Convergence Rates for AFEM: General Theory

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Outline

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Adaptive Finite Element Method (AFEM)

Adaptive loop:

$$\text{AFEM}: \quad \text{SOLVE} \quad \rightarrow \quad \text{ESTIMATE} \quad \rightarrow \quad \text{MARK} \quad \rightarrow \quad \text{REFINE}$$

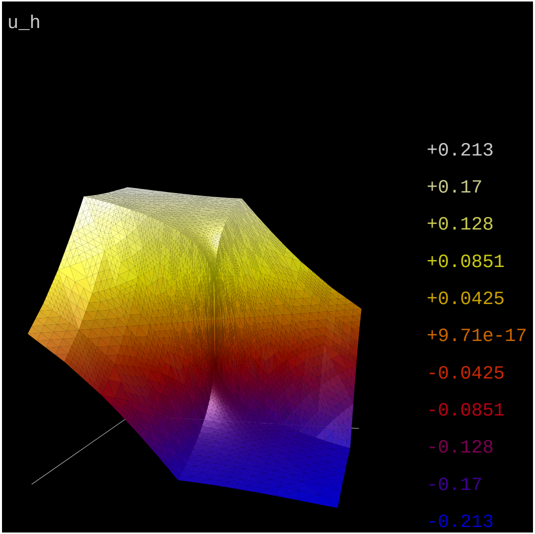
$$k \geq 0 \text{ loop counter} \quad \Rightarrow \quad (\mathcal{T}_k, \mathbb{V}(\mathcal{T}_k), U_k)$$

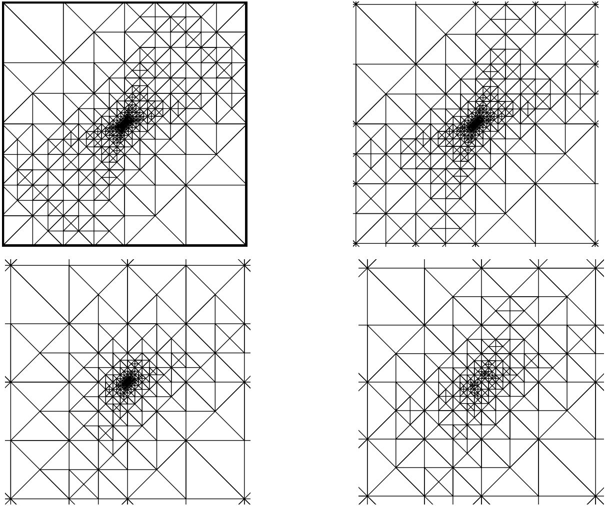
Questions:

- Convergence: This is not an asymptotic result for meshsize $h \rightarrow 0$!
  - Marking: what are minimal conditions?
  - Refinement: what refinements and meshes are admissible?
  - Problems: class and norms.
- Contraction:
  - What quantities such as norms are reduced by AFEM?
  - What quantities are good condidates for a contraction?
- Rates: Is the performance of AFEM better than classical FEM?
  - Performance: measured as error vs number of degrees of freedom
  - Nonlinear approximation theory: regularity W_p^2 with $p > 1$ for dimension $d = 2$ and polynomial degree $n = 1$, instead of classical regularity H^2.

Example (Kellogg' 75): Checkerboard discontinuous coefficients

$u \approx r^{0.1} \Rightarrow u \in H^{1.1}(\Omega) \Rightarrow |u - U_k|_{H^1(\Omega)} \approx \#\mathcal{T}_k^{-0.05}$ ($\mathcal{T}_k$ quasi-uniform)





Discontinuous coefficients: Final graded grid (full grid with < 2000 nodes) (top left), and 3 zooms $(\times 10^3, 10^6, 10^9)$; decay rate $N^{-1/2}$. Uniform grid would require $N \approx 10^{20}$ elements for a similar resolution.

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Warm-up: $1d$ Example

Question: given a continuous function $u : [0,1] \to \mathbb{R}$, a partition $\mathcal{T}_N = \{x_n\}_{n=0}^N$ with $x_0 = 0, x_N = 1$, and a pw constant approximation U_N of u over $\mathcal{T}_N$, what is the best decay rate of $\|u - U_N\|_{L^\infty(0,1)}$?

Answer 1: W^1_∞-Regularity. Let $u \in W^1_\infty(0,1)$ and $\mathcal{T}_N$ be quasi-uniform. Then $U_N(x) = u(x_{n-1})$ for $x_{n-1} \le x < x_n$ satisfies

$$|U_n(x) - u(x)| = |u(x_{n-1}) - u(x)| \le \int_x^{x_{n-1}} |u'(s)| ds \preccurlyeq \frac{1}{N} \|u'\|_{L^\infty(0,1)}.$$

Answer 2: W^1_1-Regularity. Let $u \in W^1_1(0,1)$. If x_n is defined by

$$\int_{x_{n-1}}^{x_n} |u'(s)| ds = \frac{1}{N} \|u'\|_{L^1(0,1)},$$

then

$$|U_n(x) - u(x)| = |u(x_{n-1}) - u(x)| \le \int_x^{x_{n-1}} |u'(s)| ds \le \frac{1}{N} \|u'\|_{L^1(0,1)}.$$

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Sobolev Number

Let $\omega \subset \mathbb{R}^d$ be Lipschitz and bounded, $k \in \mathbb{N}$, $1 \le p \le \infty$. The *Sobolev number* of $W_p^k(\omega)$ is

$$\operatorname{sob}(W_p^k) := k - \frac{d}{p}.$$

Remark 1. This number governs the scaling properties of seminorm $|v|_{W_p^k(\omega)}$: consider $\hat{x} = \frac{1}{h}x$ which transforms ω into $\widehat{\omega}$ and note

$$|\hat{v}|_{W_p^k(\widehat{\omega})} = h^{\operatorname{sob}(W_p^k)} |v|_{W_p^k(\omega)} \quad \forall v \in W_p^k(\omega).$$

Remark 2. Let $d = 1$ and $\omega = (0,1)$. Then $W_\infty^1(\omega)$ is the linear (and usual) Sobolev scale of $L^\infty(\omega)$, but $W_1^1(\omega)$ is in the *nonlinear* scale of $L^\infty(\omega)$, i.e.

$$\operatorname{sob}(W_1^1) = 1 - \frac{1}{1} = 0 - \frac{1}{\infty} = \operatorname{sob}(L^\infty).$$

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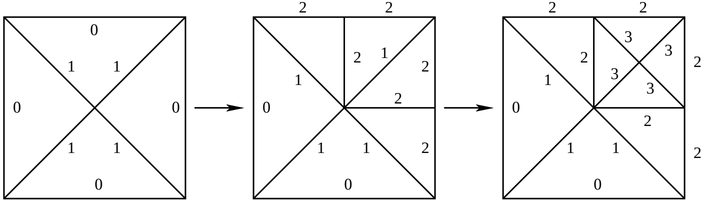
Conforming Meshes: The Bisection Method and REFINE

- Labeling of a sequence of conforming refinements $\mathcal{T}_0 \le \mathcal{T}_1 \le \mathcal{T}_2$ for $d = 2$ (similar but much more intricate for $d > 2$)

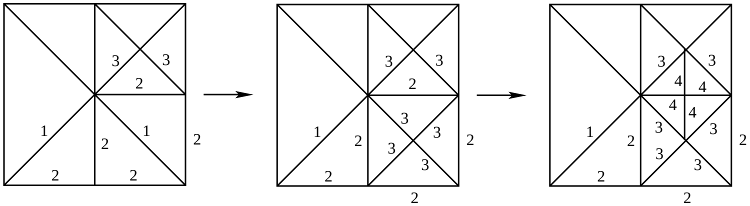


- Shape regularity: the shape-regularity constant of any $\mathcal{T} \in \mathbb{T}$ solely depends on the shape-regularity constant of $\mathcal{T}_0$.
- Nested spaces: refinement leads to $\mathbb{V}(\mathcal{T}) \subset \mathbb{V}(\mathcal{T}_*)$ because $\mathcal{T} \le \mathcal{T}_*$.
- Monotonicity of meshsize function $h_{\mathcal{T}}$: if $h_{\mathcal{T}|T} := h_T := |T|^{1/d}$, then $h_{\mathcal{T}_*} \le h_{\mathcal{T}}$ for $\mathcal{T}_* \ge \mathcal{T}$, and reduction property with $b \ge 1$ bisections

$$h_{\mathcal{T}_*|T} \le 2^{-b/d} h_{\mathcal{T}|T} \quad \forall T \in \mathcal{T} \setminus \mathcal{T}_*.$$

Complexity of REFINE

- Recursive bisection of $\mathcal{T}_3$ (sequence of compatible bisection patches)



- Naive estimate is NOT valid with Λ_0 independent of refinement level

$$\#\mathcal{T}_* - \#\mathcal{T} \leq \Lambda_0 \ \#\mathcal{M}$$

- Complexity of REFINE (Binev, Dahmen, DeVore '04 ($d = 2$), and Stevenson' 07 ($d > 2$)): If $\mathcal{T}_0$ has a suitable labeling, then there exists a constant $\Lambda_0 > 0$ only depending on $\mathcal{T}_0$ and d such that for all $k \geq 1$

