A Stochastic Roe-Scheme and Anisotropic Multi-Resolution Schemes for Uncertain Conservation Law Problems

O. Le Maître[1,2,3], J. Tryoen[4], A. Ern[5] and M. Ndjinga[6]

1- Duke University, Durham, North Carolina
2- KAUST, Saudi-Arabia
3- LIMSI CNRS, Orsay, France
4- INRIA, Bordeaux, France
5- CERMICS-ENPC, Marne la Vallée, France
6- DEN-CEA, Saclay, France

38th Conf. Dutch-Flemish NA Communities, Woudschoten

Generalized PC expansion

$$U(\omega) \approx \sum_{k=0}^{P} u_k \Psi_k(\boldsymbol{\xi}(\omega))$$

Truncated spectral expansions

Askey scheme [Xiu and Karniadakis, 2003]

Distribution of ξ_i	Polynomial family
Gaussian	Hermite
Uniform	Legendre
Exponential	Laguerre
β-distribution	Jacobi

- Convention $\Psi_0 \equiv 1$: mean mode.
- **Expectation** of U : $\mathbb{E}\{U\} \equiv \int_\Omega U(\omega) d\mu(\omega) \approx \sum_{k=0}^{P} u_k \int_\Xi \Psi_k(\boldsymbol{\xi}) p(\boldsymbol{\xi}) d\boldsymbol{\xi} = u_0$.
- **Variance** of U :

$$V[U] = \mathbb{E}\left\{U^2\right\} - \mathbb{E}\{U\}^2 \approx \sum_{k=1}^{P} u_k^2 \langle \Psi_k, \Psi_k \rangle .$$

- Estimation of other statistics (moment, pdfs, ANOVA) by sampling of $\boldsymbol{\xi}$.
- Extension to **random vectors & stochastic processes** :

$$\begin{pmatrix} U_1 \\ \vdots \\ U_m \end{pmatrix} (\omega, \boldsymbol{x}, t) \approx \sum_{k=0}^{P} \begin{pmatrix} u_1 \\ \vdots \\ u_m \end{pmatrix} (\boldsymbol{x}, t)\, \Psi_k(\boldsymbol{\xi}(\omega)).$$

Piecewise polynomial expansion

Instead of spectral expansion over Ξ one can use **piecewise polynomial approximation** on a mesh Σ of Ξ

- Stochastic mesh :

$$\Xi = \bigcup_{SE \in \Sigma} \Xi_{SE}, \quad \Xi_{SE} \cap \Xi_{SE'} = \emptyset \text{ for } SE \neq SE'$$

- Associated stochastic space : with dimension
- We can still construct orthogonal bases for $\mathbb{S}_\Sigma$ such that

$$U(\xi) = \sum_{k=0}^{\mathrm{P}} u_k \Psi_k(\xi), \quad \langle \Psi_k, \Psi_{k'} \rangle = \langle \Psi_k, \Psi_k \rangle \, \delta_{k,k'}.$$

Possible choices are :

1. **Stochastic multi-element method** [Deb *et al*, 2001], [Wang and Karniadakis, 2005]
   Each function Ψ_k has its support in a unique element Ξ_{SE} :
   Fully decouple the Galerkin problem between elements
2. **Stochastic Multiwavelet method** [olm *et al*, 2004]
   Hierarchical decomposition on meshes constructed by dyadic partitions, with some overlapping of the support over different :
   Coupled Galerkin problem, but hierarchical approximation well suited for multiresolution analysis

Content :

Hyperbolic systems : deterministic case

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{\nabla} \cdot \boldsymbol{f}(\boldsymbol{u}) = 0, \quad \boldsymbol{u}(\boldsymbol{x}, t=0) = \boldsymbol{u}^0(\boldsymbol{x}), \quad BCs$$

- $\boldsymbol{u} \in \mathcal{A}_{\boldsymbol{u}} \subset \mathbb{R}^m$ (conservative variables)
- $\boldsymbol{f} : \mathcal{A}_{\boldsymbol{u}} \mapsto \mathbb{R}^m$ (flux function)
- if $\nabla_{\boldsymbol{u}} \boldsymbol{f} \in \mathbb{R}^{m \times m}$ is $\mathbb{R}$-diagonalizable on $\mathcal{A}_{\boldsymbol{u}} \Longrightarrow$ hyperbolic
- $\boldsymbol{u}$ can develop shocks / discontinuities in finite time

Classical discretization (Finite Volume in 1-D)

$$\frac{\boldsymbol{u}_i^{n+1} - \boldsymbol{u}_i^n}{\Delta t} + \frac{\widetilde{\boldsymbol{f}}(\boldsymbol{u}_i^n, \boldsymbol{u}_{i+1}^n) - \widetilde{\boldsymbol{f}}(\boldsymbol{u}_{i-1}^n, \boldsymbol{u}_i^n)}{\Delta x} = 0$$

where $\boldsymbol{u}_i^n = \int_{\Delta x} \boldsymbol{u}(x, t_n) \mathrm{d}x$ and $\widetilde{\boldsymbol{f}}(,)$ is the numerical flux function (having had-hoc properties).

Uncertain hyperbolic problems :

- Uncertain initial & boundary conditions
- Uncertain physical parameters in $\boldsymbol{f}$
- Treat these uncertainties in a probabilistic framework :

$$\mathcal{P} \equiv (\Theta, \Sigma, \mathrm{d}\mu) : \boldsymbol{u} \to \boldsymbol{U}(\boldsymbol{x}, t, \theta \in \Theta)$$

- Stochastic Hyperbolic Problem

$$\frac{\partial \boldsymbol{U}(\boldsymbol{x}, t, \theta)}{\partial t} + \boldsymbol{\nabla} \cdot \boldsymbol{F}(\boldsymbol{U}; \theta) = 0, \quad \boldsymbol{U}(\boldsymbol{x}, t = 0, \theta) = \boldsymbol{U}^0(\boldsymbol{x}, \theta) \quad (a.s.)$$

We assume

1. $\boldsymbol{U}(\boldsymbol{x}, t, \theta) \in \mathcal{A}_{\boldsymbol{U}}$ a.s.
2. $\nabla_{\boldsymbol{U}} \boldsymbol{F}(\boldsymbol{U}; \theta)$ a.s. $\mathbb{R}$-diagonalizable for $\boldsymbol{U} \in \mathcal{A}_{\boldsymbol{U}}$
3. all random quantities have finite variance ($\in L^2(\Theta, \mathrm{d}\mu)$).

Stochastic spectral basis :

- Let $\boldsymbol{\xi}(\theta) = \{\xi_1(\theta), \ldots; \xi_{\mathrm{N}}(\theta)\}$ a set of N iid random variables with uniform distribution on $\Xi = [0, 1]^{\mathrm{N}}$
- Reformulate the problem on $\mathcal{P}_{\boldsymbol{\xi}} = (\Xi, \mathcal{B}^{\mathrm{N}}, 1)$:

$$\frac{\partial \boldsymbol{U}(\boldsymbol{x}, t, \boldsymbol{\xi})}{\partial t} + \nabla \cdot \boldsymbol{F}(\boldsymbol{U}; \boldsymbol{\xi}) = 0, \quad \boldsymbol{U}(\boldsymbol{x}, t = 0, \boldsymbol{\xi}) = \boldsymbol{U}^0(\boldsymbol{x}, \boldsymbol{\xi}) \quad (a.s.)$$

- Let $\{\Psi_0, \Psi_1, \ldots, \Psi_{\mathrm{P}}\}$ the set of orthonormal polynomials in $\boldsymbol{\xi}$ with degree less or equal to No : (fully tensorized)

$$\langle \Psi_\alpha, \Psi_\beta \rangle = \int_\Xi \Psi_\alpha(\boldsymbol{\xi}) \Psi_\beta(\boldsymbol{\xi}) \mathrm{d}\boldsymbol{\xi} = \delta_{\alpha,\beta}, \quad 0 \leq \alpha, \beta \leq \mathrm{P} = (\mathrm{No} + 1)^{\mathrm{N}} - 1$$

