High Frequency Solution Behaviour in Wave Scattering Problems

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Context

My talks apply (particularly) to acoustic waves — my background is in outdoor noise propagation and noise barriers.

My talks concern new numerical-asymptotic methods for high frequency wave scattering, that combine numerical analysis ideas and tools with high frequency asymptotics, see


This talk is the HF asymptotics – numerical methods are talk two!
Context

My talks apply (particularly) to **acoustic waves** — my background is in outdoor noise propagation and noise barriers.

My talks concern new **numerical-asymptotic methods** for **high frequency** wave scattering, that combine **numerical analysis ideas and tools** with **high frequency asymptotics**, see

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**Motivation.** Want to factor unknown oscillatory functions into sums of products of **known oscillatory functions** and **unknown non-oscillatory functions**.
Overview

1. Wave Equation and Helmholtz Equation
2. Basic Concept of high frequency asymptotic approximations of GO and GTD
3. Reflection - canonical problems and high frequency GO approximations
4. Diffraction - canonical problems and high frequency GTD approximations
5. The HF Kirchhoff Approximation
6. Preparing for NA: Quantifying Non-Oscillatoriness!
1. WAVE EQUATION AND HELMHOLTZ EQUATION
The Wave Equation and Helmholtz Equation

$$\Delta U = \frac{1}{c^2} \frac{\partial^2 U}{\partial t^2} \left( \Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2} \right).$$

If time-dependence is time harmonic, i.e., where \( x = (x_1, x_2, x_3), \)

$$U(x, t) = A(x) \cos(\phi(x) - \omega t),$$

for some \( \omega = 2\pi f > 0, \) with \( f = \) frequency, then

$$U(x, t) = \Re \left( u(x)e^{-i\omega t} \right)$$

where \( u(x) = A(x)\exp(i\phi(x)) \) satisfies the Helmholtz equation

$$\Delta u + k^2 u = 0,$$

with \( k = \omega/c \) the wavenumber.
If time-dependence is **time harmonic** then

\[ U(x, t) = \Re \left( u(x)e^{-i\omega t} \right) \]

for some \( \omega = 2\pi f > 0 \), with \( f = \text{frequency} \), where \( u \) satisfies the Helmholtz equation

\[ \Delta u + k^2 u = 0, \]

with \( k = \omega/c \) the **wavenumber**. E.g. if

\[ u(x) = e^{ikx \cdot d}, \]

for some **unit vector** \( d \), then

\[ U(x, t) = \Re \left( u(x)e^{-i\omega t} \right) = \cos(kx \cdot d - \omega t) \]

is a **plane wave** travelling in direction \( d \) with **wavelength**

\[ \lambda = \frac{2\pi}{k} = \frac{c}{f}. \]
2. GO AND GTD: THE BASIC CONCEPT
Geometrical Optics/Geometrical Theory of Diffraction (GTD)

source

\[ u(x) = \sum_j u_j(x) \]

where sum is over \textbf{rays} passing through \( x \), with

\[
\arg u_j(x) = \text{optical length} \text{ of ray path} = ks_j
\]

\[
|u_j(x)| = \text{amplitude} \text{ determined by energy conservation,}
\]

but with multiplication of \( u_j(x) \) by \textbf{coefficients} accounting for \textbf{reflection}, \textbf{refraction}, and \textbf{diffraction} events.
Geometrical Optics/Geometrical Theory of Diffraction (GTD)

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\[ u(x) = \sum_{j} u_j(x) \]

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but with multiplication of \( u_j(x) \) by coefficients accounting for reflection, refraction, and diffraction events.

The concept/theory/idea of GO/GTD is that at high frequency these interaction events are local and so these coefficients can be computed by solving canonical problems, with simple geometries and incident fields.
3. REFLECTION: CANONICAL PROBLEMS AND HIGH FREQUENCY GO APPROXIMATIONS\textsuperscript{a}