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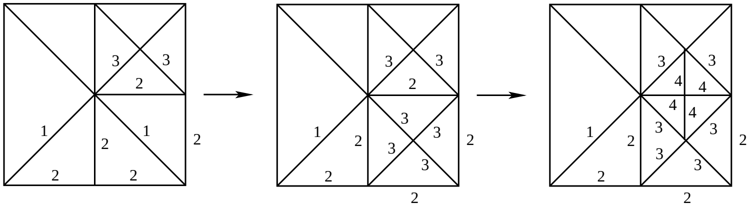

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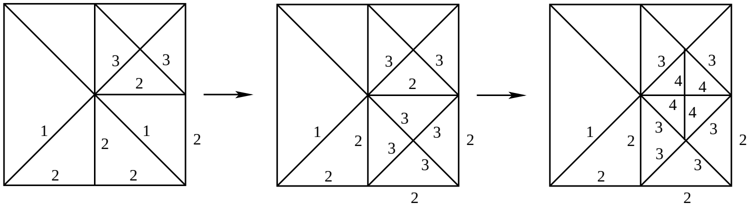

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Piecewise Polynomial Interpolation

Quasi-local error estimate: if $0 \leq t \leq s \leq n+1$ ($n \geq 1$ polynomial degree) and $1 \leq p, q \leq \infty$ satisfy $\mathrm{sob}(W_p^s) > \mathrm{sob}(W_q^t)$, then for all $T \in \mathcal{T}$

$$\|D^t(v - I_{\mathcal{T}}v)\|_{L^q(T)} \lesssim h_T^{\mathrm{sob}(W_p^s)-\mathrm{sob}(W_q^t)} \|D^s v\|_{L^p(\mathcal{N}_{\mathcal{T}}(T))},$$

where $\mathcal{N}_{\mathcal{T}}(T)$ is a discrete neighborhood of T and $I_{\mathcal{T}}$ is a quasi interpolation operator (Clement or Scott-Zhang). If $\mathrm{sob}(W_p^s) > 0$, then v is Hölder continuous, $I_{\mathcal{T}}$ can be replaced by the Lagrange interpolation operator, and $\mathcal{N}_{\mathcal{T}}(T) = T$.

- **Quasi-uniform meshes:** if $1 \leq s \leq n+1$ and $u \in H^s(\Omega)$, then

$$\|\nabla(v - I_{\mathcal{T}}v)\|_{L^2(\Omega)} \preccurlyeq |v|_{H^s(\Omega)} (\#\mathcal{T})^{-\frac{s-1}{d}}.$$

- **Optimal error decay:** If $s = n+1$ (linear Sobolev scale), then

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Adaptive Approximation (Binev, Dahmen, DeVore, Petrushev '02)

Question: can one achieve the same decay rate with lower regularity?

- Let $n = 1, d = 2$ and note that $H^2(\Omega) \subset \mathcal{A}_{1/2}$ where

$$\mathcal{A}_{1/2} = \{v \in H_0^1(\Omega) : \inf_{\#\mathcal{T} - \#\mathcal{T}_0 \le N} |v - I_{\mathcal{T}} v|_{H^1(\Omega)} \preccurlyeq N^{-1/2}\}$$

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$$\mathrm{sob}(W_p^2) = 2 - \frac{2}{p} > 1 - \frac{2}{2} = 0 = \mathrm{sob}(H^1).$$

- **Theorem 1.** Given any $\delta > 0$, the following algorithm THRESHOLD

  THRESHOLD$(\mathcal{T}, \delta)$
  while $\mathcal{M} := \{T \in \mathcal{T} : \|\nabla(v - I_{\mathcal{T}} v)\|_{L^2(T)} > \delta\} \neq \emptyset$
  $\quad\mathcal{T} :=$ REFINE$(\mathcal{T}, \mathcal{M})$
  end while
  return$(\mathcal{T})$

  terminates if $v \in W_p^2(\Omega; \mathcal{T}_0) \cap H_0^1(\Omega)$ with $p > 1$ and its output satisfies

$$|v - I_{\mathcal{T}} v|_{H^1(\Omega)} \lesssim \delta (\#\mathcal{T})^{1/2}, \qquad \#\mathcal{T} - \#\mathcal{T}_0 \lesssim \delta^{-1} |\Omega|^{1-1/p} \|D^2 v\|_{L^p(\Omega; \mathcal{T}_0)}.$$

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Remarks on Adaptive Approximation

- Let $v \in H_0^1(\Omega) \cap W_p^2(\Omega; \mathcal{T}_0)$, $n = 1$, $d = 2$, $p > 1$. For $N > \#\mathcal{T}_0$ there exists $\mathcal{T} \in \mathbb{T}$ such that

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  Choose $\delta = |\Omega|^{1-1/p} \, \|D^2 v\|_{L^p(\Omega)} N^{-1}$ in algorithm THRESHOLD.
- $W_p^2(\Omega; \mathcal{T}_0) \subset \mathcal{A}_{1/2}$ for $d = 2$ and $p > 1$. All geometric singularities for $d = 2$ (corner and interfaces) satisfy this (Nicaise' 94).
- For arbitrary $n \geq 1$, $d \geq 2$, comparing Sobolev numbers yields

$$n + 1 - \frac{d}{p} > \mathrm{sob}(H^1) = 1 - \frac{d}{2} \quad \Rightarrow \quad p > \frac{2d}{2n + d}.$$

  This may give $p < 1$ and corresponding Besov space $B_p^{n+1}(L^p(\Omega))$. Theorem 1 holds for any $n \geq 1$ (Gaspoz and Morin '11). Regularity theory for elliptic PDE is incomplete for $p < 1$.
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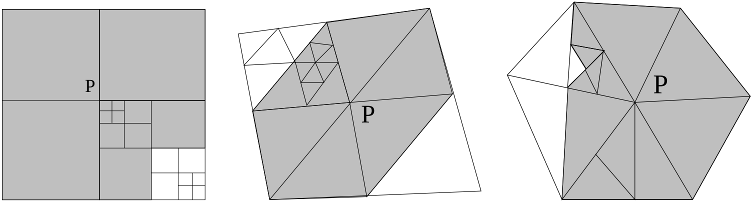

- Fixed level of nonconformity: domains of influence are comparable with elements contained in them (Ex: one hanging node per edge for quadrilaterals).

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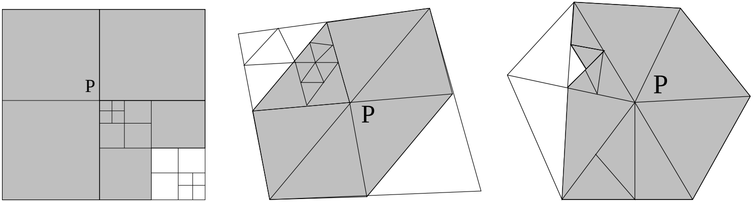

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Model Problem: Basic Assumptions

Consider model problem

$$-\mathrm{div}(A\nabla u) = f \quad \text{in} \quad \Omega, \qquad u|_{\partial\Omega} = 0,$$

with

- Ω polygonal domain in $\mathbb{R}^d$, $d \geq 2$;
- $\mathcal{T}_0$ is a conforming mesh made of simplices with compatible labeling;
- $A(x)$ is symmetric and positive definite for all $x \in \Omega$ with eigenvalues $\lambda(x)$ satisfying

$$0 < a_{\min} \leq \lambda_i(x) \leq a_{\max}, \quad x \in \Omega;$$

- A is piecewise Lipschitz in $\mathcal{T}_0$;
- $f \in L^2(\Omega)$ ($f \in H^{-1}(\Omega)$ in Extensions);
- $\mathbb{V}(\mathcal{T})$ space of continuous elements of degree $\leq n$ over a conforming refinement $\mathcal{T}$ of $\mathcal{T}_0$ (by bisection).
- Exact numerical integration.