- Denote $\mathbb{S}^{\mathrm{No}} = \mathrm{span}\{\Psi_0, \ldots, \Psi_{\mathrm{P}}\}$:

$$\lim_{\mathrm{No} \to \infty} \mathbb{S}^{\mathrm{No}} = L^2(\Xi).$$

Stochastic expansion of the solution :

- Since $\boldsymbol{U} \in L^2(\Xi)$ it has a convergent expansion :

$$\boldsymbol{U}(\boldsymbol{x}, t, \boldsymbol{\xi}) = \sum_{\alpha} \boldsymbol{u}_{\alpha}(x, t) \Psi_{\alpha}(\boldsymbol{\xi})$$

- We denote $\boldsymbol{U}^{\mathrm{P}}$ the approximation of $\boldsymbol{U}$ in $\mathbb{S}^{\mathrm{No}}$
- Stochastic Galerkin projection of the hyperbolic problem : for $\alpha = 0, \dots, \mathrm{P}$

$$\frac{\partial \boldsymbol{u}_{\alpha}(\boldsymbol{x}, t)}{\partial t} + \boldsymbol{\nabla} \cdot \boldsymbol{f}_{\alpha}(\boldsymbol{u}_0, \dots, \boldsymbol{u}_{\mathrm{P}}) = 0$$

$$\boldsymbol{f}_{\alpha}(\boldsymbol{u}_0, \dots, \boldsymbol{u}_{\mathrm{P}}) \equiv \left\langle \boldsymbol{F}(\boldsymbol{U}^{\mathrm{P}}; \boldsymbol{\xi}), \Psi_{\alpha} \right\rangle$$

$$\boldsymbol{u}_{\alpha}(\boldsymbol{x}, t = 0) = \left\langle \boldsymbol{U}^{0}(\boldsymbol{x}), \Psi_{\alpha} \right\rangle$$

$(\mathrm{P} + 1)$-coupled problems for the solution modes

Galerkin problem : (system form)

$$\frac{\partial}{\partial t}\begin{pmatrix} \boldsymbol{u}_0 \\ \vdots \\ \boldsymbol{u}_{\mathrm{P}} \end{pmatrix} + \boldsymbol{\nabla}\cdot\begin{pmatrix} \boldsymbol{f}_0(\boldsymbol{u}_0,\dots,\boldsymbol{u}_{\mathrm{P}}) \\ \vdots \\ \boldsymbol{f}_{\mathrm{P}}(\boldsymbol{u}_0,\dots,\boldsymbol{u}_{\mathrm{P}}) \end{pmatrix} = 0$$

$$\frac{\partial \mathcal{U}}{\partial t} + \boldsymbol{\nabla}\cdot\mathcal{F}(\mathcal{U}) = 0$$

- $\mathcal{U} \in \mathbb{R}^{m\times(\mathrm{P}+1)}$
- $\mathcal{F} : \mathbb{R}^{m\times(\mathrm{P}+1)} \mapsto \mathbb{R}^{m\times(\mathrm{P}+1)}$
- Is the Galerkin problem hyperbolic ?
- ($\nabla_{\mathcal{U}}\mathcal{F}$ $\mathbb{R}$-diagonalizable ?)
- What is the admissible domain $\mathcal{A}_{\mathcal{U}}$?

Jacobian of the Galerkin problem

$$\nabla_{\mathcal{U}}\mathcal{F} = \begin{pmatrix} \mathcal{F}'_{0,0} & \cdots & \mathcal{F}'_{0,\mathrm{P}} \\ \vdots & \ddots & \vdots \\ \mathcal{F}'_{\mathrm{P},0} & \cdots & \mathcal{F}'_{\mathrm{P},\mathrm{P}} \end{pmatrix}, \quad \mathcal{F}'_{\alpha,\beta} = \left\langle \nabla_{\boldsymbol{u}}\boldsymbol{F}(\boldsymbol{U}^{\mathrm{P}};\xi), \Psi_\alpha\Psi_\beta \right\rangle \in \mathbb{R}^{m,m}$$

- If $\nabla_u \boldsymbol{F}$ is symmetric (a.s.), $\nabla_{\mathcal{U}}\mathcal{F}$ is $\mathbb{R}$-diagonalizable
- In particular, scalar problems ($m = 1$) yield hyperbolicity
- If $\nabla_{\boldsymbol{u}}\boldsymbol{F} = \boldsymbol{L}\boldsymbol{D}(\xi)\boldsymbol{R}$, where $\boldsymbol{L}$ and $\boldsymbol{R}$ are deterministic, the Galerkin problem is hyperbolic
- Properties extend to $\neq$ truncature rules
- Note that strict hyperbolicity is **not** to be expected even when $\nabla_{\boldsymbol{u}}\boldsymbol{F}$ has (a.s.) distinct eigenvalues.

[J. Tryoen et al, JCP 2010]

General case

Let $\{\xi^{(i)}\}$, and $\{w^{(i)}\}$, $i = 0, \ldots, \mathrm{P}$ the points and weights of the (fully tensored) Gauss' quadrature rule over Ξ :

$$\int_{\Xi} f(\xi)\mathrm{d}\xi = \sum_{i=0}^{\mathrm{P}} f(\xi^{(i)})w^{(i)}, \quad \forall f \in \mathbb{S}^{2\mathrm{No}+1}$$

Define

$$\left(\overline{\nabla_{\mathcal{U}}\mathcal{F}}\right)_{\alpha,\beta} = \sum_{i=0}^{\mathrm{P}} \boldsymbol{\nabla_{u}F}\left(\boldsymbol{U}^{\mathrm{P}}(\xi^{(i)}); \xi^{(i)}\right) \Psi_\alpha\left(\xi^{(i)}\right) \Psi_\beta\left(\xi^{(i)}\right) w^{(i)} \approx \mathcal{F}'_{\alpha,\beta}$$

- Let $\{\Lambda^l(\xi)\}_{l=1}^{l=m}$, the stochastic Eigenvalues of $\nabla \boldsymbol{F}$
  $\{\Lambda^l_i \equiv \Lambda^l(\xi^{(i)})\}$ **are the eigenvalues of** $\overline{\nabla_{\mathcal{U}}\mathcal{F}}$
- $\Rightarrow \overline{\nabla_{\mathcal{U}}\mathcal{F}}$ is $\mathbb{R}$-diagonalizable
- For sufficient smoothness, $\lim_{\mathrm{No}\to\infty} \overline{\nabla_{\mathcal{U}}\mathcal{F}} = \nabla_{\mathcal{U}}\mathcal{F}$
- Hyperbolicity of Galerkin problem for large enough No
- If $\Lambda^l(\xi)$ are known : approximate spectrum of $\nabla_{\mathcal{U}}\mathcal{F}$

[J. Tryoen et al, JCP 2010, JCAM 2010]

Approximate Roe solver

$$\mathcal{U}_i^{n+1} = \mathcal{U}_i^n - \frac{\Delta t}{\Delta x}\left[\phi(\mathcal{U}_i^n, \mathcal{U}_{i+1}^n) - \phi(\mathcal{U}_{i-1}^n, \mathcal{U}_i^n)\right]$$

where the numerical flux Φ is chosen as

$$\phi(\mathcal{U}_L, \mathcal{U}_R) = \frac{1}{2}\left[\mathcal{F}(\mathcal{U}_L) + \mathcal{F}(\mathcal{U}_R)\right] - a\frac{\mathcal{U}_R - \mathcal{U}_L}{2}$$

with $a \in \mathbb{R}^{m(\mathrm{P}+1)\times m(\mathrm{P}+1)}$ a non-negative upwind matrix

Theorem : if the hyperbolic problem possesses a stochastic Roe matrix A^{Roe} almost surely, and $a^{\mathrm{Roe}}(\mathcal{U}_L, \mathcal{U}_R)_{\alpha,\beta} \equiv \left\langle A^{\mathrm{Roe}}, \Psi_\alpha \Psi_\beta \right\rangle$ is $\mathbb{R}$-diagonalizable, then

a^{Roe} is a Roe matrix for the Galerkin problem

i.e. has properties of consistency and conservativity through shocks.

