Time Integration for Adaptive *hp*-FEM

- Should be higher-order.
- Explicit methods not suitable ($h \to 0$).
- Method of Lines not suitable (ODE system changes in time).
- Rothe's method
  - Implicit Euler:
$$\frac{u^{n+1} - u^n}{\Delta t} - \Delta u^{n+1} = f^{n+1}$$
  - Crank-Nicolson:
$$\frac{u^{n+1} - u^n}{\Delta t} - \Delta \frac{u^{n+1} + u^n}{2} = \frac{f^{n+1} + f^n}{2}$$
  - Second-order BDF:
$$\frac{3u^{n+2} - 4u^{n+1} + u^n}{2\Delta t} - \Delta u^{n+2} = f^{n+2}$$

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  - Each new method = new weak formulation = new code.

Runge-Kutta Methods

P. Solin, L. Korous: Adaptive Higher-Order Finite Element Methods for Transient PDE Problems Based on Embedded Higher-Order Implicit Runge-Kutta Methods. Journal of Computational Physics, Volume 231, Issue 4, 20 February 2012, pp. 1635-1649.

For an ordinary differential equation of the form

$$\frac{\mathrm{d}y}{\mathrm{d}t} = f(t, y), \tag{1}$$

an s-stage Runge-Kutta method is written as follows,

$$y_{n+1} = y_n + \Delta t \sum_{j=1}^{s} b_j k_j \tag{2}$$

where

$$\begin{pmatrix} k_1 \\ k_2 \\ \vdots \\ k_s \end{pmatrix} = \begin{pmatrix} f\left(t_n + \Delta t c_1, y_n + \Delta t \sum_{j=1}^{s} a_{1j} k_j\right) \\ f\left(t_n + \Delta t c_2, y_n + \Delta t \sum_{j=1}^{s} a_{2j} k_j\right) \\ \vdots \\ f\left(t_n + \Delta t c_s, y_n + \Delta t \sum_{j=1}^{s} a_{sj} k_j\right) \end{pmatrix}$$

Butcher's Tables

A compact way to write Runge-Kutta (and many other) methods:

$$
\begin{array}{c|cccc}
c_1 & a_{11} & a_{12} & \dots & a_{1s} \\
c_2 & a_{21} & a_{22} & \dots & a_{2s} \\
\vdots & \vdots & \vdots & & \vdots \\
c_s & a_{s1} & a_{s2} & \dots & a_{ss} \\
\hline
 & b_1 & b_2 & \dots & b_s
\end{array}
$$

Of particular interest are embedded methods:

$$
\begin{array}{c|cccc}
c_1 & a_{11} & a_{12} & \dots & a_{1s} \\
c_2 & a_{21} & a_{22} & \dots & a_{2s} \\
\vdots & \vdots & \vdots & & \vdots \\
c_s & a_{s1} & a_{s2} & \dots & a_{ss} \\
\hline
 & b_1 & b_2 & \dots & b_s \\
 & b_1^* & b_2^* & \dots & b_s^*
\end{array}
$$

PDE Case

$$M\frac{\mathrm{d}Y}{\mathrm{d}t} = F(t, Y),$$

PDE Case

$$M\frac{\mathrm{d}Y}{\mathrm{d}t} = F(t, Y),$$

$$\begin{pmatrix} M & 0 & \dots & 0 \\ 0 & M & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & M \end{pmatrix} \begin{pmatrix} K_1 \\ K_2 \\ \vdots \\ K_s \end{pmatrix} = \begin{pmatrix} F\left(t_n + \Delta t c_1, Y_n + \Delta t \sum_{j=1}^{s} a_{1j} K_j\right) \\ F\left(t_n + \Delta t c_2, Y_n + \Delta t \sum_{j=1}^{s} a_{2j} K_j\right) \\ \vdots \\ F\left(t_n + \Delta t c_s, Y_n + \Delta t \sum_{j=1}^{s} a_{sj} K_j\right) \end{pmatrix}$$

