The Relationship between Discrete Calculus Methods and other Mimetic Approaches

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Background

Hardware:
GPUs, FPGAs, HPC, Algorithms

Numerical Methods:
Unstructured Staggered mesh methods, Fractional step methods, Discrete Calculus Methods.

Turbulence Modeling:
Turbulent Potentials, Eddy Collision Model

Applications:
Wind Turbines, DNS, Super-hydrophobic surfaces, droplets.

Mimetic Methods

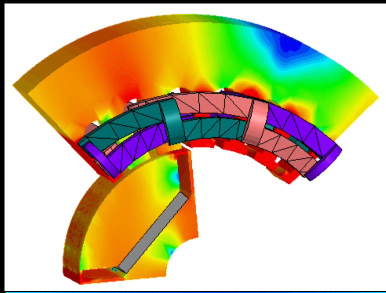
FE: Raviart-Thomas/Nedelec/Whitney
- Algebraic Topology
- Electromagnetics

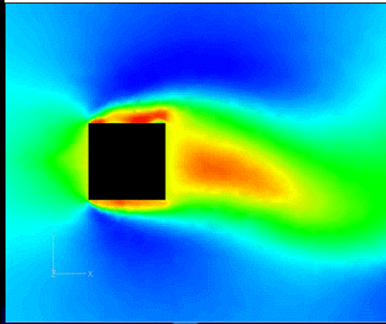
FV: Staggered Mesh Methods
- Many local conservation properties
- Fluid Dynamics

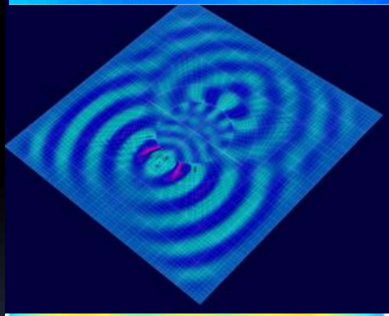
FD: Keller Box
- Multi-symplectic
- Wave Eqns