[J. Tryoen et al, JCP 2010]

Roe solver (continued)

Theorem : if the stochastic problem possesses a Roe state $\boldsymbol{U}_{L,R}^{\mathrm{Roe}}$ such that

$$A^{\mathrm{Roe}}(\boldsymbol{U}_L, \boldsymbol{U}_R; \xi) = \nabla_{\boldsymbol{U}} \boldsymbol{F}(\boldsymbol{U}_{L,R}^{\mathrm{Roe}}) \quad \text{almost surely,}$$

then

$$\exists\, \mathcal{U}_\alpha^{\mathrm{Roe}} \doteq \left\langle \boldsymbol{U}_{L,R}^{\mathrm{Roe}}, \Psi_\alpha \right\rangle, \quad a^{\mathrm{Roe}}(\mathcal{U}_L, \mathcal{U}_R) = \nabla_{\mathcal{U}} \mathcal{F}(\mathcal{U}^{\mathrm{Roe}})$$

so the Galerkin system has also a Roe state. We will take

$$\phi(\mathcal{U}_L, \mathcal{U}_R) = \frac{1}{2}\left[\mathcal{F}(\mathcal{U}_L) + \mathcal{F}(\mathcal{U}_R)\right] - \left|\nabla_{\mathcal{U}} \mathcal{F}(\mathcal{U}^{\mathrm{Roe}})\right| \frac{\mathcal{U}_R - \mathcal{U}_L}{2}$$

where $|A| = |LDR| = L\,|D|\,R$ for a $\mathbb{R}$-diagonalizable matrix

[J. Tryoen et al, JCP 2010]

Fast approximation of the upwind matrix

Computation of $\left|\nabla_{\mathcal{U}}\mathcal{F}(\mathcal{U}^{\mathrm{Roe}})\right|$ through $|A| = |LDR| = L\,|D|\,R$ requires the decomposition of a matrix of $\mathbb{R}^{m(\mathrm{P}+1)\times m(\mathrm{P}+1)}$ for each interface and time-step

$\Longrightarrow$ too expensive for large P

Instead, use a polynomial transformation :

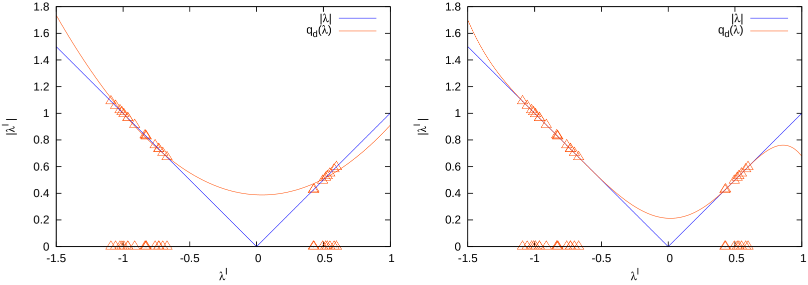
- recall $q(LDR) = Lq(D)R$
- $|\nabla_{\mathcal{U}}\mathcal{F}| \approx q_d\left(\nabla_{\mathcal{U}}\mathcal{F}\right)$, where $q_d \in \mathbb{P}_d$ minimizes

$$J = \sum_{i,l}\left[q_d\left(\Lambda_i^l\right) - \left|\Lambda_i^l\right|\right]^2$$

  with $\Lambda_i^l \approx \Lambda^l\left(\boldsymbol{U}_{LR}^{\mathrm{Roe}}(\boldsymbol{\xi}^{(i)})\right)$

- In practice $d \sim 6$ is sufficient
- Compute directly $q_d\left(\nabla_{\mathcal{U}}\mathcal{F}\right)\frac{\mathcal{U}_R-\mathcal{U}_L}{2}$

Fast approximation of the upwind matrix : illustration



Approximation polynomial q_d for $d = 2$ (left) and $d = 6$ (right).

Summary :

$$\mathcal{U}_i^{n+1} = \mathcal{U}_i^n - \frac{\Delta t}{\Delta x}\left[\phi(\mathcal{U}_i^n, \mathcal{U}_{i+1}^n) - \phi(\mathcal{U}_{i-1}^n, \mathcal{U}_i^n)\right]$$

where

$$\phi(\mathcal{U}_L, \mathcal{U}_R) = \frac{1}{2}\left[\mathcal{F}(\mathcal{U}_L) + \mathcal{F}(\mathcal{U}_R)\right] - q_d\left(\nabla_{\mathcal{U}}\mathcal{F}(\mathcal{U}^{\mathrm{Roe}})\right)\frac{\mathcal{U}_R - \mathcal{U}_L}{2}$$

- Upwinding w.r.t. the actual Galerkin Jacobian waves
- Applies conditionally to partially tensored stochastic basis
- May need Entropy corrector [J. Tryoen et al, JCAM 2010]
- Assume $\boldsymbol{U}(\xi)$ smooth and sufficient stochastic discretization
- **But solutions are not smooth in general !**

Call for piecewise polynomial approximations to allow for discontinuities at the stochastic level

Content :

① **Non-linearities :** need large polynomial orders for global approximation over full uncertainty range.

② **Discontinuous** solutions w.r.t. uncertain parameters prevent spectral convergence.

③ **Gibbs phenomenon** due to oscillating character of the spectral polynomials

Multi-Resolution System

✓ Piecewise polynomial.

✓ Convergence in polynomial order **and** resolution level.

✓ Discontinuous dependences.

✓ Local control of the resolution.

✓ Adaptive strategy.

Wiener-type orthogonal expansion (multiwavelets)

`[olm, Knio, Najm and Ghanem, JCPs 2004)].`

1-D Multi-resolution space

For $\mathrm{No} = 0, 1, \ldots$ and $k = 0, 1, \ldots,$ $\mathbf{V}_k^{\mathrm{No}}$ is the space of **piecewise polynomial functions** $f : \xi \in [0, 1] \mapsto \mathbb{R}$:

$$\mathbf{V}_k^{\mathrm{No}} \equiv \Big\{ f: \text{ the restriction of } f \text{ on } (2^{-k} l, 2^{-k}(l+1)) \in \mathbb{P}_{\mathrm{No}} \text{ for } l = 0, \ldots, 2^k - 1 \Big\},$$

where $\mathbb{P}_{\mathrm{No}}$ is the space of polynomials with degree $\leq \mathrm{No}$.
We have :

- $Dim(\mathbf{V}_k^{\mathrm{No}}) = (\mathrm{No}+1)(2^k)$,
- $\mathbf{V}_0^{\mathrm{No}} \subset \mathbf{V}_1^{\mathrm{No}} \subset \cdots \subset \mathbf{V}_k^{\mathrm{No}} \subset \ldots$
- $\mathbf{V}^{\mathrm{No}} \equiv \overline{\bigcup}_{k \geq 0} \mathbf{V}_k^{\mathrm{No}}$ is dense in $L^2([0, 1])$ with the inner product

$$\langle f, g \rangle = \int_0^1 f(\xi) g(\xi) d\xi.$$

Multi-wavelet space

Let us denote $\mathbf{W}_k^{\mathrm{No}}$, $k = 0, 1, 2, \ldots$, the orthogonal complement of $\mathbf{V}_k^{\mathrm{No}}$ in $\mathbf{V}_{k+1}^{\mathrm{No}}$:

$$\mathbf{V}_k^{\mathrm{No}} \oplus \mathbf{W}_k^{\mathrm{No}} = \mathbf{V}_{k+1}^{\mathrm{No}}, \quad \mathbf{W}_k^{\mathrm{No}} \perp \mathbf{V}_k^{\mathrm{No}},$$

so

$$\mathbf{V}_0^{\mathrm{No}} \bigoplus_{k \geq 0} \mathbf{W}_k^{\mathrm{No}} = L^2([0, 1]).$$

