Image Deblurring with Krylov Subspace Methods

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For some details:
see extended abstract.

For more details and references:
see my books.

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Image Deblurring

Sharp image

Forward problem

Blurred image

Reconstruction

Inverse Problem

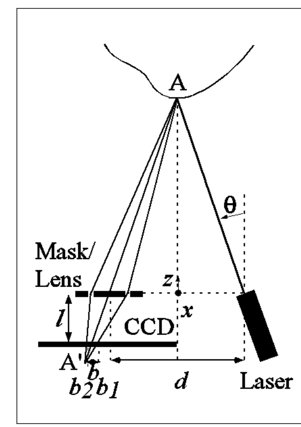
<u>This talk:</u>

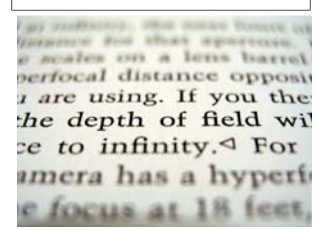
- Regularization
- Projection
- CGLS
- Other iterations
- Noise propagation
- Subspace precond.

Image deblurring is an inverse problem; hence it is ill posed:

- small perturbation in data $\rightarrow$
- large errors in reconstruction

Sources of Blurred Images





The Deblurring Problem

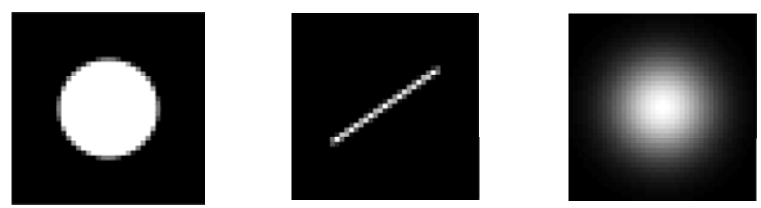
Fredholm integral equation of the first kind:

$$\int_0^1 \int_0^1 K(x,y;x',y')\, f(x,y)\, dx\, dy = g(x',y') \ , \qquad 0 \le x',y' \le 1.$$

Think of f as an unknown sharp image, and g as the blurred version.

Think of K as a model for the point spread function.

Examples of point spread functions



Discretization yields a LARGE system of linear equations: $A\, x = b$.

But the matrix A is very ill conditioned, and therefore

$\boxed{\textbf{Do not solve } A\, x = b \textbf{ !}}$

Regularize!

We must apply regularization in order to deal with the ill conditioning of the problem and suppress the influence of the noise in the data.

<u>Tikhonov regularization:</u>

$$\min_x \left\{ \|A\,x - b\|_2^2 + \lambda^2\,\|L\,x\|_2^2 \right\}$$

The choice of smoothing norm, together with the choice of λ, forces x to be effectively dominated by components in a low-dimensional subspace, determined by the GSVD of (A, L) – or the SVD of A if $L = I$.

<u>Regularization by projection:</u>

$$\min_x \|A\,x - b\|_2 \quad \text{subject to} \quad x \in \mathcal{W}_k$$

where $\mathcal{W}_k$ is a k-dimensional subspace – works well if this subspace is spanned by desirable basis vectors (think of TSVD: $\mathcal{W}_k = \mathrm{span}\{v_1, v_2, \ldots, v_k\}$).

The Projection Method

A more practical formulation of regularization by projection.

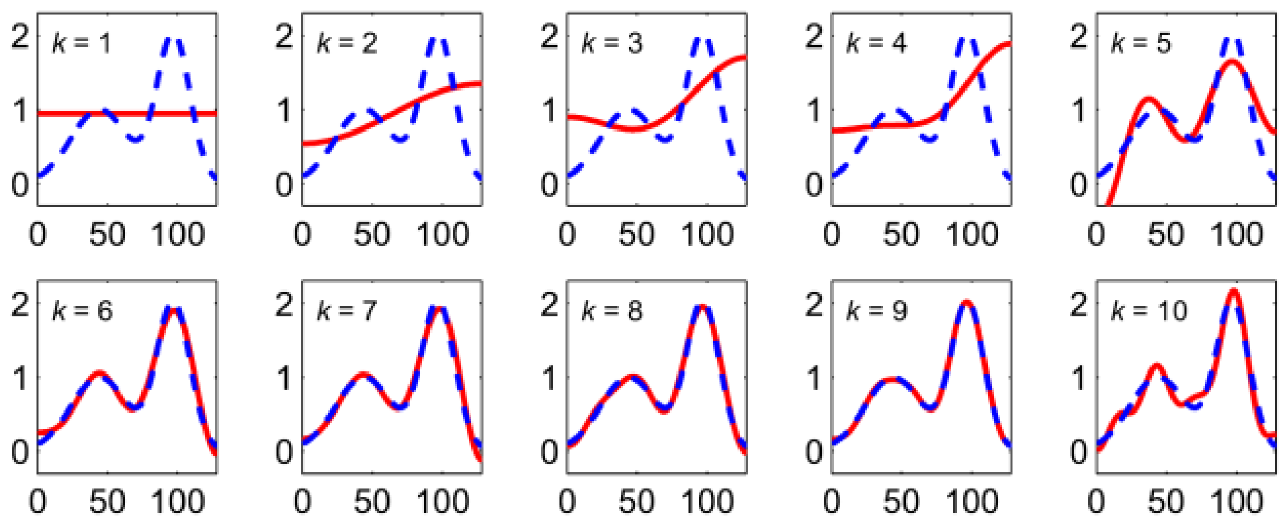
We are given the matrix $W_k = (w_1, \dots, w_k) \in \mathbb{R}^{n \times k}$ such that $\mathcal{W}_k = \mathcal{R}(W_k)$.