\textsuperscript{a}``Über die Ausbreitung der Wellen in der drahtlosen Telegraphie'', A. Sommerfeld, \textit{Ann. Phys.} (1909)
A Canonical Reflection Problem

\[ \Delta u + k^2 u = 0 \]

\[ u^{\text{inc}}(x) = e^{ikx \cdot d} \]

\[ u^{\text{ref}}(x) = Re^{ikx \cdot d'} \]

\[ \frac{\partial u}{\partial x_2} + ik\beta u = 0 \]

d = \(d_1, d_2\) is direction of **incident** plane wave,
d' = \(d_1, -d_2\) is direction of **reflected** plane wave,
\(\beta\) is the **admittance** in the **impedance boundary condition**,
\(\theta_0\) is the **angle of incidence**,
\(R\) is the **reflection coefficient**.
A Canonical Reflection Problem

\[ \Delta u + k^2 u = 0 \]

\[ u^{\text{inc}}(x) = e^{ik x \cdot d} \]

\[ u^{\text{ref}}(x) = R e^{ik x \cdot d'} \]

The total field is

\[ u(x) = u^{\text{inc}}(x) + u^{\text{ref}}(x) = e^{ik x \cdot d} + R e^{ik x \cdot d'} \]

where

\[ R = \text{reflection coefficient} = \frac{\cos \theta_0 - \beta}{\cos \theta_0 + \beta}. \]
A Canonical Reflection Problem

\[
\Delta u + k^2 u = 0
\]

\[
\begin{align*}
\text{inc} \ (x) &= e^{i k x \cdot d} \\
\text{ref} \ (x) &= R e^{i k x \cdot d'}
\end{align*}
\]

\[
R = \text{reflection coefficient} = \frac{\cos \theta_0 - \beta}{\cos \theta_0 + \beta},
\]

in particular

\[
R = 1 \text{ if } \beta = 0 \text{ (sound hard/Neumann),}
\]

\[
R = -1 \text{ if } \beta \to \infty \text{ (sound soft/Dirichlet).}
\]
A Canonical Reflection Problem

\[ \Delta u + k^2 u = 0 \]

\[ u^{\text{inc}}(x) = e^{i k x \cdot d} \]

\[ u^{\text{ref}}(x) = R e^{i k x \cdot d'} \]

\[ \frac{\partial u}{\partial x_2} + i k \beta u = 0 \]

\[ R = \text{reflection coefficient} = \frac{\cos \theta_0 - \beta}{\cos \theta_0 + \beta}. \]

and note that

\[ |R| \leq 1 \iff \Re \beta \geq 0. \]
A Canonical Reflection Problem

\[ \Delta u + k^2 u = 0 \]

\[ u^{\text{inc}}(x) = e^{ikx \cdot d} \]

\[ u^{\text{ref}}(x) = R e^{ikx \cdot d'} \]

\[ R = \text{reflection coefficient} = \frac{\cos \theta_0 - \beta}{\cos \theta_0 + \beta}, \]

and

\[ \frac{\partial u}{\partial x_2} = (1 - R) \frac{\partial u^{\text{inc}}}{\partial x_2} = 2 \frac{\partial u^{\text{inc}}}{\partial x_2} \text{ if } \beta \to \infty \text{ (sound soft/Dirichlet)}. \]
The Ray Perspective

\[ u(x) = e^{i k x \cdot d} + R e^{i k x \cdot d'} = \exp(i k s) + R \exp(i k s'), \]

\[ s = \text{distance along direct magenta ray}, \]
\[ s' = \text{distance along reflected blue ray}. \]
Geometrical Optics (and the Geometrical Theory of Diffraction)

\[ u(x) = \sum_j u_j(x) \]

where sum is over rays passing through \( x \), with

\[ \text{arg } u_j(x) = \text{optical length of ray path } = k s_j \]

\[ |u_j(x)| = \text{amplitude determined by energy conservation,} \]

but with multiplication of \( u_j(x) \) by coefficients accounting for \text{reflection, refraction, and diffraction} events, these events depending only on the local geometry.
Geometrical Optics (and the Geometrical Theory of Diffraction)

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but with multiplication of \( u_j(x) \) by \textit{coefficients} accounting for \textit{reflection}, \textit{refraction}, and \textit{diffraction} events, these events depending only on the local geometry. \textbf{In general only valid for} \( k \) \textbf{large}.
In this example the (2D) incident field is a cylindrical wave generated by a monopole point source at \( y = (y_1, y_2) \),

\[
\Phi(x, y) := \frac{i}{4} H_0^{(1)}(kR)
\]

where \( R = |x - y| \) and \( H_0^{(1)} \) is Hankel function of 1st kind of order 0.
In this example the (2D) incident field is a cylindrical wave generated by a monopole point source at \( y = (y_1, y_2) \),

\[
\mathcal{R} u^{\text{inc}}(x) = \Phi(x, y) := \frac{i}{4} H_0^{(1)}(k \mathcal{R}).
\]

