Convergence Rates for AFEM: PDE on Parametric Surfaces

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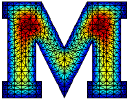

Joint work with

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P. Morin (Santa Fe, Argentina)
S. Walker (Lousianna)

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Outline

Motivation: Geometric PDE

The Laplace-Beltrami Operator and Standard Adaptivity

Adaptive Finite Element Methods (AFEM) on Parametric Surfaces

Convergence Rates of AFEM

Discontinuous Coefficients

Comments and Conclusions

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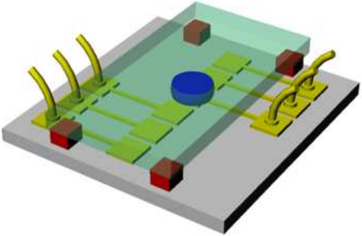
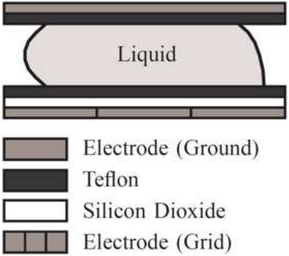
Electrowetting on Dielectric: Modeling (w. A. Bonito and S. Walker)





Mixed Formulation (u velocity, p pressure, H curvature, λ multiplier)

$$\alpha \frac{\partial u}{\partial t} + \beta u + \nabla p = 0 \qquad \text{in } \Omega$$

$$\text{div } u = 0 \qquad \text{in } \Omega$$

$$p = H + \underbrace{E}_{\text{electric actuation}} + \underbrace{P_0 \text{sign}\,(u \cdot \nu)}_{\lambda\text{(contact line pinning)}} + \underbrace{D_{visc} u \cdot \nu}_{\text{viscous damping}} \qquad \text{on } \Gamma$$

Interface Motion

$$u \cdot \nu = \partial_t X \cdot \nu \quad \text{on } \Gamma$$

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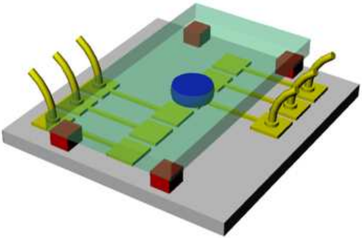

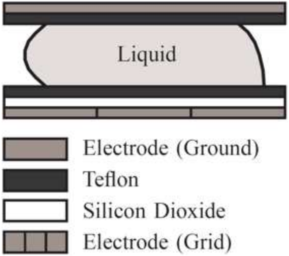

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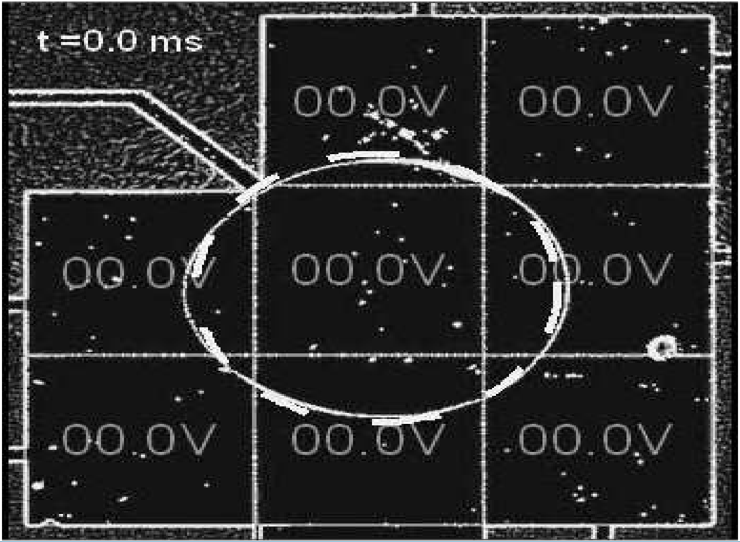
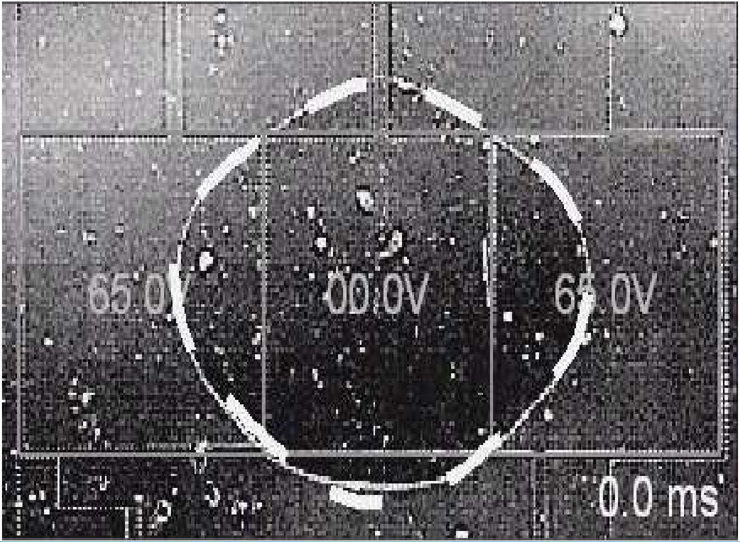
Electrowetting on Dielectric: Experiments vs Simulations

Moving droplet stirred around by varying voltages



Electrowetting on Dielectric: Experiments vs Simulations

Splitting of glycerin droplet due to voltage actuation



Biomembranes: Modeling (w. A. Bonito and M.S. Pauletti)

- Bending (Willmore) energy: $J(\Gamma) = \frac{1}{2}\int_\Gamma H^2, \quad H$ mean curvature

- Geometric Gradient Flow (with area and volume constraint):

$$\mathbf{v} = -\delta_\Gamma J = -\Big(\Delta_\Gamma H + \frac{1}{2}H^3 - 2\kappa H\Big)\boldsymbol{\nu} - \Big(\lambda H\boldsymbol{\nu} + p\boldsymbol{\nu}\Big)$$

where Δ_Γ is the Laplace-Beltrami operator on Γ.