Galerkin Method

- **Function space:** $\mathbb{V} := H_0^1(\Omega)$.
- **Bilinear form:** $\mathcal{B} : \mathbb{V} \times \mathbb{V} \to \mathbb{R}$

$$\mathcal{B}(v, w) := \int_\Omega A\nabla v \cdot \nabla w \quad \forall v, w \in \mathbb{V}.$$

Then solution u of model problem satisfies

$$u \in \mathbb{V} : \quad \mathcal{B}(u, v) = \langle f, v \rangle \quad \forall v \in \mathbb{V}.$$

- **Finite element space:** If $\mathbb{P}_n(T)$ denote polynomials of degree $\leq n$ over T, then

$$\mathbb{V}(\mathcal{T}) := \{v \in H_0^1(\Omega) : v|_T \in \mathbb{P}_n(T) \quad \forall T \in \mathcal{T}\}.$$

- **Galerkin solution:** The discrete solution $U = U_\mathcal{T}$ satisfies

$$U \in \mathbb{V}(\mathcal{T}) : \quad \mathcal{B}(U, V) = \langle f, V \rangle \quad \forall V \in \mathbb{V}(\mathcal{T}).$$

Galerkin Method (Continued)

- **Residual:** $\mathcal{R} \in V^* = H^{-1}(\Omega)$ is given by

$$\langle \mathcal{R},\, v\rangle := \langle f,\, v\rangle - \mathcal{B}(U, v) = \mathcal{B}(u - U, v) \quad \forall v \in V.$$

- **Galerkin Orthogonality:** $\langle \mathcal{R},\, V\rangle = 0 \quad \forall V \in \mathbb{V}(\mathcal{T})$.
- **Quasi-Best Approximation (Céa Lemma):** $\alpha_1 \le \alpha_2$ coercivity and continuity constants of $\mathcal{B}$

$$\begin{aligned}\alpha_1 \|u - U\|_{\mathbb{V}}^2 &\le \mathcal{B}(u - U, u - U) = \mathcal{B}(u - U, u - V)\\ &\le \alpha_2 \|u - U\|_{\mathbb{V}} \|u - V\|_{\mathbb{V}} \quad \forall V \in \mathbb{V}(\mathcal{T}).\end{aligned}$$

$$\Rightarrow \quad \|u - U\|_{\mathbb{V}} \le \frac{\alpha_2}{\alpha_1} \inf_{V \in \mathbb{V}(\mathcal{T})} \|u - V\|_{\mathbb{V}}.$$

- **Approximation Class $\mathcal{A}_s$:** Let $0 < s \le n/d$ $(n \ge 1)$ and

$$\mathcal{A}_s := \left\{ v \in \mathbb{V} : |u|_s := \sup_{N > 0} \left(N^s \inf_{\#\mathcal{T} - \#\mathcal{T}_0 \le N} \inf_{V \in \mathbb{V}(\mathcal{T})} \|v - V\|_{\mathbb{V}} \right) \right\}$$

$$\Rightarrow \quad \exists\, \mathcal{T} \in \mathbb{T} : \quad \#\mathcal{T} - \#\mathcal{T}_0 \le N, \quad \inf_{V \in \mathbb{V}(\mathcal{T})} \|v - V\|_{\mathbb{V}} \le |v|_s N^{-s}.$$

A Priori Error Analysis

If $u \in \mathcal{A}_s$, $0 < s \leq n/d$, there exists $\mathcal{T} \in \mathbb{T}$ with $\#\mathcal{T} - \#\mathcal{T}_0 \leq N$ and

$$\|u - U\|_{\mathbb{V}} \leq \frac{\alpha_2}{\alpha_1} |u|_s N^{-s}.$$

- If $n = 1, d = 2, p > 1$, and $u \in \mathbb{V} \cap W_p^2(\Omega; \mathcal{T}_0)$, then THRESHOLD shows that $|u|_{1/2} \preccurlyeq \|D^2 u\|_{L^p(\Omega;\mathcal{T}_0)}$ whence (optimal estimate)

$$\exists\, \mathcal{T} \in \mathbb{T}: \quad \#\mathcal{T} - \#\mathcal{T}_0 \leq N, \quad \|u - U\|_{\mathbb{V}} \preccurlyeq \|D^2 u\|_{L^p(\Omega;\mathcal{T}_0)} N^{-1/2}.$$

- THRESHOLD needs access to the element interpolation error E_T and so to the unknown u. It is thus not practical.
- The a posteriori error analysis provides a tool to extract this missing information from the residual $\mathcal{R}$. This is discussed next.
- The a priori analysis is valid for a bilinear for $\mathcal{B}$ on a Hilbert space $\mathbb{V}$ that is continuous and satisfies a discrete inf-sup condition

$$|\mathcal{B}(v, w)| \leq \alpha_1 \|v\|_{\mathbb{V}} \|w\|_{\mathbb{V}} \quad \forall v, w \in \mathbb{V};$$

$$\alpha_2 \|V\|_{\mathbb{V}} \leq \sup_{W \in V} \frac{\mathcal{B}(V, W)}{\|W\|_{\mathbb{V}}} \quad \forall V \in \mathbb{V}(\mathcal{T}).$$

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Outline

Error-Residual Equation (Babuška-Miller' 87)

- Since $\langle \mathcal{R}, v\rangle = \langle f, v\rangle - \mathcal{B}(U,v) = \mathcal{B}(u-U,v)$ for all $v \in \mathbb{V}$, we deduce

$$\|u-U\|_{\mathbb{V}} \le \frac{1}{\alpha_1}\|\mathcal{R}\|_{\mathbb{V}^*} \le \frac{\alpha_2}{\alpha_1}\|u-U\|_{\mathbb{V}}.$$

- **Residual representation:** elementwise integration by parts yields

$$\langle \mathcal{R}, v\rangle = \sum_{T\in\mathcal{T}} \int_T \underbrace{f + \mathrm{div}_{\mathcal{T}}(A\nabla U)}_{=r} v + \sum_{S\in\mathcal{S}} \int_S \underbrace{[A\nabla U]\cdot\nu}_{=j} v \quad \forall v \in \mathbb{V}$$

where $r = r(U), j = j(U)$ are the interior and jump residuals.

- **Localization:** The Courant (hat) basis $\{\phi_z\}_{z\in\mathcal{N}(\mathcal{T})}$ satisfy the partition of unity property $\sum_{z\in\mathcal{N}(\mathcal{T})}\phi_z = 1$. Therefore, for all $v \in \mathbb{V}$,

$$\langle \mathcal{R}, v\rangle = \sum_{z\in\mathcal{N}(\mathcal{T})} \langle \mathcal{R}, v\phi_z\rangle = \sum_{z\in\mathcal{N}(\mathcal{T})} \Big(\int_{\omega_z} rv\phi_z + \int_{\gamma_z} jv\phi_z\Big).$$

- **Galerkin orthogonality:** $\int_{\omega_z} r\phi_z + \int_{\gamma_z} j\phi_z = 0 \quad \forall z \in \mathcal{N}_0(\mathcal{T})$

Reliability: Global Upper A Posteriori Bound

- Exploit Galerkin orthogonality

$$\langle \mathcal{R}, v\rangle = \sum_{z\in\mathcal{N}(\mathcal{T})} \Big(\int_{\omega_z} r(v-\alpha_z(v))\phi_z + \int_{\gamma_z} j(v-\alpha_z(v))\phi_z \Big)$$

and take $\alpha_z(v) := \frac{\int_{\omega_z} v\phi_z}{\int_{\omega_z}\phi_z}$ if z is interior and $\alpha_z(v)=0$ if $z\in\partial\Omega$.

- Use Poincaré inequality in ω_z

$$\|v-\alpha_z(v)\|_{L^2(\omega_z)} \le C_0 h_z \|\nabla v\|_{L^2(\omega_z)} \quad \forall z\in\mathcal{N}(\mathcal{T})$$

and a scaled trace lemma, to deduce

$$\big|\langle \mathcal{R}, v\phi_z\rangle\big| \preccurlyeq \Big(h_z\|r\phi_z^{1/2}\|_{L^2(\omega_z)} + h_z^{1/2}\|j\phi_z^{1/2}\|_{L^2(\gamma_z)}\Big)\|\nabla v\|_{L^2(\omega_z)}.$$

- Sum over $z\in\mathcal{N}(\mathcal{T})$ and use $\sum_{z\in\mathcal{N}(\mathcal{T})}\|\nabla v\|^2_{L^2(\omega_z)} \preccurlyeq \|\nabla v\|^2_{L^2(\Omega)}$ to get

$$\|\mathcal{R}\|_{\mathbb{V}^*} \preccurlyeq \Big(\sum_{z\in\mathcal{N}(\mathcal{T})} h_z^2\|r\phi_z^{1/2}\|^2_{L^2(\omega_z)} + h_z\|j\phi_z^{1/2}\|^2_{L^2(\gamma_z)}\Big)^{1/2}.$$

Reliability: Global Upper A Posteriori Bound

- Exploit Galerkin orthogonality

$$\langle \mathcal{R}, v \rangle = \sum_{z \in \mathcal{N}(\mathcal{T})} \Big(\int_{\omega_z} r(v - \alpha_z(v))\phi_z + \int_{\gamma_z} j(v - \alpha_z(v))\phi_z \Big)$$

and take $\alpha_z(v) := \frac{\int_{\omega_z} v\phi_z}{\int_{\omega_z} \phi_z}$ if z is interior and $\alpha_z(v) = 0$ if $z \in \partial\Omega$.