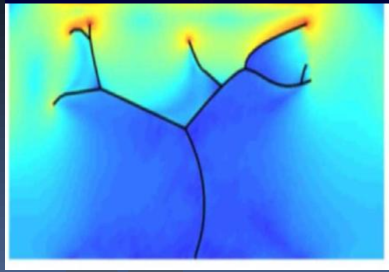
FD: SOM Box
- Robust
- Heat Eqn

NN: Non-Sibsonian Meshless methods
- Time-dependent domains
- Solid mechanics

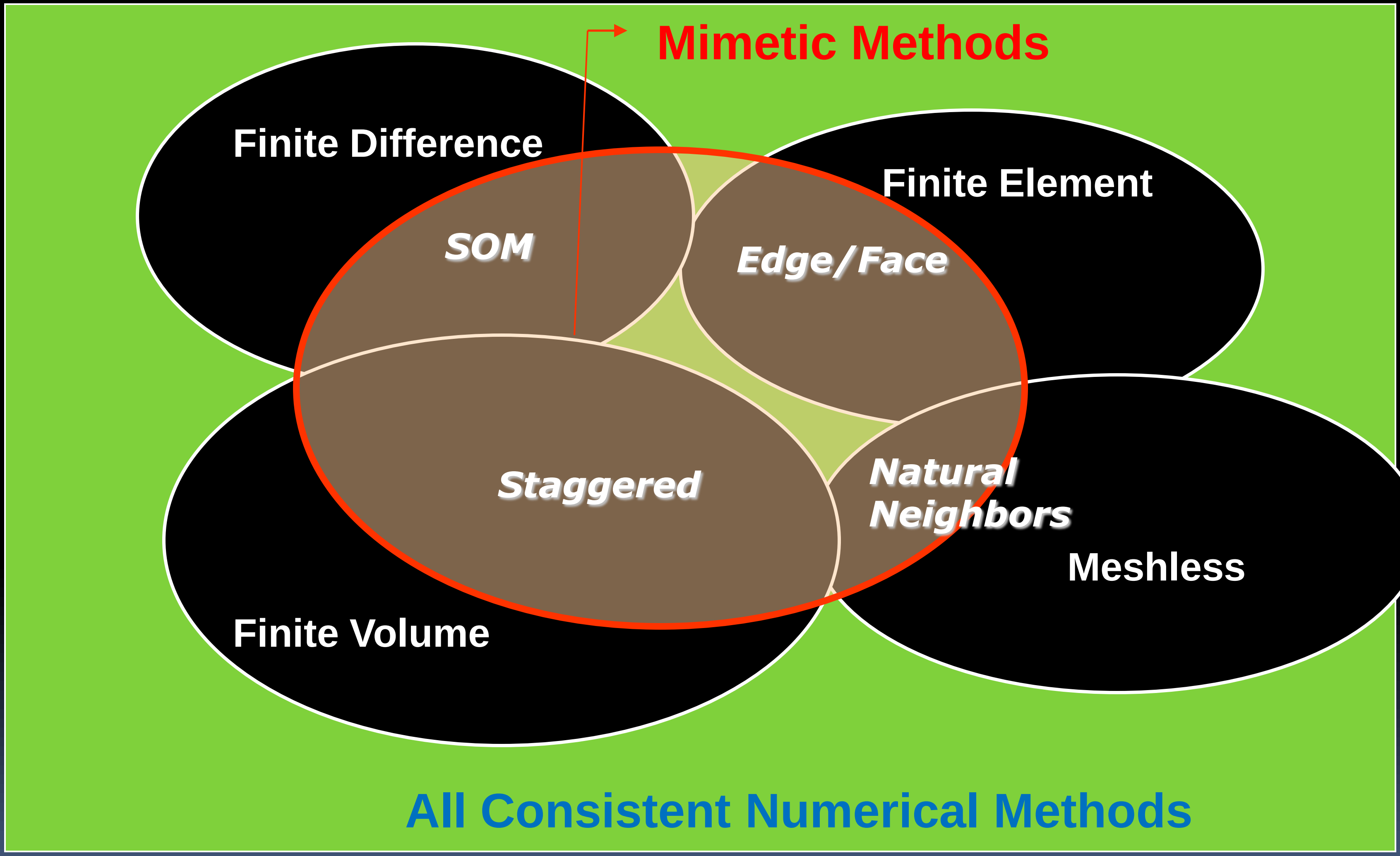








Numerical Methods



Question

Is there any relationships between the various mimetic methods?

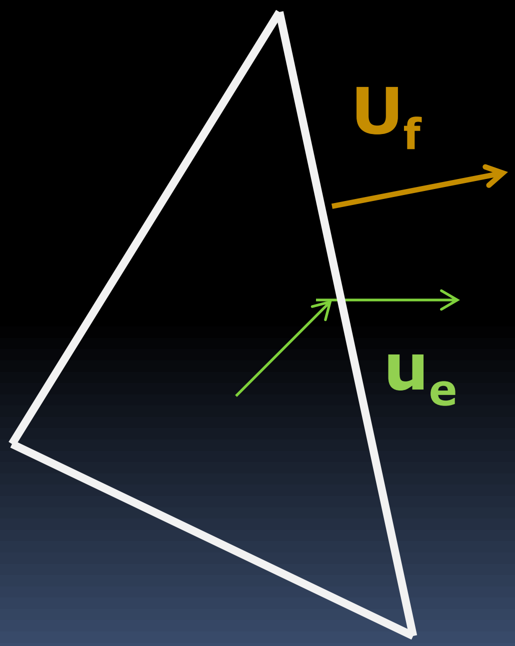
(1) Yes – many (all ?) can be derived as discrete calculus methods.

(2) Yes – they tend to use the same basis functions.

