Stochastic Spectral Methods for Uncertainty Quantification

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Content :

The simulation of a physical system involves the basic steps :

1. **Definition of a mathematical model :** express basic physical principles (conservation laws, modeling of elementary processes, formulation, ...)

2. **Prescription of the system investigated :** definition of model inputs (geometry, forcing, ICs, BCs, model constants, ...)

3. **Construction & resolution of the numerical model :** discretization method, algorithms, approximate solvers, (parallel) implementation, ...

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   Usually cannot *exactly specified*, *e.g.* because of limited measurements / data, identification procedures, inherent variabilities,... **Uncertainty representation**

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   To obtain an *approximation* of the model solution within some prescribed accuracy. **Verification & error analysis**

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Reality : all these basic steps are mixed and strongly interrelated in engineering practice.

Input-data uncertainty D

- Inherent variability (*e.g.* industrial processes, natural systems, variable environments,...)
- Epistemic uncertainty (*e.g.* incomplete identification/calibration, unobserved quantities, non-measurable model constants,...)

  May not be reducible, even theoretically
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Uncertainty propagation

Given the random input $D \in (\Theta, \mathcal{A}, d\mu)$, and a model $\mathcal{M}(\cdot|D)$, the objective is to determine the model solution U :

- U is random and defined on $(\Theta, \mathcal{A}, d\mu)$: $U = U(\theta \in \Theta)$.
- U and D are dependent random quantities, related by the model :

$$\mathcal{M}(U(\theta)|D(\theta)) = 0, \quad \forall \theta \in \Theta.$$

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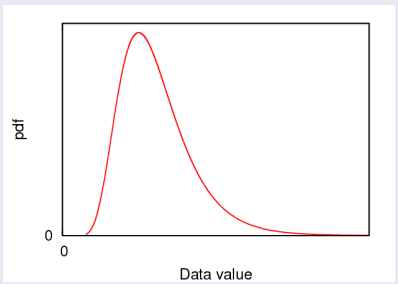
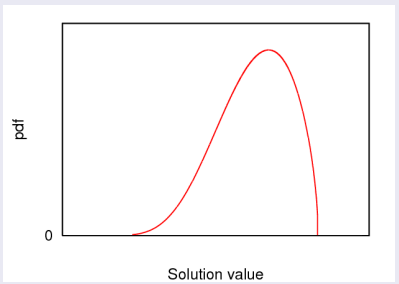
Input distribution



Model

Solution density



- Variability in U
- Predictability assessment
- Uncertainty analysis : moments, confidence intervals, full density, sensitivity analysis,...

Computational approaches (I) :

Deterministic Methods

- Local sensitivity analysis : adjoint, AD, ...
- Perturbation techniques : limited to low variance of the input.
- Neumann expansions : limited to low expansion order.
- Moments method : limited characterization of S, closure problem (non-Gaussian / non-linear problems).

Stochastic Methods — Monte-Carlo

1. Generate at random a sample set of inputs : $S_D = \{D(\theta_1), D(\theta_2), \ldots\}$
2. Compute the corresponding sample set of model ouputs : $S_U = \{U(\theta_1), U(\theta_2), \ldots\}$

$$\mathcal{M}(U(\theta_i)|D(\theta_i)) = 0.$$

3. Use the sample set S_U to estimate moments, correlations, ...

$$\mathbb{E}\{S\} = \frac{1}{M}\sum_{i=1}^{M} U(\theta_i), \ldots$$

Robust and generic approach, no assumption regarding to U or D, reuse deterministic codes, but slow convergence of the random estimates with the sample set size (M).

Computational approaches (II) :

Solution spaces

We assume that $U(\theta) \in \mathbb{V}$, almost surely. The Hilbert space $\mathbb{V}$ is equipped with the inner product and norm

$$(U, V)_{\mathbb{V}}, \quad \|U\| = (U, U)_{\mathbb{V}}^{1/2} \quad U, V \in \mathbb{V}.$$

Since U is random, $\|U\|$ is a random variable : $\|U\|_{\mathbb{V}}(\theta)$
We further assume that the solution norm is a second order random variable :

$$\mathbb{E}\left\{\|U\|_{\mathbb{V}}^2\right\} = \int_{\Theta} \|U(\theta)\|_{\mathbb{V}}^2 d\mu(\theta) < \infty.$$

We denote $L_2(\Theta, d\mu)$ the space of second order random variables defined on $(\Theta, \mathcal{A}, d\mu)$, equipped with the inner product and norm

$$\langle u, v\rangle = \mathbb{E}\{uv\} = \int_{\Theta} u(\theta)v(\theta)d\mu(\theta), \quad \|u\|_{L_2} = \langle u, u\rangle^{1/2} \quad u, v \in L_2(\Theta, d\mu).$$

Functional representation

The stochastic solution can be sought as an element of the tensored Hilbert space $X = L_2(\Theta, d\mu) \otimes \mathbb{V}$:

$$V \in X \Rightarrow V(\theta) = \sum_{\alpha} v_\alpha(\theta) V_\alpha, \quad v_\alpha \in L_2(\Theta, d\mu), \; V_\alpha \in \mathbb{V}.$$

- $v_\alpha(\theta)$ are random variables,
- V_α are the deterministic functions.

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For computational purposes, we need

1. to discretize the abstract probability space ($L_2(\Theta, d\mu)$),
2. to discretize the deterministic space $\mathbb{V}$,
3. to define the functional expansion,
4. to introduce computational strategies for its determination.

The interest in the functional expansion comes from

- possibly high convergence with the number of modes : low storage, low computational cost, ...
- explicit mapping $D(\theta) \mapsto U(\theta)$: detailed UQ, sensitivity analysis
- cheap (re)sampling of the stochastic solution : surrogate model for subsequent use.

Content :

Representation of Input Uncertainty

We assume the availability of a parametrization of the uncertain inputs in terms of a set of N independent real-valued second order random variables $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_N)$ defined on $(\Theta, \mathcal{A}, d\mu)$:

$$D(\theta) = D(\boldsymbol{\xi}(\theta)), \quad p_{\boldsymbol{\xi}}(\boldsymbol{y}) = \prod_{i=1}^{N} p_i(y_i),$$

where p_ξ is the joint density function of $\boldsymbol{\xi}$, with support $\Xi \subseteq \mathbb{R}^N$.

- transformation of random variables,
- principal orthogonal decompositions,
- principal component analysis.

Example : Karhunen-Loeve decomposition of a Gaussian random field

$$\kappa(\boldsymbol{x}, \theta) \approx \mathbb{E}\{\kappa\}(x) + \sum_{i=1}^{N} \phi_i(\boldsymbol{x})\xi_i(\theta), \quad \xi_i \sim N(0,1), \quad \mathbb{E}\{\xi_i \xi_j\} = \delta_{ij}.$$

Then, the solution $U(\theta)$ of $\mathcal{M}(U|D(\theta)) = 0$ can be identified with $\tilde{U}(\boldsymbol{\xi})$, the solution of

$$\mathcal{M}(\tilde{U}(\boldsymbol{\xi})|D(\boldsymbol{\xi})) = 0.$$

In addition (dropping the tilde),

$$U(\boldsymbol{\xi}) = \sum_{\alpha} \Psi_\alpha(\boldsymbol{\xi}) u_\alpha \in L_2(\Xi, p_{\boldsymbol{\xi}}) \otimes \mathbb{V},$$

where $\Psi_\alpha(\boldsymbol{\xi}) \in L_2(\Xi, p_{\boldsymbol{\xi}})$ and $u_\alpha \in \mathbb{V}$.

Polynomial Chaos expansions (I)

[Wiener,38], [Cameron and Martin,47]

PC expansion corresponds to the case where, in

$$U(\boldsymbol{\xi}) = \sum_{\alpha} \Psi_{\alpha}(\boldsymbol{\xi}) u_{\alpha},$$

the stochastic functions Ψ_{α} are orthogonal multivariate polynomials in $\boldsymbol{\xi}$.
For $i = 1, \ldots, \mathrm{N}$, consider the family of orthogonal polynomials $\{\psi_0^i, \psi_1^i, \ldots\}$ such that

$$\left\langle \psi_{\alpha}^i, \psi_{\beta}^i \right\rangle_i = \int \psi_{\alpha}^i(y) \psi_{\beta}^i(y) p_i(y) dy = \left\langle \psi_{\alpha}^i, \psi_{\alpha}^i \right\rangle \delta_{\alpha,\beta}, \quad \psi_{\alpha}^i \in \Pi_{\alpha}(\mathbb{R}).$$

Classical distributions [Xiu and Karniadakis, 2002]

- p_i Gaussian : Hermite polynomials,
- p_i Uniform : Legendre polynomials,
- p_i Exponential : Laguerre polynomials.

Let $\boldsymbol{\alpha} = (\alpha_1 \ldots \alpha_{\mathrm{N}}) \in \mathbb{N}^{\mathrm{N}}$ be a multi-index, and define

$$\Psi_{\boldsymbol{\alpha}}(\boldsymbol{\xi}) = \prod_{i=1}^{\mathrm{N}} \psi_{\alpha_i}^i(\xi_i),$$

such that

$$\left\langle \Psi_{\boldsymbol{\alpha}}, \Psi_{\boldsymbol{\beta}} \right\rangle = \langle \Psi_{\boldsymbol{\alpha}}, \Psi_{\boldsymbol{\alpha}} \rangle \, \delta_{\boldsymbol{\alpha}\boldsymbol{\beta}} = \prod_{i=1}^{\mathrm{N}} \left\langle \psi_{\alpha_i}^i, \psi_{\alpha_i}^i \right\rangle \delta_{\alpha_i \beta_i}.$$

Then,

$$L_2(\Xi, p_{\boldsymbol{\xi}}) = \mathrm{span}\{\Psi_{\boldsymbol{\alpha}}, \ \boldsymbol{\alpha} \in \mathbb{N}^{\mathrm{N}}\} \quad \text{and} \quad U(\boldsymbol{\xi}) = \sum_{\boldsymbol{\alpha} \in \mathbb{N}^{\mathrm{N}}} \Psi_{\boldsymbol{\alpha}}(\boldsymbol{\xi}) U_{\boldsymbol{\alpha}}.$$

Polynomial Chaos expansions (II)

In practice, the PC expansion must be truncated

$$U(\boldsymbol{\xi}) = \sum_{\boldsymbol{\alpha}\in\mathbb{N}^{\mathrm{N}}} \Psi_{\boldsymbol{\alpha}}(\boldsymbol{\xi}) u_{\boldsymbol{\alpha}} \approx \sum_{\boldsymbol{\alpha}\in\mathcal{A}} \Psi_{\boldsymbol{\alpha}}(\boldsymbol{\xi}) u_{\boldsymbol{\alpha}} = U^{|\mathcal{A}|}(\boldsymbol{\xi}) \in \mathbb{S}^{\mathcal{A}} \otimes \mathbb{V} \subset L_2(\Xi, p_{\boldsymbol{\xi}}) \otimes \mathbb{V}.$$

Classical choice for the finite multi-index set $\mathcal{A}$ are

- partial degree truncation $\mathcal{A}^p_{\mathrm{No}} = \{\boldsymbol{\alpha} \in \mathbb{N}^{\mathrm{N}}, \max_i \alpha_i \leq \mathrm{No}\}$,
- total degree truncation $\mathcal{A}^t_{\mathrm{No}} = \{\boldsymbol{\alpha} \in \mathbb{N}^{\mathrm{N}}, |\boldsymbol{\alpha}| = \sum_i \alpha_i \leq \mathrm{No}\}$,
- hyperbolic cross truncation $\mathcal{A}^h_{\mathrm{No},q} = \{\boldsymbol{\alpha} \in \mathbb{N}^{\mathrm{N}}, \sum_i \alpha_i^q \leq \mathrm{No}^q\}$,

the objective being to control the increase of the basis dimension with both the expansion order No and N :

$$|\mathcal{A}^p_{\mathrm{No}}| = (\mathrm{No}+1)^{\mathrm{N}} \geq |\mathcal{A}^p_{\mathrm{No}}| = \frac{(\mathrm{No}+\mathrm{N})!}{\mathrm{No}!\mathrm{N}!} \geq |\mathcal{A}^h_{\mathrm{No},q<1}|.$$

Having Defined the stochastic basis, it remains to determine the deterministic functions $U_{\boldsymbol{\alpha}}$, $\boldsymbol{\alpha} \in \mathcal{A}$.