We can write the requirement as $x = W_k\, y$, leading to the formulation

$$x^{(k)} = W_k\, y^{(k)}, \qquad y^{(k)} = \mathrm{argmin}_y \, \|(A\, W_k)\, y - b\|_2 .$$

Projected problem

Example:
DCT basis



Operations often do not require W_k explicitly.

Some Thought on the Basis Vectors

The DCT basis (and similar bases that define fast transforms):

- computationally convenient to work with, but
- may not be well suited for the particular problem.

The SVD basis (or GSVD basis if $L \neq I$) gives an "optimal" basis for representation of the matrix A, but ...

- it is computationally expensive, and
- it does not involve information about the rhs b.

Is there a basis that is computationally attractive and also involves information about b and thus the given problem?

$\rightarrow$ Krylov subspaces!

Regularizing Iterations

Apply CG to the normal equations for the least squares problem

$$\min \|A\,x - b\|_2 \ .$$

This algorithm, called CGLS, produces a sequence of iterates $x^{(k)}$ which solve

$$\min \|A\,x - b\|_2 \qquad \text{subject to} \qquad x \in \mathcal{K}_k \ ,$$

where $\mathcal{K}_k$ is the k-dimensional Krylov subspace

$$\mathcal{K}_k = \text{span}\{A^T b\,,\ A^T A\,A^T b\,,\ (A^T A)^2 A^T b\,,\ \ldots\} \ .$$

These methods are referred to as *regularizing iterations*.

Iterative methods are based on multiplications with A and A^T (blurring).
How come repeated blurings can lead to reconstruction?
$\rightarrow$ CGLS constructs a polynomial approximation to $(A^T A + \lambda^2 I)^{-1} A^T$.

The Behavior of of CGLS

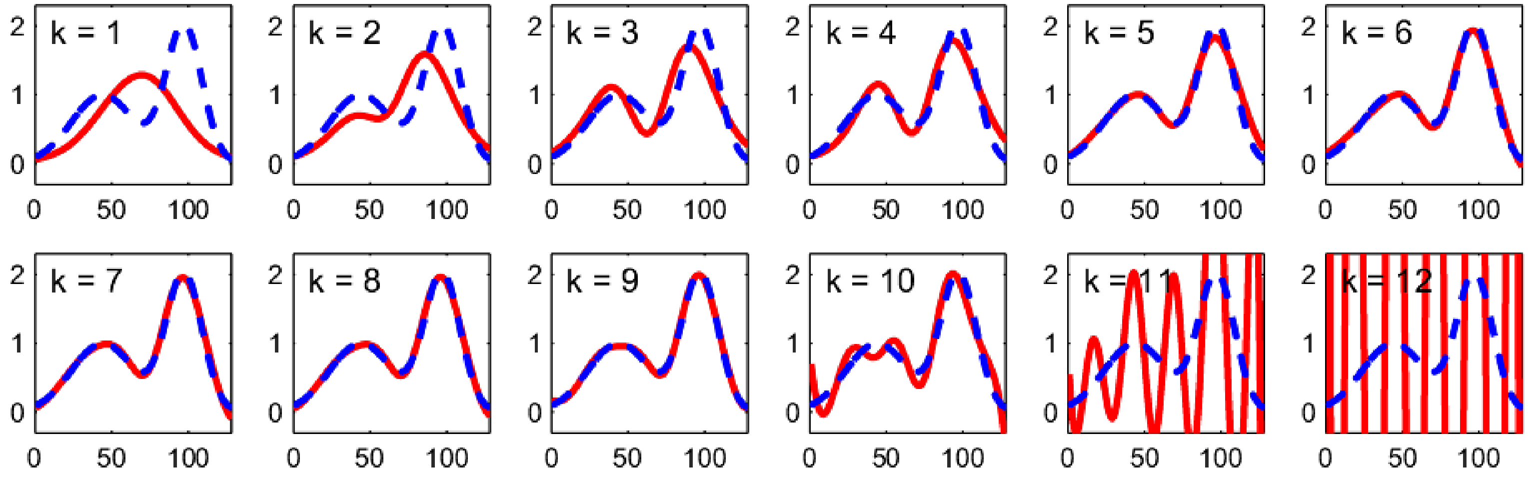
CGLS algorithm solves the problem without forming the Krylov basis explicitly.