Incompressible Fluid Dynamics

$$u_{\tilde{e}}^{n+1} - u_{\tilde{e}}^{n} = \int_{t^n}^{t^{n+1}} dt \int (\nabla \cdot \mathbf{F}) \cdot d\mathbf{l} - \mathbf{G}\,\overline{p}_{\tilde{c}}$$

$$\mathbf{D}U_f^{n+1} = 0$$



Need to relate these two

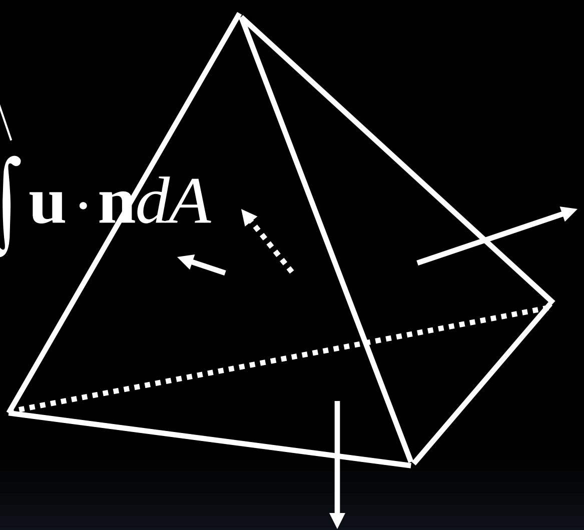
$$U_f = \int \mathbf{u} \cdot \mathbf{n}\, dA$$

$$u_{\tilde{e}} = \int \mathbf{u} \cdot d\mathbf{l}$$

FE Basis Functions

Heat Flux
Magnetic Flux
Velocity Flux

$$U_f = \int \mathbf{u} \cdot \mathbf{n} \, dA$$



Face Elements
Nedelec/RT/Whitney

$$\vec{\mathbf{u}}^h(\mathbf{x}) = \vec{\mathbf{u}}^0 + \tfrac{D}{n} \vec{\mathbf{x}}$$

Interpolant with continuity of the normal flux

$$\mathbf{u}^h \cdot \mathbf{n}^f = \mathbf{u}^0 \cdot \mathbf{n}^f + \tfrac{D}{n} L_\perp^f$$

Constant normal velocity on each face

Constant divergence

$$\nabla \cdot \mathbf{u}^h = D$$

FE Hodge

2-forms to 1 -forms

$$\vec{\mathbf{u}}^h(\mathbf{x}) = \vec{\mathbf{u}}^0 + \tfrac{D}{n}\vec{\mathbf{x}}$$

Find the constants given the data (4x4)

$$\mathbf{u}^0 \cdot \mathbf{n}^{f1} + \frac{L_\perp^{f1}}{n} D = \frac{1}{A_{f1}} U_{f1}$$

$$u_{\tilde{e}} = \int \mathbf{u}^h \cdot d\mathbf{l}$$

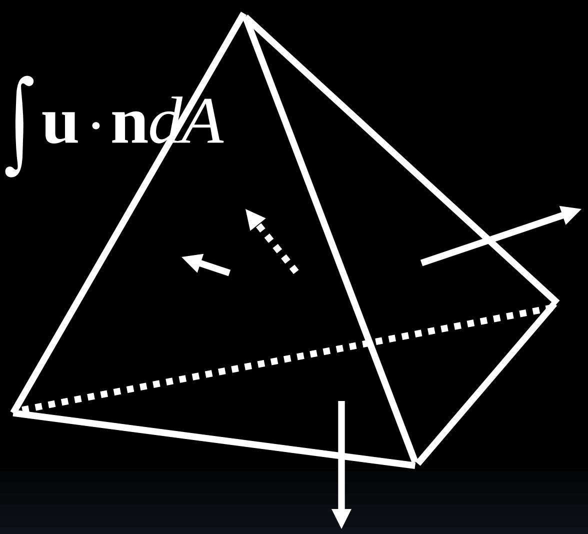
Evaluate the integral

The basis function determines the relationship between the two velocities

StagMesh Interpolation

$$\int (u_{i,i}x_j + u_j)dV = \int (u_i x_j)_{,i}\, dV = \int (x_j) u_i n_i dA$$

$$U_f = \int \mathbf{u}\cdot\mathbf{n} dA$$



- **Gauss' Theorem (Again)**
- **Assume constant flux on face**
- **Assume constant divergence**

$$\bar{\mathbf{u}}_c V = \sum_{faces} U_f (\mathbf{x}_f^{cg} - \mathbf{x}_c^{cg})$$