\( H_0^{(1)}(z) \) is analytic except for log singularity at 0 and

\[
H_0^{(1)}(z) \sim \text{const } e^{iz} z^{-1/2} \quad \text{as} \quad z \to \infty,
\]

so

\[
u^{\text{inc}}(x) \sim \text{const } \frac{e^{i k \mathcal{R}}}{\sqrt{k \mathcal{R}}} \quad \text{as} \quad |x| \to \infty
\]
is locally like a plane wave – key to GTD working.
A First High Frequency (HF) Geometrical Optics Approximation

\[ \Delta u + k^2 u = 0 \]

source \( y \)

image \( y' \)

\[ \frac{\partial u}{\partial x_2} + ik\beta u = 0 \]

\[ u^{\text{inc}}(x) = \Phi(x, y) := \frac{i}{4} H_0^{(1)}(k|R|). \]
A First High Frequency (HF) Geometrical Optics Approximation

\[ \Delta u + k^2 u = 0 \]

\[ \frac{\partial u}{\partial x_2} + i k \beta u = 0 \]

\[ u(x) \approx \Phi(x, y) + R \Phi(x, y') = \frac{i}{4} H_0^{(1)}(kR) + R \frac{i}{4} H_0^{(1)}(kR') , \]

where

\[ R = \text{reflection coefficient} = \frac{\cos \theta_0 - \beta}{\cos \theta_0 + \beta} . \]

This is accurate provided \( kR' \gg 1 \) and \( kR' |\beta + \cos \theta_0|^2 \gg 1 \).
A First High Frequency (HF) Geometrical Optics Approximation

\[ \Delta u + k^2 u = 0 \]

This HF Geometrical Optics approximation is **accurate provided** $kR' \gg 1$ and $kR' |\beta + \cos \theta_0|^2 \gg 1$.

Unfortunately in outdoor sound propagation often $|\beta| \ll 1$ and $\cos_0 \theta \ll 1$ so this approximation poor.
A First High Frequency (HF) Geometrical Optics Approximation

\[ \Delta u + k^2 u = 0 \]

source \( y \)

image \( y' \)

\[ \frac{\partial u}{\partial x_2} + i k \beta u = 0 \]

Exact solution as highly oscillatory Fourier integral and its uniform asymptotic expansion for \( kR' \gg 1 \) via a steepest descent path method modified for pole near saddle point are given in C-W and Hothersall (1995a).
A First High Frequency (HF) Geometrical Optics Approximation

\[ \Delta u + k^2 u = 0 \]

Source \( y \) \hspace{2cm} Image \( y' \)

Exact solution and its \textbf{uniform asymptotic expansion} for \( kR' \gg 1 \) via a \textbf{steepest descent path method modified for pole near saddle point} are given in C-W and Hothersall (1995a).\(^a\)

\(^a\)Extending “Die Sattelpunktsmethode in der Umgebung eines Pols. Mit Anwendungen auf die Wellenoptik und Akustik”, H. Ott, Annalen Physik, (1943)!
A First High Frequency (HF) Geometrical Optics Approximation

\[ \Delta u + k^2 u = 0 \]

Efficient and accurate numerical method, a numerical steepest descent path method, given in C-W and Hothersall (1995b). These are case studies relevant to Daan’s first lecture yesterday!
4. DIFFRACTION: CANONICAL PROBLEMS AND HIGH FREQUENCY GTD APPROXIMATIONS

A Canonical Diffraction Problem

\[ u^{\text{inc}}(x) = e^{i k x \cdot d} \]

\[ u(x) = E(r, \theta - \alpha) + E(r, \theta + \alpha) \]

where \( E(r, \psi) = \exp(-i k r \cos \psi) F(-\sqrt{2 k r} \cos(\psi/2)) \) and the Fresnel integral

\[ F(t) = c_1 \int_t^\infty e^{is^2} ds = c_2 e^{it^2} \int_{-\infty}^\infty \frac{e^{-t^2u^2}}{1 + iu^2} du, \quad t > 0. \]
The Fresnel Integral - Example relevant to Daan’s 1st Lecture