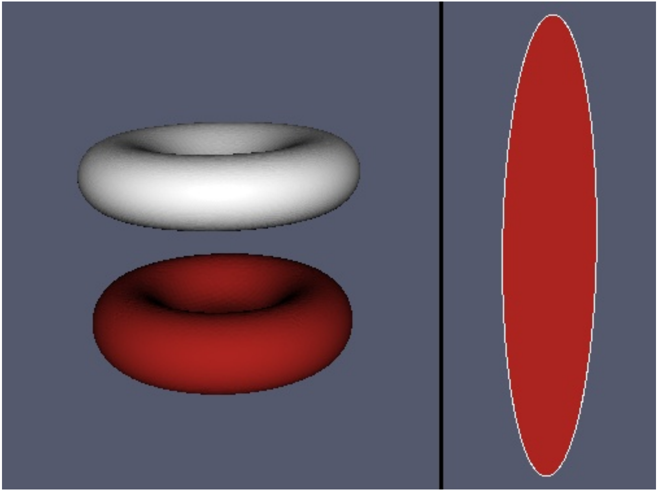
- Fluid-Membrane Interaction (with area constraint):

$$\rho D_t\mathbf{v} - \text{div}\,\underbrace{\left(-p\mathbf{I} + \mu D(\mathbf{v})\right)}_{\Sigma} = \mathbf{b} \qquad \text{in } \Omega_t,$$

$$\text{div}\,\mathbf{v} = 0 \qquad \text{in } \Omega_t,$$

$$[\boldsymbol{\Sigma}]\boldsymbol{\nu} = \delta_\Gamma J \qquad \text{on } \Gamma_t$$

Biomembrane: Geometric vs Fluid Red Blood Cell



play

Director Fields on Flexible Surfaces (w. S. Bartels and G. Dolzmann)

- Coupling of mean curvature $H = \operatorname{div}_\Gamma \nu$ with a director field n via

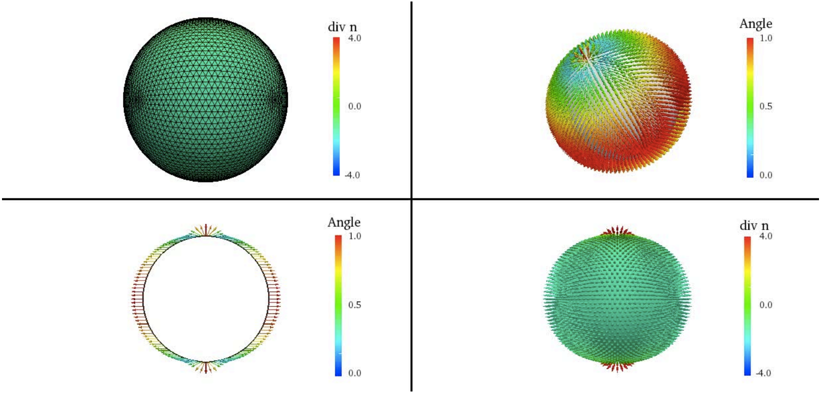
$$J(\Gamma, n) = \frac{1}{2}\int_\Gamma |\operatorname{div}_\Gamma \nu - \delta \operatorname{div}_\Gamma n|^2 \mathrm{d}\sigma + \frac{\lambda}{2}\int_\Gamma |\nabla_\Gamma n|^2 \mathrm{d}\sigma + \frac{1}{2}\int_\Gamma \mu(|n|^2 - 1)\mathrm{d}\sigma + \frac{1}{2\varepsilon}\int_\Gamma f(n\cdot\nu)\mathrm{d}\sigma$$

- μ the Lagrange multiplier for the rigid constraint $|n| = 1$
- $f(x) = (x^2 - \xi_0^2)^2$ with $\xi_0 \in [0,1]$ penalizes the deviation of the angle between ν and n from $\arccos \xi_0$
- Spontaneous curvature $H_0 = \delta \operatorname{div}_\Gamma n$ induced by director field n
- Relaxation dynamics (L^2- gradient flow): V normal velocity of Γ

$$V = -\delta_\Gamma J(\Gamma, n), \qquad \partial_t n = -\delta_n J(\Gamma, n)$$

Coupling of Director Fields and Flexible Surfaces: Simulations

Cone-like structure near positive degree-one defects pointing outwards $\Rightarrow$ stomatocyte shape



The Laplace-Beltrami Operator and Curvature

- Vector curvature: $\mathbf{H} = H\boldsymbol{\nu} = -\Delta_\Gamma \mathbf{X}$, $\quad \mathbf{X} =$ identity on Γ (Dziuk' 91)
- Semi-implicit Time Discretization $(t_n \to t_{n+1})$: explicit geometry $(\Gamma = \Gamma_n, \quad \nabla_\Gamma = \nabla_{\Gamma^n}, \quad \boldsymbol{\nu} = \boldsymbol{\nu}^n)$

$$\int_{\Gamma^n} \mathbf{H}^{n+1} \cdot \boldsymbol{\Psi} = \int_{\Gamma^n} \nabla_{\Gamma^n} \mathbf{X}^{n+1} : \nabla_{\Gamma^n} \boldsymbol{\Psi}, \qquad \mathbf{X}^{n+1} = \mathbf{X}^n + \tau^n \mathbf{V}^{n+1}$$

- $\int_{\Gamma^n} \mathbf{H}^{n+1} \cdot \boldsymbol{\Psi} - \tau^n \int_{\Gamma^n} \nabla_{\Gamma^n} \mathbf{V}^{n+1} : \nabla_{\Gamma^n} \boldsymbol{\Psi} = \int_{\Gamma^n} \nabla_{\Gamma^n} \mathbf{X}^n : \nabla_{\Gamma^n} \boldsymbol{\Psi}$
- Mixed Method: operator splitting
  - Velocity (gradient flow or Navier-Stokes)
  - Curvature (Laplace-Beltrami)

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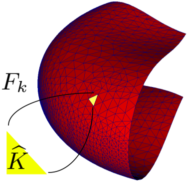
The Laplace-Beltrami Problem

$$-\Delta_\gamma u = f \qquad \text{on } \gamma, \qquad u = 0 \qquad \text{on } \partial\gamma.$$

- γ is a parametric Lipschitz surface, piecewise 'smooth', with Lipschitz boundary $\partial\gamma$.
- $f \in L^2(\gamma)$ (see Cohen-DeVore-Nochetto for $H^{-1}(\gamma)$ data).
- $\nabla_\gamma u = \nabla \tilde{u} - (\nabla \tilde{u})\nu \cdot \nu, \quad \Delta_\gamma = \text{div}_\gamma \cdot \nabla_\gamma.$