SM = implicit basis functions

$$\bar{\mathbf{u}}_c = \tfrac{1}{V_c}\mathbf{R}U_f$$

StagMesh Hodge

2-forms to 1 -forms

Average Cell velocity $$\bar{\mathbf{u}}_c = \tfrac{1}{V_c}\mathbf{R}U_f$$

$$u_e = \mathbf{R}^T\bar{\mathbf{u}}_c$$ **for incompressible**

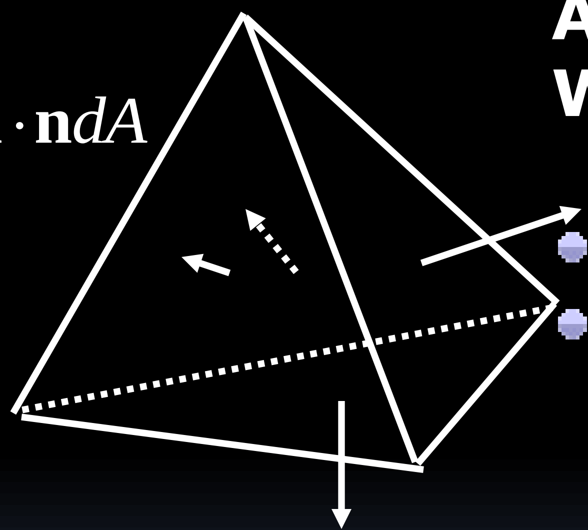
$$u_e = (\mathbf{R}^T \tfrac{1}{V_c}\mathbf{R})U_f$$

- **Explicit Formula for the same matrix relationship (Hodge*) as FE**
- **Symmetric**
- **Generalizable to polyhedra**
- **The intermediate is a (cell average) velocity <u>vector</u>. (Momentum, KE)**

FD Interpolation

SOM

$$\int \mathbf{u} \cdot \mathbf{n} dA$$



Use 3 face values at each vertex
Average vectors to center
Works on (almost) any polygon

- **Assumes constant on face**
- **Assumes constant divergence**

CoVolume

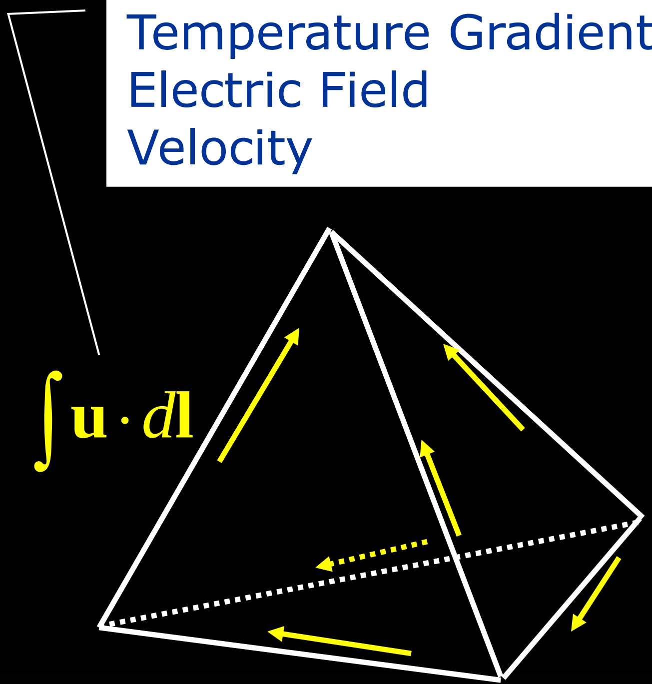
Use Least Squares $\mathbf{N}\vec{\mathbf{u}}_c = \frac{1}{A_f} U_f$