\[ F(t) = c_1 \int_t^\infty e^{is^2} \, ds = c_2 e^{it^2} \int_{-\infty}^\infty \frac{e^{-t^2u^2}}{1 + iu^2} \, du. \]

oscillatory integral \hspace{1cm} SDP integral

Use **Watson’s lemma** or **numerical method of steepest descent**!
The Fresnel Integral - Example relevant to Daan’s 1st Lecture

\[ F(t) = c_1 \int_t^\infty e^{is^2} \, ds = c_2 \int_{-\infty}^\infty \frac{e^{-t^2u^2}}{1 + iu^2} \, du. \]

oscillatory integral  \hspace{1cm}  SDP integral

Use Watson’s lemma or numerical method of steepest descent!
Alternatively, based on contour integral arguments dating back to Turing (1945), Alazah, C-W, La Porte, *Numer Math* (2014) propose the modified truncated midpoint rule

\[ F(t) \approx F_N(t) := \frac{1}{2} + \frac{i}{2} \tan \left( \pi t e^{i\pi/4}/h_N \right) + \frac{t}{\pi} e^{i(t^2 + \pi/4)} h_N \sum_{k=1}^N \frac{e^{-s_k^2}}{t^2 + i s_k^2} \]

where \( s_k = (k - 1/2) h_N \) and \( h_N = \sqrt{\pi/(N + 1/2)} \), and show

\[ \frac{|F(t) - F_N(t)|}{|F(t)|} < 11 e^{-\pi N}, \quad t \in \mathbb{R}. \]
The Fresnel Integral - Example relevant to Daan’s 1st Lecture

\[ F(t) = c_1 \int_{t}^{\infty} e^{is^2} \, ds \quad = \quad c_2 \int_{-\infty}^{\infty} \frac{e^{-t^2u^2}}{1 + iu^2} \, du. \]

\( \text{oscillatory integral} \quad \text{SDP integral} \)

Alternatively, based on contour integral arguments dating back to Turing (1945)\(^a\), Alazah, C-W, La Porte, *Numer Math* (2014) propose the modified truncated midpoint rule

\[ F(t) \approx F_N(t) := \frac{1}{2} + \frac{i}{2} \tan \left( \pi te^{i\pi/4}/h_N \right) + \frac{t}{\pi} e^{i(t^2+\pi/4)} h_N \sum_{k=1}^{N} \frac{e^{-s_k^2}}{t^2 + is_k^2}. \]

The GTD Approach to this Knife-Edge Problem

\[ u^{\text{inc}}(x) = e^{ikx \cdot d} \]

\[ u(x) \approx \begin{cases} 
  u^{\text{inc}}(x) + u^{\text{ref}}(x) + u^d(x), & x \text{ in Zone 1} \\
  u^{\text{inc}}(x) + u^d(x), & x \text{ in Zone 2} \\
  u^d(x), & x \text{ in Zone 3} 
\end{cases} \]

where

\[ u^d(x) = u^{\text{inc}}(z) \mathcal{D} \frac{e^{ikr}}{\sqrt{kr}} \]

and \( \mathcal{D} \) is a diffraction coefficient.
5. THE HF KIRCHHOFF APPROXIMATION
Green’s Representation Theorem and the Kirchhoff Approximation

\[ \nabla u^{inc} \quad \Delta u + k^2 u = 0 \]

\[ u - u^{inc} \text{ satisfies S.R.C.} \]

**Theorem**

\[ u(x) = u^{inc}(x) + \int_{\Gamma} \left( \frac{\partial u}{\partial n}(y) \Phi(x, y) - u(y) \frac{\partial \Phi(x, y)}{\partial n(y)} \right) \, ds(y), \quad x \in D. \]

**Proof.** Green’s theorem – see Ivan’s talk.

**N.B.** We only need the **Cauchy data** \( u, \frac{\partial u}{\partial n} \) on \( \Gamma \) to compute \( u \) in \( D \).
Green’s Representation Theorem and the Kirchhoff Approximation

\[ \nabla^2 u^{\text{inc}} + k^2 u = 0 \]

\[ u - u^{\text{inc}} \] satisfies S.R.C.

