics Approximation



Exact solution as **highly oscillatory Fourier integral** and its **uniform asymptotic expansion** for $k\mathcal{R}' \gg 1$ via a **steepest descent path method modified for pole near saddle point** are given in C-W and Hothersall (1995a).

.

A First High Frequency (HF) Geometrical Optics Approximation

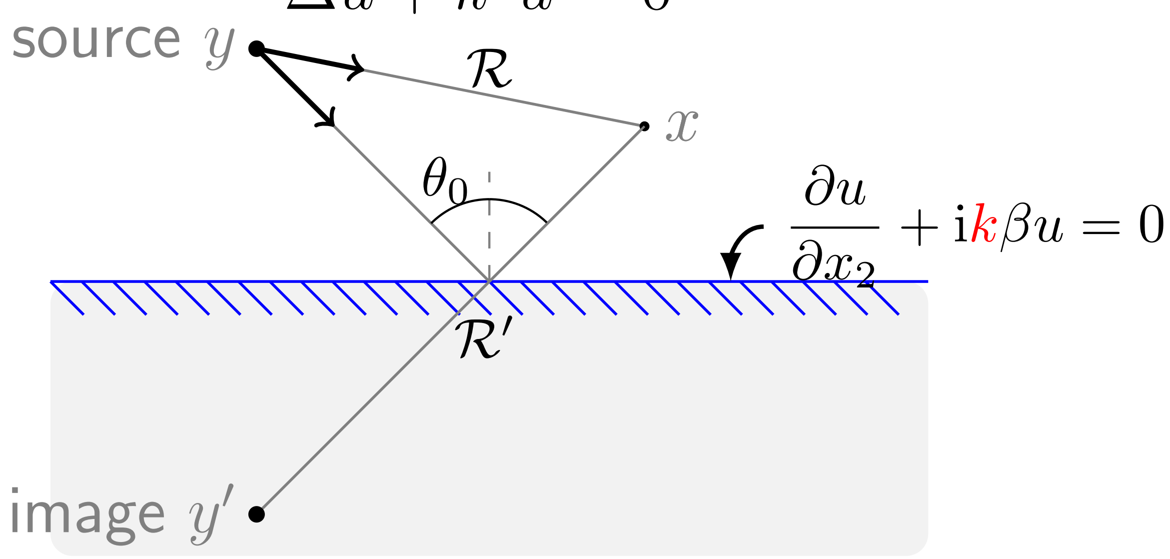


Exact solution and its **uniform asymptotic expansion** for $k\mathcal{R}' \gg 1$ via a **steepest descent path method modified for pole near saddle point** are given in C-W and Hothersall (1995a).[a]

[a] Extending "Die Sattelpunktsmethode in der Umgebung eines Pols. Mit Anwendungen auf die Wellenoptik und Akustik", H. Ott, *Annalen Physik*, (1943)!

A First High Frequency (HF) Geometrical Optics Approximation

$$\Delta u + k^2 u = 0$$



Efficient and accurate numerical method, a **numerical steepest descent path method**, given in C-W and Hothersall (1995b).

These are **case studies** relevant to Daan's first lecture yesterday!

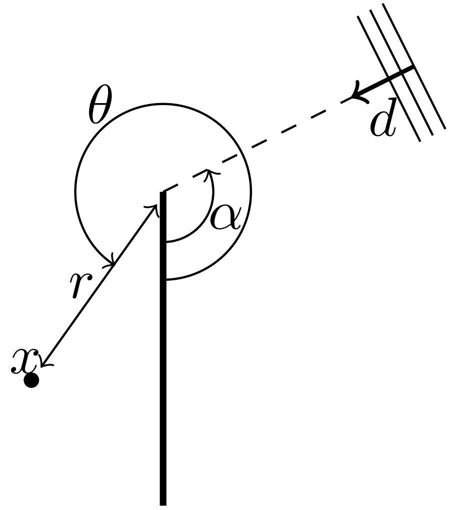
.

.