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- Sum over $z \in \mathcal{N}(\mathcal{T})$ and use $\sum_{z \in \mathcal{N}(\mathcal{T})} \|\nabla v\|^2_{L^2(\omega_z)} \preccurlyeq \|\nabla v\|^2_{L^2(\Omega)}$ to get

$$\|\mathcal{R}\|_{\mathbb{V}^*} \preccurlyeq \Big(\sum_{z \in \mathcal{N}(\mathcal{T})} h_z^2 \|r\phi_z^{1/2}\|^2_{L^2(\omega_z)} + h_z \|j\phi_z^{1/2}\|^2_{L^2(\gamma_z)} \Big)^{1/2}.$$

Reliability: Global Upper A Posteriori Bound

- Exploit Galerkin orthogonality

$$\langle \mathcal{R},\, v\rangle = \sum_{z\in\mathcal{N}(\mathcal{T})} \Big(\int_{\omega_z} r(v-\alpha_z(v))\phi_z + \int_{\gamma_z} j(v-\alpha_z(v))\phi_z \Big)$$

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Upper A Posteriori Bound (Element Oriented)

- Use that $h_z \preccurlyeq h(x)$ for all $x \in \omega_z$, and $\sum_{z \in \mathcal{N}(\mathcal{T})} \phi_z = 1$, to derive

$$\|\mathcal{R}\|_{V^*} \preccurlyeq \left(\|hr\|^2_{L^2(\Omega)} + \|h^{1/2} j\|^2_{L^2(\Gamma)} \right)^{1/2}$$

in terms of weighted (and computable) L^2 norms of the residuals.

- **Upper bound**: Introduce element indicators $\mathcal{E}_\mathcal{T}(U,T)$

$$\mathcal{E}_\mathcal{T}(U,T)^2 = h_T^2 \|r\|^2_{L^2(T)} + h_T \|j\|^2_{L^2(\partial T)}$$

and error estimator $\mathcal{E}_\mathcal{T}(U)^2 = \sum_{T \in \mathcal{T}} \mathcal{E}_\mathcal{T}(U,T)^2$. Then

$$\|u - U\|_{\mathbb{V}} \leq \frac{1}{\alpha_1} \|\mathcal{R}\|_{V^*} \preccurlyeq \frac{1}{\alpha_1} \mathcal{E}_\mathcal{T}(U).$$

Efficiency: Local Lower A Posteriori Bound ($n = 1$) (Verfürth'89)

- **Local dual norms:** for $v \in H_0^1(\omega)$ we have

$$\langle \mathcal{R}, v \rangle = \mathcal{B}(u-U, v) \le \alpha_2 \|u-U\|_{\mathbb{V}} \|v\|_{\mathbb{V}} \quad \Rightarrow \quad \|\mathcal{R}\|_{H^{-1}(\omega)} \le \alpha_2 \|u-U\|_{\mathbb{V}}$$

- **Interior residual:** take $\omega = T \in \mathcal{T}$ and note $\langle \mathcal{R}, v \rangle = \int_T rv$. Then

$$\|\mathcal{R}\|_{H^{-1}(T)} = \|r\|_{H^{-1}(T)}$$

- **Overestimation:** Poincaré inequality yields $\|r\|_{H^{-1}(T)} \preccurlyeq h_T \|r\|_{L^2(T)}$

$$\int_T rv \le \|r\|_{L^2(T)} \|v\|_{L^2(T)} \preccurlyeq h_T \|r\|_{L^2(T)} \|\nabla v\|_{L^2(T)}$$

- **Pw constant** r**:** Let $\eta \in H_0^1(T)$, $|T| \preccurlyeq \int_T \eta, \|\nabla \eta\|_{L^\infty(T)} \preccurlyeq h_T^{-1}$. Then

$$\begin{aligned}\|r\|_{L^2(T)}^2 &\preccurlyeq \int_T r(r\eta) \le \|r\|_{H^{-1}(T)} \|r\|_{L^2(T)} \|\nabla \eta\|_{L^\infty(T)} \\ &\preccurlyeq h_T^{-1} \|r\|_{H^{-1}(T)} \|r\|_{L^2(T)} \quad \Rightarrow \quad h_T \|r\|_{L^2(T)} \preccurlyeq \|r\|_{H^{-1}(T)}\end{aligned}$$

Efficiency: Local Lower A Posteriori Bound ($n=1$) (Verfürth'89)

- **Local dual norms:** for $v \in H_0^1(\omega)$ we have

$$\langle \mathcal{R},\, v\rangle = \mathcal{B}(u{-}U, v) \le \alpha_2 \|u{-}U\|_{\mathbb{V}} \|v\|_{\mathbb{V}} \quad \Rightarrow \quad \|\mathcal{R}\|_{H^{-1}(\omega)} \le \alpha_2 \|u{-}U\|_{\mathbb{V}}$$

- **Interior residual:** take $\omega = T \in \mathcal{T}$ and note $\langle \mathcal{R},\, v\rangle = \int_T rv$. Then

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- **Overestimation:** Poincaré inequality yields $\|r\|_{H^{-1}(T)} \preccurlyeq h_T \|r\|_{L^2(T)}$

$$\int_T rv \le \|r\|_{L^2(T)} \|v\|_{L^2(T)} \preccurlyeq h_T \|r\|_{L^2(T)} \|\nabla v\|_{L^2(T)}$$

- **Pw constant r:** Let $\eta \in H_0^1(T)$, $|T| \preccurlyeq \int_T \eta$, $\|\nabla \eta\|_{L^\infty(T)} \preccurlyeq h_T^{-1}$. Then

$$\|r\|_{L^2(T)}^2 \preccurlyeq \int_T r(r\eta) \le \|r\|_{H^{-1}(T)} \|r\|_{L^2(T)} \|\nabla \eta\|_{L^\infty(T)}$$

$$\preccurlyeq h_T^{-1} \|r\|_{H^{-1}(T)} \|r\|_{L^2(T)} \quad \Rightarrow \quad h_T \|r\|_{L^2(T)} \preccurlyeq \|r\|_{H^{-1}(T)}$$

Efficiency: Local Lower A Posteriori Bound ($n = 1$) (Verfürth'89)

- **Local dual norms:** for $v \in H_0^1(\omega)$ we have

$$\langle \mathcal{R}, v \rangle = \mathcal{B}(u-U, v) \le \alpha_2 \|u-U\|_{\mathbb{V}} \|v\|_{\mathbb{V}} \quad \Rightarrow \quad \|\mathcal{R}\|_{H^{-1}(\omega)} \le \alpha_2 \|u-U\|_{\mathbb{V}}$$

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Lower A Posteriori Bound: Oscillation

- **Oscillation of r:** $h_T\|r-\bar{r}_T\|_{L^2(T)}$ with meanvalue $\bar{r}_T$. Then

$$h_T\|r\|_{L^2(T)} \preccurlyeq \|\mathcal{R}\|_{H^{-1}(T)} + h_T\|r-\bar{r}_T\|_{L^2(T)}$$

- **Data oscillation:** if A is pw constant, then $r=f$ and

$$h_T\|r-\bar{r}_T\|_{L^2(T)} = h_T\|f-\bar{f}_T\|_{L^2(T)} = \operatorname{osc}_{\mathcal{T}}(f,T)$$

- **Oscillation of j:** likewise $h_S\|j-\bar{j}_S\|_{L^2(S)}$ with meanvalue $\bar{j}_S$ and

$$h_S^{1/2}\|j\|_{L^2(S)} \preccurlyeq \|\mathcal{R}\|_{H^{-1}(\omega_S)} + h_S^{1/2}\|j-\bar{j}_S\|_{L^2(S)} + h_S\|r\|_{L^2(\omega_S)}$$

  where $\omega_S = T_1\cup T_2$ with $T_1\cap T_2 = S$ and $T_1,T_2\in\mathcal{T}$.

- **Local lower bound:** let $\omega_T = \cup_{S\in\partial T}\omega_S$ and the local oscillation be $\operatorname{osc}_{\mathcal{T}}(U,\omega_T) := \|h(r-\bar{r})\|_{L^2(\omega_T)} + \|h^{1/2}(j-\bar{j})\|_{L^2(\partial T)}$. Then

$$\mathcal{E}_{\mathcal{T}}(U,T) \preccurlyeq \alpha_2\|\nabla(u-U)\|_{L^2(\omega_T)} + \operatorname{osc}_{\mathcal{T}}(U,\omega_T).$$

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where $\omega_S = T_1\cup T_2$ with $T_1\cap T_2 = S$ and $T_1,T_2\in\mathcal{T}$.