HINT :

- Minimize L_2-distance $\mathbb{E}\left\{\|U(\boldsymbol{\xi}) - U^{|\mathcal{A}|}(\boldsymbol{\xi})\|^2_{\mathbb{V}}\right\}$ — NISP, "Regression"
- Verify $\|U(\boldsymbol{\xi}) - U^{|\mathcal{A}|}(\boldsymbol{\xi})\|_{\mathbb{V}} = 0$ over a set of selected points — Collocation
- Reformulate model equations for the $U_{\boldsymbol{\alpha}}$ — Stochastic Galerkin projection

Non-intrusive methods

Basics

Use code as a black-box

- Compute/estimate spectral coefficients *via* a set of deterministic model solutions
- Requires a deterministic solver only

1. $\mathcal{S}_\Xi \equiv \{\boldsymbol{\xi}^{(1)}, \ldots, \boldsymbol{\xi}^{(m)}\}$ sample set of $\boldsymbol{\xi}$
2. Let $s^{(i)}$ be the solution of the deterministic problem
$$\mathcal{M}\left(U^{(i)} | D(\boldsymbol{\xi}^{(i)})\right) = 0$$
3. $\mathcal{S}_U \equiv \{U^{(1)}, \ldots, U^{(m)}\}$ sample set of model solutions
4. Estimate expansion coefficients $u_{\boldsymbol{\alpha}}$ from this sample set.

- Complex models, reuse of determinsitic codes, planification, ...
- Error control and computational complexity (curse of dimensionality), ...

Least square fit

"Regression" [Bervillier et al, 2006]

- Best approximation is defined by minimizing a (weighted) sum of squares of residuals :

$$R^2\left(\{U_{\boldsymbol{\alpha}}, \boldsymbol{\alpha} \in \mathcal{A}\}\right) \equiv \sum_{i=1}^{m} w_i \left\| U^{(i)} - \sum_{\boldsymbol{\alpha} \in \mathcal{A}} u_{\boldsymbol{\alpha}} \Psi_{\boldsymbol{\alpha}} \left(\boldsymbol{\xi}^{(i)}\right) \right\|_{\mathbb{V}}^2 .$$

Advantages/issues

- Stability with ratio m to $|\mathcal{A}|$
- Convergence with number of regression points m
- Selection of the regression points and "regressors" $\Psi_{\boldsymbol{\alpha}}$
- Error estimate, ...
- $\Rightarrow$ current trend toward Bayesian identification of the U_α : regularized version of the least square problem [Rizzi et al, 2012]

Non intrusive spectral projection :

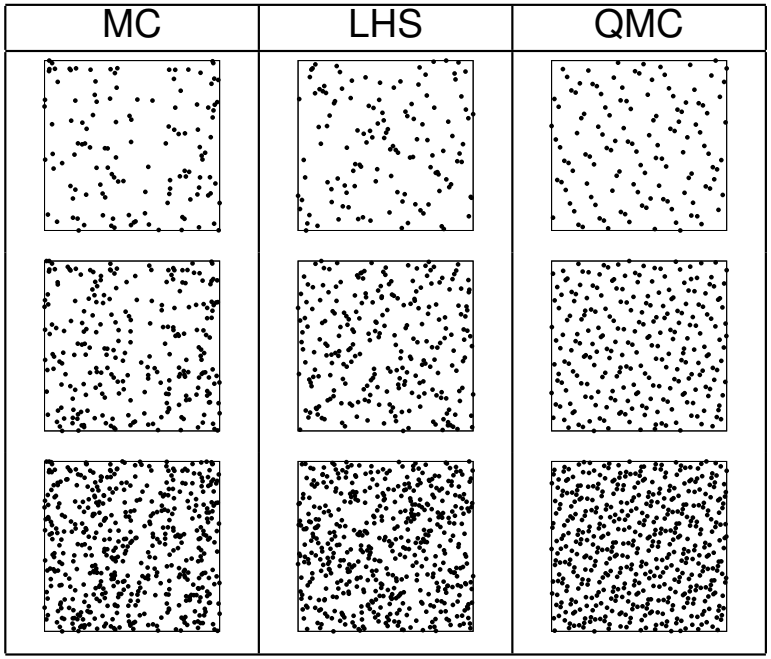
NISP [OLM et al, 2001]

Exploit the orthogonality of the spectral basis :

$$\mathbb{E}\left\{\Psi_{\boldsymbol{\alpha}}^2\right\} u_{\boldsymbol{\alpha}} = \langle U, \Psi_{\boldsymbol{\alpha}} \rangle = \int_{\Xi} U(\boldsymbol{\xi}) \Psi_{\boldsymbol{\alpha}}(\boldsymbol{\xi}) p_{\boldsymbol{\xi}}(\boldsymbol{\xi}) d\boldsymbol{\xi}.$$

Estimate integrals from a random sample sets (MC and variants) :

$$\langle U, \Psi_{\boldsymbol{\alpha}} \rangle \approx \frac{1}{m} \sum_{i=1}^{m} U(\boldsymbol{\xi}^{(i)}) \Psi_{\boldsymbol{\alpha}}(\boldsymbol{\xi}^{(i)})$$



- Convergence rate
- Error estimate
- Optimal sampling strategy

Non intrusive projection

Deterministic Quadratures

Approximate integrals by N-dimensional quadratures :

$$\langle U, \Psi_{\boldsymbol{\alpha}} \rangle \approx \sum_{i=1}^{N_Q} w^{(i)} U^{(i)} \Psi_{\boldsymbol{\alpha}} \left(\boldsymbol{\xi}^{(i)} \right) .$$

Owing to the product structure of Ξ the quadrature points $\boldsymbol{\xi}^{(i)}$ and weights $w^{(i)}$ can be obtained by

- full tensorization of n points 1-D quadrature (*i.e.* Gauss) :

$$\boxed{N_Q = n^{\mathrm{N}}}$$

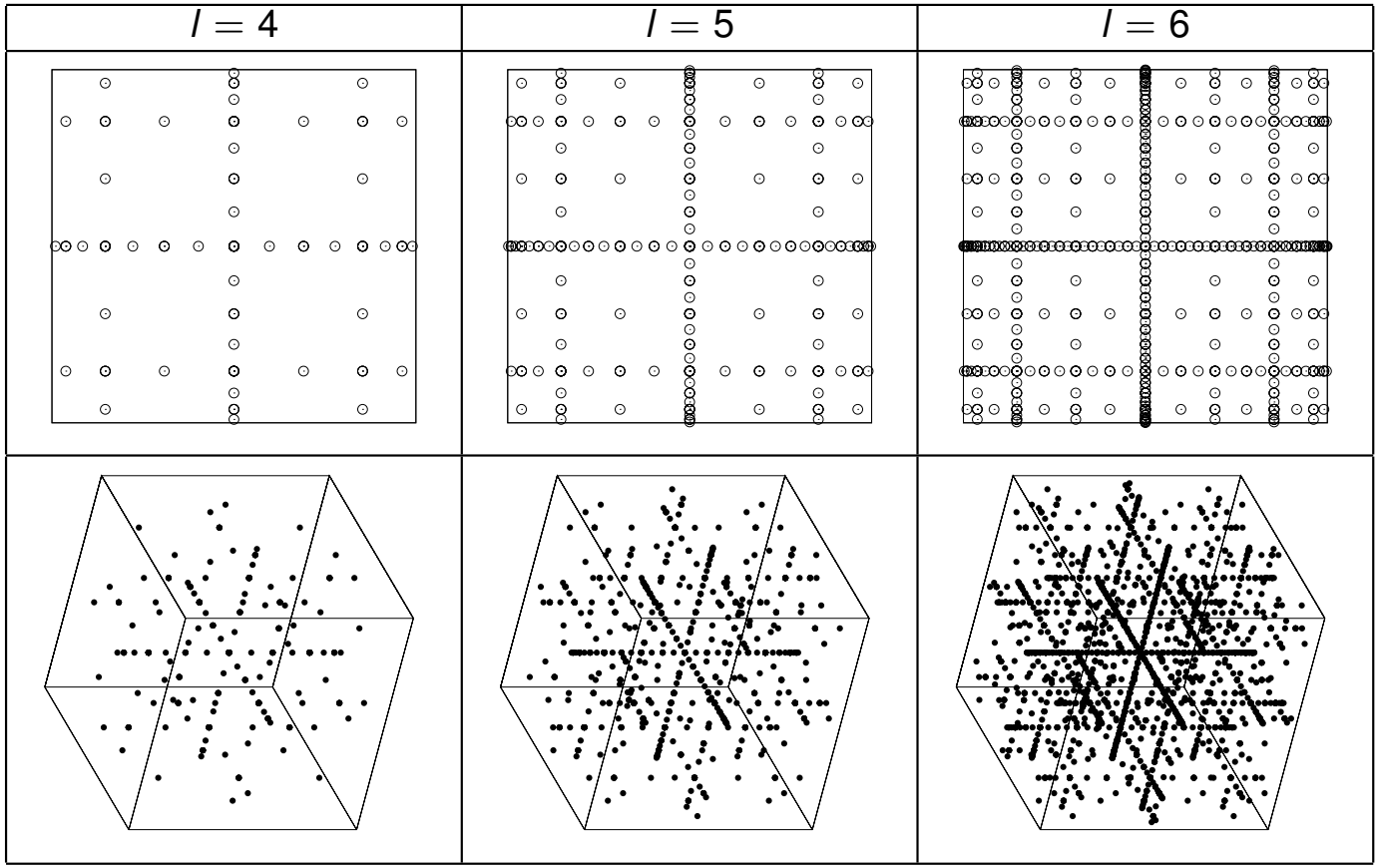
- partial tensorization of nested 1-D quadrature formula (Féjer, Clenshaw-Curtis) using Smolyak formula : [Smolyak, 63]

$$\boxed{N_Q << n^{\mathrm{N}}}$$

The partial tensorization results in so-called Sparse-Grid cubature formula, that can be constructed adaptively to the integrants (anisotropic formulas) in order to account for variable behaviors along the stochastic directions. [Gerstner and Griebel, 2003]

Non intrusive projection

Deterministic Quadratures



- Important development of sparse-grid methods
- Anisotropy and adaptivity
- Also (sparse grid) collocation methods (N-dimensional interpolation) [Mathelin and Hussaini, 2003], [Nobile et al, 2008]

Galerkin projection

- Weak solution of the stochastic problem $\mathcal{M}(U(\boldsymbol{\xi}); D(\boldsymbol{\xi})) = 0$
- Needs **adaptation of deterministic codes**
- Potentially more efficient than NI techniques.