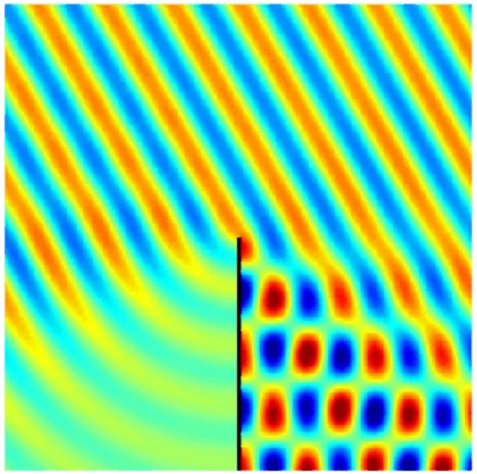
4. DIFFRACTION: CANONICAL PROBLEMS AND HIGH FREQUENCY GTD APPROXIMATIONS[a]

[a] "Goemetrical theory of diffraction", J.B. Keller, *JOSA* (1962), "Mathematische Theorie der Diffraction", A. Sommerfeld, *Math. Ann.*, (1896), "The Computation of Conical Diffraction Coefficients in High-Frequency Acoustic Wave Scattering", B.D. Bonner, I.G. Graham, and V.P. Smyshlyaev, *SINUM* (2005)

A Canonical Diffraction Problem



$$u^{\mathrm{inc}}(x) = \mathrm{e}^{\mathrm{i}kx\cdot d}$$



$\Re\, u(x)$

$$u(x) = E(r, \theta - \alpha) + E(r, \theta + \alpha)$$

where $E(r, \psi) = \exp(-\mathrm{i}kr\cos\psi)\, F(-\sqrt{2kr}\, \cos(\psi/2))$ and the **Fresnel integral**

$$F(t) \quad = \quad c_1 \int_t^\infty \mathrm{e}^{\mathrm{i}s^2}\, ds = c_2\, \mathrm{e}^{\mathrm{i}t^2} \int_{-\infty}^{\infty} \frac{\mathrm{e}^{-t^2u^2}}{1 + \mathrm{i}u^2}\, du, \qquad t > 0.$$

The Fresnel Integral - Example relevant to Daan's 1st Lecture

$$F(t) \quad = \quad c_1 \quad \underbrace{\int_t^\infty \mathrm{e}^{\mathrm{i}s^2}\,ds}_{\textbf{oscillatory integral}} \quad = c_2\,\mathrm{e}^{\mathrm{i}t^2}\underbrace{\int_{-\infty}^{\infty}\frac{\mathrm{e}^{-t^2u^2}}{1+\mathrm{i}u^2}\,du}_{\textbf{SDP integral}}\,.$$

Use **Watson's lemma** or **numerical method of steepest descent**!

.

The Fresnel Integral - Example relevant to Daan's 1st Lecture

$$F(t) \quad = \quad c_1 \quad \underbrace{\int_t^\infty e^{is^2}\,ds}_{\textbf{oscillatory integral}} \quad = c_2\, e^{it^2} \underbrace{\int_{-\infty}^{\infty} \frac{e^{-t^2u^2}}{1+iu^2}\,du}_{\textbf{SDP integral}}\,.$$

Use **Watson's lemma** or **numerical method of steepest descent**! Alternatively, based on contour integral arguments dating back to Turing (1945), Alazah, C-W, La Porte, *Numer Math* (2014) propose the **modified truncated midpoint rule**

$$F(t) \approx F_N(t) := \frac{1}{2} + \frac{i}{2}\tan\left(\pi t e^{i\pi/4}/h_N\right) + \frac{t}{\pi}\, e^{i(t^2+\pi/4)}\, h_N \sum_{k=1}^{N} \frac{e^{-s_k^2}}{t^2 + is_k^2}$$

where $s_k = (k - 1/2)\, h_N$ and $h_N = \sqrt{\pi/(N+1/2)}$, and show

$$\frac{|F(t) - F_N(t)|}{|F(t)|} < 11\, e^{-\pi N}, \quad t \in \mathbb{R}.$$

The Fresnel Integral - Example relevant to Daan's 1st Lecture

$$F(t) \quad = \quad c_1 \qquad \underbrace{\int_t^\infty \mathrm{e}^{\mathrm{i}s^2}\, ds}_{\textbf{oscillatory integral}} \qquad = c_2\, \mathrm{e}^{\mathrm{i}t^2} \underbrace{\int_{-\infty}^{\infty} \frac{\mathrm{e}^{-t^2u^2}}{1+\mathrm{i}u^2}\, du}_{\textbf{SDP integral}}\,.$$

