Introduction to Saddle Point Problems

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Motivation and goals

Let A be a real, symmetric, $n \times n$ matrix and let $f \in \mathbf{R}^n$ be given.

Let $\langle \cdot, \cdot \rangle$ denote the standard inner product in $\mathbf{R}^n$.

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Many algorithms exist for solving SPD linear systems: Cholesky, Preconditioned Conjugate Gradients, AMG, etc.

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Standard approach: Introduce Lagrange multipliers, $p \in \mathbf{R}^m$

- Lagrangian $\mathcal{L}(u, p) = \frac{1}{2}\langle Au, u\rangle - \langle f, u\rangle + \langle p, Bu - g\rangle$
- First-order optimality conditions: $\nabla_u \mathcal{L} = 0$, $\nabla_p \mathcal{L} = 0$

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System (1) is a saddle point problem. Its solutions (u^*, p^*) are saddle points for the Lagrangian $\mathcal{L}(u, p)$:

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Gil Strang calls (1) “the fundamental problem of scientific computing.”

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- Linear elasticity
- Mixed FEM formulations of 2nd- and 4th-order elliptic PDEs
- PDE-constrained optimization (e.g., variational data assimilation)
- SQP and IP methods for nonlinear constrained optimization
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For flow problems, we would like to have solvers that converge fast regardless of mesh size, viscosity, etc. Moreover, the cost per iteration should be linear in the number of unknowns.

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In particular, $\mathcal{A}$ is invertible if A is SPD and B has full rank ("standard case").

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Another important example is the discretization of the Reissner–Mindlin plate model in linear elasticity. In this case β is related to the thickness of the plate; the limit case $\beta = 0$ can be seen as a reformulation of the biharmonic problem.

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lies entirely in the open right-half plane $Re(z) > 0$. Moreover, if A is SPD and the following condition holds:

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then $\mathcal{A}_-$ is diagonalizable with real positive eigenvalues. In this case, there exists a non-standard inner product on $\mathbf{R}^{n+m}$ in which $\mathcal{A}_-$ is self-adjoint and positive definite, and a corresponding conjugate gradient method (B./Simoncini, NM 2006).

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where $\langle \cdot, \cdot \rangle$ denotes the L^2 inner product.

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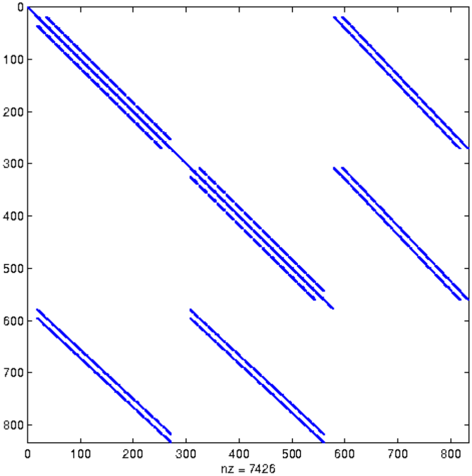
Robust, optimal solvers have been developed for this problem: Cahouet–Chabard for $\alpha > 0$; Silvester–Wathen for $\alpha = 0$.

Sparsity pattern: 2D stokes (Q1-P0)



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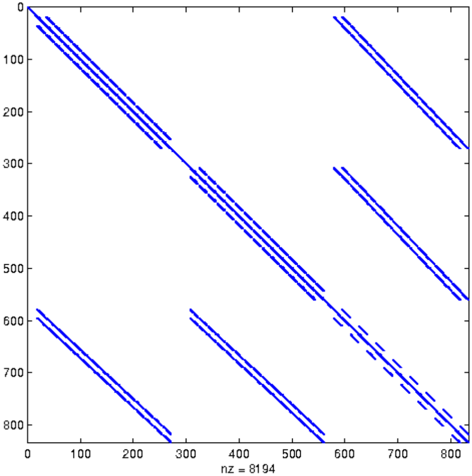

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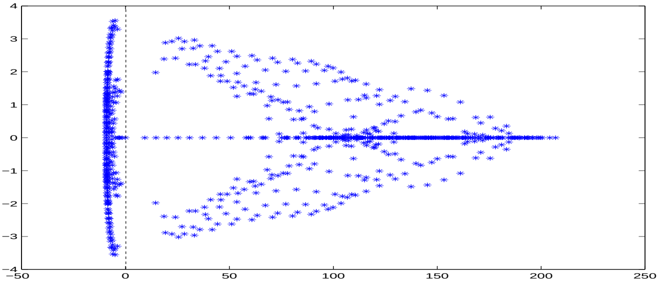
As we will see, this problem can be very challenging to solve, especially for small values of the viscosity ν and on stretched meshes.