Let $\{\psi_0, \psi_1, \ldots, \psi_{\mathrm{No}}\}$ be an orthonormal basis of $\mathbf{W}_0^{\mathrm{No}}$:

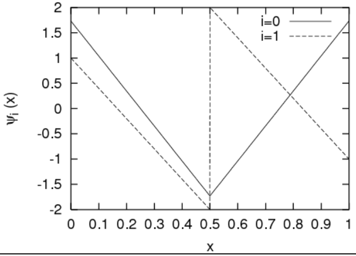
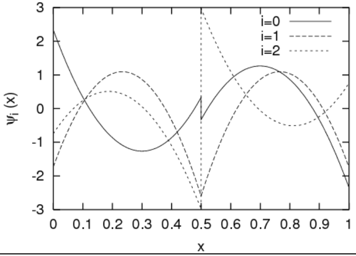
$$\langle \psi_i(\xi), \psi_j(\xi) \rangle = \delta_{ij},$$

and since $\mathbf{W}_0^{\mathrm{No}} \perp \mathbf{V}_0^{\mathrm{No}}$ we have

$$\left\langle \psi_j, \xi^i \right\rangle = 0, \quad 0 \leq i, j \leq \mathrm{No}.$$

Multi-wavelet space

The ψ_j are the **generating functions** of the MRA system.

$\mathrm{No} = 1$	$\mathrm{No} = 2$
	

Multi-wavelet space

The ψ_j are the **generating functions** of the MRA system.

Multi-wavelets ψ_{jl}^k

$$\psi_{jl}^k(\xi) = 2^{k/2}\psi_j(2^k\xi - l), \quad j = 0, \dots, \mathrm{No}, \text{ and } l = 0, \dots, 2^k - 1.$$

- $\mathrm{Supp}(\psi_{jl}^k) = [2^{-k}l, 2^{-k}(l+1)]$.
- $\left\langle \psi_{il}^k, \psi_{jm}^{k'} \right\rangle = \delta_{ij}\delta_{lm}\delta_{kk'}$.

Basis of $\mathbf{V}_0^{\mathrm{No}}$ — Legendre polynomials

$$\phi_i(\xi) = \frac{\mathcal{L}e_i(2\xi - 1)}{L_i}, \quad i = 0, 1, \dots, \mathrm{No},$$
$$\langle \phi_i(\xi), \phi_j(\xi) \rangle = \delta_{ij} \text{ for } i, j = 0, \dots, \mathrm{No}.$$

Projection on $\mathbf{V}_{\mathrm{Nr}}^{\mathrm{No}}$

Let us denote $f^{\mathrm{No},\mathrm{Nr}}$ the projection of f on $\mathbf{V}_{\mathrm{Nr}}^{\mathrm{No}}$:

$$f^{\mathrm{No},\mathrm{Nr}}(\xi) \equiv \mathcal{P}_{\mathrm{Nr}}^{\mathrm{No}}[f] = \sum_{i=0}^{\mathrm{No}} f_i \phi_i(\xi) + \sum_{k=0}^{\mathrm{Nr}-1} \sum_{l=0}^{2^k-1} \left(\sum_{i=0}^{\mathrm{No}} \delta f_{il}^k \psi_{il}^k(\xi) \right),$$

where

$$f_i = \langle f, \phi_i \rangle, \text{and } \delta f_{il}^k = \left\langle f, \psi_{il}^k \right\rangle.$$

For $f \in L^2([0,1])$, the projection error can be made arbitrarily small by increasing the expansion order No and/or resolution level Nr.

Application of MRA to UQ

One-dimensional case

- ξ : RV with density with uniform density on $[0, 1]$.
- $U(\xi) \in L^2([0, 1]) \Rightarrow U(\xi) = \sum_k u_k W_k(\xi)$.
  W_k elements of the orthonormal 1-D MRA system.

N-dimensionnal case

- Proceed by (sparse) **tensorization** of 1-D MRA system.
- $U(\boldsymbol{\xi}) \equiv U(\xi_1, \dots, \xi_{\mathrm{N}}) \approx \sum_{\boldsymbol{k}} u_{\boldsymbol{k}} \mathcal{M}^w{}_{\boldsymbol{k}}(\xi_1, \dots, \xi_{\mathrm{N}})$.
- $\mathcal{M}^w{}_{\boldsymbol{k}}(\boldsymbol{\xi}) = W_{k_1}(\xi_1) \times \cdots \times W_{k_{\mathrm{N}}}(\xi_{\mathrm{N}})$.

Summary

- Expansion in terms of random variables $\boldsymbol{\xi} \sim U(0, 1)^{\mathrm{N}}$.
- **Piecewise polynomial approximation**.
- Error reduction through p (No) or h (Nr) **refinement**.
- **Fast increase with No, Nr and N of approximation space's dimension** (calls for adaptive techniques).

Euler equations (Sod Shock Tube)

$\rho = 1$	$\rho = 0.125$
$\mathrm{v} = 0$	$\mathrm{v} = 0$
$\mathrm{p} = 1$	$\mathrm{p} = 0.125$

$$\frac{\partial U}{\partial t} + \frac{\partial F(U)}{\partial x} = 0, \quad U = (\rho, q, E)^T$$

$$F(U) = (\rho v, \rho v^2 + p, v(E+p))^T$$

$$v = \frac{q}{\rho} \quad p = (\gamma - 1)\left(E - \frac{1}{2}\rho v^2\right)$$

$$\boxed{\gamma(\xi) = 1.4 + 0.2\,\xi \quad \xi \sim \mathcal{U}[0,1]}$$

Computation of Galerkin flux and Jacobian matrix

Use **pseudo-spectral approximations**

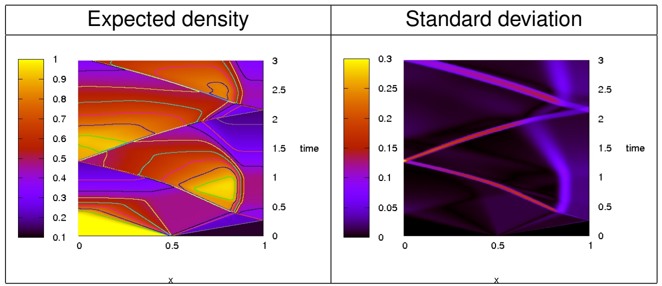
[Debusschere et al, 04]

- Spectral product $a * b = \sum_{\alpha=0}^{\mathrm{P}} (a * b)_\alpha \Psi_\alpha$ with

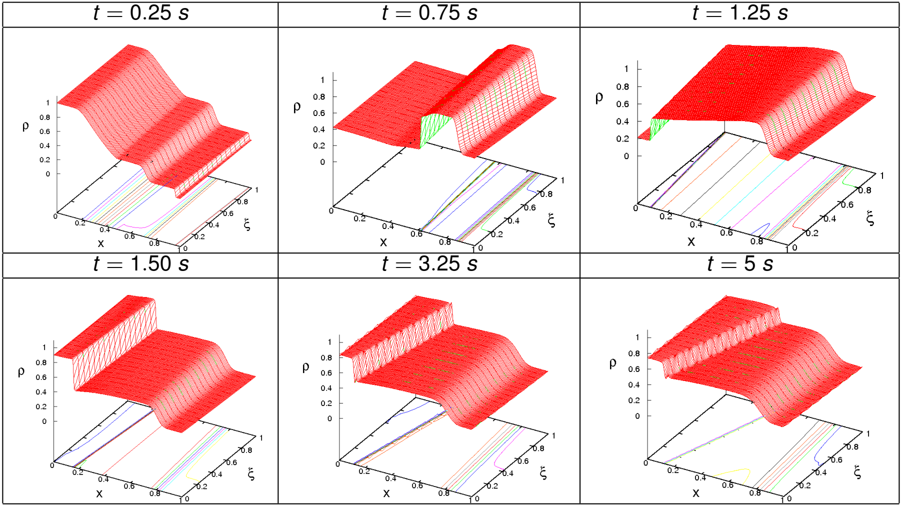
$$(a * b)_\alpha = \sum_{\beta,\delta=0}^{\mathrm{P}} a_\beta b_\gamma \mathcal{M}_{\alpha\beta\gamma}, \quad \mathcal{M}_{\alpha\beta\gamma} = \left\langle \Psi_\alpha, \Psi_\beta \Psi_\gamma \right\rangle$$