Newton's Method

$$\begin{pmatrix} M & 0 & \dots & 0 \\ 0 & M & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & M \end{pmatrix} \begin{pmatrix} K_1 \\ K_2 \\ \vdots \\ K_s \end{pmatrix} - \begin{pmatrix} F\left(t_n + \Delta t c_1, Y_n + \Delta t \sum_{j=1}^{s} a_{1j} K_j\right) \\ F\left(t_n + \Delta t c_2, Y_n + \Delta t \sum_{j=1}^{s} a_{2j} K_j\right) \\ \vdots \\ F\left(t_n + \Delta t c_s, Y_n + \Delta t \sum_{j=1}^{s} a_{sj} K_j\right) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

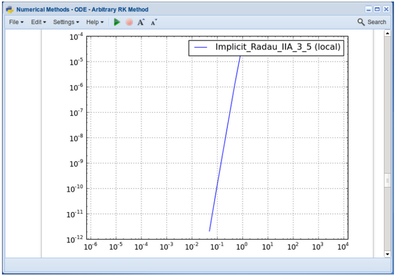
Jacobian matrix:

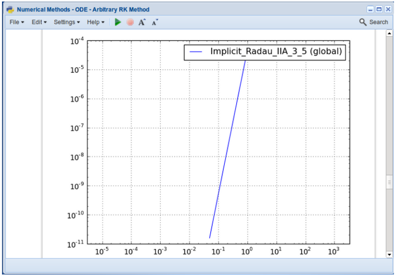
$$J_{ij} = \delta_{ij} M - a_{ij} \Delta t J(t_n + \Delta t c_i, Y_n + \Delta t \sum_{j=1}^{s} a_{ij} K_j).$$

Input data:

- Jacobian and residual of the right-hand side of the PDE.
- Arbitrary Butcher's table.

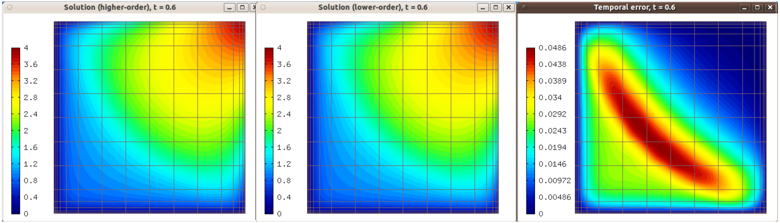
30+ Verified Butcher's Tables in NCLab



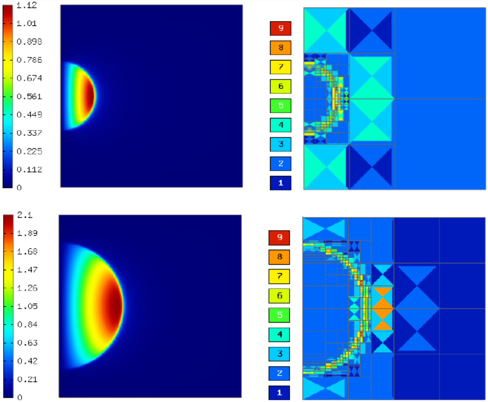


Temporal Error is a Function

(Estimate is based on a pair of embedded RK methods)



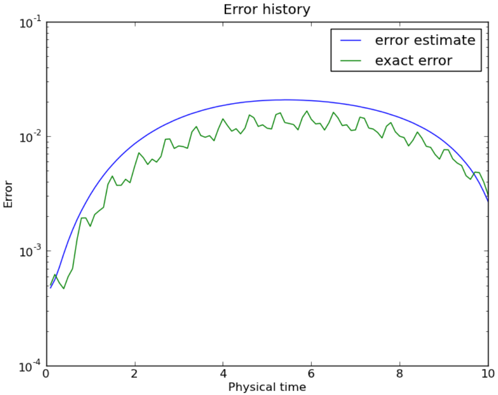
Benchmark for Adaptive FEM with Dynamical Meshes



More details in: P. Solin, L. Korous: Adaptive Higher-Order Finite Element Methods for Transient PDE Problems Based on Embedded Higher-Order Implicit Runge-Kutta Methods. Journal of Computational Physics, Volume 231, Issue 4, 20 February 2012, pp. 1635-1649.