Let $\mathbb{S}^{\mathrm{P}} \subset L^2(\Xi, p_\xi)$ defined as

$$\mathbb{S}^{\mathrm{P}} = \mathrm{span}\{\Psi_0, \ldots, \Psi_{\mathrm{P}}\},$$

where the $\{\Psi_k\}$ are orthogonal functionals in $\boldsymbol{\xi}$, *e.g.* a PC basis truncated to an order No.
We seek for the approximate stochastic model solution in $\mathbb{V} \otimes \mathbb{S}^{\mathrm{P}}$.

$$U(\boldsymbol{\xi}) \approx U^{\mathrm{P}}(\boldsymbol{\xi}) = \sum_{k=0}^{\mathrm{P}} u_k \Psi_k(\boldsymbol{\xi}).$$

Inserting U^{P} in the weak formulation yields the stochastic residual $\mathcal{R}(U^{\mathrm{P}}) = \mathcal{M}(U^{\mathrm{P}}(\boldsymbol{\xi})|D(\boldsymbol{\xi})) \in \mathbb{V}'$.

Galerkin projection [Ghanem & Spanos, 1991]
In general, one cannot find $U^{\mathrm{P}} \in \mathbb{V} \otimes \mathbb{S}^{\mathrm{P}}$ such that

$$\mathcal{R}(U^{\mathrm{P}}) = 0$$

It is then required that $\mathcal{R}(U^{\mathrm{P}})$ is orthogonal to the stochastic approximation space :

$$\left\langle \left\langle \mathcal{M}(U^{\mathrm{P}}(\boldsymbol{\xi})|D(\boldsymbol{\xi})), v \right\rangle_{\mathbb{V}'}, \beta(\boldsymbol{\xi}) \right\rangle = 0 \quad \forall \beta \in \mathbb{S}^{\mathrm{P}} \text{ and } v \in \mathbb{V}.$$

- This weak formulation corresponds to the stochastic Galerkin formulation.
- The actual formulation is obtained in practice by projecting all model equations on $\mathbb{S}^{\mathrm{P}}$.

The Galerkin projection results in a set of $\mathrm{P}+1$ coupled problems for the stochastic modes u_k of the solution.

Find $\{u_k, k=0,\ldots,\mathrm{P}+1\} \in \mathbb{V}^{\mathrm{P}+1}$ such that

$$\left\langle \left\langle \mathcal{M}\left(\sum_{k=0}^{\mathrm{P}} u_k \Psi_k(\boldsymbol{\xi}) | D(\boldsymbol{\xi})\right), v\right\rangle_{\mathbb{V}'}, \Psi_l(\boldsymbol{\xi})\right\rangle = 0, \quad \forall v \in \mathbb{V} \text{ and } l=0,\ldots,\mathrm{P}.$$

- The size of the Galerkin problem increases with the stochastic basis dimension $\mathrm{P}+1$.
- Recall that $\mathrm{P}+1 = (\mathrm{N}+\mathrm{No})!/\mathrm{N}!\mathrm{No}!$ for polynomial truncation at order No.
- This can be costly for complex problems requiring large parametrization and large expansion order.
- $\mathrm{P}+1 < N_Q$ so complexity potentially less than for NISP and other non-intrusive approaches.
- Galerkin solvers and algorithms targeting a complexity scaling with $\mathrm{P}+1$, whenever possible.
- Projections on the Ψ_l of the model equations can be problematic for **non-linear models**.

Case of linear models Linear problems are of practical importance in scientific computing, whether as stand-alone mathematical problems or as ingredients of numerical methods (*e.g.* iterative techniques for the resolution of non-linear problems).

We analyze the structure of the Galerkin problem arising from the projection of linear models, and examine implications regarding suitable solution strategies.

Consider a linear problem discretized at the deterministic level and recast in the matrix form

$$[A](\boldsymbol{\xi})\boldsymbol{U}(\boldsymbol{\xi}) = \boldsymbol{B}(\boldsymbol{\xi}).$$

Seeking the solution $\boldsymbol{U}(\boldsymbol{\xi})$ in $\mathbb{R}^m \otimes \mathbb{S}^{\mathrm{P}}$, the Galerkin projection gives :

$$\sum_{i=0}^{\mathrm{P}} \langle \Psi_k, [A]\Psi_i \rangle \, \boldsymbol{u}_i = \langle \Psi_k, \boldsymbol{B} \rangle \,, \quad k \in \{0, \ldots, \mathrm{P}\}.$$

equivalent to the larger (block) system of linear equations

$$\begin{bmatrix} [A]_{00} & \cdots & [A]_{0\mathrm{P}} \\ \vdots & \ddots & \vdots \\ [A]_{\mathrm{P}0} & \cdots & [A]_{\mathrm{PP}} \end{bmatrix} \begin{pmatrix} \boldsymbol{u}_0 \\ \vdots \\ \boldsymbol{u}_{\mathrm{P}} \end{pmatrix} = \begin{pmatrix} \boldsymbol{b}_0 \\ \vdots \\ \boldsymbol{b}_{\mathrm{P}} \end{pmatrix}.$$

$[A]_{ij}$ the $(m \times m)$ matrix given by $[A]_{ij} := \left\langle \Psi_i, [A]\Psi_j \right\rangle$, and $\boldsymbol{b}_i := \langle \Psi_i, \boldsymbol{B} \rangle$.

- The linear Galerkin problem couples all the stochastic modes $\boldsymbol{u}_i \in \mathbb{R}^m$ of the stochastic solution.
- It is not possible in general to compute independently the components $\boldsymbol{u}_i$.
- The size of the spectral problem is large : $m \times \dim \mathbb{S}^{\mathrm{P}} = m \times (\mathrm{P}+1)$.
- Resolution of the linear Galerkin system can be demanding.
- An understanding of the block structured system is instructive to design and apply well-suited numerical methods.

In general, the matrix $[A]$ has a PC expansion

$$[A](\boldsymbol{\xi}) = \sum_{i=0}^{\mathrm{P}} [A]_i \Psi_i(\boldsymbol{\xi}) \;\Rightarrow\; [A]_{ij} = \left\langle \Psi_i, [A]\Psi_j \right\rangle = \sum_{k=0}^{\mathrm{P}} [A]_k \left\langle \Psi_i, \Psi_j \Psi_k \right\rangle ,$$

and the Galerkin system can be conveniently recast as

$$\begin{bmatrix} [\overline{A}]_{00} & \dots & [\overline{A}]_{0\mathrm{P}} \\ \vdots & \ddots & \vdots \\ [\overline{A}]_{\mathrm{P}0} & \dots & [\overline{A}]_{\mathrm{PP}} \end{bmatrix} \begin{pmatrix} \boldsymbol{u}_0 \\ \vdots \\ \boldsymbol{u}_{\mathrm{P}} \end{pmatrix} = \begin{pmatrix} \overline{\boldsymbol{b}}_0 \\ \vdots \\ \overline{\boldsymbol{b}}_{\mathrm{P}} \end{pmatrix} ,$$

where $\overline{\boldsymbol{b}}_i := \langle B, \Psi_i \rangle \,/\, \left\langle \Psi_i^2 \right\rangle$ and

$$[\overline{A}]_{ij} := \sum_{k=0}^{\mathrm{P}} [A]_k \, C_{kji} , \qquad C_{ijk} := \frac{\left\langle \Psi_i \Psi_j \Psi_k \right\rangle}{\left\langle \Psi_k \Psi_k \right\rangle} .$$

The third-order tensor C_{ijk} plays a fundamental role in stochastic Galerkin methods, especially in non-linear problems.

- C_{ijk} is symmetric w.r.t. the two first indices, $C_{ijk} = C_{jik}$.
- It induces block-symmetry in the spectral problem, $[\overline{A}]_{ij} = [\overline{A}]_{ji}$
- Many of the $(\mathrm{P}+1)^3$ entries are zero with many simplifications.
- For instance the first block of the Galerkin system reduces to

$$[\overline{A}]_{00} = \sum_{k=0}^{\mathrm{P}} [A]_k \, C_{k00} = [A]_0$$

  and the sum for the upper-right block (and lower-left block) actually reduces to $[\overline{A}]_{0\mathrm{P}} = [A]_{\mathrm{P}} / \left\langle \Psi_{\mathrm{P}}^2 \right\rangle$.
- Many other simplifications occur.

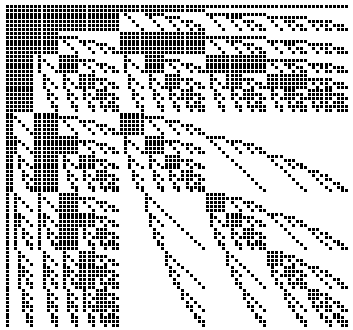
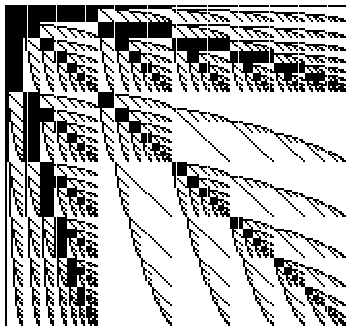
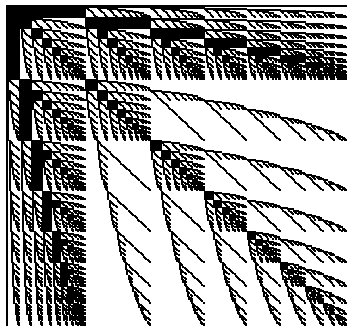
$\mathrm{N} = 4$-dim $\mathbb{S}^{\mathrm{P}} = 35$-$S = 0.58$	$\mathrm{N} = 6$-dim $\mathbb{S}^{\mathrm{P}} = 84$-$S = 0.41$
	
$\mathrm{N} = 8$-dim $\mathbb{S}^{\mathrm{P}} = 165$-$S = 0.31$	$\mathrm{N} = 10$-dim $\mathbb{S}^{\mathrm{P}} = 286$-$S = 0.23$
	

Illustration of the sparse structure of the matrices of the linear spectral problem for different dimensions, N, with $\mathrm{No} = 3$. Matrix blocks $[\overline{A}]_{ij}$ that are generally non-zero appear as black squares.