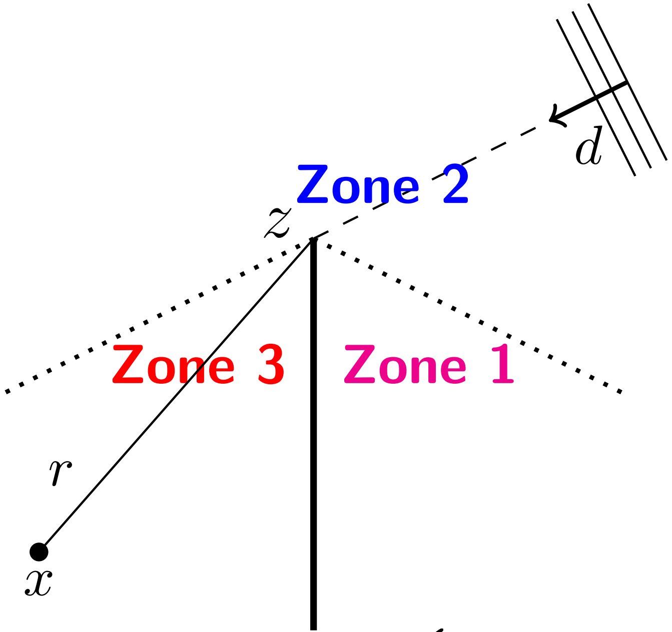
Alternatively, based on contour integral arguments dating back to Turing (1945)[a], Alazah, C-W, La Porte, *Numer Math* (2014) propose the **modified truncated midpoint rule**

$$F(t) \approx F_N(t) := \frac{1}{2} + \frac{\mathrm{i}}{2}\tan\left(\pi t \mathrm{e}^{\mathrm{i}\pi/4}/h_N\right) + \frac{t}{\pi}\, \mathrm{e}^{\mathrm{i}(t^2+\pi/4)}\, h_N \sum_{k=1}^{N} \frac{\mathrm{e}^{-s_k^2}}{t^2+\mathrm{i}s_k^2}.$$

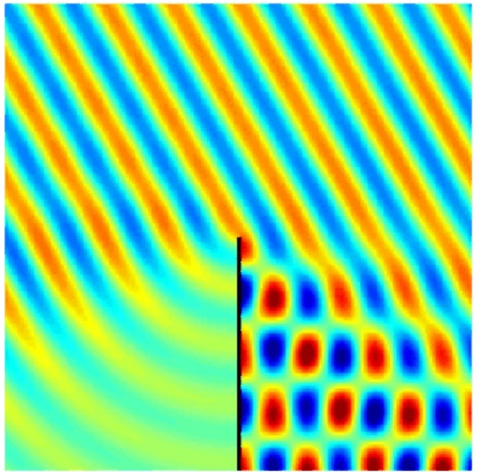
.

[a] "A method for the calculation of the zeta-function", A.M. Turing, *Proc. London Math. Soc.* (1945), and cf. Trefethen and Weideman, *SIAM Rev.* (2014)

