The Relationship between Discrete Calculus Methods and other Mimetic Approaches

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Background

Hardware:
- GPUs, FPGAs, HPC, Algorithms

Numerical Methods:
- Unstructured Staggered mesh methods, Fractional step methods, Discrete Calculus Methods.

Turbulence Modeling:
- Turbulent Potentials, Eddy Collision Model

Applications:
- Wind Turbines, DNS, Super-hydrophobic surfaces, droplets.
Mimetic Methods

FE: Raviart-Thomas/Nedelec/Whitney
  • Algebraic Topology
  • Electromagnetics

FV: Staggered Mesh Methods
  • Many local conservation properties
  • Fluid Dynamics

FD: Keller Box
  • Multi-symplectic
  • Wave Eqns

FD: SOM Box
  • Robust
  • Heat Eqn

NN: Non-Sibsonian Meshless methods
  • Time-dependent domains
  • Solid mechanics
Numerical Methods

Finite Difference

Finite Element

Meshless

Finite Volume

SOM

Edge/Face

Natural Neighbors

Staggered

Mimetic Methods

All Consistent Numerical Methods
Question

Is there any relationships between the various mimetic methods?

(1) Yes – many (all ?) can be derived as discrete calculus methods.

(2) Yes – they tend to use the same basis functions.
Incompressible Fluid Dynamics

\[ u_{e}^{n+1} - u_{e}^{n} = \int_{t^{n}}^{t^{n+1}} dt \int (\nabla \cdot F) \cdot dl - G \cdot p \]

\[ DU_{f}^{n+1} = 0 \]

Need to relate these two

\[ U_{f} = \int u \cdot n dA \]

\[ u_{e} = \int u \cdot dl \]
FE Basis Functions

Heat Flux
Magnetic Flux
Velocity Flux

Face Elements
Nedelec/RT/Whitney

Constant normal velocity on each face

\[ U_f = \int u \cdot n \, dA \]

Interpolant with continuity of the normal flux

\[ \vec{u}^h(x) = \vec{u}^0 + \frac{D}{n} \hat{x} \]

Constant divergence

\[ \nabla \cdot \vec{u}^h = D \]
FE Hodge

\[ \mathbf{u}^h(x) = \mathbf{u}^0 + \frac{D}{n} \mathbf{X} \]

Find the constants given the data (4x4)

\[ \mathbf{u}^0 \cdot \mathbf{n}^{f_1} + \frac{L_{f_1}^{f_1}}{n} D = \frac{1}{A_{f_1}} U_{f_1} \]

\[ \mathbf{u}_{\mathbf{e}} = \int \mathbf{u}^h \cdot d\mathbf{l} \]

Evaluate the integral

The basis function determines the relationship between the two velocities
StagMesh Interpolation

$$\int (u_{i,i} x_j + u_j) \, dV = \int (u_i x_j)_{,i} \, dV = \int (x_j) u_i n_i \, dA$$

- Gauss’ Theorem (Again)
- Assume constant flux on face
- Assume constant divergence

$$U_f = \int u \cdot n \, dA$$

$$\bar{u}_c V = \sum_{\text{faces}} U_f (x_{cf}^g - x_{c}^g)$$

$$\bar{u}_c = \frac{1}{V_c} \mathbf{R} U_f$$
StagMesh Hodge

Average Cell velocity

\[ \bar{u}_c = \frac{1}{V_c} R U_f \]

\[ u_e = R^T \bar{u}_c \quad \text{for incompressible} \]

\[ u_e = (R^T \frac{1}{V_c} R) U_f \]

- Explicit Formula for the same matrix relationship (Hodge*) as FE
- Symmetric
- Generalizable to polyhedra
- The intermediate is a (cell average) velocity vector. (Momentum, KE)
FD Interpolation

Use 3 face values at each vertex
Average vectors to center
Works on (almost) any polygon

- Assumes constant on face
- Assumes constant divergence

\[ \int \mathbf{u} \cdot n \, dA \]

Use Least Squares

\[ \mathbf{N} \mathbf{u}_c = \frac{1}{A_f} \mathbf{U}_f \]

- Also same cell velocity
Temperature Gradient
Electric Field
Velocity

**Edge Elements**
Nedelec/RT/Witney

\[
\mathbf{u}^h(\mathbf{x}) = \mathbf{u}^0 + \frac{1}{n-1} \mathbf{W} \times \mathbf{x}
\]

Interpolant with continuity of the tangential components

- Constant tangential velocity on each edge
- Constant vorticity

\[
\nabla \times \mathbf{u}^h = \mathbf{w}
\]
StagMesh Interpolation

\[(a \cdot x)u \rightarrow \int (u \times n + xw \cdot n) dA = \int xu \cdot dl\]

- Stokes’ Theorem
- Assume constant along edge
- Assume constant vorticity

\[u_e = \int u \cdot dl\]

\[\vec{u}_f \times \vec{n}_f A_f = \sum_{edges} u_e (\vec{x}_e^{cg} - \vec{x}_f^{cg})\]

SM = Rampant use of Stokes’ Theorem

\[\int \nabla \times \vec{u} \vec{X} dV = -\int \vec{u} \times \vec{n} dA\]
Summary: Basis Functions

• FE uses Explicit Basis Functions

• SM uses Stokes’ Theorem
  This approach can be applied to arbitrary polygons.

