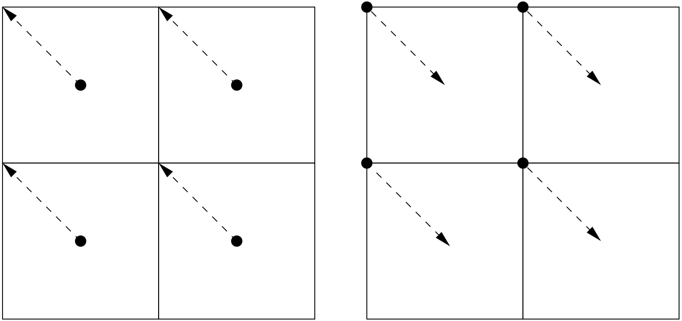
Adaptive *hp*-FEM in image compression

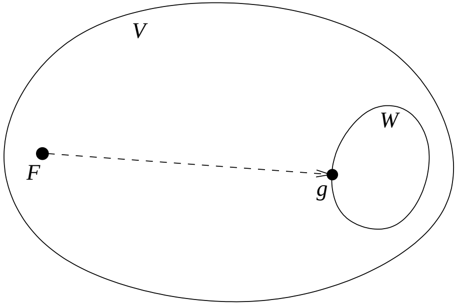
Image = pixel-wise constant function.

Making it continuous

Piecewise-bilinear approximation:



Hilbert space setting



V ... large Hilbert space (for example, L^2 or H^1).
$W \subset V$ ... piecewise-polynomial subspace.
$F \in V$ ... raw image.
$g \in W$ ... (unknown) best approximation of F in W.

Best approximation problem

Best approximation = orthogonal projection:

$$(F - g, v_i)_V = 0 \quad \text{for all } v_1, v_2, \dots, v_N, \quad N = \dim(W).$$

Express

$$g = \sum_{j=1}^{N} y_j v_j.$$

Thus we obtain

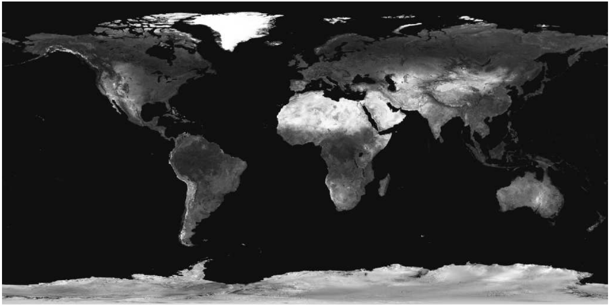
$$\sum_{j=1}^{N} \underbrace{(v_j, v_i)_V}_{a(v_j, v_i)} y_j = \underbrace{(F, v_i)_V}_{l(v_i)} \quad \text{for all } v_1, v_2, \dots, v_N.$$

Discrete problem in matrix form:

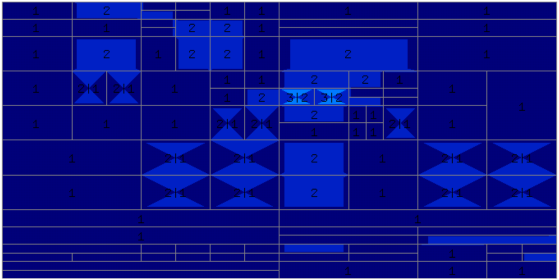
$$MY = b.$$

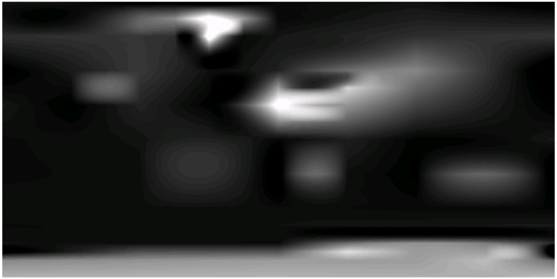
Mass matrix M is SPD, well conditioned.