**Theorem**

\[ u(x) = u^{\text{inc}}(x) + \int_{\Gamma} \left( \frac{\partial u}{\partial n}(y) \Phi(x, y) - u(y) \frac{\partial \Phi(x, y)}{\partial n(y)} \right) \, ds(y), \quad x \in D. \]

**Proof.** Green’s theorem – see Ivan’s talk.

**N.B.** These **Cauchy data** \( u, \frac{\partial u}{\partial n} \) can be obtained from B.C. + boundary integral equation on \( \Gamma \).
Green’s Representation Theorem and the Kirchhoff Approximation

\[ \nabla \cdot u^{\text{inc}} + \Delta u + k^2 u = 0 \]

\[ u - u^{\text{inc}} \text{ satisfies S.R.C.} \]

**Theorem**

\[ u(x) = u^{\text{inc}}(x) + \int_{\Gamma} \left( \frac{\partial u}{\partial n}(y) \Phi(x, y) - u(y) \frac{\partial \Phi(x, y)}{\partial n(y)} \right) ds(y), \quad x \in D. \]

**Proof.** Green’s theorem – see Ivan’s talk.

**N.B.** If \( \Gamma \) is convex then GO can be used, e.g. if \( u = 0 \) on \( \Gamma \) then

\[ \frac{\partial u}{\partial n} \approx \begin{cases} 2 \frac{\partial u^{\text{inc}}}{\partial n}, & \text{on illuminated part} \\ 0, & \text{on part of } \Gamma \text{ in shadow} \end{cases} \]
Green’s Representation Theorem and the Kirchhoff Approximation

\[ \nabla^2 u^{\text{inc}} + k^2 u = 0 \]

\[ u - u^{\text{inc}} \text{ satisfies S.R.C.} \]

\[ u(x) \approx u^{\text{K.O.}}(x) := u^{\text{inc}}(x) + 2 \int_{\Gamma_{\text{illum}}} \frac{\partial u^{\text{inc}}}{\partial n}(y) \Phi(x,y) \, ds(y), \quad x \in D. \]

\[ \text{oscillatory integral - call Daan!} \]

**N.B.** If \( \Gamma \) is convex then GO can be used, e.g. if \( u = 0 \) on \( \Gamma \) then

\[ \frac{\partial u}{\partial n} \approx \begin{cases} 2 \frac{\partial u^{\text{inc}}}{\partial n}, & \text{on illuminated part} \\ 0, & \text{on part of } \Gamma \text{ in shadow} \end{cases} \]
6. PREPARING FOR NA: QUANTIFYING NON-OSCILLATORARINESS!
Oscillatory and Non-Oscillatory Functions on $(0, \infty)$

**Motivation.** I want to factor unknown oscillatory functions into (maybe sums of) products of known oscillatory functions and unknown non-oscillatory functions.

To make a theory of this I need a definition.
Oscillatory and Non-Oscillatory Functions on \((0, \infty)\)

**Motivation.** I want to factor **unknown oscillatory functions** into (maybe sums of) products of **known oscillatory functions** and **unknown non-oscillatory functions**.

**Definition.** Call \(F \in C^\infty(0, \infty)\) **non-oscillatory** if, for some \(p_0 > -1\) and \(p_\infty < 0\), it holds for \(n = 0, 1, \ldots\) that

\[
F^{(n)}(t) = \begin{cases} 
O(t^{p_0-n}), & t \to 0, \\
O(t^{p_\infty-n}), & t \to \infty.
\end{cases}
\]
Oscillatory and Non-Oscillatory Functions on $(0, \infty)$

**Definition.** Call $F \in C^\infty(0, \infty)$ non-oscillatory if, for some $p_0 > -1$ and $p_\infty < 0$, it holds for $n = 0, 1, \ldots$ that

$$F^{(n)}(t) = \begin{cases} O(t^{p_0-n}), & t \to 0, \\ O(t^{p_\infty-n}), & t \to \infty. \end{cases}$$

**Are these examples??**

(i) $F(t) = t^{-1/2}$  
(ii) $F(t) = t^{-1/2}e^{it}$  
(iii) $F(t) = H_0^{(1)}(t)$  
(iv) $F(t) = e^{-it} H_0^{(1)}(t)$
Oscillatory and Non-Oscillatory Functions on \((0, \infty)\)

**Definition.** Call \(F \in C^\infty(0, \infty)\) non-oscillatory if, for some \(p_0 > -1\) and \(p_\infty < 0\), it holds for \(n = 0, 1, \ldots\) that

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O(t^{p_0-n}), & t \to 0, \\
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\end{cases}
\]

Are these examples??