• SOM uses Discrete Inner Products
  highly discontinuous/anisotropic materials and arbitrary polygons

Many approaches to achieving the same underlying interpolation (Hodge *).
Test Functions

Dual mesh is not unique.
FV = top hat
FE = tent functions (unique for Galerkin)
FE / FV relationship

FE Discrete Calculus

Weighted Exact Discretization

\[ \int w(\nabla \cdot u) dV = \int wu \cdot ndA - \int (\nabla w) \cdot udV = \int u \cdot (-\nabla w) dV \]

Smeared Flux

\[ D^w = \sum_{faces} U^w_f \]

Compare

\[ D = \int \nabla \cdot u dV = \sum_{faces} \int u \cdot ndA = \sum_{faces} U^f \]
FE Exact Discretization

\[ \int \nabla \cdot u dV = \sum_{\text{faces}} \int u \cdot n dA \]

\[ \int (\nabla \times v) \cdot n dA = \sum_{\text{edges}} \int v \cdot d l \]

\[ \int \nabla T \cdot d l = T_2 - T_1 \]

\[ \int w \nabla \cdot u dV = \sum_{\text{cells}} \int u \cdot (-\nabla w) dV \]

\[ \int (\nabla \times v) \cdot (-\nabla w) dV = \sum_{\text{faces}} \int v \cdot (n \times \nabla w) dA \]

\[ \int \nabla T \cdot (n \times \nabla w) dA = \sum_{\text{edges}} \int T (\nabla w) \cdot d l \]
Higher-Order Exact Gradient

\[
\begin{align*}
\int_{\text{edge}} \nabla T \cdot dl &= T_{n2} - T_{n1} \\
\int_{\text{edge}} x \nabla T \cdot dl &= (x_{n2}T_{n2} - x_{n1}T_{n1}) - t_e \int_{n1}^{n2} Tdl \\
\int_{\text{face}} n \times \nabla T dA &= \sum_{\text{edges}} t_e \int Tdl
\end{align*}
\]

20 outputs / tet  
10 inputs / tet
Higher-Order Exact Curl

\[
\int (\nabla \times \mathbf{v}) \cdot \mathbf{n} \, dA = \sum_{\text{edges}} \int \mathbf{v} \cdot d\mathbf{l}
\]
\[
\int \mathbf{x}(\nabla \times \mathbf{v}) \cdot \mathbf{n} \, dA = \sum_{\text{edges}} \int \mathbf{xv} \cdot d\mathbf{l} + \int \mathbf{n} \times \mathbf{v} \, dA
\]
\[
\int \nabla \times \mathbf{v} \, dV = \sum_{\text{faces}} \int \mathbf{n} \times \mathbf{v} \, dA
\]

15 outputs / tet

20 inputs / tet

Note: \( C^{[2]} G^{[2]} = 0 \)

Mimetic

\[
\begin{bmatrix}
\int_{\text{face}} (\nabla \times \mathbf{v}) \cdot \mathbf{n} \, dA \\
\int_{\text{face}} \mathbf{x}(\nabla \times \mathbf{v}) \cdot \mathbf{n} \, dA \\
\int_{\text{cell}} (\nabla \times \mathbf{v}) \, dV
\end{bmatrix} = C^{[2]}
\begin{bmatrix}
\int_{\text{edge}} \mathbf{v} \cdot d\mathbf{l} \\
\int_{\text{edge}} \mathbf{xv} \cdot d\mathbf{l} \\
\int_{\text{face}} \mathbf{n} \times \mathbf{v} \, dA
\end{bmatrix}
\]
Review

FE:
Explicit basis functions
Fixed geometry – precise proofs
Implicit dual mesh
Semi-implicit Hodge*

StagMesh:
Implicit basis functions
Arbitrary polygons.
Explicit dual mesh
Explicit Factored Hodge*
Summary

- Underlying assumptions about the solution (basis functions) are often the same.

- The Test Functions are different. Test functions affect the metric (geom).

- Higher Order is achieved by noticing that you can exactly discretize different ways and with different moments.

- Anyone can do it.

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