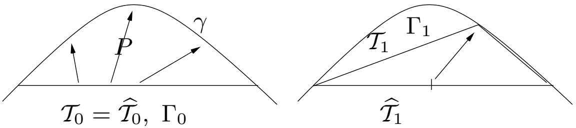
Weak Formulation

$$\text{Seek } u \in H_0^1(\gamma) : \qquad \int_\gamma \nabla_\gamma u \cdot \nabla_\gamma v = \int_\gamma f\ v, \qquad \forall v \in H_0^1(\gamma).$$

$$\int_K \nabla_\gamma u \cdot \nabla_\gamma v = \int_{\widehat{K}} \nabla(u \circ F_k)^T G_K^{-1} \nabla(v \circ F_K) \sqrt{\det(G_K)}, \qquad G_K = DF_K^T DF_K$$

The Finite Element Method on Surfaces

- $\Gamma_0 = \cup_{i=1}^n \Gamma_0^i$ is an initial polyhedral domain with nodes lying on γ
- $P^i : \Gamma_0^i \to \gamma^i$ parametrization of γ (globally Lipschitz, C^1 on Γ_0^i).
- Assume, for simplicity, one such Γ_0^i (i.e. $n = 1$) and drop the index i.
- A sequence of refinements of Γ_0 is obtained as follows:
  (1) subdivide Γ_0 onto $\widehat{\mathcal{T}}_k$;
  (2) Determine $\mathcal{T}_k$ using map P to place the new nodes on γ.
  Hence, the new polygonal approximation is given by $\Gamma_k = \text{Range}(I_{\widehat{\mathcal{T}}_k} P)$.
  $I_{\widehat{\mathcal{T}}_k}$: Lagrange interpolant onto continuous piecewise linears



- $\mathbb{V}(\widehat{\mathcal{T}}_k)$ is the space of continuous piecewise linear polynomials subordinate to $\widehat{\mathcal{T}}_k$ and $\mathbb{V}_k := \mathbb{V}(\mathcal{T}_k)$ is its lift using $I_{\widehat{\mathcal{T}}_k} P$.

Discrete Formulation: the geometry changes with iteration counter k

$$\text{Seek } U_k \in \mathbb{V}_k : \qquad \int_{\Gamma_k} \nabla_{\Gamma_k} U_k \cdot \nabla_{\Gamma_k} V = \int_{\Gamma_k} f \frac{q}{Q_k} V, \qquad \forall V \in \mathbb{V}_k.$$

Standard Adaptive Algorithm

SOLVE : Compute the solution $U_k \in \mathbb{V}_k := \mathbb{V}(\mathcal{T}_k)$ of the discrete problem.

ESTIMATE : Compute a local estimator $\eta_k(U_k, K)$, $K \in \mathcal{T}_k$, for the error in terms of the discrete solution U_k and given data.

MARK : Use the estimator to mark a subset $\mathcal{M}_k \subset \mathcal{T}_k$ for refinement $\sum_{K\in\mathcal{M}_k} \eta_k(U_k, K)^2 \geq \theta^2 \sum_{K\in\mathcal{T}_k} \eta_k(U_k, K)^2$. (Dörfler marking)

REFINE : Refine the marked subset $\mathcal{M}_k$ to obtain $\mathcal{T}_{k+1}$, conforming or with hanging nodes, increment k and go to step SOLVE.

Quasi-Optimal Algorithm

If $f \in L^2(\Omega)$ and the decay rate for the best approximation of u is

$$\inf_{\#\mathcal{T}-\#\mathcal{T}_0\leq N} \inf_{V\in\mathbb{V}(\mathcal{T})} \|\nabla_\gamma(u-V)\|_{L^2(\gamma)} \leq C_1 N^{-s}, \qquad 0 < s \leq 1/d$$

then the finite element method delivers the same rate

$$||\nabla_\gamma(u-U_k)||_{L^2(\gamma)} \leq C_2(\#\mathcal{T}_k)^{-s}$$

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$$||\nabla_\gamma(u-U_k)||_{L^2(\gamma)} \leq C_2(\#\mathcal{T}_k)^{-s}$$

Quasi-Optimal Adaptive Finite Element Methods (AFEM)

- Babuska, Vogelius, 1986 (1D problem).
- Binev, Dahmen, DeVore, 2004 (2D problem, coarsening).
- Stevenson, 2006 (marking by oscillation).
- Kreuzer, Cascon, Nochetto, Siebert, 2008.
- Bonito, Nochetto, 2010 (dG).
- Cohen, DeVore, Nochetto, 2011 (H^{-1} data and approximation classes).
- Diening, Kreuzer, Stevenson, 2013 (maximum strategy).

Sufficient Condition (for best approximation of u to decay with rate $N^{-1/d}$)

$$\text{sob }(W^2_p) > \text{sob }(H^1) \quad \Rightarrow \quad 2-\frac{d}{p} > 1-\frac{d}{2} \quad \Rightarrow \quad p > \frac{2d}{2+d}$$

Main Ingredients

Orthogonality (Pythagoras):

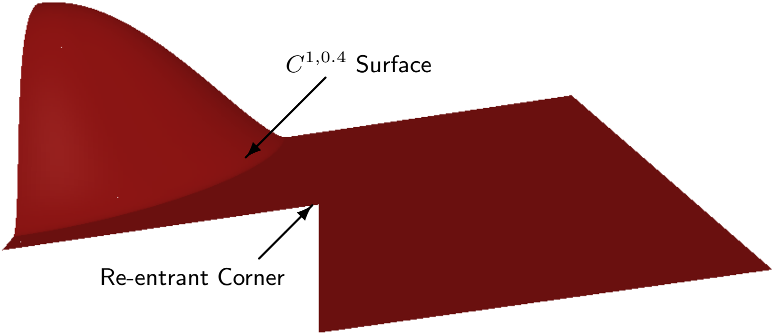
$$||\nabla(u-U_k)||^2_{L^2(\Omega)} = ||\nabla(u-U_{k-1})||^2_{L^2(\Omega)} - ||\nabla(U_k-U_{k-1})||^2_{L^2(\Omega)}.$$