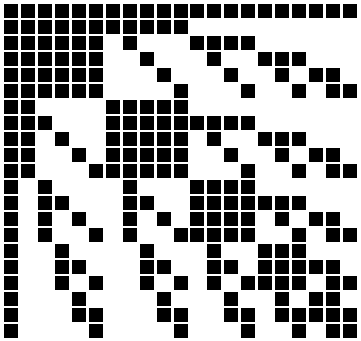
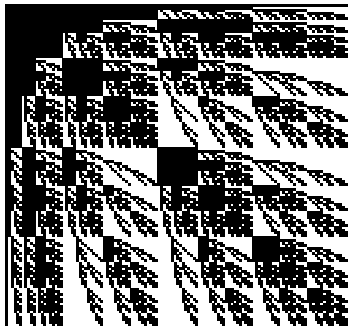
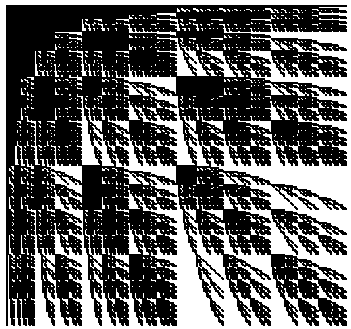
$\mathrm{No} = 2\text{-dim}\,\mathbb{S}^{\mathrm{P}} = 21\text{-}S = 0.52$	$\mathrm{No} = 3\text{-dim}\,\mathbb{S}^{\mathrm{P}} = 56\text{-}S = 0.49$
	
$\mathrm{No} = 4\text{-dim}\,\mathbb{S}^{\mathrm{P}} = 126\text{-}S = 0.54$	$\mathrm{No} = 5\text{-dim}\,\mathbb{S}^{\mathrm{P}} = 252\text{-}S = 0.55$
	

Illustration of the sparse structure of the matrices of the linear spectral problem for different expansion orders No, with $\mathrm{N} = 5$. Matrix blocks $[\overline{A}]_{ij}$ that are generally non-zero appear as black squares.

- The main difficulty in solving discrete linear spectral problems is the size of the system.
- The structure and sparsity of the linear Galerkin problem suggests iterative solution strategies.
- Iterative solvers (*e.g.* conjugate gradient techniques for symmetric systems, and Krylov subspace methods) can be used.
- The efficiency of iterative solvers depends on the availability of appropriate preconditioners which need be adapted to the Galerkin problem.
- Construction of preconditioners to exploit the structure of the linear Galerkin problem.

Convection dispersion equation

Proposed by : J.-M. Martinez (CEA/DEN/DM2S/LGLS) and A. Cartalade (CEA/DEN/DM2S).

1-D Convection dispersion

Model equation — A. Cartalade (CEA)

- Concentration $C(x,t)$

$$\phi \frac{\partial C}{\partial t} = -\frac{\partial}{\partial x}\left[qy - (\phi D_0 + \lambda |q|)\frac{\partial C}{\partial x}\right].$$

- IC and BC : $C(x, t=0) = 0,\ C(x=0, t) = 1.$
- Parameters : $q > 0$ the Darcy velocity (1m/day), ϕ the fluid fraction (given in $]0,1[$), D_0 the molecular diffusivity ($<< 1$), and λ **uncertain the hydrodynamic dispersion coefficient**.

Uncertainty model

- λ follows an uncertain power-law :

$$\boxed{\lambda = a\phi^b}.$$

- a and b independent random variables.
- $\log_{10}(a) \sim U[-4,-2]$ and $b \sim U[-3.5,-1]$.

$$a(\xi_1) = \exp(\mu_1 + \sigma_1\xi_1), \quad b = \mu_2 + \sigma_2\xi_2, \quad \xi_1, \xi_2 \sim U[-1,1].$$

$$\boxed{\lambda(\xi_1,\xi_2) = \exp(\mu_1 + \sigma_1\xi_1)\phi^{\mu_2+\sigma_2\xi_2} \approx \sum_k \lambda_k \Psi_k(\xi_1,\xi_2)}$$

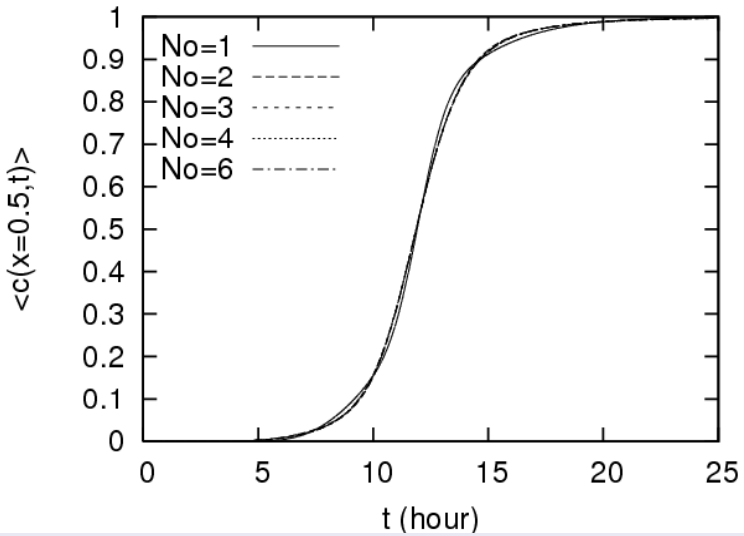
[Debusschere et al, 2004]

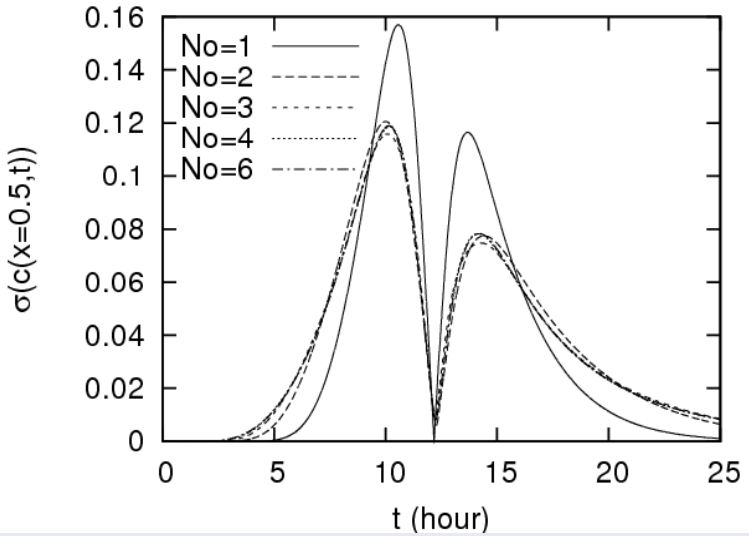
Convection dispersion equation — results

Solution method

- Wiener-Legendre expansion and Galerkin projection :
  $C(x,t,\xi_1,\xi_2) = \sum_{k=0}^{P} C_k(x,t)\Psi_k(\xi_1,\xi_2)$.
- Finite volume deterministic discretization $\mathcal{O}(\Delta x^2)$.
- Implicit time scheme $\mathcal{O}(\Delta t^2)$ (block tri-diagonal system, mean operator preconditionner).
- upwind stabilization of convection term (velocity is certain).

Expectation & standard deviation at $x = 0.5$

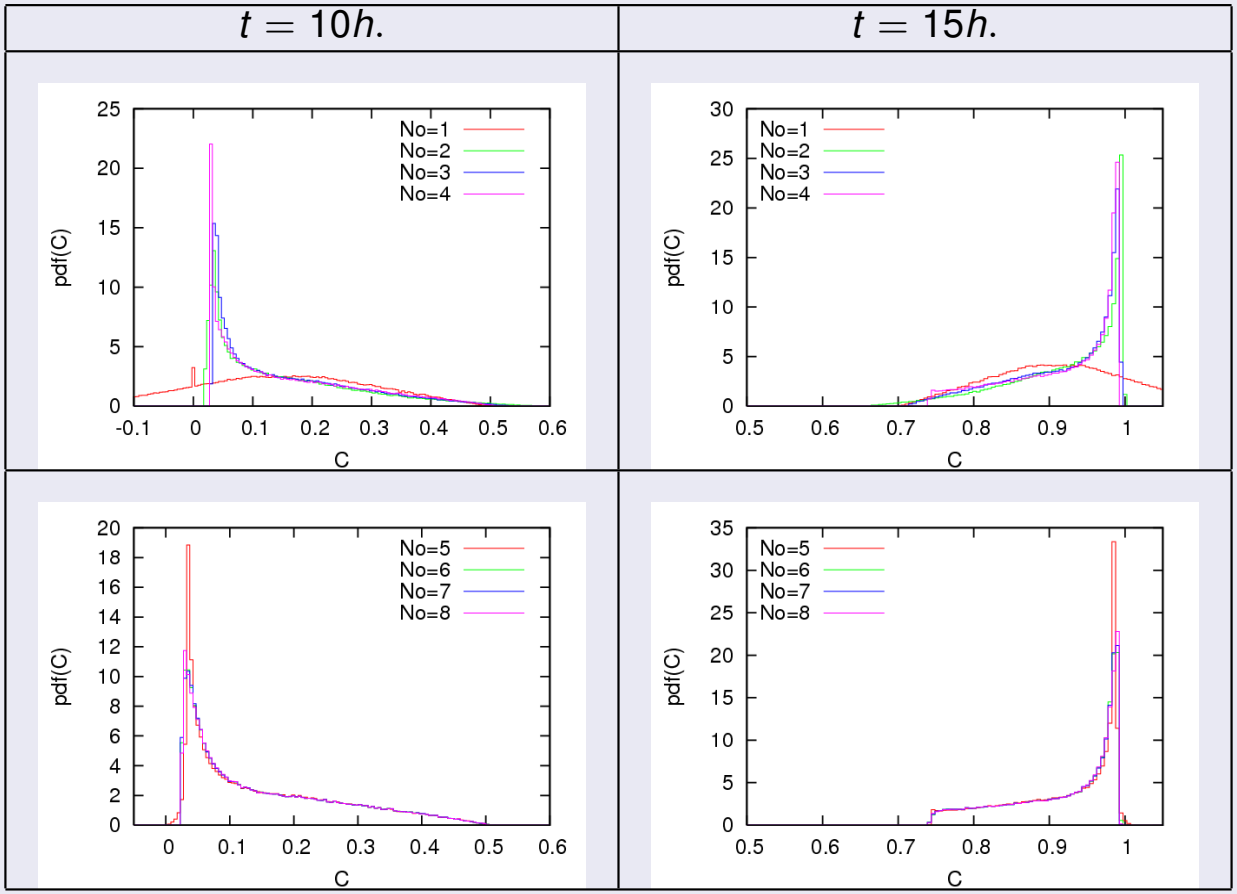




$\mathrm{No} = 1 \rightarrow \mathrm{P}+1 = 3, \mathrm{No} = 6 \rightarrow \mathrm{P}+1 = 145.$