The GTD Approach to this Knife-Edge Problem



$$u^{\mathrm{inc}}(x) = \mathrm{e}^{\mathrm{i}kx\cdot d}$$



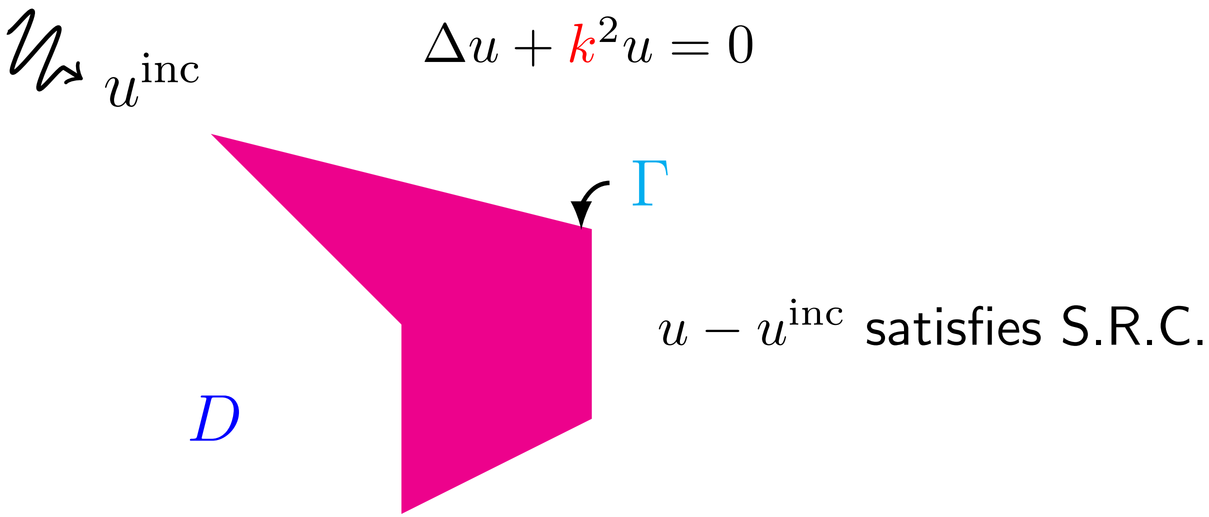
$\Re\, u(x)$

$$u(x) \approx \begin{cases} u^{\mathrm{inc}}(x) + u^{\mathrm{ref}}(x) + u^{\mathrm{d}}(x), & x \text{ in Zone 1} \\ u^{\mathrm{inc}}(x) + u^{\mathrm{d}}(x), & x \text{ in Zone 2} \\ u^{\mathrm{d}}(x), & x \text{ in Zone 3} \end{cases}$$

where $u^{\mathrm{d}}(x) = u^{\mathrm{inc}}(z)\,\mathcal{D}\,\dfrac{\mathrm{e}^{\mathrm{i}kr}}{\sqrt{kr}}$ and $\mathcal{D}$ is a **diffraction coefficient**.

5. THE HF KIRCHHOFF APPROXIMATION

Green's Representation Theorem and the Kirchhoff Approximation



Theorem

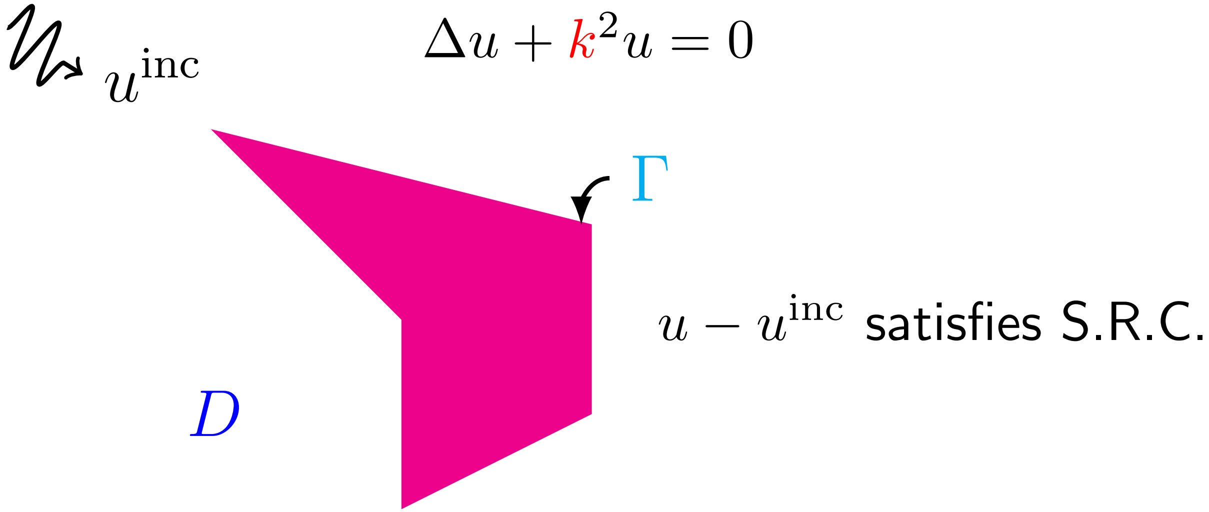
$$u(x) = u^{\text{inc}}(x) + \int_\Gamma \left(\frac{\partial u}{\partial n}(y)\Phi(x,y) - u(y)\frac{\partial \Phi(x,y)}{\partial n(y)} \right) ds(y), \quad x \in D.$$

Proof. Green's theorem – see Ivan's talk.