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$$\mathcal{E}_{\mathcal{T}}(U,T) \preccurlyeq \alpha_2\|\nabla(u-U)\|_{L^2(\omega_T)} + \operatorname{osc}_{\mathcal{T}}(U,\omega_T).$$

Lower A Posteriori Bound (Continued)

- **Higher order:** we expect $\mathrm{osc}_{\mathcal{T}}(U, \omega_T) \ll \|\nabla(u-U)\|_{L^2(\omega_T)}$ as $h_T \to 0$.
- **Marking:** if $\mathcal{E}_{\mathcal{T}}(U,T) \preccurlyeq \|\nabla(u-U)\|_{L^2(\omega_T)}$ and $\mathcal{E}_{\mathcal{T}}(U,T)$ is large relative to $\mathcal{E}_{\mathcal{T}}(U)$, then T contains a large portion of the error. To improve the solution U effectively, such T must be split giving rise to a procedure that tries to equidistribute errors.
- **Global lower bound:** we have $\mathcal{E}_{\mathcal{T}}(U) \preccurlyeq \alpha_2 \|u-U\|_{\mathbb{V}} + \mathrm{osc}_{\mathcal{T}}(U)$ where

$$\mathrm{osc}_{\mathcal{T}}(U) = \|h(r-\bar{r})\|_{L^2(\Omega)} + \|h^{1/2}(j-\bar{j})\|_{L^2(\Gamma)}.$$

- **Discrete local lower bound (Dörfler'96, Morin, N, Siebert'00):**

$$\mathcal{E}_{\mathcal{T}}(U,T) \preccurlyeq \alpha_2 \|\nabla(U_* - U)\|_{L^2(\omega_T)} + \mathrm{osc}_{\mathcal{T}}(U, \omega_T).$$

provided the interior of T and each of its sides contain a node of $\mathcal{T}_* \geq \mathcal{T}$ (interior node property).

Lower A Posteriori Bound (Continued)

- **Higher order:** we expect $\operatorname{osc}_{\mathcal{T}}(U, \omega_T) \ll \|\nabla(u-U)\|_{L^2(\omega_T)}$ as $h_T \to 0$.

- **Marking:** if $\mathcal{E}_{\mathcal{T}}(U, T) \preccurlyeq \|\nabla(u-U)\|_{L^2(\omega_T)}$ and $\mathcal{E}_{\mathcal{T}}(U, T)$ is large relative to $\mathcal{E}_{\mathcal{T}}(U)$, then T contains a large portion of the error. To improve the solution U effectively, such T must be split giving rise to a procedure that tries to equidistribute errors.

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- **Global lower bound:** we have $\mathcal{E}_{\mathcal{T}}(U) \preccurlyeq \alpha_2\|u-U\|_{\mathbb{V}} + \operatorname{osc}_{\mathcal{T}}(U)$ where

$$\operatorname{osc}_{\mathcal{T}}(U) = \|h(r-\bar{r})\|_{L^2(\Omega)} + \|h^{1/2}(j-\bar{j})\|_{L^2(\Gamma)}.$$

- **Discrete local lower bound (Dörfler'96, Morin, N, Siebert'00):**

$$\mathcal{E}_{\mathcal{T}}(U,T) \preccurlyeq \alpha_2\|\nabla(U_*-U)\|_{L^2(\omega_T)} + \operatorname{osc}_{\mathcal{T}}(U,\omega_T).$$

provided the interior of T and each of its sides contain a node of $\mathcal{T}_* \geq \mathcal{T}$ (interior node property).

Outline

Adaptive Finite Element Method (AFEM)

AFEM : SOLVE $\rightarrow$ ESTIMATE $\rightarrow$ MARK $\rightarrow$ REFINE

$k \geq 0$ loop counter $\Rightarrow$ $(\mathcal{T}_k, \mathbb{V}(\mathcal{T}_k), U_k)$

- $U_k = \text{SOLVE}(\mathcal{T}_k)$ computes the exact Galerkin solution $U_k \in \mathbb{V}(\mathcal{T}_k)$
  - dealing with L^2
  - exact linear algebra
- $\mathcal{E}_k = \text{ESTIMATE}(\mathcal{T}_k, U_k, f)$ computes local error indicators $e(z)$
  - localization of global H^{-1} norms to stars ω_z for $z \in \mathcal{N}_k = \mathcal{N}(\mathcal{T}_k)$
  - computation of residuals in weighted L^2 norms
- $\mathcal{M}_k = \text{MARK}(\mathcal{E}_k, \mathcal{T}_k)$ selects $\mathcal{M}_k \subset \mathcal{T}_k$ using Dörfler marking
  - $\mathcal{E}_k(\mathcal{M}_k) \geq \theta \mathcal{E}_k(\mathcal{T}_k)$ for $0 < \theta < 1$ (bulk chasing)
  - marked set $\mathcal{M}_k$ must be minimal for optimal rates
- $\mathcal{T}_{k+1} = \text{REFINE}(\mathcal{T}_k, \mathcal{M}_k)$ refines the marked elements $\mathcal{M}_k$ and outputs a conforming mesh $\mathcal{T}_{k+1}$ refinement of $\mathcal{T}_k$
  - uses $b \geq 1$ newest vertex bisection (Mitchell) for $d = 2$ to refine each $T \in \mathcal{M}_k$ so that each element $T \in \mathcal{T}_k$ is bisected at least once.

AFEM: Main Results

- Convergence of AFEM: $U_k \to u$ as $k \to \infty$ without assuming that meshsize goes to zero, and with minimal assumptions regarding underlying problem and MARK (Morin, Siebert, Veeser '08).

- Contraction property of AFEM: there exist $0 < \alpha < 1$ and $\gamma > 0$ so that

$$\|u - U_{k+1}\|_\Omega^2 + \gamma \mathcal{E}_{k+1}^2 \leq \alpha^2 \Big(\|u - U_k\|_\Omega^2 + \gamma \mathcal{E}_k^2 \Big).$$

- Quasi-optimal convergence rates (for total error): if

$$\inf_{\#\mathcal{T} - \#\mathcal{T}_0 \leq N} \inf_{V \in \mathbb{V}(\mathcal{T})} \Big(\|u - V\|_\Omega + \operatorname{osc}_{\mathcal{T}}(V, \mathcal{T}) \Big) \preccurlyeq N^{-s}$$
$$\Rightarrow \quad \|u - U_k\|_\Omega + \operatorname{osc}_{\mathcal{T}_k}(U_k, \mathcal{T}_k) \preccurlyeq (\#\mathcal{T}_k - \#\mathcal{T}_0)^{-s}.$$

- Sufficient conditions on (u, f, A) for total error decay N^{-s}.

Module ESTIMATE: Basic Properties

Reliability: Upper Bounds (Babuška-Miller, Stevenson)

- **Upper bound:** there exists a constant $C_1 > 0$, depending solely on the initial mesh $\mathcal{T}_0$ and the smallest eigenvalue $a_{\min}$ of A, such that

$$|||u - U|||_\Omega^2 \le C_1 \mathcal{E}_\mathcal{T}(U, \mathcal{T})^2$$

- **Localized upper bound:** if $U_* \in \mathbb{V}(\mathcal{T}_*)$ is the Galerkin solution for a conforming refinement $\mathcal{T}_*$ of $\mathcal{T}$, and $\mathcal{R} = \mathcal{R}_{\mathcal{T}\to\mathcal{T}_*}$ (refined set), then

$$|||U - U_*|||_\Omega^2 \le C_1 \mathcal{E}_\mathcal{T}(U, \mathcal{R})^2$$

Efficiency: Lower Bound (Babuška-Miller, Verfürth)
There exists a constant $C_2 > 0$, depending only on the shape regularity constant of $\mathcal{T}_0$ and the largest eigenvalue $a_{\max}$, such that

$$C_2 \mathcal{E}_\mathcal{T}(U, \mathcal{T})^2 \le |||u - U|||_\Omega^2 + \operatorname{osc}_\mathcal{T}(U, \mathcal{T})^2.$$

- **Reduction of Estimator:** For $\lambda = 1 - 2^{-b/d}$, $\mathcal{T}_* = \text{REFINE}(\mathcal{T}, \mathcal{M})$, and all $V \in \mathbb{V}(\mathcal{T})$ we have

$$\mathcal{E}^2_{\mathcal{T}_*}(V, \mathcal{T}_*) \le \mathcal{E}^2_{\mathcal{T}}(V, \mathcal{T}) - \lambda \mathcal{E}^2_{\mathcal{T}}(V, \mathcal{M}).$$

- **Lipschitz Property:** The mapping $V \mapsto \mathcal{E}_{\mathcal{T}}(V, \mathcal{T})$ satisfies

$$|\mathcal{E}_{\mathcal{T}}(V, \mathcal{T}) - \mathcal{E}_{\mathcal{T}}(W, \mathcal{T})| \le C_0 \|V - W\|_\Omega \qquad \forall V, W \in \mathbb{V}(\mathcal{T})$$

  with a constant C_0 depending on $\mathcal{T}_0$, A, d and n.

This implies that for all $\delta > 0$

$$\mathcal{E}^2_{\mathcal{T}_*}(V_*, \mathcal{T}_*) \le (1+\delta)\big(\mathcal{E}^2_{\mathcal{T}}(V, \mathcal{T}) - \lambda \mathcal{E}^2_{\mathcal{T}}(V, \mathcal{M})\big) + (1+\delta^{-1}) C_0^2 \|V_* - V\|^2_\Omega.$$

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- **Pythagoras:** $\|u - U_*\|^2_\Omega = \|u - U\|^2_\Omega - \|U - U_*\|^2_\Omega$

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Module MARK: Dörfler Marking

- Given a mesh $\mathcal{T}$, indicators $\{\mathcal{E}_{\mathcal{T}}(U_{\mathcal{T}}, T)\}_{T\in\mathcal{T}}$, and a parameter $\theta \in (0,1]$, we select a subset $\mathcal{M}$ of $\mathcal{T}$ of marked elements such that

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- The marked set $\mathcal{M}$ is minimal (this is crucial for optimal cardinality).