Convection dispersion equation — results

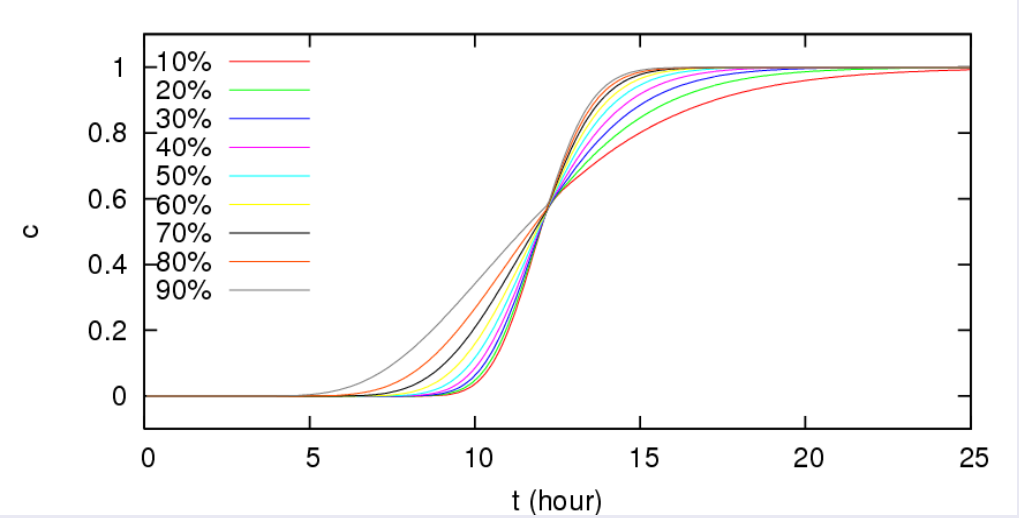
Convergence of pdfs at $x = 0.5$

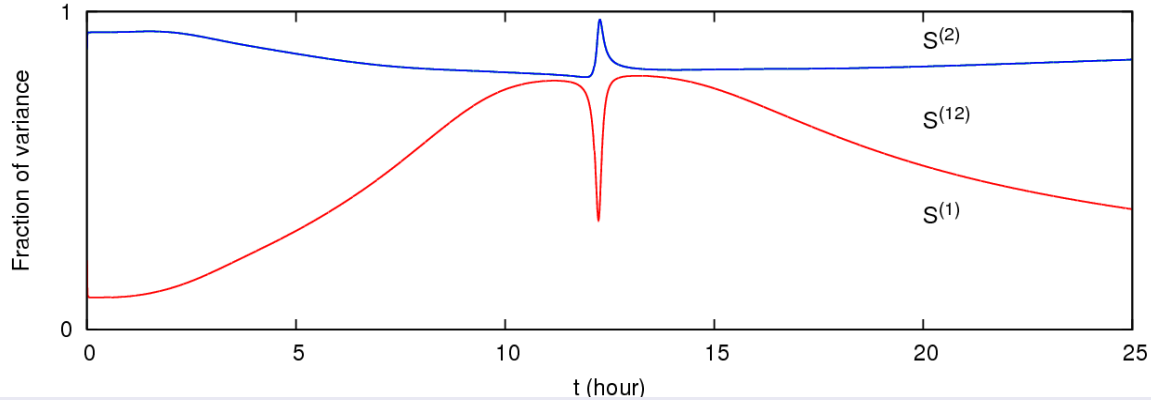


Convection dispersion equation

results

Further uncertainty analysis : quartiles & ANOVA (Sobol)





Stochactic spectral solvers
for incompressible Navier-Stokes equations

Stochastic Navier-Stokes equations with uncertain viscosity and source term :

$$\frac{\partial \boldsymbol{U}}{\partial t} + \boldsymbol{U}\boldsymbol{\nabla}\boldsymbol{U} = -\boldsymbol{\nabla}P + \nu\nabla^2\boldsymbol{U} + \boldsymbol{F}$$
$$\boldsymbol{\nabla}\cdot\boldsymbol{U} = 0.$$

PC expansion of the solution :

$$(\boldsymbol{U}, P) \approx (\boldsymbol{U}^{\mathrm{P}}, P^{\mathrm{P}}) = \sum_{k=0}^{\mathrm{P}} (\boldsymbol{u}_k, p_k)\Psi_k.$$

And the Galerkin projection on Ψ_α gives

$$\frac{\partial \boldsymbol{u}_k}{\partial t} + \sum_{i,j=0}^{P} C_{kij}\boldsymbol{u}_i\boldsymbol{\nabla}\boldsymbol{u}_j = -\boldsymbol{\nabla}p_k + \sum_{i=0}^{\mathrm{P}} \frac{\langle \nu\Psi_i, \Psi_k\rangle}{\langle \Psi_k, \Psi_k\rangle}\nabla^2\boldsymbol{u}_i + \frac{\langle F, \Psi_k\rangle}{\langle \Psi_k, \Psi_k\rangle}$$
$$\boldsymbol{\nabla}\cdot\boldsymbol{u}_k = 0.$$

Galerkin projection of the Navier-Stokes Equation :

$$\frac{\partial \boldsymbol{u}_k}{\partial t} + \sum_{l,m} C_{klm} \boldsymbol{u}_l \boldsymbol{\nabla} \boldsymbol{u}_m = -\boldsymbol{\nabla} p_k + \sum_{l,m} C_{klm} \nu_l \nabla^2 \boldsymbol{u}_m + \boldsymbol{f}_k, \quad \boldsymbol{\nabla} \cdot \boldsymbol{u}_k = 0$$

where $C_{klm} := \frac{\langle \Psi_l \Psi_m, \Psi_k \rangle}{\langle \Psi_k, \Psi_k \rangle}$

Deterministic viscosity and explicit treatment of the non-linearity Decoupling !

$$\frac{1}{\Delta t} \boldsymbol{u}_k^{n+1} - \nu \nabla^2 \boldsymbol{u}_k^{n+1} + \boldsymbol{\nabla} p_k^{n+1} = \frac{1}{\Delta t} \boldsymbol{u}_k^n - \sum_{i,j=0}^{P} C_{kij} \boldsymbol{u}_i^n \boldsymbol{\nabla} \boldsymbol{u}_j^n + \frac{\langle F, \Psi_k \rangle}{\langle \Psi_k, \Psi_k \rangle}$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{u}_k^{n+1} = 0.$$

Other treatments of the convective part :

- semi-implicit, $\boldsymbol{u}_l^n \boldsymbol{\nabla} \boldsymbol{u}_m^{n+1}$, $\longrightarrow$ set of linear non-symmetric coupled problems : stabilization, ?
- alternative semi-implicit form :

$$\left(\sum_{l,m} C_{klm} \boldsymbol{u}_l \boldsymbol{\nabla} \boldsymbol{u}_m \right)^{n+1} \approx \boldsymbol{u}_0^n \boldsymbol{\nabla} \boldsymbol{u}_k^{n+1} + \sum_{l>0,m} C_{klm} \boldsymbol{u}_l^n \boldsymbol{\nabla} \boldsymbol{u}_m^n$$

$\longrightarrow$ mean-flow based stabilization (*e.g. upwinding*).

Steady problem

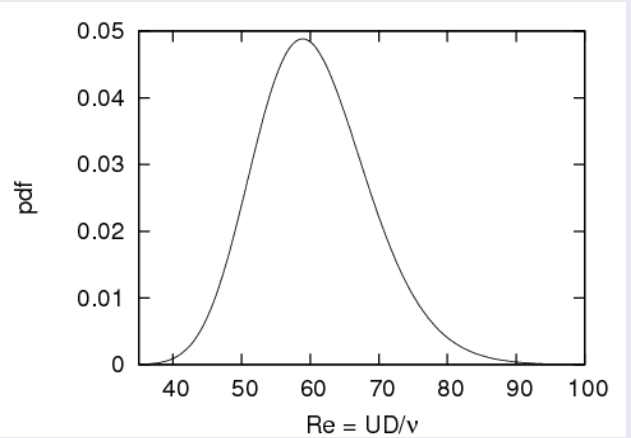
Solve the nonlinear set of equations

$$\sum_{l,m} C_{klm} \left(\boldsymbol{u}_l \boldsymbol{\nabla} \boldsymbol{u}_m - \nu_l \nabla^2 \boldsymbol{u}_m \right) + \boldsymbol{\nabla} p_k = \boldsymbol{f}_k, \quad \boldsymbol{\nabla} \cdot \boldsymbol{u}_k = 0.$$

- Large problem
- Iterative approach mandatory (Newton-like)
- Construction of approximate tangent operator (matrix-free)
- Derive appropriate preconditioners, *e.g.* based on time-stepper [olm, 2009]

Steady Flow around a circular cylinder - Vorticity formulation

Uncertain Reynolds : $\mathrm{Re} = \mathrm{Re}(\xi) \sim LN$
(Median above critical value)
stochastic basis :
Wiener-Hermite



Numerical Method :
Newton Iterations (with Unstd. stoch. Stokes prec.)
$\psi - \omega$ formulation + influence matrix for BCs

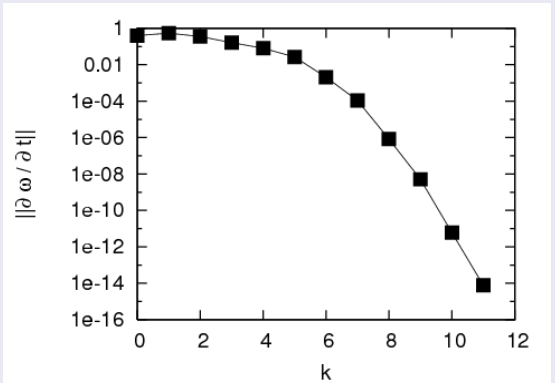
$$\boxed{\boldsymbol{u}(\xi)\boldsymbol{\nabla}\omega(\xi) - \frac{1}{\mathrm{Re}(\xi)}\nabla^2\omega(\xi) = 0.}$$

Centered Finite differences $\mathcal{O}(\Delta x^2)$
Uniform mesh (512×360) and direct FFT-based solvers