- $1/a \approx a^{-*}$ obtained by solving $a * a^{-*} = 1$
- $\sqrt{a} \approx a^{*/2}$ obtained by solving $(a^{*/2}) * (a^{*/2}) = a$
- Example $p^* = (\gamma - 1) * (E - ((q * q) * \rho^{-*})/2)$

Mean and standard deviation of density

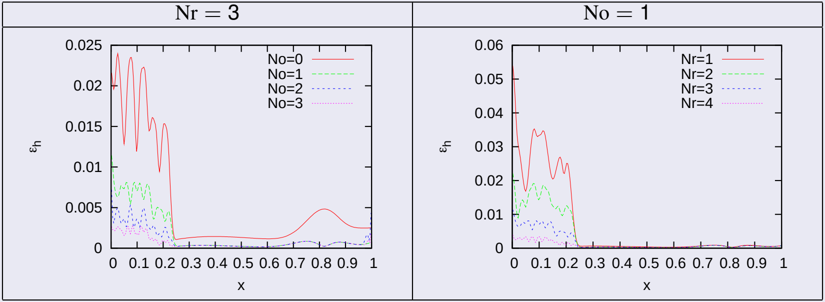


Computations with $\mathrm{Nr} = 3$, $\mathrm{No} = 2$, and $N_c = 250$



Stochastic density as a function of (x, ξ) ; $\mathrm{Nr} = 3$ and $\mathrm{No} = 2$

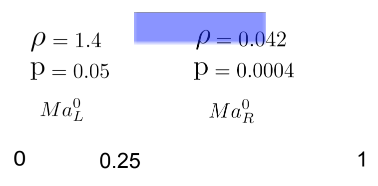
Convergence assessment



Stochastic error $\epsilon_h(x, t = 6.5)$ for various No and Nr ; $N_c = 250$

$$\epsilon_h(x,t) := \left(\frac{1}{M} \sum_{i=1}^{M} \left(\rho_h^{\mathrm{No},\mathrm{Nr}}(x,t,\xi^{(i)}) - \rho_h^{\mathrm{MC}}(x,t,\xi^{(i)}) \right)^2 \right)^{1/2}$$

Euler equations with sonic points



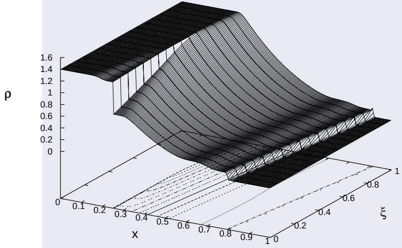
$$Ma^0(x,\xi) = \begin{cases} Ma_L^0(\xi) & x \in [0, 1/4) \\ Ma_R^0(\xi) & x \in (1/4, 1] \end{cases} \qquad \xi \sim \mathcal{U}[0,1]$$

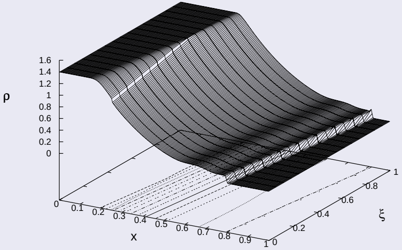
$\exists$ sonic points for $\xi \in [0, 0.6]$

Entropy corrector

- Adaptation of **non-parametrized entropy corrector** proposed by [Dubois & Mehlmann 96] for Roe solvers in deterministic case
- Use **approximate eigenvalues and eigenvectors** of $a^{\text{Roe}}(u_i^n, u_{i+1}^n)$
- **Mean-value averaged criterium** to improve CPU times

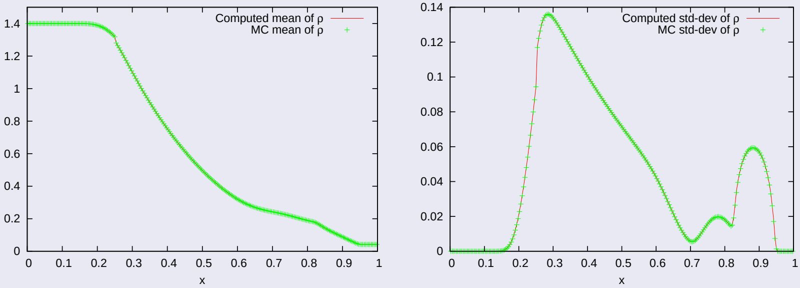
Necessity for entropy corrector





Stochastic density $\rho(x, t, \xi)$ at $t = 1$ obtained without (left) and with (right) entropy corrector ; $\mathrm{Nr} = 3$ and $\mathrm{No} = 2$

Accuracy assessment

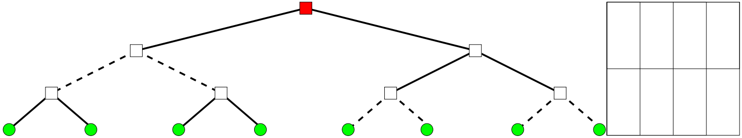


Comparison of mean and standard deviation of density at $t = 1$, computed with Galerkin (using $\mathrm{Nr} = 3$ and $\mathrm{No} = 2$) and MC methods

Content :

Binary trees for piecewise polynomial space

$\mathbb{S} = \mathbb{S}(\mathtt{T})$



- Dyadic partitions of a node along a prescribe direction $\mathrm{d} : \mathtt{p} \to (\mathtt{c}^-, \mathtt{c}^+)$
- Piecewise-polynomial with **fixed** order No on each leaf of $\mathtt{T}$.

1. Union of local modal basis : SE-basis

   [Deb et al, 2001], [Karniadakis et al]

   Uncoupled application of the Roe scheme over different leafs

2. Hierarchical global basis over Ξ : MW-Basis

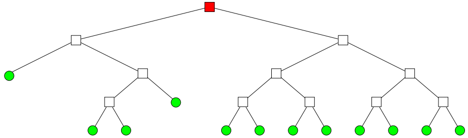
   [OLM et al, 2004]

   Hierarchical sequence of details, suited for adaptive scheme

Adaptivity

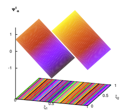
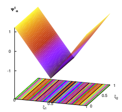
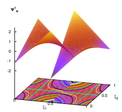
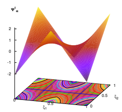
Singularity curves are localized in Ξ : **stochastic adaptivity**

- **Incomplete and anisotropic binary trees**



Operators for multi-resolution analysis :

- **Prediction operator** : define the solution in a stochastic space larger than the current one (add new leafs and L^2-injection).
- **Restriction operator** : define the solution in a stochastic space smaller one the current one (remove leafs and L^2-projection).
- Rely on recursive application of **elementary (directional) operators**, full exploitation of the tree structure.

$\alpha = 1$	$\alpha = 2$	$\alpha = 3$	$\alpha = 4$
			

Mother wavelets $\tilde{\Psi}^{\mathrm{d}}_{\alpha}$ for $\mathrm{N} = 2$, $\mathrm{No} = 1$ in direction $\mathrm{d} = 1$.