Module REFINE: Bisection

Binev, Dahmen, DeVore ($d = 2$), Stevenson ($d > 2$): If $\mathcal{T}_0$ has a suitable labeling, then there exists a constant $\Lambda_0 > 0$ only depending on $\mathcal{T}_0$ and d such that for all $k \geq 1$

$$\#\mathcal{T}_k - \#\mathcal{T}_0 \leq \Lambda_0 \sum_{j=0}^{k-1} \#\mathcal{M}_j .$$

Module SOLVE: Multilevel Solvers

Bramble, Pasciak, Xu,and others; Chen, Nochetto, Xu'10: Optimal multigrid and BPX preconditioners for graded bisection grids, any polynomial degree $n \geq 1$, and any dimension $d \geq 2$.

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Contraction Property of AFEM

Vanishing Oscillation (Morin, N, Siebert'00)

We assume $\operatorname{osc}_{\mathcal{T}_k}(U_k) = 0$. If $\mathcal{T}_{k+1}$ satisfies an interior node property wrt $\mathcal{T}_k$, then we have the discrete lower bound

$$C_2 \mathcal{E}_k(U_k, \mathcal{M}_k)^2 \leq \|U_{k+1} - U_k\|_\Omega^2$$

Therorem 2 (Contraction) For $\alpha := (1 - \theta^2 \frac{C_2}{C_1})^{1/2} < 1$ there holds

$$\|u - U_{k+1}\|_\Omega \leq \alpha \|u - U_k\|_\Omega,$$

Proof: Recall Pythagoras

$$\|u - U_{k+1}\|_\Omega^2 = \|u - U_k\|_\Omega^2 - \|U_{k+1} - U_k\|_\Omega^2.$$

Combine the discrete lower bound with Dörfler marking and upper bound

$$\|U_{k+1} - U_k\|_\Omega^2 \geq C_2 \mathcal{E}_k(U_k, \mathcal{M}_k)^2 \geq C_2 \theta^2 \mathcal{E}_k(U_k)^2 \geq \frac{C_2}{C_1} \theta^2 \|u - U_k\|_\Omega^2$$

$$\Rightarrow \quad \|u - U_{k+1}\|_\Omega^2 \leq \left(1 - \frac{C_2}{C_1}\theta^2\right) \|u - U_k\|_\Omega^2.$$

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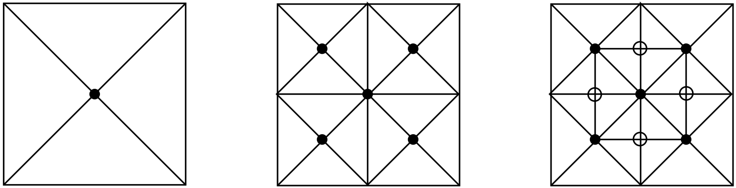
General Data: Contracting Quantities

- Energy error: $\|U_k - u\|_\Omega$ is monotone, but not strictly monotone (e.g. $U_{k+1} = U_k$).



$$\Omega = (0,1)^2, A = I, f = 1 \quad \Rightarrow \quad U_0 = U_1 = \frac{1}{12}\,\phi_0, \quad U_2 \neq U_1$$

- Residual estimator: $\mathcal{E}_k(U_k, \mathcal{T}_k)$ is not reduced by AFEM, and is not even monotone. But, if $U_{k+1} = U_k$, then $\mathcal{E}_k(U_k, \mathcal{T}_k)$ decreases strictly

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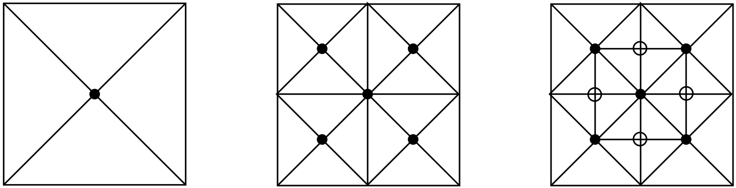

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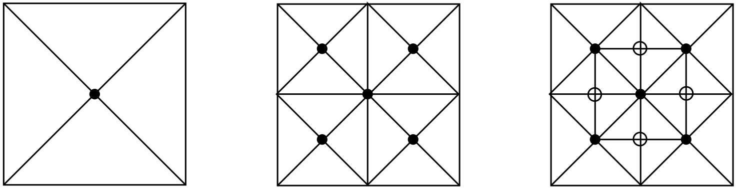

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Contraction Property (Cascón, Kreuzer, Nochetto, Siebert' 08)

Theorem 3. There exist constants $\gamma > 0$ and $0 < \alpha < 1$, depending on the shape regularity constant of $\mathcal{T}_0$, the eigenvalues of A, and θ, such that

$$|||u - U_{k+1}|||_\Omega^2 + \gamma\,\mathcal{E}_{k+1}^2 \le \alpha^2 \left(|||u - U_k|||_\Omega^2 + \gamma\,\mathcal{E}_k^2 \right).$$

Main ingredients of the proof:

- Pythagoras: $|||U_{k+1} - u|||_\Omega^2 = |||U_k - u|||_\Omega^2 - |||U_k - U_{k+1}|||_\Omega^2$;
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Proof of Theorem 3

Error orthogonality $|||u - U_{k+1}|||_\Omega^2 = |||u - U_k|||_\Omega^2 - |||U_k - U_{k+1}|||_\Omega^2$ yields

$$|||u - U_{k+1}|||_\Omega^2 + \gamma \mathcal{E}_{k+1}^2(U_{k+1}, \mathcal{T}_{k+1}) \le |||u - U_k|||_\Omega^2 - |||U_k - U_{k+1}|||_\Omega^2 + \gamma \mathcal{E}_{k+1}^2(U_{k+1}, \mathcal{T}_{k+1})$$

Estimator reduction property implies

$$|||u - U_{k+1}|||_\Omega^2 + \gamma \mathcal{E}_{k+1}^2(U_{k+1}, \mathcal{T}_{k+1}) \le |||u - U_k|||_\Omega^2 - |||U_k - U_{k+1}|||_\Omega^2 + \gamma(1+\delta)\big(\mathcal{E}_k^2(U_k, \mathcal{T}_k) - \lambda \mathcal{E}_k^2(U_k, \mathcal{M}_k)\big) + \gamma(1+\delta^{-1})C_0^2 |||U_k - U_{k+1}|||_\Omega^2.$$

Choose $\gamma := \frac{1}{(1+\delta^{-1})\, C_0^2}$ to cancel $|||U_k - U_{k+1}|||_\Omega$:

$$|||u - U_{k+1}|||_\Omega^2 + \gamma \mathcal{E}_{k+1}^2(U_{k+1}, \mathcal{T}_{k+1}) \le |||u - U_k|||_\Omega^2 + \gamma(1+\delta)\mathcal{E}_k^2(U_k, \mathcal{T}_k) - \gamma(1+\delta)\lambda \mathcal{E}_k^2(U_k, \mathcal{M}_k).$$

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$$|||u - U_{k+1}|||_\Omega^2 + \gamma \mathcal{E}_{k+1}^2(U_{k+1}, \mathcal{T}_{k+1}) \le |||u - U_k|||_\Omega^2 - |||U_k - U_{k+1}|||_\Omega^2 + \gamma \mathcal{E}_{k+1}^2(U_{k+1}, \mathcal{T}_{k+1})$$

Estimator reduction property implies

$$|||u - U_{k+1}|||_\Omega^2 + \gamma \mathcal{E}_{k+1}^2(U_{k+1}, \mathcal{T}_{k+1}) \le |||u - U_k|||_\Omega^2 - |||U_k - U_{k+1}|||_\Omega^2 + \gamma(1+\delta)\big(\mathcal{E}_k^2(U_k, \mathcal{T}_k) - \lambda \mathcal{E}_k^2(U_k, \mathcal{M}_k)\big) + \gamma(1+\delta^{-1}) C_0^2 |||U_k - U_{k+1}|||_\Omega^2.$$

Choose $\gamma := \frac{1}{(1+\delta^{-1})\, C_0^2}$ to cancel $|||U_k - U_{k+1}|||_\Omega$:

$$|||u - U_{k+1}|||_\Omega^2 + \gamma \mathcal{E}_{k+1}^2(U_{k+1}, \mathcal{T}_{k+1}) \le |||u - U_k|||_\Omega^2 + \gamma(1+\delta)\mathcal{E}_k^2(U_k, \mathcal{T}_k) - \gamma(1+\delta)\lambda \mathcal{E}_k^2(U_k, \mathcal{M}_k).$$

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$$\begin{aligned}|\!|\!|u - U_{k+1}|\!|\!|_\Omega^2 + \gamma\mathcal{E}_{k+1}^2(U_{k+1}, \mathcal{T}_{k+1}) &\le |\!|\!|u - U_k|\!|\!|_\Omega^2 - |\!|\!|U_k - U_{k+1}|\!|\!|_\Omega^2 \\ &\quad + \gamma\mathcal{E}_{k+1}^2(U_{k+1}, \mathcal{T}_{k+1})\end{aligned}$$