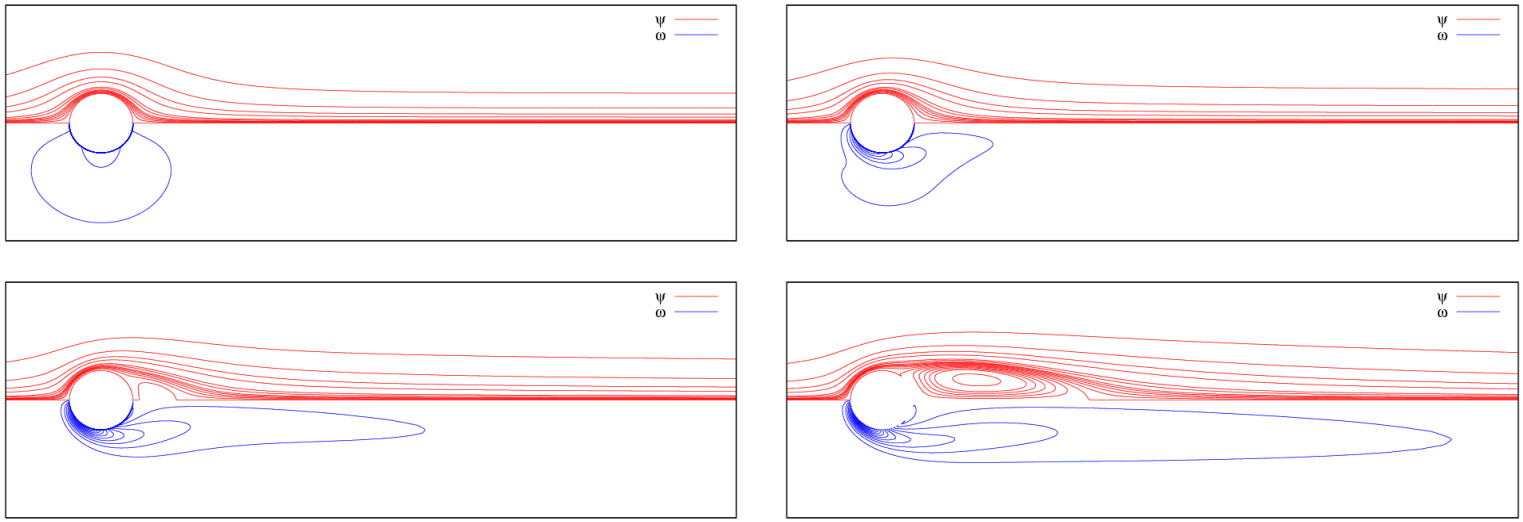
Convergence of Newton iterates

Wiener-Hermite $\mathrm{No} = 4$
L_2 Residual of stochastic equation :

$$\boxed{\boldsymbol{u}(\xi)\boldsymbol{\nabla}\omega(\xi) - \frac{1}{\mathrm{Re}(\xi)}\nabla^2\omega(\xi) = 0.}$$

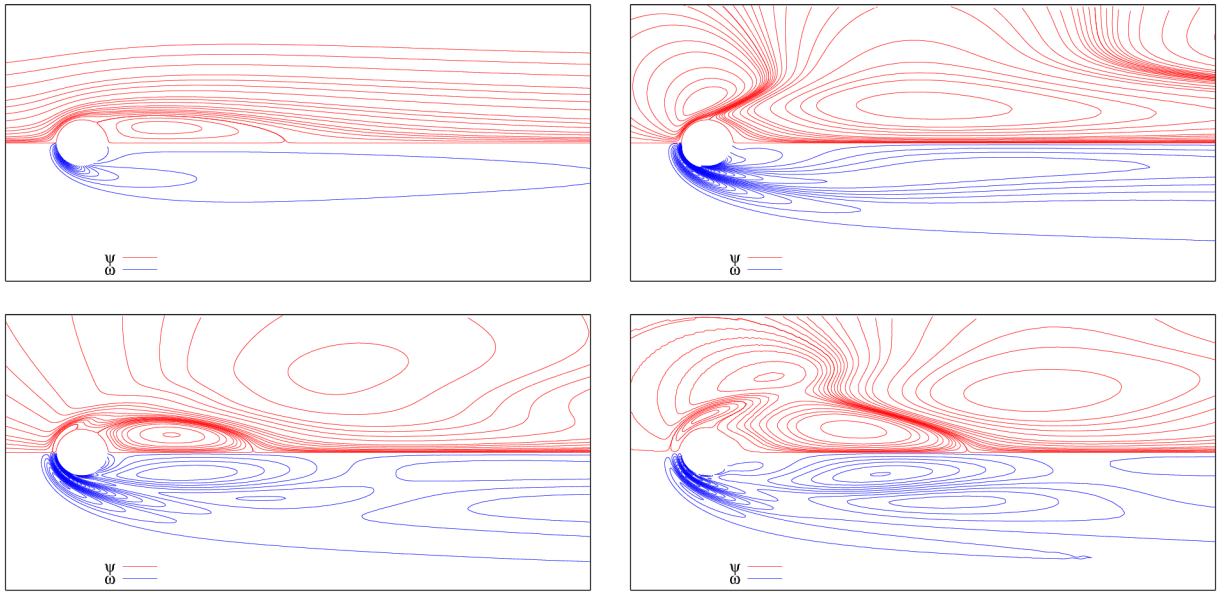


Convergence of the mean mode : (first 4 iterations)

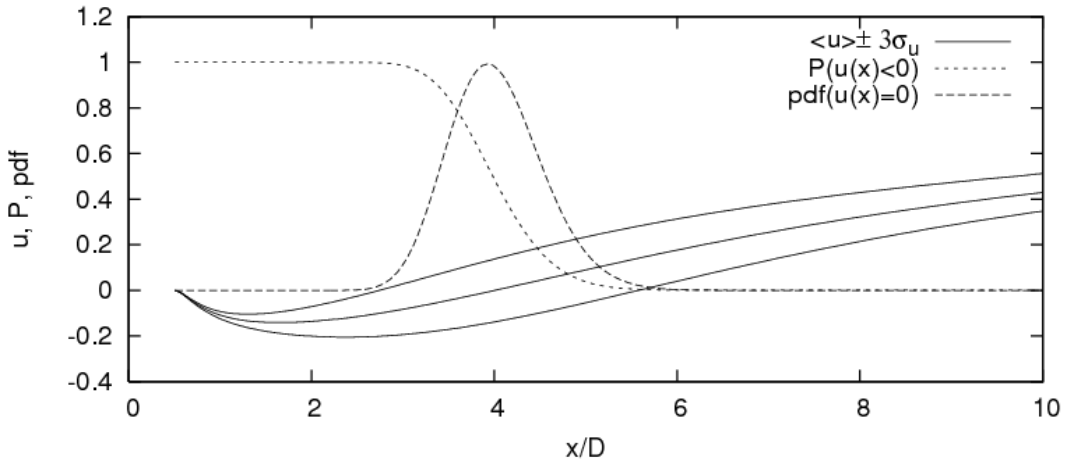


First 4 stoch. modes :

$$\omega(\boldsymbol{x},\xi) = \sum_k \omega_k(\boldsymbol{x})\Psi_k(\xi)$$



Near wake statistics :



Content :

Functional expansions

Recall that the stochastic solution is sought in $L_2(\Xi, p_\xi) \otimes \mathbb{V}$:

$$U(\boldsymbol{\xi}) = \sum_\alpha f_\alpha(\boldsymbol{\xi}) u_\alpha(\boldsymbol{\xi}), \quad f_\alpha \in L_2(\Xi, p_{\boldsymbol{\xi}}) \text{ and } u_\alpha \in \mathbb{V}.$$

The PC methods consist in :

1. Discretization of $L_2(\Xi, p_{\boldsymbol{\xi}})$ introducing a subspace $\mathbb{S}^{|\mathcal{A}|}$ spanned by a basis $\{\Psi_{\boldsymbol{\alpha}}, \boldsymbol{\alpha} \in \mathcal{A}\}$,
2. Determination of the "best" approximation $U^{|\mathcal{A}|}$ of $U(\boldsymbol{\xi})$ in the subspace $\mathbb{S}^{|\mathcal{A}|} \otimes \mathbb{V}$.

The *a priori* selection of the stochastic basis yields the question of the optimality of the PC approximation. Can we do better ?

Instead, we could consider the stochastic subspace space $\mathbb{S}^{|\mathcal{A}|}$, and define the generic separated representation as

$$U^{|\mathcal{A}|}(\boldsymbol{\xi}) \approx U^m(\boldsymbol{\xi}) = \sum_{\alpha=1}^{m} f_\alpha(\boldsymbol{\xi}) u_\alpha(\boldsymbol{\xi}), \quad f_\alpha \in \mathbb{S}^{|\mathcal{A}|} \text{ and } u_\alpha \in \mathbb{V},$$

where known of the U_α and λ_α are selected *a priori*, and we expect $m \ll |\mathcal{A}|$ if the approximation is reducible.
The reduced solution $U^m(\boldsymbol{\xi})$ is called the rank-m approximation, while $\{\lambda_1, \ldots, \lambda_m\}$ and $\{u_1, \ldots, u_m\}$ are the stochastic and deterministic reduced bases.
The questions are then :

- how to define the (deterministic or stochastic) reduced basis ?
- how to compute the reduced basis and the m-th order PGD of U ?

Optimal L^2-spectral decomposition : POD, KL decomposition

If we define the rank-m approximation as the minimizer of

$$R^2(U^m(\boldsymbol{\xi})) = \mathbb{E}\left\{ \left\| U(\boldsymbol{\xi}) - \sum_{\alpha=1}^{m} \lambda_\alpha(\boldsymbol{\xi}) u_\alpha \right\|_{\mathbb{V}}^2 \right\},$$

it is well known that the optimum is the KL decomposition of U, that is the u_α are the *m* dominant eigenmodes of correlation operator of U, $C : \mathbb{V} \mapsto \mathbb{V}$ (symmetric positive operator) :

$$\forall v \in \mathbb{V}, \quad (Cu_\alpha, v)_{\mathbb{V}} = \gamma_\alpha (u_\alpha, v), \quad \lambda_\alpha(\boldsymbol{\xi}) \|u_\alpha\|_{\mathbb{V}}^2 = (U, u_\alpha)_{\mathbb{V}} .$$

But the solution is needed to perform the decomposition !

- Solve the Galerkin problem in $\mathbb{V} \otimes \mathbb{S}^{|\mathcal{A}'|}$, with $|\mathcal{A}'| < |\mathcal{A}|$, to construct a deterministic basis $\{u_\alpha, \alpha = 1, \ldots, m\}$; then solve the reduced problem for the $\left\{ \lambda_{\alpha=1,\ldots,m} \in \mathbb{S}^{|\mathcal{A}|} \right\}$.
- Solve the Galerkin problem in $\mathbb{V}^H \otimes \mathbb{S}^{|\mathcal{A}|}$ to construct $\{\lambda_{\alpha=1,\ldots,m}\}$; then solve the reduced problem for the $\left\{ u_{\alpha=1,\ldots,m} \in \mathbb{V}^h \right\}$ with $\dim \mathbb{V}^H \ll \dim \mathbb{V}^h$.

See works by the groups of Ghanem and Matthies.

Observe : the above approaches mix L_2 and Galerkin optimality definitions !

An alternative definition of the optimality

Consider the family of variational problems : given $\boldsymbol{\xi} \in \Xi$, find $u \in \mathbb{V}$ such that

$$a(u, v|\boldsymbol{\xi}) = b(v|\boldsymbol{\xi}) \quad \forall v \in \mathbb{V},$$

where the bilinear form $a(\cdot, \cdot|\boldsymbol{\xi})$ is a.s. symmetric and positive definite.