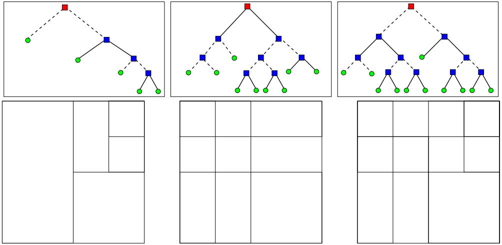
Adaptivity :

Singularity curves are localized in x and t

- **Each spatial cell** carries its own adapted stochastic discretization
- Flux computation,

$$\phi(\mathcal{U}_L,\mathcal{U}_R) = \frac{\mathcal{F}(\mathcal{U}_L)+\mathcal{F}(\mathcal{U}_R)}{2} - \left|a^{\mathrm{Roe}}(\mathcal{U}_L,\mathcal{U}_R)\right| \frac{\mathcal{U}_R-\mathcal{U}_L}{2},$$

  with $\mathcal{U}_R$ and $\mathcal{U}_L$ known on **different stochastic spaces**
- **Union operator** : given two stochastic spaces, construct the minimal stochastic space containing the two :



Adaptive Algorithm :

1. Loop over all interfaces of the spatial mesh :
   - Construct the union space of the left and right cells
   - **Enrich** this space
   - Predict left and right states of the interface
   - Evaluate the numerical flux (App. Roe scheme)
2. Loop over all cells of the spatial mesh :
   - Construct the union space of the cell's interfaces
   - Predict cell's fluxes on the union space
   - Compute fluxes difference and update cell's solution
   - Restrict cell's solution by **thresholding**
3. Repeat for the next time step

Two indicators needed : based on multiwavelet details of nodes.

- for Enrichment : anticipate emergence of new stochastic details,
- for Thresholding : remove unnecessary/negligible details.

Thresholding criterion :

Let us denote

- $\mathtt{T}$ a binary tree and $\mathbb{S}(\mathtt{T})$ the corresponding stochastic approximation space
- $\mathtt{n} \in \mathcal{N}(\mathtt{T})$ a node of the tree, and $\widehat{\mathcal{N}}(\mathtt{T})$ set set of nodes having children
- Nr the maximal depth allowed in a direction
- $\mathtt{T}_{[\mathrm{NNr}]}$ the maximal tree given Nr

We define for $U \in \mathbb{S}(\mathtt{T}_{[\mathrm{NNr}]})$ and $\eta > 0$ the subset of $\mathcal{N}(\mathtt{T}_{[\mathrm{NNr}]})$

$$\mathcal{D}(\eta) := \left\{ \mathtt{n} \in \widehat{\mathcal{N}}(\mathtt{T}_{[\mathrm{NrN}]}); \|\tilde{\boldsymbol{u}}^{\mathtt{n}}\|_{\ell^2} \leq 2^{-|\mathtt{n}|/2} \frac{\eta}{\sqrt{\mathrm{NNr}}} \right\},$$

where $\tilde{\boldsymbol{u}}^{\mathtt{n}} := (\tilde{u}^{\mathtt{n}}_{\alpha})_{1 \leq \alpha \leq \mathrm{P}}$ are the MW coefficients of $\mathtt{n}$.

Then

$$\| U^{\mathtt{T}_{[\mathrm{NNr}]}} - U^{\mathtt{T}_{[\mathrm{NNr}]} \setminus \mathcal{D}} \| \leq \eta.$$

Coarsening strategy :

Two sisters **leafs** $(\mathtt{c}^-, \mathtt{c}^+)$ of a parent $\mathtt{p}(\mathtt{c}^-)$ are removed from the discretization if

$$\|\tilde{\boldsymbol{u}}^{\mathtt{p}(\mathtt{c}^-)}\|_{\ell^2} \leq 2^{-|\mathtt{n}|/2} \frac{\eta}{\sqrt{\mathrm{NNr}}}$$

Note : the coarsening is applied to the class of equivalent trees.

Enrichment strategy :

Enrichment is necessary to anticipate emergence of new-stochastic details.

- 1-D enrichment criterion : if U is (locally) smooth enough $\tilde{u}_\alpha^{\mathrm{n}}$ of a generic node n can be bounded as

$$|\tilde{u}_\alpha^{\mathrm{n}}| = \inf_{P \in \mathbb{P}_{\mathrm{No}}[\xi]} |\langle (U - P), \Psi_\alpha^{\mathrm{n}} \rangle| \leq C |S(\mathrm{n})|^{\mathrm{No}+1} \|U\|_{H^{\mathrm{No}+1}(S(\mathrm{n}))},$$

  where $|S(\mathrm{n})| = 2^{-|\mathrm{n}|}$ is the volume of the node.

- Therefore

$$\|\tilde{\boldsymbol{u}}^{\mathrm{n}}\|_{\ell^2} \sim 2^{-(\mathrm{No}+1)} \|\tilde{\boldsymbol{u}}^{\mathrm{p}(\mathrm{n})}\|_{\ell^2},$$

  and a leaf l is refined if

$$\|\tilde{\boldsymbol{u}}^{\mathrm{p}(\mathrm{l})}\|_{\ell^2} \geq 2^{\mathrm{No}+1} 2^{-|\mathrm{l}|/2} \eta / \sqrt{\mathrm{Nr}} \quad \text{and} \quad |\mathrm{l}| < \mathrm{Nr}.$$

Enrichment strategy :

Extension of to the N-dimensional case :

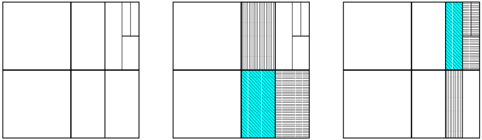
- Isotropic enrichment is not an option for $\mathrm{N} > 2, 3$
- Using the decay estimation

$$|\tilde{u}^{\mathtt{n}}_{\alpha}| = \inf_{P \in \mathbb{P}^{\mathrm{N}}_{\mathrm{No}}[\xi]} \left| \left\langle (U - P), \Psi^{\mathtt{n},d}_{\alpha} \right\rangle \right| \leq C \mathrm{diam}(S(\mathtt{n}))^{\mathrm{No}+1} \|U\|_{H^{\mathrm{No}+1}(S(\mathtt{n}))},$$

- A leaf $\mathtt{l}$ is partitioned in direction d if

$$\|\tilde{\boldsymbol{u}}^{\mathtt{p}^d(\mathtt{l})}\|_{\ell^2} \geq \left(\frac{\mathrm{diam}(S(\mathtt{p}^d(\mathtt{l})))}{\mathrm{diam}(S(\mathtt{l}))} \right)^{\mathrm{No}+1} 2^{-|\mathtt{l}|/2} \eta / \sqrt{\mathrm{NNr}} \quad \text{and} \quad |S(\mathtt{l})|_d > 2^{-\mathrm{Nr}}.$$

- Requires construction of the virtual sister and parent of $\mathtt{l}$ in arbitrary direction d



- A sharper directional criterion has been proposed using N families of 1-d analysis functions [Tryoen, LM and Ern, SISC; 2012].

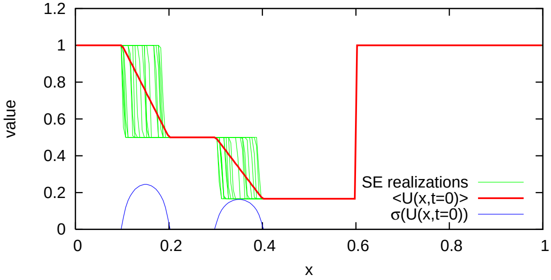
Burgers equation

$$\frac{\partial U}{\partial t} + \frac{\partial F(U)}{\partial x} = 0, \quad F(U) = \frac{U^2}{2}$$

Uncertain initial condition $U^0(x, \xi)$:

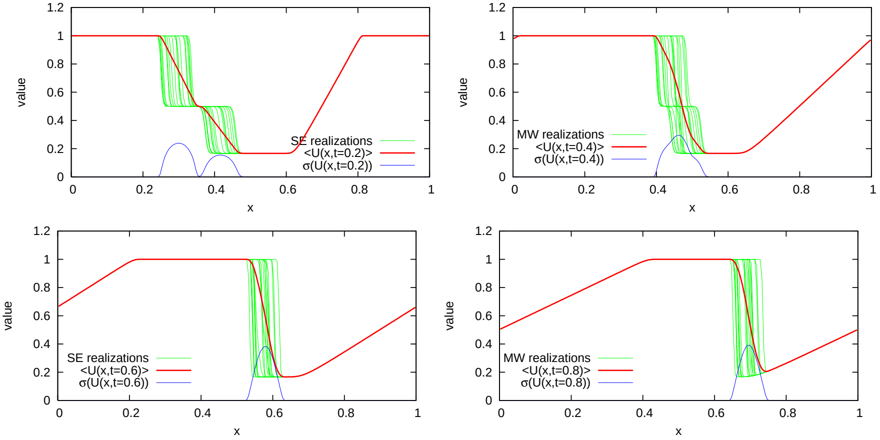
$$X_{1,2} = 0.1 + 0.1\xi_1, \quad X_{2,3} = 0.3 + 0.1\xi_2, \quad \xi_1, \xi_2 \sim \mathcal{U}[0, 1]$$

2 stochastic dimensions.