Eigenvalues of discrete Oseen problem ($\nu = 0.01$), indefinite form

Eigenvalues of Oseen matrix $\mathcal{A} = \begin{pmatrix} A & B^T \\ B & O \end{pmatrix}$, MAC discretization.



Note the different scales in the x and y axes.

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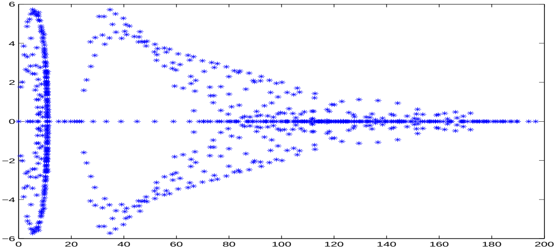

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Much effort has been put into developing preconditioners, with optimality and robustness w.r.t. parameters as the ultimate goals. Parallelizability also needs to be taken into account.

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- Efficient parallelization is possible

Example: Silvester-Wathen preconditioner

Number of MINRES iterations needed for 10^{-6} reduction in residual for locally stabilized $Q1 - P0$ mixed finite elements for Stokes flow in a cavity. Block diagonal preconditioner: $\widehat{A}$ is one multigrid V-cycle with 1,1 relaxed Jacobi smoothing and $\widehat{M}_p$ is the diagonal pressure mass matrix. The cpu time (in seconds) is on a Sun sparcv9 502 MHz processor with 1024 Mb of memory. The cpu time is also given for a sparse direct solve (UMFPACK in MATLAB).

grid	n	m	iterations	cpu time	sparse direct cpu
64×64	8450	4096	38	14.3	6.8
128×128	33282	16384	37	37.7	48.0
256×256	132098	65536	36	194.6	897
512×512	526339	263169	35	6903	out of memory

Note: the lack of scalability for finer meshes is caused by memory management issues.

But what about more difficult problems?

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where $S = -BA^{-1}B^T$ (Schur complement).

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then

- The spectrum of $\mathcal{P}_D^{-1}\mathcal{A}$ is $\sigma(\mathcal{P}_D^{-1}\mathcal{A}) = \left\{1, \frac{1\pm\sqrt{5}}{2}\right\}$

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- GMRES converges in three iterations with $\mathcal{P}_D$, and in two iterations with $\mathcal{P}_T$.

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- Approximating S is more involved, except in special situations; for example, in the case of Stokes we can use the pressure mass matrix ($\widehat{S} = M_p$) or its diagonal, assuming the LBB condition holds. This is the Silvester–Wathen preconditioner.
- For the Oseen problem this does not work, except for very small Reynolds.

Block preconditioners (cont.)

Recall that $S = -BA^{-1}B^T$ is a discretization of the operator

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This preconditioner performs well for small or moderate Reynolds numbers.

Results for Kay, Loghin and Wathen preconditioner

Test problems: steady Oseen, homogeneous Dirichlet BCs, two choices of the wind function.

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A preconditioning step requires two convection-diffusion solves (three in 3D) and one Poisson solve at each iteration, plus some mat-vecs.

Results for Kay, Loghin, and Wathen preconditioner (cont.)

Results for KLW preconditioner.

mesh size h	viscosity ν				
	1	0.1	0.01	0.001	0.0001
constant wind					
1/16	6 / 12	8 / 16	12 / 24	30 / 34	100 / 80
1/32	6 / 10	10 / 16	14 / 24	24 / 28	86 / 92
1/64	6 / 10	8 / 14	16 / 24	22 / 32	64 / 66
1/128	6 / 10	8 / 12	16 / 26	24 / 36	64 / 58
rotating vortex					
1/16	6 / 8	10 / 12	30 / 40	$>$ 400 / 188	
1/32	6 / 8	10 / 12	30 / 40	$>$ 400 / 378	
1/64	4 / 6	8 / 12	26 / 40	$>$ 400 / $>$ 400	
1/128	4 / 6	8 / 10	22 / 44	228 / $>$ 400	

Number of Bi-CGSTAB iterations
(**Note**: exact solves used throughout. Stopping criterion: $\|\mathbf{b} - \mathcal{A}\mathbf{x}_k\|_2 < 10^{-6}\|\mathbf{b}\|_2$).

Results with Vanka-type MG preconditioner

Results for Vanka-MG-BiCGStab approach, isoP2-P0 FEM.

mesh size h	viscosity $= \nu$				
	1	0.1	0.01	10^{-3}	10^{-4}
constant wind					
$1/16$	4	4	3	4	5
$1/32$	4	4	3	4	4
$1/64$	4	4	4	3	5
$1/128$	4	4	4	3	4
rotating vortex					
$1/16$	4	5	5	8	14
$1/32$	4	4	6	9	17
$1/64$	4	4	5	8	40
$1/128$	4	4	4	9	> 400

Better than KLW preconditioner, but still not robust for low viscosity. Can we do better than this?

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