Upper and Lower Bounds:

$$||\nabla(u-U_k)||^2_{L^2(\Omega)} \preceq \eta^2_k(U_k,\mathcal{T}_k) \preceq ||\nabla(u-U_k)||^2_{L^2(\Omega)} + \mathsf{osc}^2(U_k,\mathcal{T}_k).$$

Monotonicity of Estimator: $\quad \eta_{k+1}(U_k,\mathcal{T}_{k+1}) \le \eta_k(U_k,\mathcal{T}_k).$

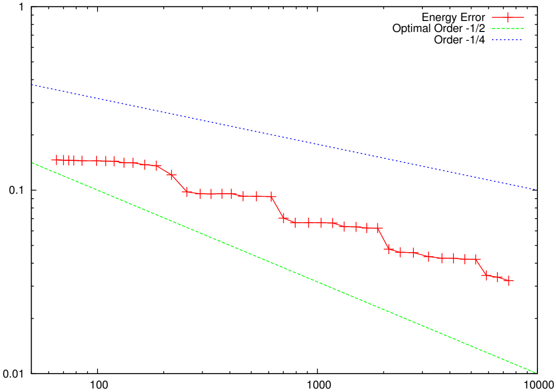
$C^{1,0.4}$ surface with Lipschitz boundary



Standard AFEM

Error indicator: $\eta_{\mathcal{T}}^2(U,T) := h_T^2 ||f||_{L^2(T)}^2 + \sum_{\substack{S \text{ side} \\ S \subset \partial T}} h_S ||[\nabla_{\Gamma_k} U]||_{L^2(S)}^2$

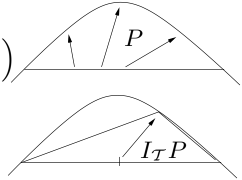
Dörfler parameter in MARK: $\theta = 10\%$ (quite conservative).



$\Rightarrow$ suboptimal decay rate

Residual Based A-Posteriori Estimators on Surfaces

Refs: Demlow-Dziuk (2007), Demlow (2009), Mekchay-Morin-Nochetto (2011)



Upper Bound For $U \in \mathbb{V}$ a discrete Galerkin solution then

$$\|\nabla_\gamma (u - U)\|^2_{L^2(\gamma)} \le C_1 \Big(\eta(U, \mathcal{T})^2 + \Lambda_1 \lambda(\mathcal{T})^2 \Big)$$

where Λ_1 depends on $\mathcal{T}_0$ and $\lambda(\mathcal{T})$ is the geometric estimator

$$\lambda(\mathcal{T}) = \max_{T \in \mathcal{T}} \|\nabla (P - I_{\mathcal{T}} P)\|_{L^\infty(\underbrace{(I_{\mathcal{T}} P)^{-1}(T)}_{=\widehat{T}})}$$

Lower Bound

$$C_2 \eta(U, \mathcal{T})^2 \le \|\nabla_\gamma (u - U)\|^2_{L^2(\gamma)} + \underbrace{\sum_{T \in \mathcal{T}} h_T^2 \Big\| f \frac{q}{Q} - \overline{f \frac{q}{Q}} \Big\|^2_{L^2(T)}}_{\text{data oscillation: } \operatorname{osc}(f, \mathcal{T})^2} + \Lambda_1 \lambda(\mathcal{T})^2 .$$

Quasi-Orthogonality

For $U \in \mathbb{V}(\mathcal{T})$, $U^* \in \mathbb{V}(\mathcal{T}^*)$ two Galerkin solutions, $\mathcal{T}^*$ a refinement of $\mathcal{T}$, then

$$\|\nabla_\gamma (u - U_*)\|^2_{L^2(\gamma)} \le \|\nabla_\gamma (u - U)\|^2_{L^2(\gamma)} - \frac{1}{2} \|\nabla_\gamma (U - U_*)\|^2_{L^2(\gamma)} + \Lambda_2 \lambda(\mathcal{T})^2 .$$

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Adaptive Finite Element Method (AFEM)

Let $\mathcal{T}_0$ be the initial triangulation of γ, $k = 0$. Given $\epsilon_0 > 0$ and $\omega > 0$

$\boxed{\mathcal{T}_k^+ = \text{GEOMETRY}(\mathcal{T}_k, \omega\epsilon_k)}$: Refine surface until $\max_{T\in\mathcal{T}} \lambda_{\mathcal{T}}(T) \leq \omega\ \epsilon_k$

$\boxed{[\mathcal{T}_{k+1}, U_{k+1}] = \text{PDE}(\mathcal{T}_k^+, \epsilon_k)}$: Use the Dörfler marking and refine until

$$\eta(U_{k+1}, \mathcal{T}_{k+1}) \leq \epsilon_k$$

Set $\epsilon_{k+1} = \epsilon_k/2$ and repeat.

Equidistribute the errors, similar to R. Stevenson (2006) for oscillations.

The Module GEOMETRY: Greedy Algorithm

Let $\mathcal{T} := \mathcal{T}_k$

$\mathcal{T}^+ =$ GEOMETRY $(\mathcal{T}, \tau)$

while $(\max_{T\in\mathcal{T}} \lambda_{\mathcal{T}}(T) > \tau)$

Refine all $T \in \mathcal{T}$ such that $\lambda_{\mathcal{T}}(T) > \tau$

Update $\mathcal{T}$

end

- Module GEOMETRY independent of the PDE
- Complexity of GEOMETRY?

The Module PDE

$[\mathcal{T}, U] = \text{PDE}\ (\mathcal{T}^+, \epsilon)$
while $(\eta(U, \mathcal{T}) > \epsilon)$

SOLVE → ESTIMATE → MARK → REFINE

Update $\mathcal{T}$
end

Quasi-monotonicity of geometric estimator

$$\lambda(\mathcal{T}) = \max_{T\in\mathcal{T}} ||\nabla(P - I_{\mathcal{T}}P)||_{L^\infty(T)} \le \max_{T\in\mathcal{T}} ||\nabla(P - I_{\mathcal{T}^+}P)||_{L^\infty(T)}$$
$$+ \max_{T\in\mathcal{T}} ||\nabla I_{\mathcal{T}}(P - I_{\mathcal{T}^+}P)||_{L^\infty(T)} \le 2 \max_{T\in\mathcal{T}^+} ||\nabla(P - I_{\mathcal{T}^+}P)||_{L^\infty(T)} = 2\lambda(\mathcal{T}^+)$$