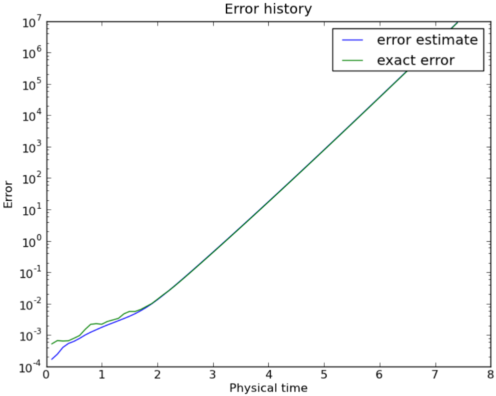
Sample Results: Cash-23

Implicit embedded RK method by Cash of orders 2 and 3



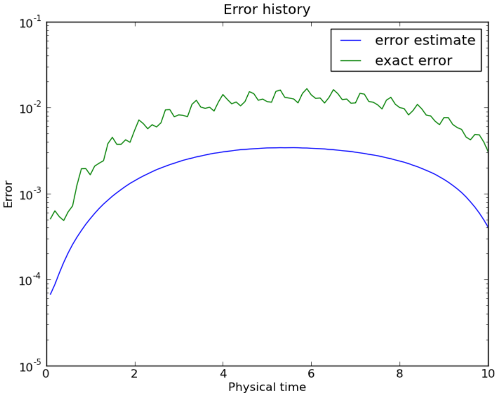
Sample Results: Billington-23

Implicit embedded RK method by Billington of orders 2 and 3



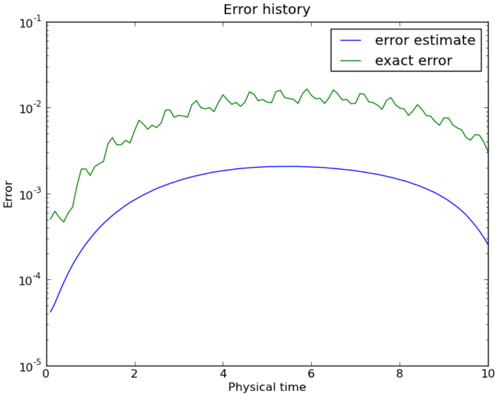
Sample Results: TRBDF2-23

Implicit embedded RK method TRBDF2 of orders 2 and 3



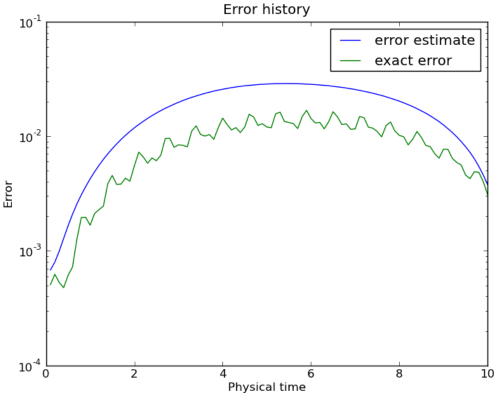
Sample Results: TRX2-23

Implicit embedded RK method TRX2 of orders 2 and 3



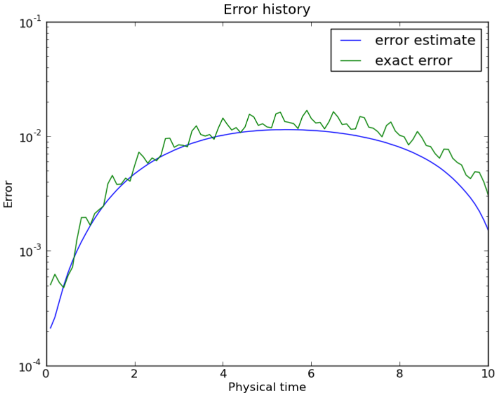
Sample Results: Cash-24

Implicit embedded RK method by Cash of orders 2 and 4



Sample Results: Cash-34

Implicit embedded RK method by Cash of orders 3 and 4



Sample Results: Ismail-45

Implicit embedded RK method by Ismail of orders 4 and 5