Finite precision: convergence slows down, but no deterioration of the solution.

The solution and residual norms are monotone functions of k:

$$\|x^{(k)}\|_2 \geq \|x^{(k-1)}\|_2, \qquad \|A\,x^{(k)} - b\|_2 \leq \|A\,x^{(k-1)} - b\|_2, \qquad k = 1, 2, \ldots$$

Same example as before: CGLS iterates



Semi-Convergence

During the first iterations, the Krylov subspace $\mathcal{K}_k$ captures the "important" information in the noisy right-hand side b.

- In this phase, the CGLS iterate $x^{(k)}$ approaches the exact solution.

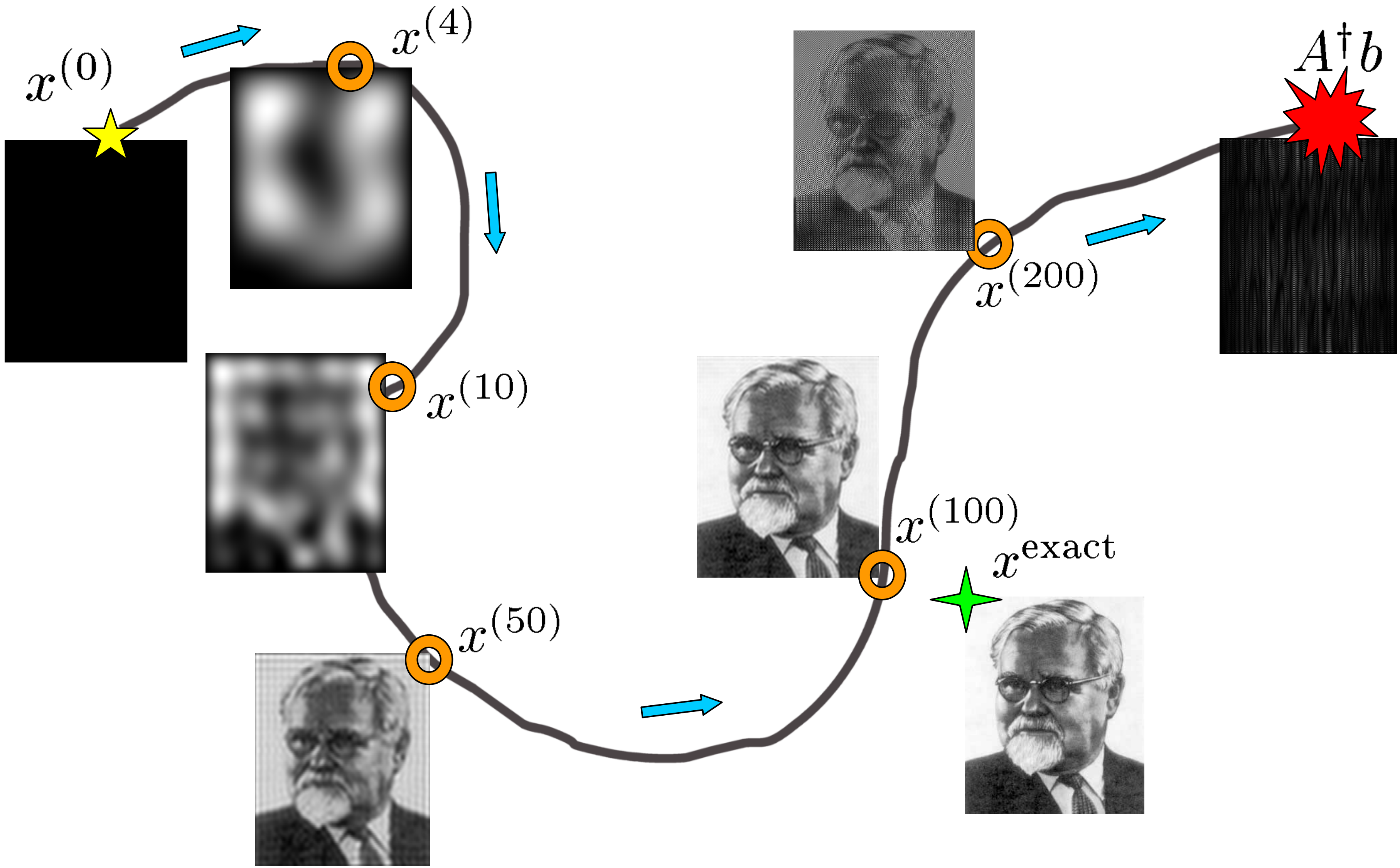
At later stages, the Krylov subspace $\mathcal{K}_k$ starts to capture undesired noise components in b.