Relation between geometric and PDE errors

$$\lambda(\mathcal{T}) \le 2\lambda(\mathcal{T}_k^+) \le 2\omega\epsilon \le 2\omega\eta(U, \mathcal{T})$$

⇒ the geometric error $\lambda(\mathcal{T})$ is small relative to $\eta(U, \mathcal{T})$ and can be controlled

Conditional Contraction Property of Module PDE

Theorem

If $\lambda(\mathcal{T}_j) \leq 2\omega\eta(U_j, \mathcal{T}_j)$ for $\omega \leq \omega_*$, then there exists constants $0 < \alpha < 1$ and $\beta > 0$ such that the inner iterates of the module PDE satisfy

$$\|\nabla_\gamma(u - U_{j+1})\|^2_{L^2(\gamma)} + \beta\eta(U_{j+1}, \mathcal{T}_{j+1})^2 \leq \alpha^2 \Big(\|\nabla_\gamma(u - U_j)\|^2_{L^2(\gamma)} + \beta\eta(U_j, \mathcal{T}_j)^2 \Big).$$

Moreover, the number J of inner iterates of PDE is uniformly bounded.

- The proof proceeds as in Cascón, Kreuzer, Nochetto, and Siebert (2008) and Bonito and Nochetto (2010), with the additional information

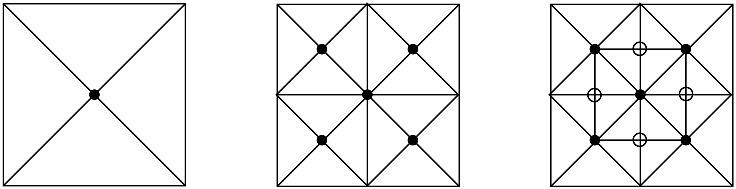
$$\lambda(\mathcal{T}) \leq 2\omega\eta(U, \mathcal{T})$$

  in the inner loops of PDE for $\omega \leq \omega_*$ sufficiently small.
- Reduction of error estimator: there exist constants $0 < \xi < 1$ and $\Lambda_2, \Lambda_3 > 0$ such that for all $\delta > 0$

$$\eta(U_*, \mathcal{T}_*)^2 \leq (1+\delta)\big(\eta(U, \mathcal{T})^2 - \xi\eta(U, \mathcal{M})^2\big) + (1+\delta^{-1})\big(\Lambda_3\|\nabla_\gamma(U_* - U)^2\|_{L^2(\gamma)} + \Lambda_2\lambda(\mathcal{T})^2\big).$$

Contracting Quantities of AFEM (in flat domains)

- Energy error: $\|U_k - u\|_\Omega$ is monotone, but not strictly monotone (e.g. $U_{k+1} = U_k$).



$$\Omega = (0,1)^2, A = I, f = 1 \quad \Rightarrow \quad U_0 = U_1 = \frac{1}{12}\phi_0, \quad U_2 \neq U_1$$

- Residual estimator: $\eta_k(U_k, \mathcal{T}_k)$ is not reduced by AFEM, and is not even monotone. But, if $U_{k+1} = U_k$, then $\eta_k(U_k, \mathcal{T}_k)$ decreases strictly

$$\eta_{k+1}^2(U_{k+1}, \mathcal{T}_{k+1}) = \eta_{k+1}^2(U_k, \mathcal{T}_{k+1}) \leq \eta_k^2(U_k, \mathcal{T}_k) - \xi\eta_k^2(U_k, \mathcal{M}_k)$$

- Heuristics: the quantity $\|U_k - u\|_\Omega^2 + \beta\eta_k^2(U_k, \mathcal{T}_k)$ might contract!

- Additional term $\lambda(\mathcal{T}_k)$ for Laplace-Beltrami, but $\lambda(\mathcal{T}_k) \leq 2\omega\eta_k(U_k, \mathcal{T}_k)$.

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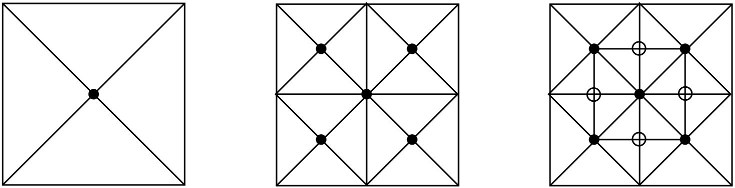

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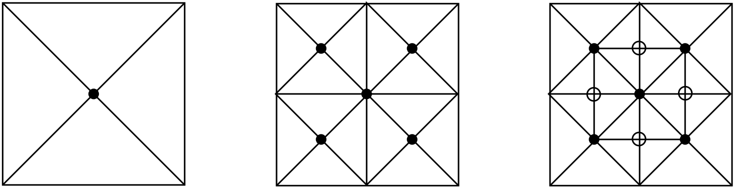

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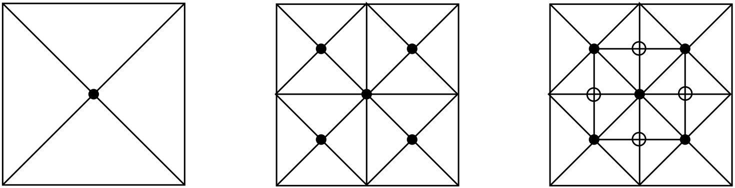

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Proof of Contraction: Step 1

$$e_j = \|\nabla_\gamma(u - U_j)\|_{L^2(\gamma)}, \quad E_j = \|\nabla_\gamma(U_{j+1} - U_j)\|_{L^2(\gamma)}, \quad \eta_j = \eta(U_j, \mathcal{T}_j), \quad \lambda_j = \lambda(\mathcal{T}_j).$$

Combine, quasi-orthogonality of energy error

$$e_{j+1}^2 \leq e_j^2 - \frac{1}{2} E_j^2 + \Lambda_2 \lambda_j^2$$

with reduction of residual error estimator:

$$\eta_{j+1}^2 \leq (1+\delta)\big(\eta_j^2 - \xi \eta_j(\mathcal{M}_j)^2\big) + (1+\delta^{-1})\big(\Lambda_3 E_j^2 + \Lambda_2 \lambda_j^2\big)$$

to get

$$e_{j+1}^2 + \beta \eta_{j+1}^2 \leq e_j^2 + \Big(-\frac{1}{2} + \beta(1+\delta^{-1})\Lambda_3 \Big) E_j^2 + \Lambda_2 \Big(1 + \beta(1+\delta^{-1})\Big) \lambda_j^2 + \beta(1+\delta)\Big(\eta_j^2 - \xi \eta_j(\mathcal{M}_j)^2\Big).$$