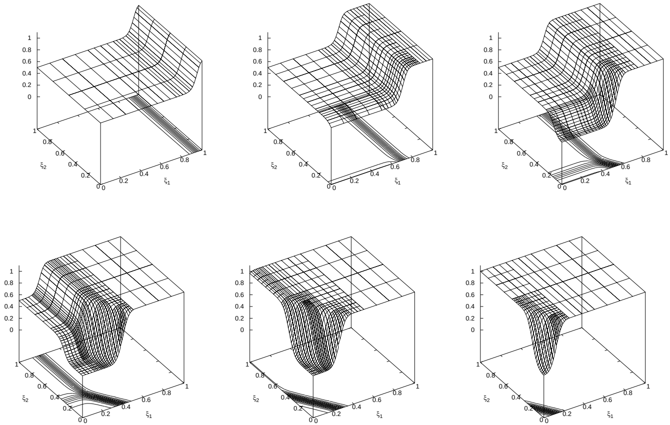
Burgers equation

$$\frac{\partial U}{\partial t} + \frac{\partial F(U)}{\partial x} = 0, \quad F(U) = \frac{U^2}{2}$$



Burgers equation

$$\frac{\partial U}{\partial t} + \frac{\partial F(U)}{\partial x} = 0, \quad F(U) = \frac{U^2}{2}$$



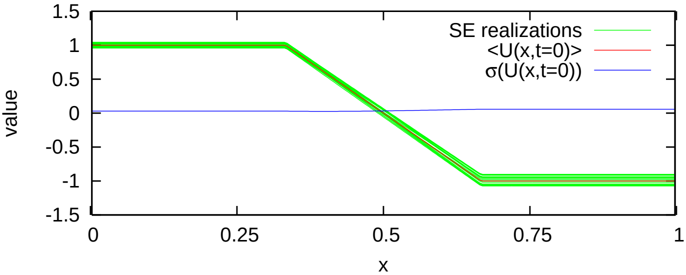
2nd test case

Continuous initial conditions : two constants stochastic states

$$U = U^+ = 1 \pm 0.05 \qquad x < 1/3,$$

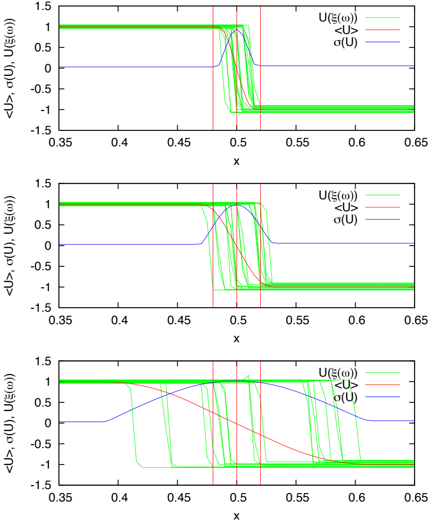
$$U = U^- = -1 \pm 0.1 \qquad x > 2/3,$$

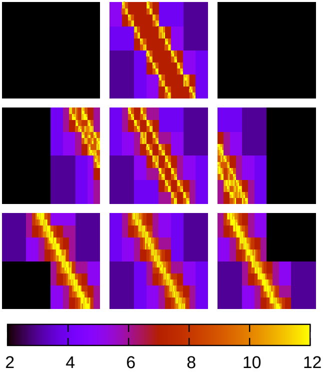
and affine variation in between. $U^+ > U^-$ a.s. and U^+ and U^- independent with uniform distribution : $U^+(\xi_1),\ U^-(\xi_2)$.



$$U^+(\xi_1) + U^-(\xi_2) \neq 0 \text{ a.s.}$$

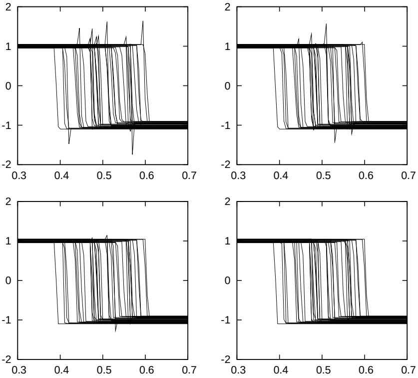
2nd test case

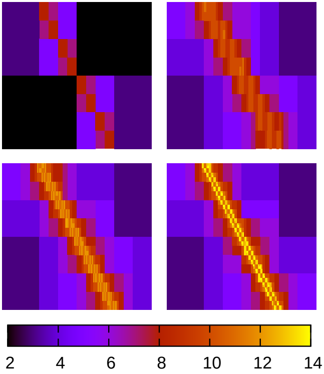




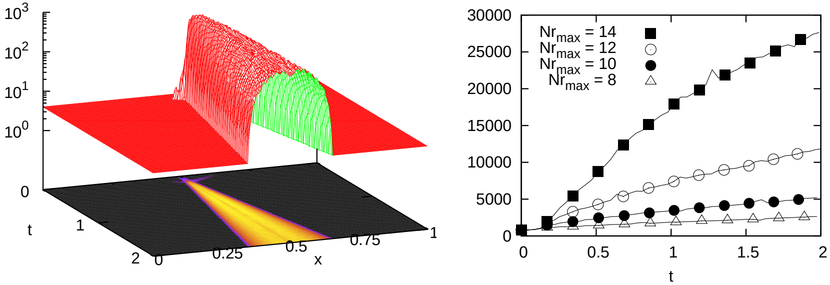
Solution with x at different times.

Convergence with max resolution level (8 to 14)



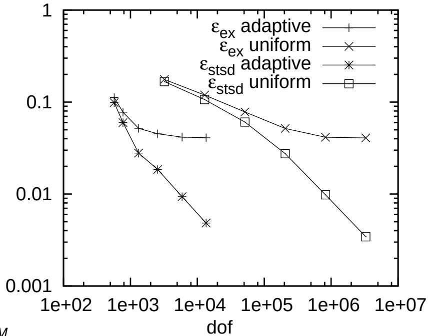


Complexity



Evolution of the # of dof in space and time (left), and time only (right).

Mean-squared error with (final) # of dof (MC estimate)



$$\epsilon_{\mathrm{ex}}^2 = \frac{\Delta x}{M} \sum_{i=1}^{N_c} \sum_{j=1}^{M} \left(U_i(\boldsymbol{\xi}^{(j)}) - U^{\mathrm{ex}}(x_i, \boldsymbol{\xi}^{(j)}) \right)^2$$ w.r.t. exact sol.

$$\epsilon_{\mathrm{stsd}}^2 = \frac{\Delta x}{M} \sum_{i=1}^{N_c} \sum_{j=1}^{M} \left(U_i(\boldsymbol{\xi}^{(j)}) - U_i^{\mathrm{stsd}}(\boldsymbol{\xi}^{(j)}) \right)^2$$ w.r.t. semi-discrete sol.