N.B. We only need the **Cauchy data** $u, \dfrac{\partial u}{\partial n}$ on Γ to compute u in D.

.

Green's Representation Theorem and the Kirchhoff Approximation



Theorem

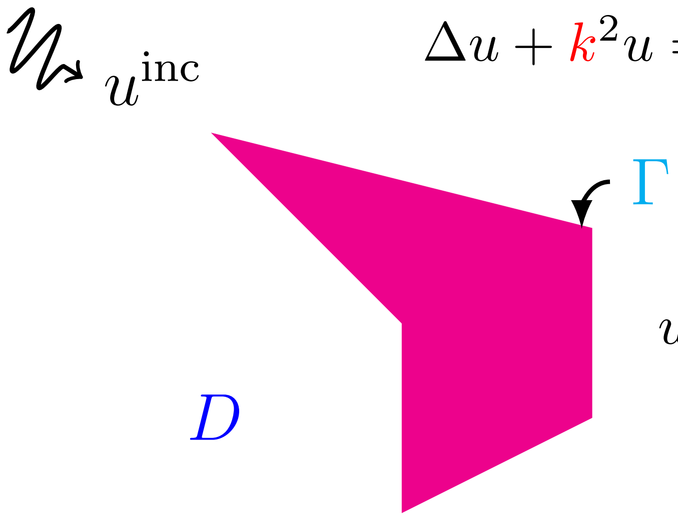
$$u(x) = u^{\text{inc}}(x) + \int_\Gamma \left(\frac{\partial u}{\partial n}(y)\Phi(x,y) - u(y)\frac{\partial \Phi(x,y)}{\partial n(y)} \right) ds(y), \quad x \in D.$$

Proof. Green's theorem – see Ivan's talk.

N.B. These **Cauchy data** $u, \dfrac{\partial u}{\partial n}$ can be obtained from B.C. + **boundary integral equation** on Γ.

Green's Representation Theorem and the Kirchhoff Approximation

$$\Delta u + k^2 u = 0$$



$u - u^{\mathrm{inc}}$ satisfies S.R.C.

Theorem

$$u(x) = u^{\mathrm{inc}}(x) + \int_\Gamma \left(\frac{\partial u}{\partial n}(y)\Phi(x,y) - u(y)\frac{\partial \Phi(x,y)}{\partial n(y)} \right) ds(y), \quad x \in D.$$

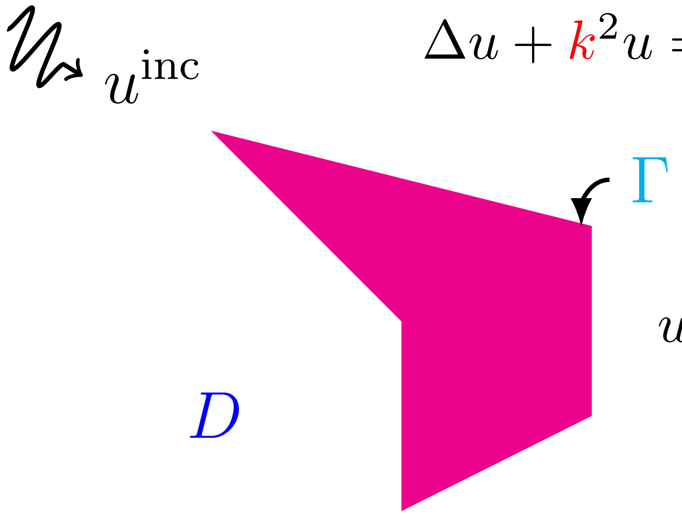
Proof. Green's theorem – see Ivan's talk.