Satellite Image of Earth

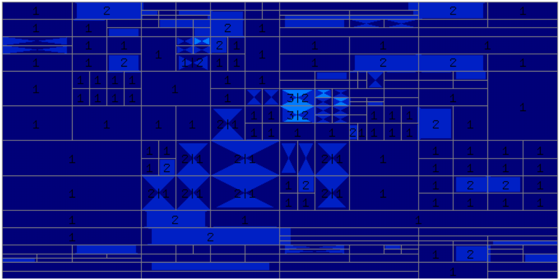


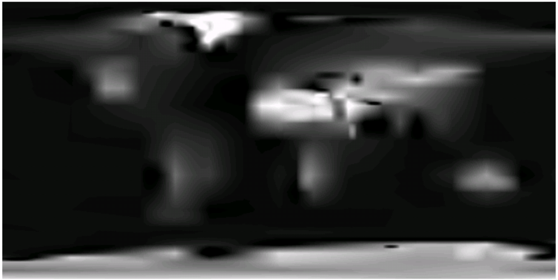
dof = 148, err = 0.2617



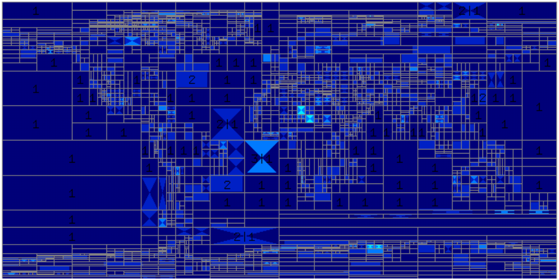


dof = 419, err = 0.182095

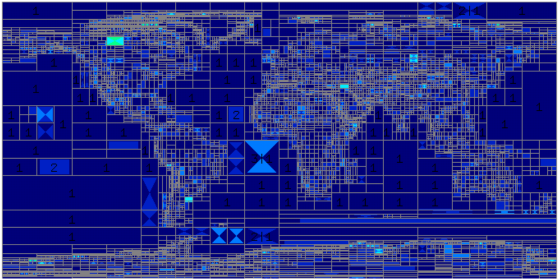


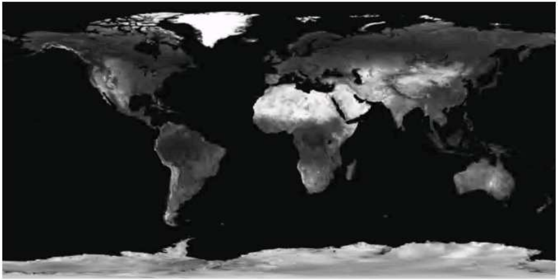


dof = 3778, err = 0.063895

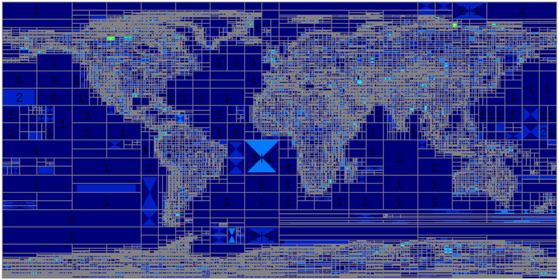


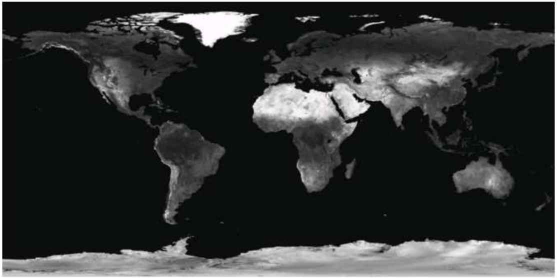
dof = 12125, err = 0.030938