Example : elliptic equation with random coefficient : $\mathbb{V} = H_0^1$

$$a(u, v|\boldsymbol{\xi}) = \int_\Omega k(\boldsymbol{\xi}) \boldsymbol{\nabla} u \boldsymbol{\nabla} v \, dx, \quad \epsilon < k(\xi) < \infty \; a.s. \text{ in } \Omega.$$

The Galerkin problem becomes : Find $U(\boldsymbol{\xi}) \in L_2(\Xi, p_{\boldsymbol{\xi}}) \otimes \mathbb{V}$ such that

$$A(U, V) = \mathbb{E}\{a(U, V|\boldsymbol{\xi})\} = B(V) = \mathbb{E}\{b(V|\boldsymbol{\xi})\} \quad \forall V \in L_2(\Xi, p_{\boldsymbol{\xi}}) \otimes \mathbb{V}.$$

Clearly, the bilinear form $A : (L_2(\Xi, p_{\boldsymbol{\xi}}) \otimes \mathbb{V}) \times (L_2(\Xi, p_{\boldsymbol{\xi}}) \otimes \mathbb{V}) \mapsto \mathbb{R}$ is symmetric positive definite, so the solution U is the minimizer of **the energy functional**

$$\mathcal{J}(V) \equiv \frac{1}{2} A(V, V) - B(V)$$

Introducing the reduced solution, U^m can be defined as

$$\mathcal{J}(U^m) = \frac{1}{2} A(U^m, U^m) - B(U^m) = \min_{\{u_{\alpha=1,\ldots,m}\}, \{\lambda_{\alpha=1,\ldots,m}\}} \mathcal{J}\left(\sum_{\alpha=1}^{m} u_\alpha \lambda_\alpha\right).$$

(detailed analysis for elliptic problems in [Nouy and Falco, 2012])

The rank-m solution U^m is defined by

$$\mathcal{J}(U^m) = \frac{1}{2}A(U^m, U^m) - B(U^m) = \min_{\{u_{\alpha=1,\dots,m}\},\{\lambda_{\alpha=1,\dots,m}\}} \mathcal{J}\left(\sum_{\alpha=1}^{m} u_\alpha \lambda_\alpha\right).$$

- The minimum is not unique (homogeneity)
- Minimizing $\mathcal{J}$ is equivalent to minimizing a Rayleigh quotient
- The reduced solution is **optimal w.r.t the A-norm** defined as

$$\|V\|_A^2 \equiv \mathbb{E}\{a(V,V)\} = A(V,V),$$

to be compared with the KL norm $\mathbb{E}\left\{\|V\|_{\mathbb{V}}^2\right\} = \mathbb{E}\{(V,V)_{\mathbb{V}}\}$.

Recursive construction :

For $i = 1, 2, 3 \dots$ Sequential construction [Nouy, 2007]

$$\mathcal{J}(U^i) = \min_{v\in\mathbb{V},\beta\in\mathbb{S}^P} \mathcal{J}\left(\beta v + \sum_{j=1}^{i-1}\lambda_j u_j\right) = \min_{v\in\mathbb{V},\beta\in\mathbb{S}^P} \mathcal{J}\left(\beta v + U^{i-1}\right)$$

The optimal couple (λ_i, u_i) **solves simultaneously** [Nouy, 2008]

- a) deterministic problem $u_i = \mathcal{D}(\lambda_i, U^{i-})$

$$A(\lambda_i u_i, \lambda_i v) = B(\lambda_i v) - A\left(U^{i-1}, \lambda_i v\right), \quad \forall v \in \mathbb{V}$$

- b) stochastic problem $\lambda_i = \mathcal{S}(u_i, U^{i-1})$

$$A(\lambda_i u_i, \beta u_i) = B(\beta u_i) - A\left(U^{i-1}, \beta u_i\right), \quad \forall \beta \in \mathbb{S}^P$$

Example of stochastic elliptic problem :

- Deterministic problem : $u_i = \mathcal{D}(\lambda_i, U^{i-1})$

$$\int_\Omega \mathbb{E}\left\{\lambda_i^2 k\right\} \boldsymbol{\nabla} u_i \cdot \boldsymbol{\nabla} v \mathrm{d}\boldsymbol{x} = \mathbb{E}\left\{ - \int_\Omega \lambda_i k \boldsymbol{\nabla} U^{i-1} \cdot \boldsymbol{\nabla} v \mathrm{d}\boldsymbol{x} + \int_\Omega \lambda_i f v \mathrm{d}\boldsymbol{x} \right\}, \quad \forall v.$$

- Stochastic problem : $\lambda_i = \mathcal{S}(u_i, U^{i-1})$

$$\mathbb{E}\left\{\lambda_i \beta \int_\Omega k \boldsymbol{\nabla} u_i \cdot \boldsymbol{\nabla} u_i \mathrm{d}\boldsymbol{x}\right\} = \mathbb{E}\left\{ -\beta \int_\Omega k \boldsymbol{\nabla} U^{i-1} \cdot \boldsymbol{\nabla} u_i \mathrm{d}\boldsymbol{x} + \int_\Omega f u_i \mathrm{d}\boldsymbol{x} \right\}, \quad \forall \beta.$$

Properties : [Nouy, 2007, 2008]

- The couple (λ_i, u_i) is a fixed point of :

$$\lambda_i = \mathcal{S} \circ \mathcal{D}(\lambda_i, \cdot), \quad u_i = \mathcal{D} \circ \mathcal{S}(u_i, \cdot)$$

- Homogeneity property :

$$\frac{\lambda_i}{c} = \mathcal{S}(c u_i, \cdot), \quad \frac{u_i}{c} = \mathcal{D}(c \lambda_i, \cdot), \quad \forall c \in \mathbb{R} \setminus \{0\}.$$

$\Rightarrow$ arbitrary normalization of one of the two elements.

Algorithms inspired from dominant subspace methods
Power-type, Krylov/Arnoldi, ...

Power Iterations

1. Set $l = 1$
2. initialize λ (*e.g.* randomly)
3. While not converged, repeat (power iterations)
   - **a)** Solve : $u = \mathcal{D}(\lambda, U^{l-1})$
   - **b)** Normalize u
   - **c)** Solve : $\lambda = \mathcal{S}(u, U^{l-1})$
4. Set $u_l = u$, $\lambda_l = \lambda$
5. $l \leftarrow l + 1$, if $l < m$ repeat from step 2

Comments :

- Convergence criteria for the power iterations (subspace with dim > 1 or clustered eigenvalues) [Nouy, 2007,2008]
- Usually few (4 to 5) inner iterations are sufficient

Power Iterations with Update

[Nouy and olm, 2009]

1. Same as Power Iterations, but after (u_l, λ_l) is obtained (step 4) update of the stochastic coefficients :
   - Orthonormalyze $\{u_1, \ldots, u_l\}$ (optional)
   - Find $\{\lambda_1, \ldots, \lambda_l\}$ s.t.

$$A\left(\sum_{i=1}^{l} u_i\lambda_i, \sum_{i=1}^{l} u_i\beta_i\right) = B\left(\sum_{i=1}^{l} u_i\beta_i\right), \quad \forall \beta_{i=1,\ldots,l} \in \times\mathbb{S}^{\mathrm{P}}$$

2. Continue for next couple

Comments :

- Improves the convergence
- Low dimensional stochastic linear system ($l \times l$)
- Cost of update increases linearly with the order l of the reduced representation

Arnoldi (Full Update version)

[Tamellini et al, 2009]

1. Set $l = 0$
2. Initialize $\lambda \in \mathbb{S}^P$
3. For $l' = 1, 2, \ldots$ (Arnoldi iterations)
   - Solve deterministic problem $u' = \mathcal{D}(\lambda, U^l)$
   - Orthogonalize : $u_{l+l'} = u' - \sum_{j=1}^{l+l'-1} (u', u_j)_\Omega$
   - If $\|u_{l+l'}\|_{\mathbb{V}} \leq \epsilon$ or $l + l' = m$ then break
   - Normalize $u_{l+l'}$
   - Solve $\lambda = \mathcal{S}(u_{l'}, U^l)$
4. $l \leftarrow l + l'$
5. Find $\{\lambda_1, \ldots, \lambda_l\}$ s.t. (Update)

$$A\left(\sum_{i=1}^{l} u_i \lambda_i, \sum_{i=1}^{l} u_i \beta_i\right) = B\left(\sum_{i=1}^{l} u_i \beta_i\right), \quad \forall \beta_{i=1,\ldots,l} \in \mathbb{S}^P$$

6. If $l < m$ return to step 2.

Summary

- Resolution of a series of deterministic elliptic problems, with elliptic coefficients $\mathbb{E}\left\{\lambda^2 k\right\}$ and modified (deflated) rhs

  dimension is $\dim \mathbb{V}^h$

- Resolution of a series of linear stochastic equations

  dimension is $\dim \mathbb{S}^{\mathrm{P}}$

- Update problems : system of linear equations for stochastic random variables

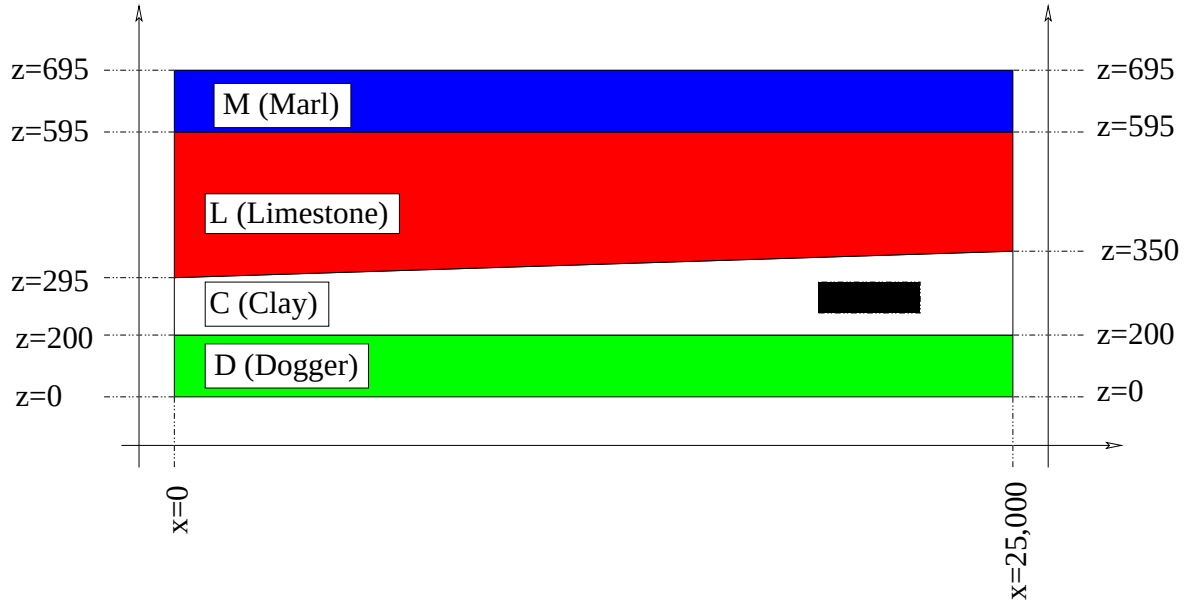
  dimension is $m \times \dim \mathbb{S}^{\mathrm{P}}$