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Proof of Theorem 3 (Continued)

Dörfler marking $\mathcal{E}_k(U_k, \mathcal{M}_k) \geq \theta \mathcal{E}_k(U_k, \mathcal{T}_k)$ yields

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Applying the Upper Bound $\|u - U_k\|_\Omega^2 \leq C_1 \mathcal{E}_k^2(U_k, \mathcal{T}_k)$ gives

$$\|u - U_{k+1}\|_\Omega^2 + \gamma \mathcal{E}_{k+1}^2(U_{k+1}, \mathcal{T}_{k+1}) \leq \Big(1 - \frac{1}{2}\gamma(1+\delta)\frac{\lambda\theta^2}{C_1}\Big)\|u - U_k\|_\Omega^2 + (1+\delta)\Big(1 - \frac{\lambda\theta^2}{2}\Big)\gamma\mathcal{E}_k^2(U_k, \mathcal{T}_k).$$

Choosing $\delta > 0$ sufficiently small so that

$$\alpha^2 := \max\Big\{1 - \frac{1}{2}\gamma(1+\delta)\frac{\lambda\theta^2}{C_1}, (1+\delta)\Big(1 - \frac{\lambda\theta^2}{2}\Big)\Big\} < 1,$$

we finally obtain the desired estimate

$$\|u - U_{k+1}\|_\Omega^2 + \gamma \mathcal{E}_{k+1}^2(U_{k+1}, \mathcal{T}_{k+1}) \leq \alpha^2\Big(\|u - U_k\|_\Omega^2 + \gamma\mathcal{E}_k^2(U_k, \mathcal{T}_k)\Big).$$

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Outline

The Total Error

- AFEM controls the new error quantity $\|u - U_k\|_\Omega^2 + \gamma \mathcal{E}_k^2(U_k, \mathcal{T}_k)$.

- Since estimator dominates oscillation

$$\operatorname{osc}_k(U_k, \mathcal{T}_k) \le \mathcal{E}_k(U_k, \mathcal{T}_k)$$

and there is a global lower bound,

$$C_2 \mathcal{E}_k^2(U_k, \mathcal{T}_k) \le \|u - U_k\|_\Omega^2 + \operatorname{osc}_k^2(U_k, \mathcal{T}_k)$$

$\|u - U_k\|_\Omega^2 + \gamma \mathcal{E}_k^2(U_k, \mathcal{T}_k)$ is equivalent to total error and error estimator:

$$\|u - U_k\|_\Omega^2 + \gamma \mathcal{E}_k^2(U_k, \mathcal{T}_k) \approx \|u - U_k\|_\Omega^2 + \operatorname{osc}_k^2(U_k, \mathcal{T}_k) \approx \mathcal{E}_k^2(U_k, \mathcal{T}_k)$$

- Total error: $E_{\mathcal{T}}(u, A, f; U) := \Big(\|u - U\|_\Omega^2 + \operatorname{osc}_{\mathcal{T}}^2(U, \mathcal{T})\Big)^{1/2}$

The Total Error: Quasi-Best Approximation

- Operator with pw constant A (eg Laplace) and polynomial degree $n = 1$:

$$E_{\mathcal{T}}^2(u, A, f; U) = \|u - U\|_\Omega^2 + \|h(f - P_0 f)\|_\Omega^2$$

- **Quasi-Best Approximation:** There exists a constant $D > 0$ only depending on oscillation of A on $\mathcal{T}_0$ and on $\mathcal{T}_0$ such that

$$E_{\mathcal{T}}(u, A, f; U) \le D \inf_{V \in \mathbb{V}(\mathcal{T})} E_{\mathcal{T}}(u, A, f; V).$$

The Total Error: Quasi-Best Approximation

- Operator with pw constant A (eg Laplace) and polynomial degree $n = 1$:

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Approximation Class (for Total Error)

The set of all conforming triangulations with at most N elements more than in $\mathcal{T}_0$ is denoted

$$\mathbb{T}_N := \{\mathcal{T} \in \mathbb{T} \mid \#\mathcal{T} - \#\mathcal{T}_0 \le N\}.$$

The quality of the best approximation to the total error in $\mathbb{T}_N$ is

$$\sigma_N(u; A, f) := \inf_{\mathcal{T} \in \mathbb{T}_N} \inf_{V \in \mathbb{V}(\mathcal{T})} E_{\mathcal{T}}(u, A, f; V)$$

For $0 < s \le n/d$ the approximation class is finally given as

$$\mathbb{A}_s := \left\{ (u, A, f) \mid \ |u, A, f|_s := \sup_{N \ge 0} \Big(N^s \sigma_N(u; A, f) \Big) < \infty \right\}.$$

Approximation of data is explicitly included in the definition of the class $\mathbb{A}_s$:

$$r(V) - P_{n-1} r(V) \quad \text{where} \quad r(V) = \operatorname{div}(A \nabla V) + f,$$

with $n \ge 1$. Nonlinear coupling between A and ∇U via oscillation!

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Characterization of Approximation Class

- For A pw constant over $\mathcal{T}_0$, $n \geq 1$, $d \geq 2$, we have the equivalence

$$|u, f, A|_s \approx |u|_{\mathcal{A}_s} + |f|_{\mathcal{B}_s}$$

where

$$\mathcal{A}_s : \quad |v|_{\mathcal{A}_s} := \sup_{N>0} \Big(N^s \inf_{\mathcal{T} \in \mathbb{T}_N} \inf_{V \in \mathbb{V}(\mathcal{T})} \|v - V\|_\Omega \Big) < \infty,$$

$$\mathcal{B}_s : \quad |g|_{\mathcal{B}_s} := \sup_{N>0} \Big(N^s \inf_{\mathcal{T} \in \mathbb{T}_N} \|h_\mathcal{T}(g - P_{n-1}\, g)\|_{L^2(\Omega)} \Big) < \infty$$

- Characterization of class $\mathbb{A}_s$ is open for variable A (nonlinear interaction between A and V in $\operatorname{osc}_\mathcal{T}(V, \mathcal{T})$).

- **Sufficient condition** (dimension $d = 2$, $u \in H_0^1(\Omega) \cap W_p^2(\Omega; \mathcal{T}_0)$ with $p > 1$, $f \in L^2(\Omega)$, A pw Lipschitz, and polynomial degree $n = 1$, imply optimal decay rate $s = 1/2$, and

$$|u, f, A|_{1/2} \lesssim \|D^2 u\|_{L^p(\Omega;\mathcal{T}_0)} + \|A\|_{W^1_\infty(\Omega;\mathcal{T}_0)} + \|f\|_{L^2(\Omega)}.$$

$\Rightarrow \quad s = 1/2$ for checkerboard discontinuous coefficients (Kellogg)

Quasi-Optimal Cardinality: Vanishing Oscillation (Stevenson' 07)

Stevenson's insight: *any marking strategy that reduces the energy error relative to the current value must contain a substantial portion of $\mathcal{E}_{\mathcal{T}}(U,\mathcal{T})$, and so it can be related to Dörfler Marking.*

Lemma 3 (Dörfler Marking). Let $\theta < \theta_* = \sqrt{\frac{C_2}{C_1}}$, and $\mu = 1 - \frac{\theta^2}{\theta_*^2}$. Let $\mathcal{T}_*$ be a conforming refinement of $\mathcal{T}$, and $U_* \in \mathbb{V}(\mathcal{T}_*)$ satisfy

$$\|u - U_*\|_\Omega^2 \le \mu \|u - U\|_\Omega^2.$$

Then the refinement set $\mathcal{R} = \mathcal{R}_{\mathcal{T}\to\mathcal{T}_*}$ satisfies Dörfler marking with θ

$$\mathcal{E}_{\mathcal{T}}(U,\mathcal{R}) \ge \theta \mathcal{E}_{\mathcal{T}}(U,\mathcal{T}).$$

Proof: Use lower bound followed by Pythagoras equality

$$\begin{aligned}(1-\mu) C_2 \mathcal{E}_{\mathcal{T}}^2(U,\mathcal{T}) &\le (1-\mu)\|u-U\|_\Omega^2 \\ &\le \|u-U\|_\Omega^2 - \|u-U_*\|_\Omega^2 = \|U-U_*\|_\Omega^2.\end{aligned}$$

Finally, resort to the discrete lower bound

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Quasi-Optimal Cardinality: Vanishing Oscillation (Continued)

Lemma 4 (Cardinality of $\mathcal{M}_k$) If Dörfler marking chooses minimal set, and $u \in \mathcal{A}_s$, then the k-th marked set $\mathcal{M}_k$ generated by AFEM satisfy

$$\#\mathcal{M}_k \preccurlyeq |u|_s^{\frac{1}{s}} \|u - U_k\|_\Omega^{-\frac{1}{s}}.$$