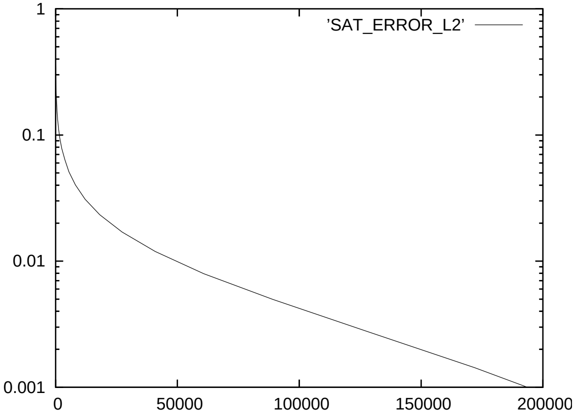


dof = 40901, err = 0.011916



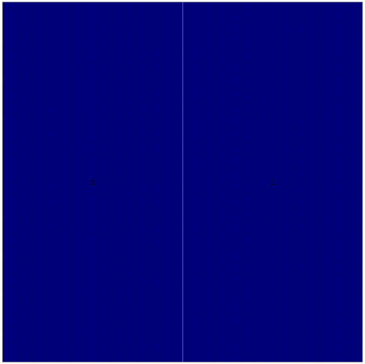


Convergence in L^2-norm

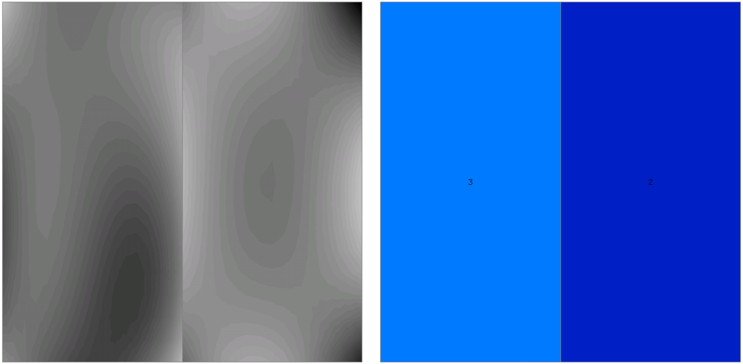


Example: Lena

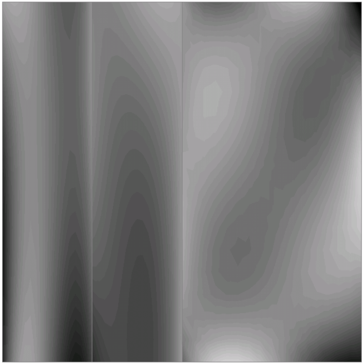
Step 1 (dof=8, err=0.331940)

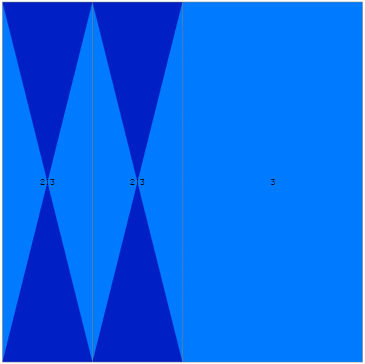


Step 2 (dof=25, err=0.303916)

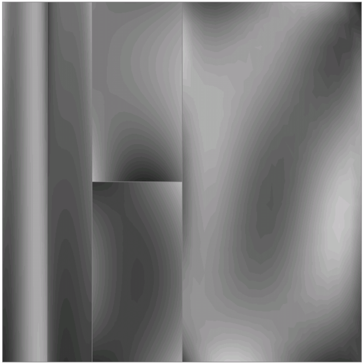


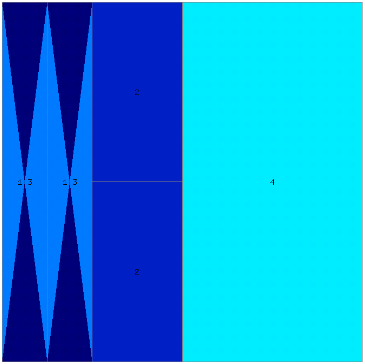
Step 3 (dof=40, err=0.281455)