N.B. If Γ is **convex** then **GO** can be used, e.g. if $u = 0$ on Γ then

$$\frac{\partial u}{\partial n} \approx \begin{cases} 2\frac{\partial u^{\mathrm{inc}}}{\partial n}, & \text{on illuminated part} \\ 0, & \text{on part of } \Gamma \text{ in shadow} \end{cases}$$

Green's Representation Theorem and the Kirchhoff Approximation

$$\Delta u + k^2 u = 0$$



$u - u^{\rm inc}$ satisfies S.R.C.

$$u(x) \approx u^{\rm K.O.}(x) := u^{\rm inc}(x) + \underbrace{2\int_{\Gamma_{\rm illum}} \frac{\partial u^{\rm inc}}{\partial n}(y)\Phi(x,y)\,ds(y)}_{\text{oscillatory integral - call Daan!}}, \quad x \in D.$$

N.B. If Γ is **convex** then **GO** can be used, e.g. if $u = 0$ on Γ then

$$\frac{\partial u}{\partial n} \approx \begin{cases} 2\frac{\partial u^{\rm inc}}{\partial n}, & \text{on illuminated part} \\ 0, & \text{on part of } \Gamma \text{ in shadow} \end{cases}$$

6. PREPARING FOR NA: QUANTIFYING NON-OSCILLATORARINESS!

Oscillatory and Non-Oscillatory Functions on $(0, \infty)$

Motivation. I want to factor **unknown oscillatory functions** into (maybe sums of) products of **known oscillatory functions** and **unknown non-oscillatory functions**.

To make a theory of this I need a **definition**.

.

Oscillatory and Non-Oscillatory Functions on $(0,\infty)$

Motivation. I want to factor **unknown oscillatory functions** into (maybe sums of) products of **known oscillatory functions** and **unknown non-oscillatory functions**.

Definition. Call $F \in C^\infty(0,\infty)$ **non-oscillatory** if, for some $p_0 > -1$ and $p_\infty < 0$, it holds for $n = 0, 1, \ldots$ that

$$F^{(n)}(t) = \begin{cases} O(t^{p_0 - n}), & t \to 0, \\ O(t^{p_\infty - n}), & t \to \infty. \end{cases}$$

.

Oscillatory and Non-Oscillatory Functions on $(0, \infty)$

Definition. Call $F \in C^\infty(0,\infty)$ **non-oscillatory** if, for some $p_0 > -1$ and $p_\infty < 0$, it holds for $n = 0, 1, ...$ that

$$F^{(n)}(t) = \begin{cases} O(t^{p_0 - n}), & t \to 0, \\ O(t^{p_\infty - n}), & t \to \infty. \end{cases}$$

Are these examples??

(i) $F(t) = t^{-1/2}$

(ii) $F(t) = t^{-1/2}\mathrm{e}^{\mathrm{i}t}$

(iii) $F(t) = H_0^{(1)}(t)$

(iv) $F(t) = \mathrm{e}^{-\mathrm{i}t} H_0^{(1)}(t)$

.

Oscillatory and Non-Oscillatory Functions on $(0, \infty)$

Definition. Call $F \in C^\infty(0,\infty)$ **non-oscillatory** if, for some $p_0 > -1$ and $p_\infty < 0$, it holds for $n = 0, 1, \ldots$ that

$$F^{(n)}(t) = \begin{cases} O(t^{p_0 - n}), & t \to 0, \\ O(t^{p_\infty - n}), & t \to \infty. \end{cases}$$

Are these examples??

(i) $F(t) = t^{-1/2}$ **Yes,** with $p_0 = p_\infty = -1/2$.

(ii) $F(t) = t^{-1/2} \mathrm{e}^{\mathrm{i}t}$ **No,** $F^{(n)}(t) \sim \mathrm{i}^n t^{-1/2} \mathrm{e}^{\mathrm{i}t}$ as $t \to \infty$.

(iii) $F(t) = H_0^{(1)}(t)$ **No,** ditto.

(iv) $F(t) = \mathrm{e}^{-\mathrm{i}t} H_0^{(1)}(t)$ **Yes,** with any $-1 < p_0 < 0$ and $p_\infty = -1/2$.

.