- To be compared with the Galerkin problem dimension

  $\dim \mathbb{V}^h \times \dim \mathbb{S}^{\mathrm{P}}$

Weak modification of existing (FE/FV) codes
(weakly intrusive)

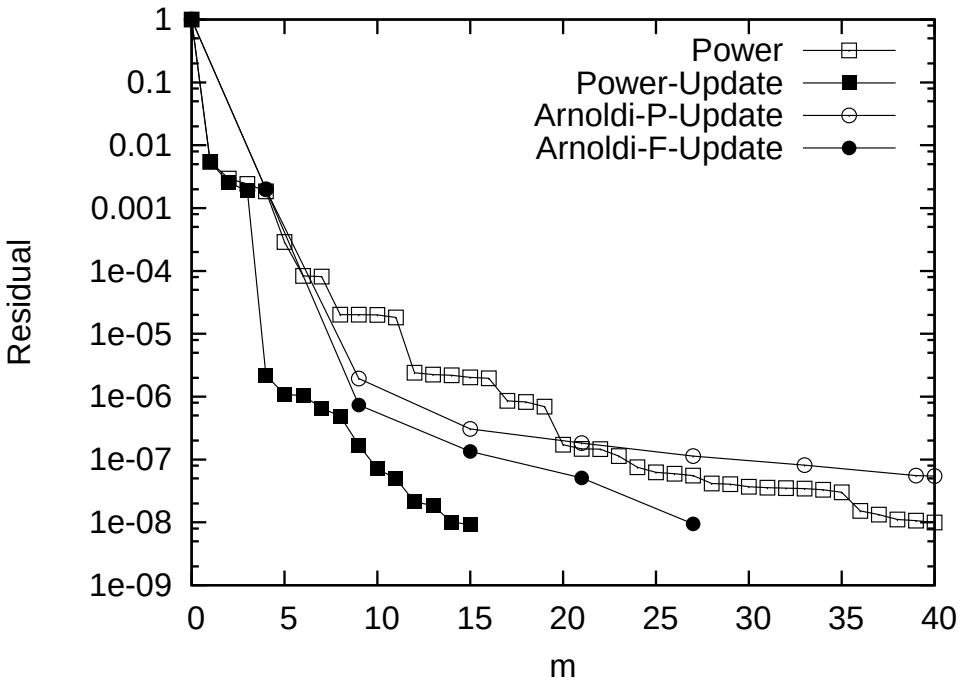
Example definition

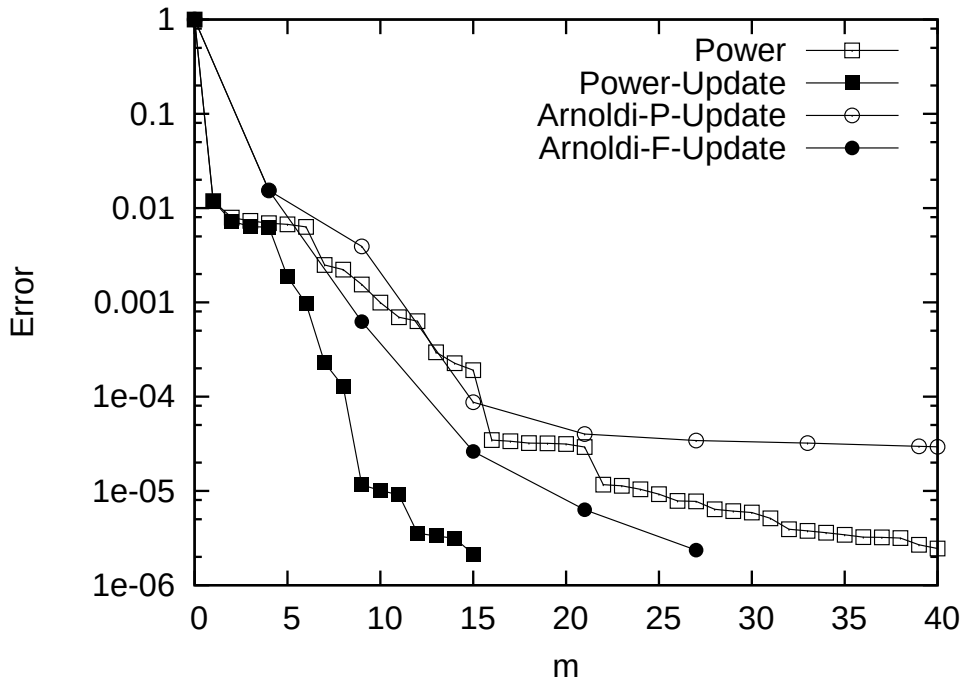


- Rectangular domain 25,000$\times$695 (m)
- 4 Geological layers with uncertain conductivities
- uncertainty on BCs

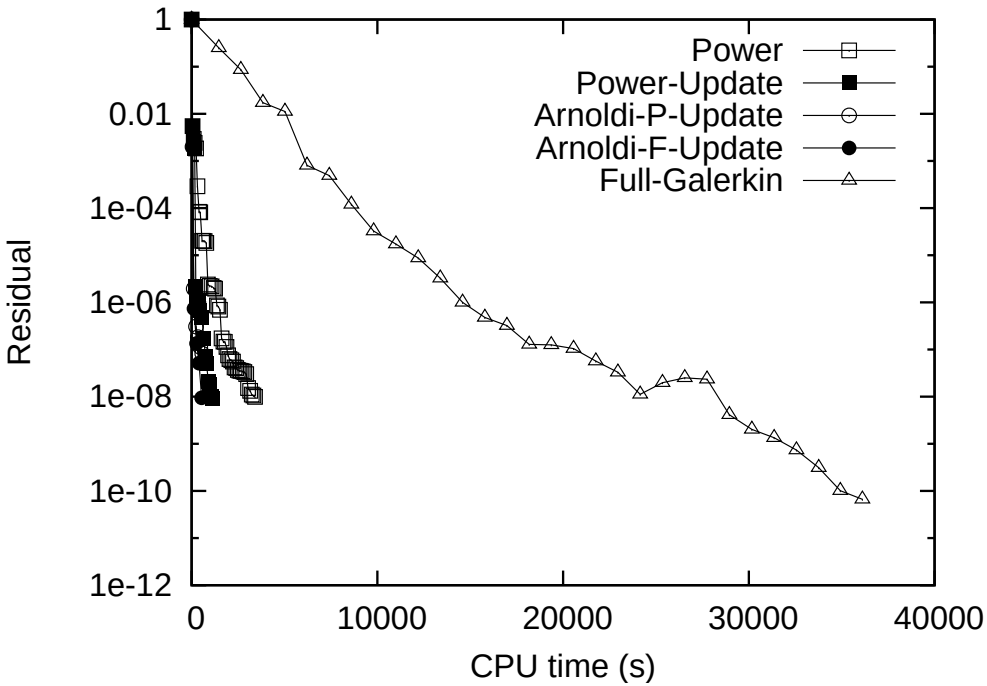
- 9 independent r.v. $\{\xi_1, \ldots, \xi_9\} \sim U[0,1]^9$
- Stochastic space $\mathbb{S}^{\mathrm{P}}$: Legendre polynomial up to order No
- $\dim \mathbb{S}^{\mathrm{P}} = \mathrm{P}+1 = (9+\mathrm{No})!/(9!\mathrm{No}!)$

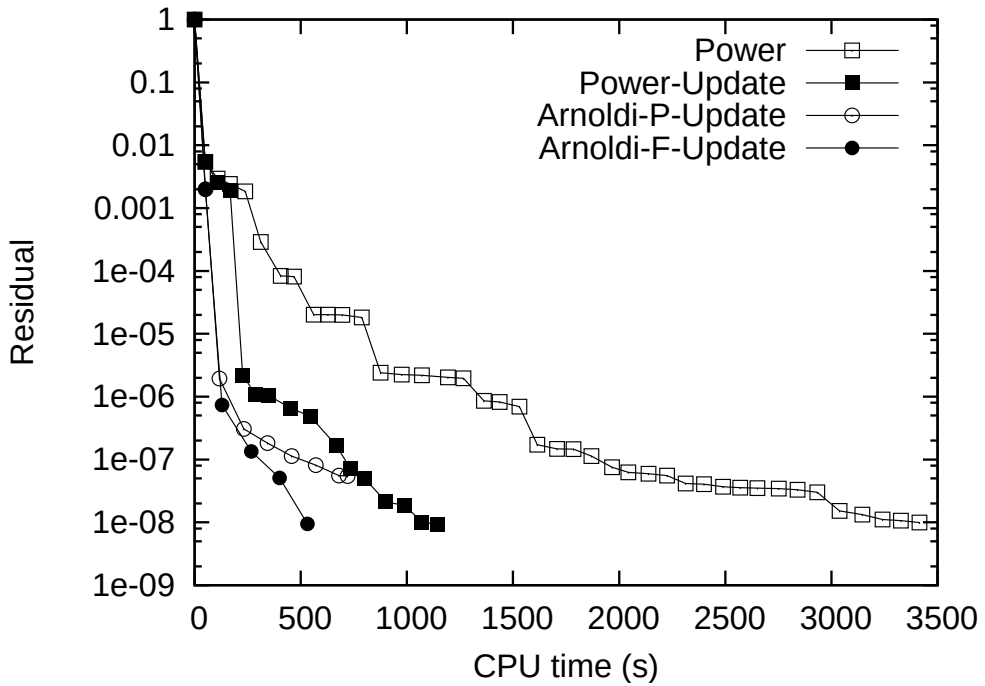
Norm of residual and error as a function of m ($\mathrm{No} = 3$)





CPU times ($\mathrm{No} = 3$)





Closing remarks

Polynomial Chaos expansions

- Fast convergence when the solution has smooth dependence with respect to ξ
- Adaptivity of the spectral basis to the solution
- Development of dedicated Galerkin solvers
- Treatment of non-linearities in Galerkin projection
- Preconditioning for non-intrusive methods
- Case of non-smooth solutions : multi-resolution.

Proper Generalized Decompositions

- Theoretically well grounded method for linear symmetric definite operators
- Yield close to optimal decomposition
- Connection with Greedy algorithms
- Non-intrusive versions possible
- Adaptation of algorithms and analysis to more general class of problems
- Application to non-linear problems [Nouy and Olm, 2010] *e.g.* Navier-Stokes [Tamellini Nouy and olm, 2013]

Thanks for your attention

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with Omar Knio

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