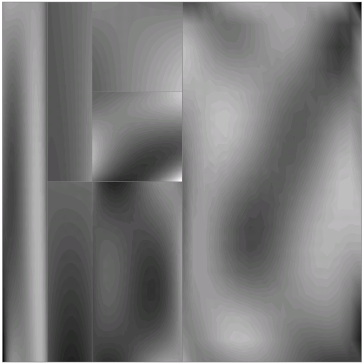


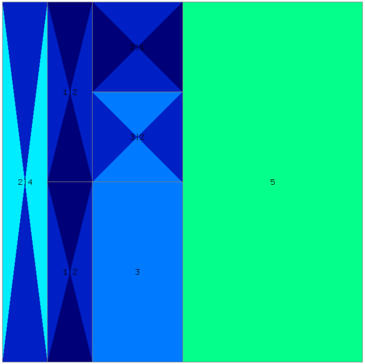
Step 4 (dof=59, err=0.259902)



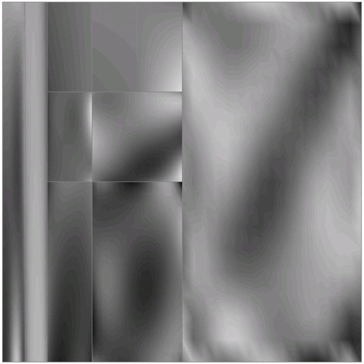


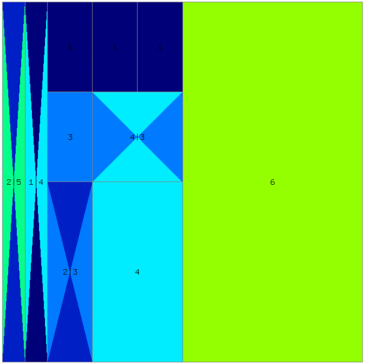
Step 5 (dof=97, err=0.241367)



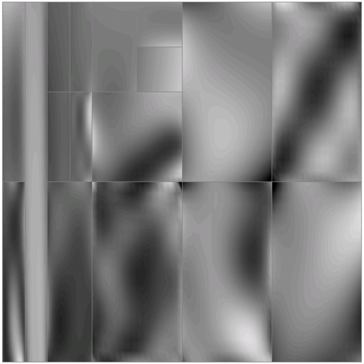


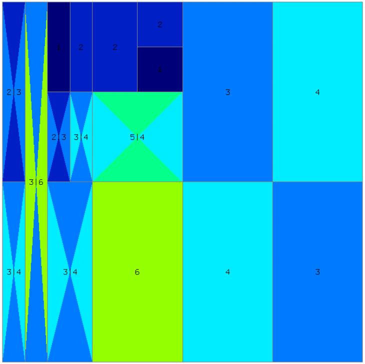
Step 6 (dof=162, err=0.226622)



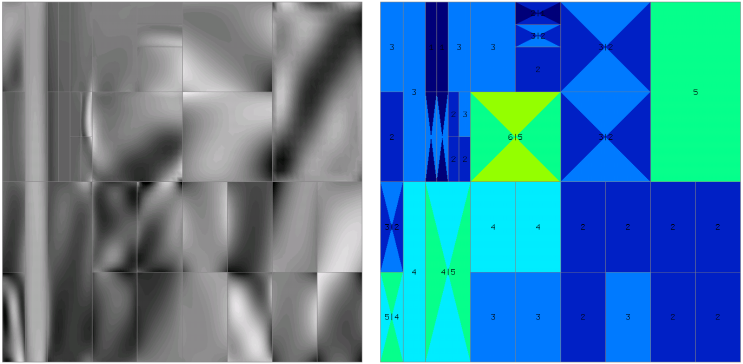


Step 7 (dof=308, err=0.203519)