Choose β, depending on δ, so that

$$\beta(1+\delta^{-1})\Lambda_3 = \frac{1}{2} \qquad \Rightarrow \qquad \beta(1+\delta) = \frac{\delta}{2\Lambda_3}.$$

This implies

$$e_{j+1}^2 + \beta \eta_{j+1}^2 \leq e_j^2 + \Lambda_2 \Big(1 + \beta(1+\delta^{-1})\Big) \lambda_j^2 + \beta(1+\delta)\Big(\eta_j^2 - \xi \eta_j(\mathcal{M}_j)^2\Big).$$

Proof of Contraction: Step 2

Use Dörfler marking $\eta_j(\mathcal{M}_j) \geq \theta\eta_j$ to derive

$$\eta_j^2 - \xi\eta_j(\mathcal{M}_j)^2 \leq \left(1 - \xi\theta^2\right)\eta_j^2.$$

Recall $\lambda_j \leq 2\lambda_+$ and properties $\lambda_+ \leq \omega\epsilon$ and $\epsilon < \eta_j$ to write

$$e_{j+1}^2 + \beta\eta_{j+1}^2 \leq e_j^2 - \beta(1+\delta)\frac{\xi\theta^2}{2}\eta_j^2$$
$$+ \beta\Big((1+\delta)\Big(1 - \frac{\xi\theta^2}{2}\Big) + \Lambda_2\Big(1 + \frac{1}{2\Lambda_3}\Big)\frac{4\omega^2}{\beta}\Big)\eta_j^2.$$

Employ upper bound

$$e_j^2 \leq C_1\left(\eta_j + \Lambda_1\lambda_j^2\right) \leq C_1\left(1 + 4\omega^2\Lambda_1\right)\eta_j^2 = C_3\eta_j^2$$

to deduce

$$e_{j+1}^2 + \beta\eta_{j+1}^2 \leq \underbrace{\Big(1 - \delta\frac{\xi\theta^2}{2\Lambda_3 C_3}\Big)}_{=\alpha_1(\delta)} e_j^2 + \underbrace{\Big((1+\delta)\Big(1 - \frac{\xi\theta^2}{2}\Big) + \Lambda_2\Big(1 + \frac{1}{2\Lambda_3}\Big)\frac{4\omega^2}{\beta}\Big)}_{=\alpha_2(\delta)}\eta_j^2$$

Choose $\delta = \frac{\xi\theta^2}{4 - 2\xi\theta^2}$ and $\beta = \frac{\xi\theta^2}{2\Lambda_3(4 - \xi\theta^2)}$ to obtain $\alpha_1, \alpha_2 < 1.$

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Motivation: Geometric PDE

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Optimal Decay Rates

Assumptions: Let $0 < s \leq 1/d$.

- The solution u and forcing f are of class $\mathbb{A}_s$, namely given an error tolerance $\epsilon > 0$ there exists a refinement $\mathcal{T}_\epsilon$ of $\mathcal{T}_0$ such that

$$\|\nabla_\gamma(u - U_\epsilon)\|_{L^2(\gamma)} + \operatorname{osc}(f, \mathcal{T}_\epsilon) \leq \epsilon, \qquad \#\mathcal{T}_\epsilon - \#\mathcal{T}_0 \lesssim |u, f|_{\mathbb{A}_s} \epsilon^{-\frac{1}{s}}.$$

- The surface is of class $\mathbb{B}_s$ and $\mathcal{T}^+ =$ GEOMETRY $(\mathcal{T}, \tau)$ is s-optimal, i.e.

$$\#\mathcal{M}^+ \lesssim |\gamma|_{\mathbb{B}_s} \tau^{-\frac{1}{s}}.$$

Theorem

Assume that (u, f) are of class $\mathbb{A}_s$, that γ is of class $\mathbb{B}_s$, and that GEOMETRY is s-optimal. Then for $\theta \leq \theta_*$ and $\omega \leq \omega_*$ sufficiently small, we have

$$||\nabla_\gamma(u - U_k)||_{L^2(\gamma)} + \omega^{-1}\lambda_{\Gamma_k} + \operatorname{osc}(f, \mathcal{T}_k) \preceq \Big(|u, f|_{\mathbb{A}_s} + |\gamma|_{\mathbb{B}_s}\Big)(\#\mathcal{T}_k - \#\mathcal{T}_0)^{-s}.$$

Ingredients of the Proof

- Localized upper bound (to the refined set)
- Minimality of set $\mathcal{M}$ in Dörfler marking
- Explicit restriction of Dörfler parameter $\theta < \theta_* < 1$
- Explicit restriction of surface parameter $\omega \leq \omega_* < 1$
- Conditional contraction property of PDE
- Complexity of REFINE (Binev-Dahmen-DeVore ($d = 2$), Stevenson ($d > 2$), for conforming meshes, and Bonito-Nochetto for non-conforming meshes ($d \geq 2$)).

Greedy Algorithm for GEOMETRY

$\mathcal{T}^+ = \text{GEOMETRY}(\mathcal{T}, \tau)$

Theorem (Quasi-optimality of GEOMETRY for continuous pw linears)

Let γ be a surface of dimension d and piecewise of class W_p^2, $p > d$ (over the initial partition $\mathcal{T}_0$). Then the greedy algorithm terminates and is $1/d$-optimal

$$\#\mathcal{M}^+ \preceq |\gamma|_{W_p^2}^d \; \tau^{-d}.$$

Therefore, $\gamma \in W_p^2$ for $p > d$ is of class $\mathbb{B}_{\frac{1}{d}}$. Note that

$$\text{sob}\;(W_p^2) > \text{sob}\;(W_\infty^1) \quad \Rightarrow \quad 2 - \frac{d}{p} > 1 - \frac{d}{\infty} \quad \Rightarrow \quad p > d$$

but NOT $\gamma \in W_\infty^2$.