- **Also same cell velocity**

FE Basis Functions

1-forms

Temperature Gradient
Electric Field
Velocity



Edge Elements
Nedelec/RT/Witney

$$\vec{\mathbf{u}}^h(\mathbf{x}) = \vec{\mathbf{u}}^0 + \frac{1}{n-1}\vec{\mathbf{w}} \times \vec{\mathbf{x}}$$

Interpolant with continuity of the tangential components

Constant tangential velocity on each edge

$$\mathbf{u}^h \cdot \mathbf{t}^e = \mathbf{u}^0 + \frac{1}{n-1}(\mathbf{w} \times \mathbf{L}_\perp^e) \cdot \mathbf{t}^e$$

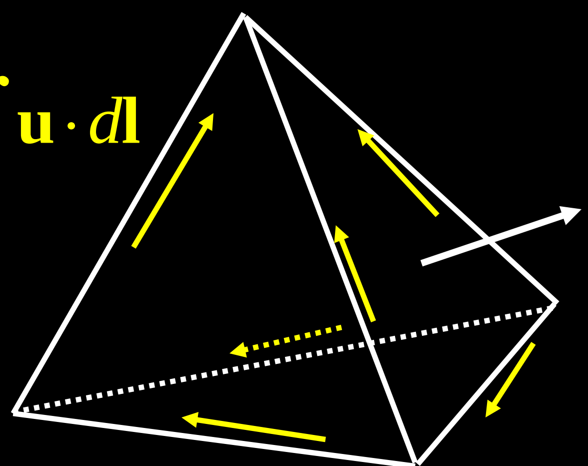
Constant vorticity

$$\nabla \times \mathbf{u}^h = \mathbf{w}$$

StagMesh Interpolation

$$(\mathbf{a}\cdot\mathbf{x})\mathbf{u} \;\Rightarrow\; \int(\mathbf{u}\times\mathbf{n}+\mathbf{x}\mathbf{w}\cdot\mathbf{n})\,dA=\int\mathbf{x}\mathbf{u}\cdot d\mathbf{l}$$

$$u_e=\int\mathbf{u}\cdot d\mathbf{l}$$



- <u>**Stokes' Theorem**</u>
- **Assume constant along edge**
- **Assume constant vorticity**

$$\vec{\mathbf{u}}_f^{cg}\times\vec{\mathbf{n}}_f A_f=\sum_{edges}u_e(\vec{\mathbf{x}}_e^{cg}-\vec{\mathbf{x}}_f^{cg})$$

SM = Rampant use of Stokes' Theorem

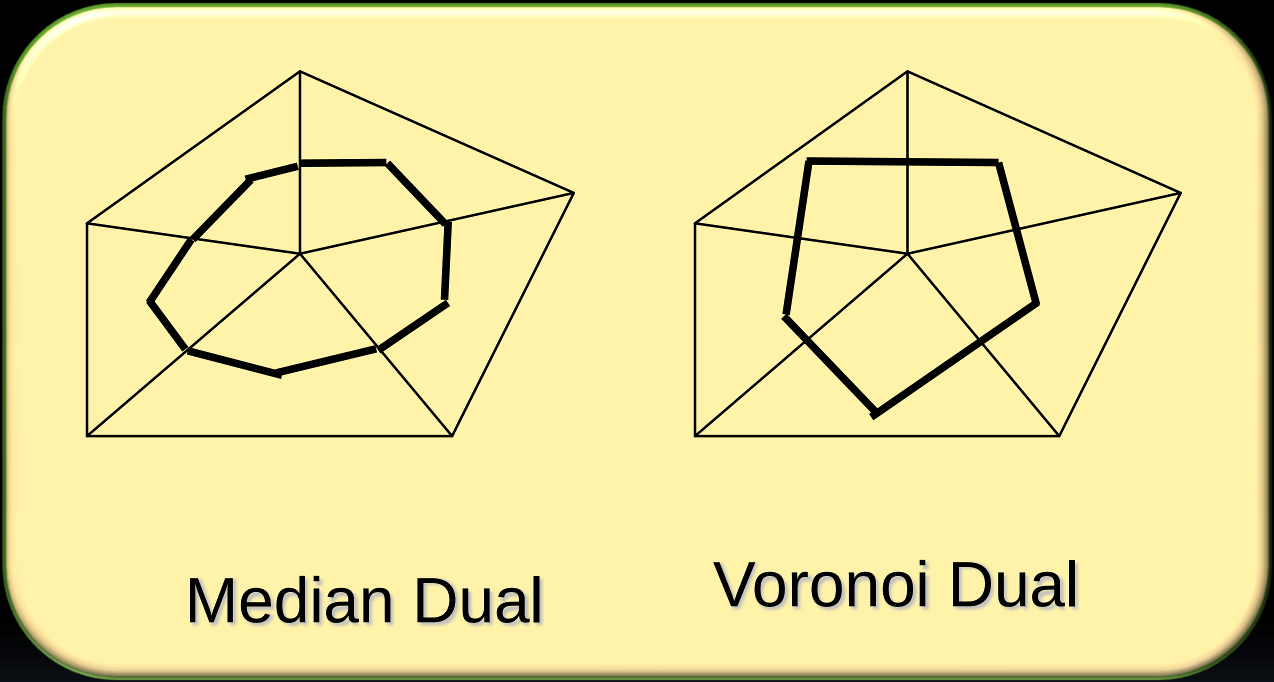
$$\int\nabla\times\vec{\mathbf{u}}\bar{X}\,dV=-\int\vec{\mathbf{u}}\times\vec{\mathbf{n}}\,dA$$