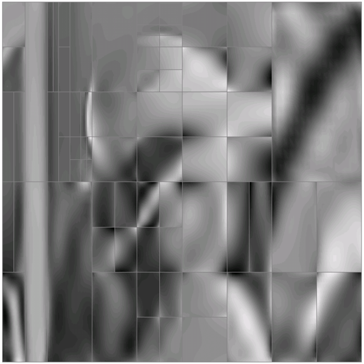


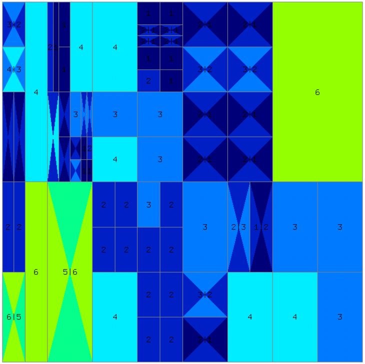


Step 8 (dof=527, err=0.173636)

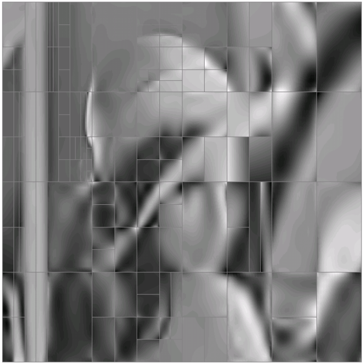


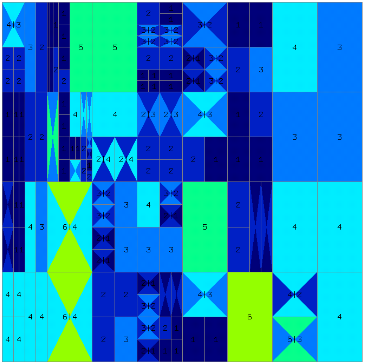
Step 9 (dof=922, err=0.153029)



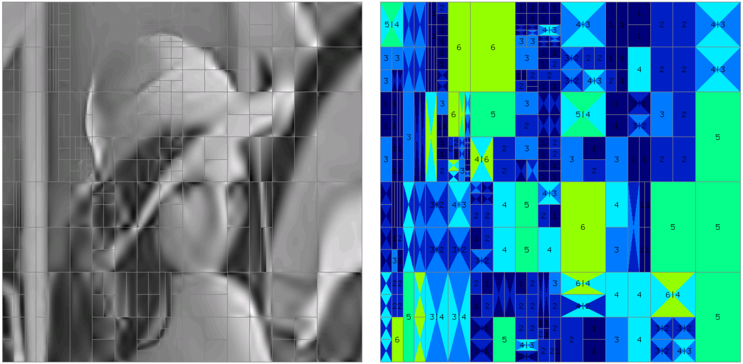


Step 10 (dof=1658, err=0.134793)

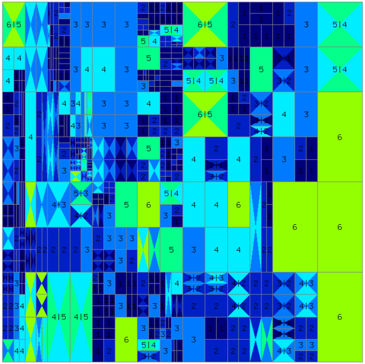




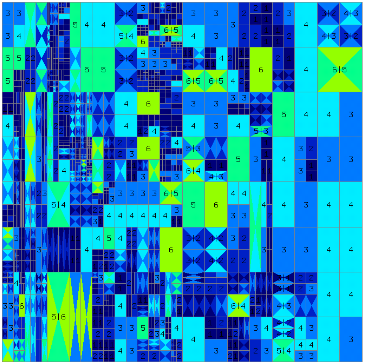
Step 11 (dof=3093, err=0.114705)



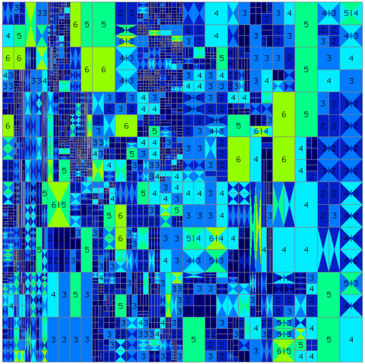
Step 12 (dof=5786, err=0.098427)



Step 13 (dof=11230, err=0.082684)



Step 14 (dof=21437, err=0.069197)



Step 15 (dof=30560, err=0.058156)