The Role of ω for Convergence Rates

Consider the example

$$-\Delta_\gamma u = 1, \quad \text{in } \gamma, \qquad u = 0, \quad \text{on } \partial\gamma,$$

where γ is the graph of class $C^{1,\alpha}$ given by

$$z(x,y) = \left(0.75 - x^2 - y^2\right)_+^{1+\alpha},$$

over the flat domain $\Omega = (0,1)^2$, and consider two cases $\alpha = 3/5$ and $\alpha = 2/5$.

It turns out that

$$\alpha = 3/5: \qquad \Rightarrow \qquad z \in \mathbb{B}_{1/2}$$
$$\alpha = 2/5: \qquad \Rightarrow \qquad z \in \mathbb{B}_t, \quad \forall t < 2/5.$$

The Role of ω for Convergence Rates: Case $\alpha = 3/5$

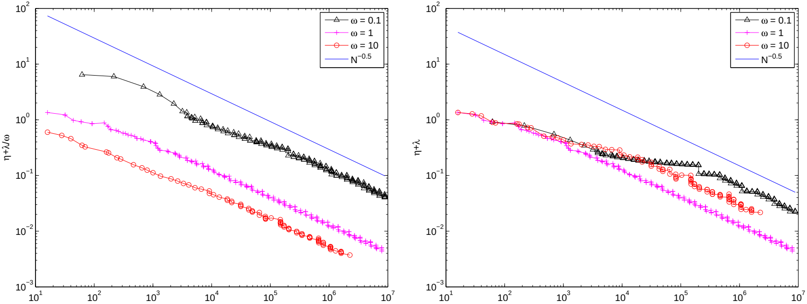

Figure: $\eta_k + \lambda_k/\omega$ (left) and $\eta_k + \lambda_k$ (right) versus the number of elements in logarithmic scale for $\omega = 0.1, 1, 10$. We observe that $\eta_k + \lambda_k/\omega$ decays as $N^{-0.5}$ right from the beginning, whereas $\eta_k + \lambda_k$ shows the same decay after the meshes have some refinement, depending on the value of ω. Our theory predicts the decay of $N^{-0.5}$ for both notions of total error if ω is sufficiently small, but the best relation between the error $\eta_k + \lambda_k$ and #DOFs is obtained for $w = 1$, which is *not so small*.

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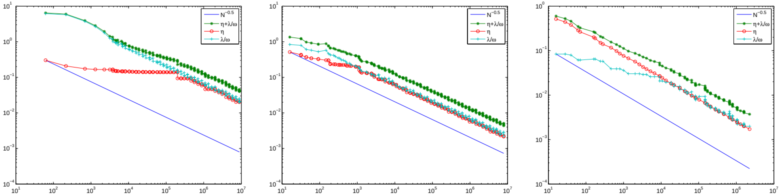


$\omega = 0.1$ $\qquad$ $\omega = 1$ $\qquad$ $\omega = 10$

Figure: η_k, λ_k/ω and $\eta_k + \lambda_k/\omega$ for $\omega = 0.1$ (left) $\omega = 1$ (middle) and $\omega = 10$ (right).

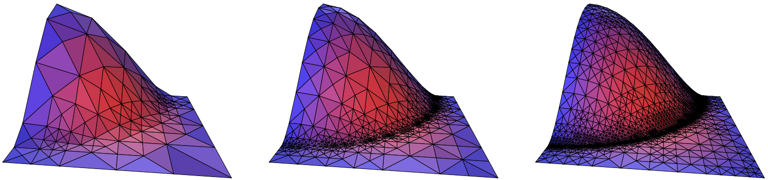
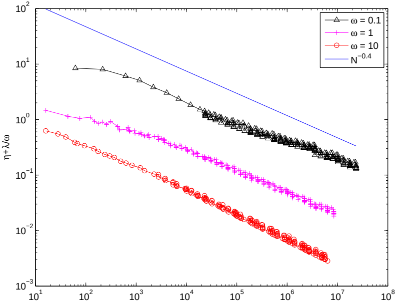


Figure: Meshes after 10, 20 and 30 refinements have been performed, $C^{1,0.6}$-surface, with $\omega = 1$. They are composed of 192, 1216 and 5564 elements, respectively.

The Role of ω for Convergence Rates: Case $\alpha = 2/5$



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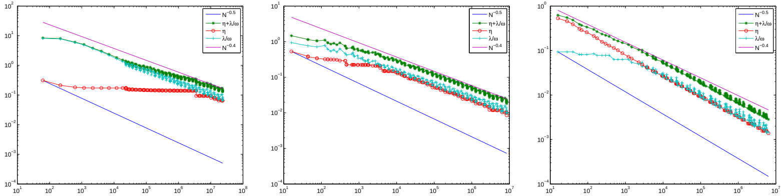
The Role of ω for Convergence Rates: Case $\alpha = 2/5$



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Discontinuous Coefficients (w. A. Bonito and R. DeVore)

Consider elliptic PDE of the form $-\operatorname{div}(A\nabla u) = f$ with

- $A = (a_{ij}(x))_{i,j=1}^d$ uniformly positive definite and bounded

$$\lambda_{\min}(A)|y|^2 \leq y^t A(x) y \leq \lambda_{\max}(A)|y|^2 \quad \forall\, x \in \Omega,\ y \in \mathbb{R}^d;$$

- The discontinuities of A are not match by the sequence of meshes $\mathcal{T}$;
- The forcing $f \in W_p^{-1}(\Omega)$ for some $p > 2$.

Goal: Design and study an AFEM able to handle such an A.

Difficulty: PDE perturbation results hinge on approximation of A in L^∞

$$\|u - \widehat{u}\|_{H_0^1(\Omega)} \leq \lambda_{\min}^{-1}(\widehat{A})\Big(\|f - \widehat{f}\|_{H^{-1}(\Omega)} + \|A - \widehat{A}\|_{L^\infty(\Omega)}\|f\|_{H^{-1}(\Omega)}\Big)$$

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Theorem (perturbation). Let $p \geq 2$, $q = 2p/(p-2) \in [2, \infty]$ and $\nabla u \in L^p(\Omega)$. Then

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Question: can we guarantee that $\nabla u \in L^p(\Omega)$ with $p > 2$?