Proof: Let $\varepsilon^2 = \mu \|u - U_k\|_\Omega^2$. Since $u \in \mathcal{A}_s$ there exist $\mathcal{T}_\varepsilon \in \mathbb{T}$ and $U_\varepsilon \in \mathbb{V}(\mathcal{T}_\varepsilon)$ such that

$$\|u - U_\varepsilon\|_\Omega^2 \le \varepsilon^2, \qquad \#\mathcal{T}_\varepsilon - \#\mathcal{T}_0 \preccurlyeq |u|_s^{\frac{1}{s}} \varepsilon^{-\frac{1}{s}}.$$

We introduce the overlay $\mathcal{T}_* = \mathcal{T}_\varepsilon \oplus \mathcal{T}_k$, and exploit that $\mathcal{T}_* \ge \mathcal{T}_\varepsilon$ to get

$$\|u - U_*\|_\Omega^2 \le \|u - U_\varepsilon\|_\Omega^2 \le \varepsilon^2 = \mu \|u - U\|_\Omega^2.$$

This implies $\mathcal{R} = \mathcal{R}_{\mathcal{T} \to \mathcal{T}_*}$ satisfies Dörfler marking with $\theta < \theta_*$. Since $\mathcal{M}_k$ is minimal, we conclude

$$\#\mathcal{M}_k \le \#\mathcal{R} \le \#\mathcal{T}_* - \#\mathcal{T}_k \le \#\mathcal{T}_\varepsilon - \#\mathcal{T}_0 \preccurlyeq |u|_s^{\frac{1}{s}} \varepsilon^{-\frac{1}{s}}.$$

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Proof: Let $\varepsilon^2 = \mu \|u - U_k\|_\Omega^2$. Since $u \in \mathcal{A}_s$ there exist $\mathcal{T}_\varepsilon \in \mathbb{T}$ and $U_\varepsilon \in \mathbb{V}(\mathcal{T}_\varepsilon)$ such that

$$\|u - U_\varepsilon\|_\Omega^2 \leq \varepsilon^2, \qquad \#\mathcal{T}_\varepsilon - \#\mathcal{T}_0 \preccurlyeq |u|_s^{\frac{1}{s}} \varepsilon^{-\frac{1}{s}}.$$

We introduce the overlay $\mathcal{T}_* = \mathcal{T}_\varepsilon \oplus \mathcal{T}_k$, and exploit that $\mathcal{T}_* \geq \mathcal{T}_\varepsilon$ to get

$$\|u - U_*\|_\Omega^2 \leq \|u - U_\varepsilon\|_\Omega^2 \leq \varepsilon^2 = \mu \|u - U\|_\Omega^2.$$

This implies $\mathcal{R} = \mathcal{R}_{\mathcal{T} \to \mathcal{T}_*}$ satisfies Dörfler marking with $\theta < \theta_*$. Since $\mathcal{M}_k$ is minimal, we conclude

$$\#\mathcal{M}_k \leq \#\mathcal{R} \leq \#\mathcal{T}_* - \#\mathcal{T}_k \leq \#\mathcal{T}_\varepsilon - \#\mathcal{T}_0 \preccurlyeq |u|_s^{\frac{1}{s}} \varepsilon^{-\frac{1}{s}}.$$

Quasi-Optimal Cardinality: General Data (Cascón, Kreuzer, Nochetto, Siebert' 08)

Lemma 5 (Dörfler Marking) Let $\theta < \theta_* = \sqrt{\frac{C_2}{1+C_1(1+C_3)}}$, with C_3 explicitly depending on A and $\mathcal{T}_0$, and $\mu = \frac{1}{2}(1 - \frac{\theta^2}{\theta_*^2})$. Let $\mathcal{T}_* \le \mathcal{T}$ and $U_* \in \mathbb{V}(\mathcal{T}_*)$ satisfy

$$|||u - U_*|||_\Omega^2 + \operatorname{osc}_{\mathcal{T}_*}^2(U_*, \mathcal{T}_*) \le \mu\Big(|||u - U|||_\Omega^2 + \operatorname{osc}_{\mathcal{T}}^2(U, \mathcal{T})\Big).$$

Then the refinement set $\mathcal{R} = \mathcal{R}_{\mathcal{T}\to\mathcal{T}_*}$ satisfies Dörfler marking with θ

$$\mathcal{E}_{\mathcal{T}}(U, \mathcal{R}) \ge \theta \mathcal{E}_{\mathcal{T}}(U, \mathcal{T}).$$

Lemma 6 (Cardinality of $\mathcal{M}_k$). If Dörfler marking chooses a minimal set $\mathcal{M}_k$, and $(u, A, f) \in \mathbb{A}_s$, then the k-th mesh $\mathcal{T}_k$ and marked set $\mathcal{M}_k$ generated by AFEM satisfy

$$\#\mathcal{M}_k \preccurlyeq |(u, A, f)|_s^{\frac{1}{s}} \Big(|||U_k - u|||_\Omega^2 + \operatorname{osc}_k^2(U_k, \mathcal{T}_k)\Big)^{-\frac{1}{2s}}.$$

Theorem 4 (Quasi-Optimal Cardinality of AFEM)

If $(u, A, f) \in \mathbb{A}_s$ for $s > 0$, then AFEM produces a sequence $\{\mathcal{T}_k, U_k\}_{k=0}^{\infty}$ of conforming bisection meshes and discrete solutions such that

$$\Big(|\!|\!| U_k - u |\!|\!|_\Omega^2 + \operatorname{osc}_k^2(U_k, \mathcal{T}_k) \Big)^{1/2} \preccurlyeq |u, A, f|_s \big(\#\mathcal{T}_k - \#\mathcal{T}_0 \big)^{-1/s}.$$

- Counting DOF (Binev, Dahmen, DeVore '04, Stevenson '06):

$$\#\mathcal{T}_k - \#\mathcal{T}_0 \preccurlyeq \sum_{j=0}^{k-1} \#\mathcal{M}_j \preccurlyeq \sum_{j=0}^{k-1} \Big(|\!|\!| U_j - u |\!|\!|_\Omega^2 + \operatorname{osc}_j^2(U_j, \mathcal{T}_j) \Big)^{-\frac{1}{2s}}.$$

- Contraction Property of AFEM:

$$|\!|\!| U_k - u |\!|\!|_\Omega^2 + \gamma \mathcal{E}_k^2(U_k, \mathcal{T}_k) \le \alpha^{2(k-j)} \Big(|\!|\!| U_j - u |\!|\!|_\Omega^2 + \mathcal{E}_j^2(U_j, \mathcal{T}_j) \Big),$$

whence

$$\#\mathcal{T}_k - \#\mathcal{T}_0 \preccurlyeq \Big(|\!|\!| U_k - u |\!|\!|_\Omega^2 + \gamma \underbrace{\mathcal{E}_k^2(U_k, \mathcal{T}_k)}_{\ge \operatorname{osc}_k^2(U_k, \mathcal{T}_k)} \Big)^{-\frac{1}{2s}} \underbrace{\sum_{j=0}^{k} \alpha^{\frac{j}{s}}}_{<(1-\alpha^{\frac{1}{s}})^{-1}}.$$

Outline

Extensions

- **Non-Residual Estimators** (Cascón, Nochetto; Kreuzer, Siebert' 10).
- **Non-conforming meshes** (Bonito, Nochetto' 10).
- **Adaptive dG (interior penalty)** (Bonito, Nochetto' 10).
- **Adaptive HDG** (Cockburn, Nochetto, Zhang '13).
- **Raviart-Thomas mixed methods** (Chen, Holst, Xu' 09).
- **Edge elements for Maxwell** (Zhong, Chen, Shu, Wittum, Xu' 10).
- **Local H^1-norm and L^2-norm** (Demlow, Stevenson' 10).
- H^{-1}**-data** (Cohen, DeVore, Nochetto' 11).
- **Laplace-Beltrami on parametric surfaces** (Bonito, Cascón, Mekchay, Morin, Nochetto '12).
- **Discontinuous coefficients** (Bonito, DeVore, Nochetto '12).
- **Instance optimality of the adaptive maximum strategy** (Diening, Kreuzer, Stevenson' 13).

Limitations

- **Pythagoras or variants:** does not apply to saddle point problems
- **Other norms** such as $L^\infty, L^p, W^1_\infty$.

Surveys

- R.H. Nochetto *Adaptive FEM: Theory and Applications to Geometric PDE*, Lipschitz Lectures, Haussdorff Center for Mathematics, University of Bonn (Germany), February 2009 (see `www.hausdorff-center.uni-bonn.de/event/2009/lipschitz-nochetto/`).
- R.H. Nochetto, K.G. Siebert and A. Veeser, *Theory of adaptive finite element methods: an introduction*, in *Multiscale, Nonlinear and Adaptive Approximation*, R. DeVore and A. Kunoth eds, Springer (2009), 409-542.
- R.H. Nochetto and A. Veeser, *Primer of adaptive finite element methods*, in *Multiscale and Adaptivity: Modeling, Numerics and Applications*, CIME Lectures, eds R. Naldi and G. Russo, Springer (to appear).