Summary: Basis Functions

- FE uses Explicit Basis Functions
- SM uses Stokes' Theorem
  This approach can be applied to arbitrary polygons.
- SOM uses Discrete Inner Products
  highly discontinuous/anisotropic materials and arbitrary polygons

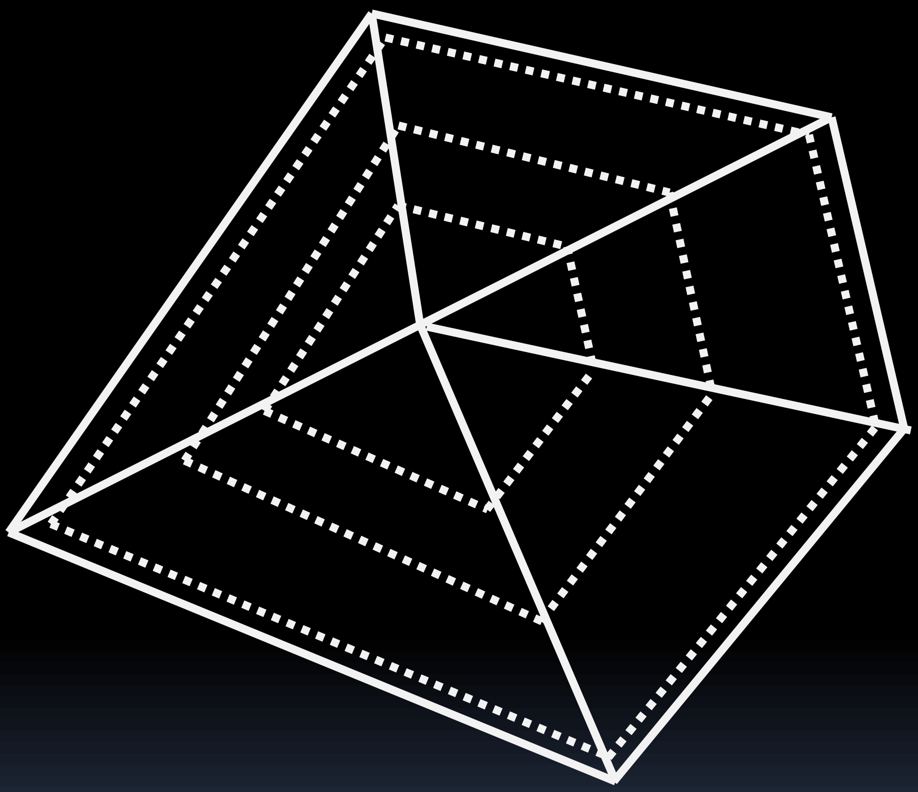
Many approaches to achieving the same underlying interpolation (Hodge *) .