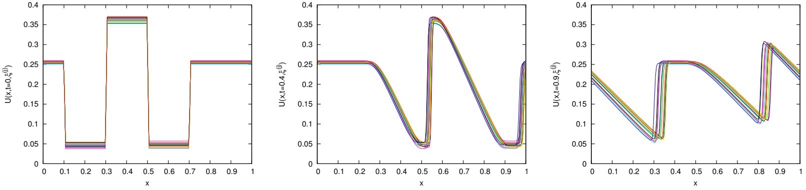
Trafic equation in periodic [0, 1]-domain

$$F(U(\xi);\xi) = A(\xi)U(\xi)(1-U(\xi)) \quad \text{1-Periodic BC.}$$

- uncertain initial density of vehicles

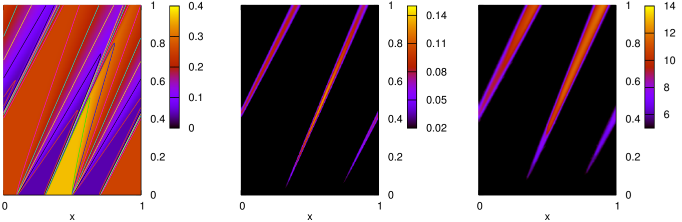
$$\begin{aligned} U^0(x,\xi) =& 0.25 + 0.01\xi_1 - \mathbb{I}_{[0.1,0.3]}(x)(0.2+0.015\xi_2) \\ &+ \mathbb{I}_{[0.3,0.5]}(x)(0.1+0.015\xi_3) - \mathbb{I}_{[0.5,0.7]}(x)(0.2+0.015\xi_4) \end{aligned}$$

- uncertain characteristic velocity $A(\xi) = 1 + 0.1\xi_5$
- 5-dimensional problem $(\xi_1, \dots, \xi_5) \sim U[0,1]^5$.

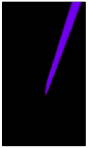
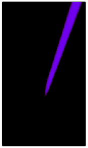
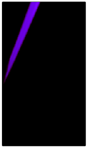
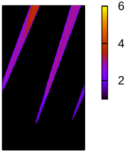


20 realizations of the initial condition (left) and solution at $t = 0.4$ (middle) and $t = 0.9$ (right) : 2 shocks and 2 rarefaction waves.

Space-time diagrams of the solution mean (left), standard deviation (center) and average depth of the leafs (right) :



Averaged number of partitions in each direction D_i and anisotropy factor ρ :

D_1	D_2	D_3	D_4	D_5	ρ
					

Hoeffding decomposition. Orthogonal hierarchical decomposition

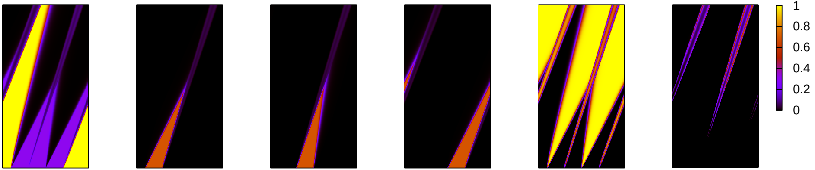
$$U(\xi_1,\ldots,\xi_N) = U_0 + \sum_{i_1=1}^{N} U_{i_1}(\xi_{i_1}) + \sum_{i_1=1}^{N}\sum_{i_2=i_1+1}^{N} U_{i_1,i_2}(\xi_{i_1},\xi_{i_2}) + \ldots + U_{1,\ldots,N}(\xi_{i_1},\ldots,\xi_{i_N}),$$

Sobol ANOVA (analysis of the variance)

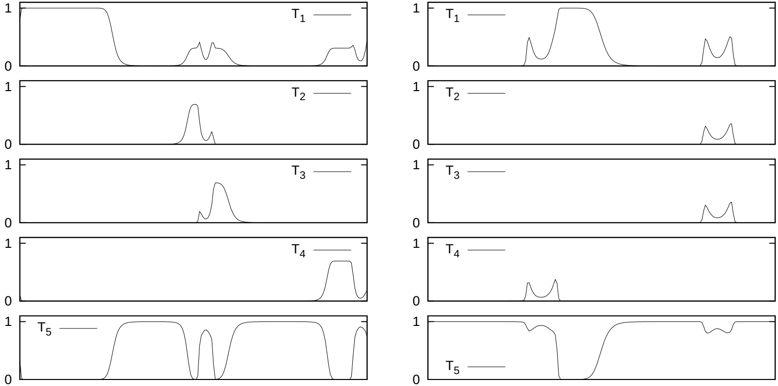
$$V(U) = \sum_{i_1=1}^{N} V_{i_1} + \sum_{i_1=1}^{N}\sum_{i_2=i_1+1}^{N} V_{i_1,i_2} + \cdots + V_{1,\ldots,N},$$

- First order sensitivity indexes : $\mathrm{S}_i = V_i/V$
- Total sensitivity indexes : $\mathrm{T}_i = \sum_{u\subseteq\{1,\ldots,N\}}^{u\ni\{i\}} V_u/V$

S_1	S_2	S_3	S_4	S_5	$1 - \sum_d S_d$

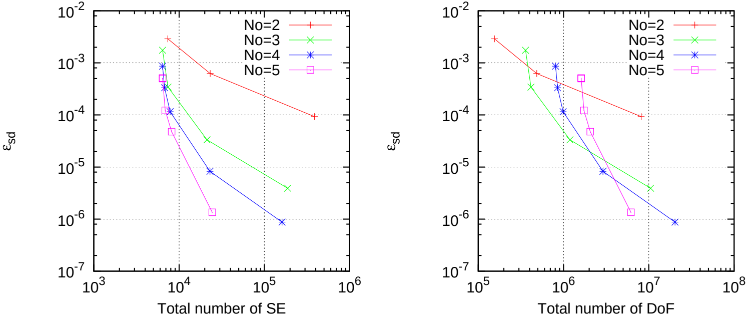


Space-time diagrams of the 1-st order sensitivity indexes S_i and contribution of higher order indexes.

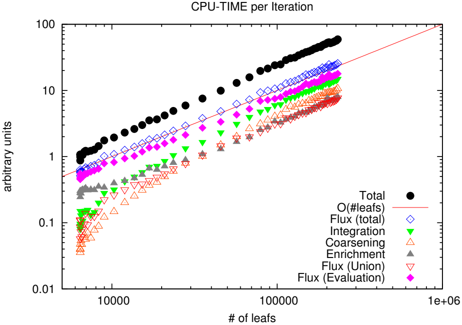


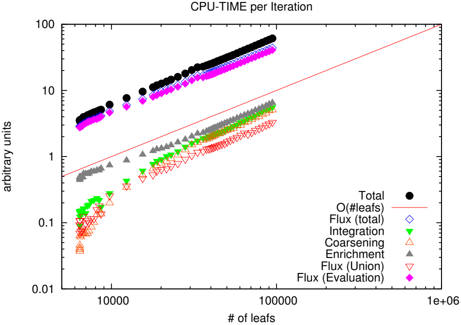
Total sensitivity indices as a function of $x \in [0, 1]$ at $t = 0.4$ (left) and $t = 0.9$ (right).

L^2-norm of stochastic error for different values of $\eta \in [10^{-2}, 10^{-5}]$ and polynomial degrees No



Left : error as a function of the total number of leafs in the final discretization ($t^n = 0.5$). Right : error as a function of the total number of degrees of freedom (number of leafs times the dimension of the local polynomial basis).





Computational time (per time-iteration) as a function of the stochastic discretization (total number of leafs) ; left : $\mathrm{No} = 2$ and $\eta = 10^{-3}$; right : $\mathrm{No} = 3$ and $\eta = 10^{-4}$.

Extension & future works

- Extension to higher-order flux approximation, limiters, . . .
- $h - p$ adaptation at the stochastic level.
- Spatial adaptation : FV mesh function of time and ξ !
- Adaptivity for systems of conservation laws.
- Higher spatial dimension.
- Parallel implementation,