(i) \(F(t) = t^{-1/2}\) \quad **Yes**, with \(p_0 = p_\infty = -1/2\).

(ii) \(F(t) = t^{-1/2}e^{it}\) \quad **No**, \(F^{(n)}(t) \sim i^n t^{-1/2}e^{it}\) as \(t \to \infty\).

(iii) \(F(t) = H_0^{(1)}(t)\) \quad **No**, ditto.

(iv) \(F(t) = e^{-it}H_0^{(1)}(t)\) \quad **Yes**, with any \(-1 < p_0 < 0\) and \(p_\infty = -1/2\).
Oscillatory and Non-Oscillatory Functions on \((0, \infty)\)

**Definition.** Call \(F \in C^\infty(0, \infty)\) **non-oscillatory** if, for some \(p_0 > -1\) and \(p_\infty < 0\), it holds for \(n = 0, 1, \ldots\) that

\[
F^{(n)}(t) = \begin{cases} 
O(t^{p_0-n}), & t \to 0, \\
O(t^{p_\infty-n}), & t \to \infty.
\end{cases}
\]

**Remark.** Non-oscillatory \(F\) with \(p_\infty < -1\), so \(F \in L^1(0, \infty)\), are easy to integrate with `quadgk`.

Compare

\[
F(t) = \frac{H_0^{(1)}(t)}{(1+t)^{3/4}} \quad \text{with} \quad F(t) = \frac{e^{-it}H_0^{(1)}(t)}{(1+t)^{3/4}}.
\]

Matlab demo ...
Oscillatory and Non-Oscillatory Functions on $(0, \infty)$

**Definition.** Call $F \in C^\infty(0, \infty)$ **non-oscillatory** if, for some $p_0 > -1$ and $p_\infty < 0$, it holds for $n = 0, 1, \ldots$ that

$$F^{(n)}(t) = \begin{cases} O(t^{p_0-n}), & t \to 0, \\ O(t^{p_\infty-n}), & t \to \infty. \end{cases}$$

**Definition.** Call $F(z)$ **strongly non-oscillatory** if it is analytic in $\Re z > 0$ and, for some $p_0 > -1, p_\infty < 0,$ and $C' > 0,$ it holds for $\Re z > 0$ that

$$|F(z)| \leq \begin{cases} C'|z|^{p_0}, & |z| < 1, \\ C'|z|^{p_\infty}, & |z| \geq 1. \end{cases}$$

**Theorem.** If $F$ is strongly non-oscillatory then it is non-oscillatory, with the same values of $p_0$ and $p_\infty.$
Oscillatory and Non-Oscillatory Functions on \((0, \infty)\)

**Definition.** Call \(F \in C^\infty(0, \infty)\) non-oscillatory if, for some \(p_0 > -1\) and \(p_\infty < 0\), it holds for \(n = 0, 1, \ldots\) that

\[
F^{(n)}(t) = \begin{cases} 
O(t^{p_0-n}), & t \to 0, \\
O(t^{p_\infty-n}), & t \to \infty.
\end{cases}
\]

**Definition.** Call \(F(z)\) strongly non-oscillatory if it is analytic in \(\Re z > 0\) and, for some \(p_0 > -1\), \(p_\infty < 0\), and \(C > 0\), it holds for \(\Re z > 0\) that

\[
|F(z)| \leq \begin{cases} 
C|z|^{p_0}, & |z| < 1, \\
C|z|^{p_\infty}, & |z| \geq 1.
\end{cases}
\]

**Theorem.** If \(F\) is strongly non-oscillatory then it is non-oscillatory, with the same values of \(p_0\) and \(p_\infty\). **Example.** \(F(z) = e^{-iz}H_0^{(1)}(z)\).
Recap

1. Wave Equation and Helmholtz Equation
2. Basic Concept of high frequency asymptotic approximations of GO and GTD
3. Reflection - canonical problems and high frequency GO approximations
4. Diffraction - canonical problems and high frequency GTD approximations
5. The HF Kirchhoff Approximation
6. Preparing for NA: Quantifying Non-Oscillatoryariness!

**Tomorrow:** use this knowledge to design Galerkin methods for boundary integral equations that combine $hp$-approximation with new oscillatory basis functions to solve (at least some classes of) HF scattering problems with $O(1)$ cost as $k \rightarrow \infty$. 
References