Test Functions



Dual mesh is not unique.
FV = top hat
FE = tent functions (unique for Galerkin)

FE / FV relationship



FV = one CV

FE = weighted average of CV

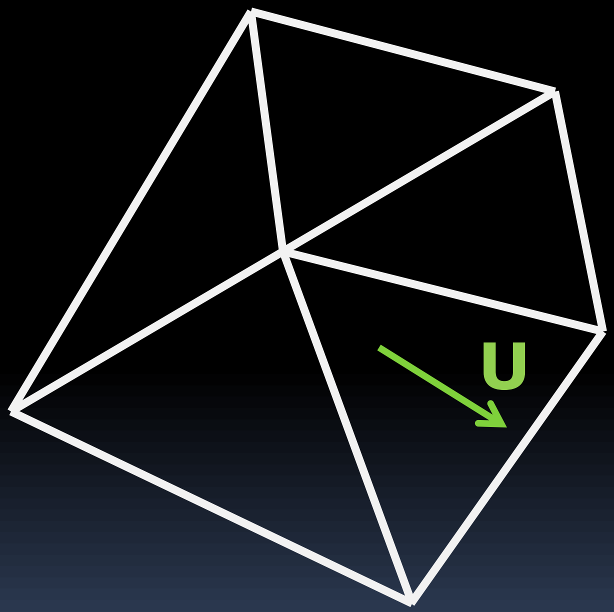
Mattiussi C, 1997, An analysis of finite volume, finite element, and finite difference methods using some concepts from algebraic topology, *J. Computational Physics*, 133: 289-309.

FE Discrete Calculus

Weighted Exact Discretization

$$\int w(\nabla\cdot\mathbf{u})dV = \int w\mathbf{u}\cdot\mathbf{n}dA - \int(\nabla w)\cdot\mathbf{u}dV = \int \mathbf{u}\cdot(-\nabla w)dV$$

Smeared Flux



$$D^w = \sum_{faces} U_f^w$$

Compare

$$D = \int \nabla\cdot\mathbf{u}dV = \sum_{faces} \int \mathbf{u}\cdot\mathbf{n}dA = \sum_{faces} U_f$$

FE Exact Discretization

$$\int \nabla \cdot \mathbf{u} dV = \sum_{faces} \int \mathbf{u} \cdot \mathbf{n} dA$$

$$\int (\nabla \times \mathbf{v}) \cdot \mathbf{n} dA = \sum_{edges} \int \mathbf{v} \cdot d\mathbf{l}$$

$$\int \nabla T \cdot d\mathbf{l} = T_2 - T_1$$

$$\int w \nabla \cdot \mathbf{u} dV = \sum_{cells} \int \mathbf{u} \cdot (-\nabla w) dV$$

$$\int (\nabla \times \mathbf{v}) \cdot (-\nabla w) dV = \sum_{faces} \int \mathbf{v} \cdot (\mathbf{n} \times \nabla w) dA$$

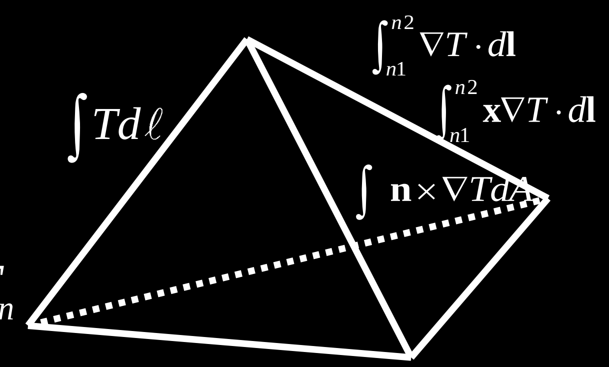
$$\int \nabla T \cdot (\mathbf{n} \times \nabla w) dA = \sum_{edges} \int T (\nabla w) \cdot d\mathbf{l}$$