Proposition (Meyers). Let $\widetilde{K} > 0$ be so that the solution $\widetilde{u}$ of the Laplacian satisfies

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DISC: AFEM for Discontinuous Diffusion Matrices

Given $\omega > 0$ explicit and $\beta < 1$, let

DISC$(\mathcal{T}_0, \epsilon_1)$
  $k = 1$
  LOOP
    $[\mathcal{T}(f)_k, f_k] =$ **RHS**$(\mathcal{T}_{k-1}, f, \omega\varepsilon_k)$
    $[\mathcal{T}(A)_k, A_k] =$ **COEFF**$(\mathcal{T}_k(f), A, \omega\varepsilon_k)$
    $[\mathcal{T}_k, U_k] =$ **PDE**$(\mathcal{T}(A)_k, A_k, f_k, \varepsilon_k/2)$
    $\epsilon_{k+1} = \beta\epsilon_k$
    $k \leftarrow k+1$
  END LOOP
END **DISC**

- $[\mathcal{T}(f)_k, f_k] =$ **RHS**$(\mathcal{T}_{k-1}, f, \omega\varepsilon_k)$ gives a mesh $\mathcal{T}_k(f) \geq \mathcal{T}_{k-1}$ and a pw polynonial approximation f_k of f on $\mathcal{T}_k^F$ such that $\|f - f_k\|_{H^{-1}(\Omega)} \leq \omega\epsilon_k$;
- $[\mathcal{T}(A)_k, A_k] =$ **COEFF**$(\mathcal{T}_k(f), A, \omega\varepsilon_k)$ gives a mesh $\mathcal{T}_k(A) \geq \mathcal{T}_k(f)$ and a pw polynomial approximation A_k of A on $\mathcal{T}_k(A)$ such that $\|A - A_k\|_{L^q(\Omega)} \leq \omega\epsilon_k$ and its eigenvalues satisfy uniformly in k

$$C^{-1}\lambda_{\min}(A) \leq \lambda(A_k) \leq C\lambda_{\max}(A).$$

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END **DISC**

- $[\mathcal{T}(f)_k, f_k] =$ **RHS**$(\mathcal{T}_{k-1}, f, \omega\varepsilon_k)$ gives a mesh $\mathcal{T}_k(f) \geq \mathcal{T}_{k-1}$ and a pw polynonial approximation f_k of f on $\mathcal{T}_k^F$ such that $\|f - f_k\|_{H^{-1}(\Omega)} \leq \omega\epsilon_k$;
- $[\mathcal{T}(A)_k, A_k] =$ **COEFF**$(\mathcal{T}_k(f), A, \omega\varepsilon_k)$ gives a mesh $\mathcal{T}_k(A) \geq \mathcal{T}_k(f)$ and a pw polynomial approximation A_k of A on $\mathcal{T}_k(A)$ such that $\|A - A_k\|_{L^q(\Omega)} \leq \omega\epsilon_k$ and its eigenvalues satisfy uniformly in k

$$C^{-1}\lambda_{\min}(A) \leq \lambda(A_k) \leq C\lambda_{\max}(A).$$

Optimality of DISC

Theorem (optimality). Assume that the right side f is in $\mathcal{B}^{s_f}(H^{-1}(\Omega))$ with $0 < s_f \le S$, and that the diffusion matrix A is positive definite, in $L_\infty(\Omega)$ and in $\mathcal{M}^{s_A}(L_q(\Omega))$ for $q := \frac{2p}{p-2}$ and $0 < s_A \le S$. Let $\mathcal{T}_0$ be the initial subdivision and $U_k \in \mathbb{V}(\mathcal{T}_k)$ be the Galerkin solution obtained at the kth iteration of the algorithm. Then, whenever $u \in \mathcal{A}^s(H_0^1(\Omega))$ for $0 < s \le S$, we have for $k \ge 1$

$$\|u - U_k\|_{H_0^1(\Omega)} \le \epsilon_k,$$

and

$$\#\mathcal{T}_k - \#\mathcal{T}_0 \lesssim \Big(|A|^{1/s_*}_{\mathcal{M}^{s_*}(L_q(\Omega))} + |f|^{1/s_*}_{\mathcal{B}^{s_*}(H^{-1}(\Omega))} + |u|^{1/s_*}_{\mathcal{A}^{s_*}(H_0^1(\Omega))}\Big)\epsilon_k^{-1/s_*},$$

with $s_* = \min(s, s_A, s_f)$.

Counterexample: s cannot be achieved if $s_A, s_f < s$.

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$$\|u - U_k\|_{H^1_0(\Omega)} \leq \epsilon_k,$$

and

$$\#\mathcal{T}_k - \#\mathcal{T}_0 \lesssim \Big(|A|^{1/s_*}_{\mathcal{M}^{s_*}(L_q(\Omega))} + |f|^{1/s_*}_{\mathcal{B}^{s_*}(H^{-1}(\Omega))} + |u|^{1/s_*}_{\mathcal{A}^{s_*}(H^1_0(\Omega))}\Big)\epsilon_k^{-1/s_*},$$

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Counterexample: s cannot be achieved if $s_A, s_f < s$.

Outline

Motivation: Geometric PDE

The Laplace-Beltrami Operator and Standard Adaptivity

Adaptive Finite Element Methods (AFEM) on Parametric Surfaces

Convergence Rates of AFEM

Discontinuous Coefficients

Comments and Conclusions

Comments and Conclusions

- **Coupling PDE-Geometry:** This is a new feature in adaptivity and leads to separate handling of geometry and PDE resolution with specific relative tolerances.
- **Convergence rates:** We show optimal convergence rates in the energy norm

$$\|\nabla(u - U_k)\|_{L^2(\gamma)} \lesssim (\#\mathcal{T}_k)^{-s}$$

  provided this is the rate of the best approximation of u in H^1 and that of γ in W^1_∞.
- **Weaker conditions on f:** We refer to Cohen, DeVore, Nochetto (2011) for convergence rates of elliptic PDE in flat domains with $f \in H^{-1}$ and A piecewise constant:

$$\operatorname{div}(A\nabla u) = f.$$

  We show that approximability of u is sufficient for a complete theory.
- **Weaker conditions on γ:** We assume γ is W^2_p with $p > d$, which implies γ is C^1. In the flat case, this corresponds to piecewise continuous A. We refer to Bonito, DeVore, Nochetto (2013) for convergence rates with weaker assumptions on A.