Oscillatory and Non-Oscillatory Functions on $(0, \infty)$

Definition. Call $F \in C^\infty(0,\infty)$ **non-oscillatory** if, for some $p_0 > -1$ and $p_\infty < 0$, it holds for $n = 0, 1, \ldots$ that

$$F^{(n)}(t) = \begin{cases} O(t^{p_0 - n}), & t \to 0, \\ O(t^{p_\infty - n}), & t \to \infty. \end{cases}$$

Remark. Non-oscillatory F with $p_\infty < -1$, so $F \in L^1(0,\infty)$, are easy to integrate with `quadgk`.

Compare

$$F(t) = \frac{H_0^{(1)}(t)}{(1+t)^{3/4}} \quad \text{with} \quad F(t) = \frac{\mathrm{e}^{-\mathrm{i}t} H_0^{(1)}(t)}{(1+t)^{3/4}}.$$

Matlab demo ...

Oscillatory and Non-Oscillatory Functions on $(0,\infty)$

Definition. Call $F \in C^\infty(0,\infty)$ **non-oscillatory** if, for some $p_0 > -1$ and $p_\infty < 0$, it holds for $n = 0, 1, \ldots$ that

$$F^{(n)}(t) = \begin{cases} O(t^{p_0 - n}), & t \to 0, \\ O(t^{p_\infty - n}), & t \to \infty. \end{cases}$$

Definition. Call $F(z)$ **strongly non-oscillatory** if it is **analytic** in $\Re z > 0$ and, for some $p_0 > -1$, $p_\infty < 0$, and $C > 0$, it holds for $\Re z > 0$ that

$$|F(z)| \le \begin{cases} C|z|^{p_0}, & |z| < 1, \\ C|z|^{p_\infty}, & |z| \ge 1. \end{cases}$$

Theorem. If F is strongly non-oscillatory then it is non-oscillatory, with the same values of p_0 and p_∞.

Oscillatory and Non-Oscillatory Functions on $(0, \infty)$

Definition. Call $F \in C^\infty(0, \infty)$ **non-oscillatory** if, for some $p_0 > -1$ and $p_\infty < 0$, it holds for $n = 0, 1, \ldots$ that

$$F^{(n)}(t) = \begin{cases} O(t^{p_0 - n}), & t \to 0, \\ O(t^{p_\infty - n}), & t \to \infty. \end{cases}$$

Definition. Call $F(z)$ **strongly non-oscillatory** if it is **analytic** in $\Re z > 0$ and, for some $p_0 > -1$, $p_\infty < 0$, and $C > 0$, it holds for $\Re z > 0$ that

$$|F(z)| \leq \begin{cases} C|z|^{p_0}, & |z| < 1, \\ C|z|^{p_\infty}, & |z| \geq 1. \end{cases}$$

Theorem. If F is strongly non-oscillatory then it is non-oscillatory, with the same values of p_0 and p_∞. **Example.** $F(z) = \mathrm{e}^{-\mathrm{i}z} H_0^{(1)}(z)$.

Recap

1. **Wave Equation and Helmholtz Equation**
2. **Basic Concept** of high frequency asymptotic approximations of **GO** and **GTD**
3. **Reflection** - canonical problems and high frequency GO approximations
4. **Diffraction** - canonical problems and high frequency GTD approximations
5. **The HF Kirchhoff Approximation**
6. **Preparing for NA: Quantifying Non-Oscillatorariness!**

Tomorrow: use this knowledge to design Galerkin methods for boundary integral equations that combine hp-approximation with new oscillatory basis functions to solve (at least some classes of) HF scattering problems with $O(1)$ cost as $k \to \infty$.

References

- "Numerical-asymptotic boundary integral methods in high-frequency acoustic scattering", C-W, I.G. Graham, S Langdon, & E.A. Spence, *Acta Numerica* (2012).
- "A uniformly valid far field asymptotic expansion for the Green function for two-dimensional propagation above a homogeneous impedance plane", C-W & D.C. Hothersall, *J. Sound Vib.* (1995).
- "Efficient calculation of the Green function for acoustic propagation above a homogeneous impedance plane", C-W & D.C. Hothersall, *J. Sound Vib.* (1995).
- "Computing Fresnel integrals with modified trapezium rules", M. Alazah, C-W, & S. La Porte, *Numer. Math.* (2014).