Higher-Order Exact Gradient

$$\int_{edge} \nabla T \cdot d\mathbf{l} = T_{n2} - T_{n1}$$

$$\int_{edge} \mathbf{x}\nabla T \cdot d\mathbf{l} = (\mathbf{x}_{n2}T_{n2} - \mathbf{x}_{n1}T_{n1}) - \mathbf{t}_e \int_{n1}^{n2} Tdl$$

$$\int_{face} \mathbf{n}\times\nabla T dA = \sum_{edges} \mathbf{t}_e \int Tdl$$



20 outputs / tet **10 inputs / tet**

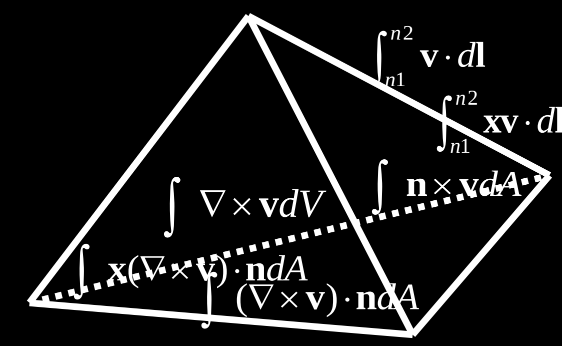
$$\begin{bmatrix} \int_{edge} \nabla T \cdot d\mathbf{l} \\ \int_{edge} \mathbf{x}\nabla T \cdot d\mathbf{l} \\ \int_{face} \mathbf{n}\times\nabla T dA \end{bmatrix} = \mathbf{G}^{[2]} \begin{bmatrix} T_n \\ \int_{edge} Tdl \end{bmatrix} = \begin{bmatrix} \mathbf{G} & \mathbf{0} \\ \mathbf{G}\mathbf{x}_n & -\mathbf{t}_e\mathbf{I} \\ \mathbf{0} & \mathbf{T} \end{bmatrix} \begin{bmatrix} T_n \\ \int_{edge} Tdl \end{bmatrix}$$

Higher-Order Exact Curl

$$\int (\nabla\times\mathbf{v})\cdot\mathbf{n}dA = \sum_{edges}\int \mathbf{v}\cdot d\mathbf{l}$$

$$\int \mathbf{x}(\nabla\times\mathbf{v})\cdot\mathbf{n}dA = \sum_{edges}\int \mathbf{xv}\cdot d\mathbf{l} + \int \mathbf{n}\times\mathbf{v}dA$$

$$\int \nabla\times\mathbf{v}dV = \sum_{faces}\int \mathbf{n}\times\mathbf{v}dA$$



15 outputs / tet **20 inputs / tet**

$$\begin{bmatrix} \int_{face}(\nabla\times\mathbf{v})\cdot\mathbf{n}dA \\ \int_{face}\mathbf{x}(\nabla\times\mathbf{v})\cdot\mathbf{n}dA \\ \int_{cell}(\nabla\times\mathbf{v})dV \end{bmatrix} = \mathbf{C}^{[2]}\begin{bmatrix} \int_{edge}\mathbf{v}\cdot d\mathbf{l} \\ \int_{edge}\mathbf{xv}\cdot d\mathbf{l} \\ \int_{face}\mathbf{n}\times\mathbf{v}dA \end{bmatrix}$$

Note: $\mathbf{C}^{[2]}\mathbf{G}^{[2]} = 0$

Mimetic

Review

FE:
Explicit basis functions
Fixed geometry – precise proofs
Implicit dual mesh
Semi-implicit Hodge*

StagMesh:
Implicit basis functions
Arbitrary polygons.
Explicit dual mesh
Explicit Factored Hodge*

Summary

- Underlying assumptions about the solution (basis functions) are often the same.
- The Test Functions are different. Test functions affect the metric (geom).
- Higher Order is achieved by noticing that you can exactly discretize different ways and with different moments
- Anyone can do it.

www.ecs.umass.edu/mie/tcfd/Publications.html