- Now the CGLS iterate $x^{(k)}$ diverges from the exact solution and approach the undesired solution $A^{\dagger}b$ to the least squares problem.

The iteration number k (= the dimension of the Krylov subspace $\mathcal{K}_k$) plays the role of the regularization parameter.

This behavior is called *semi-convergence*.

Illustration of Semi-Convergence



Advantages of the Krylov Subspace

The SVD basis vectors $v_1, v_2, \ldots$ are well suited for representation of A.

But this basis "does not know" about the given problem – it can not utilize information about the right-hand side b.

The Krylov subspace $\mathcal{K}_k$ "knows" about the right-hand side and therefore adapts itself to the given problem, through the starting vector

$$A^T b = A^T A\, x^{\mathrm{exact}} + A^T e = \textstyle\sum_{i=1}^n \sigma_i^2 \, (v_i^T x^{\mathrm{exact}})\, v_i + \sum_{i=1}^n \sigma_i \, (u_i^T e)\, v_i .$$

Hence the Krylov basis vectors are rich in those directions that are needed.

SVD analysis

$$x^{(k)} = \textstyle\sum_{i=1}^n f_i^{(k)} \, \frac{u_i^T b}{\sigma_i} \, v_i , \qquad f_i^{(k)} = \prod_{j=1}^k \frac{\theta_j^{(k)} - \sigma_i^2}{\theta_j^{(k)}}$$

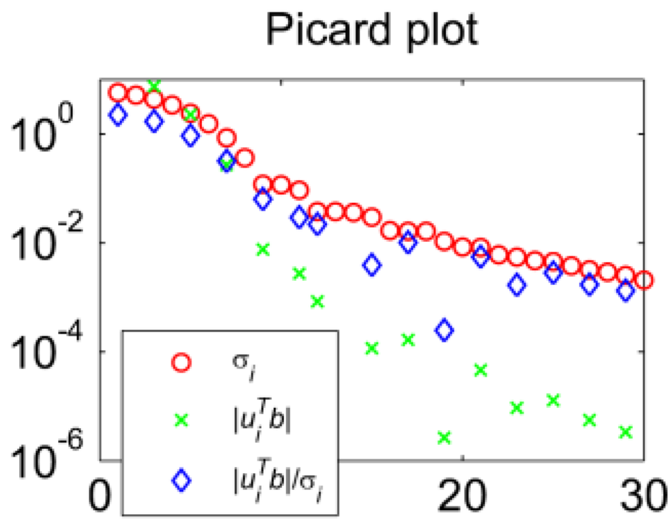
Here $\theta_j^{(k)}$ are the Ritz values, i.e., the eigenvalues of the projection of $A^T A$ on the Krylov subspace $\mathcal{K}_k$. They converge to those σ_i^2 whose corresponding SVD components $u_i^T b$ are large.

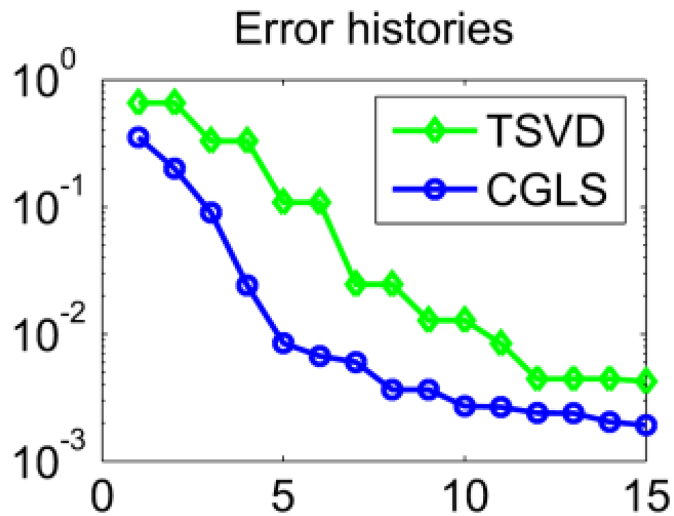
CGLS Focuses on Significant Components

Example: **phillips** (from Regularization Tools).

Exact solution has many zero SVD coefficients.

- TSVD solution x_k includes all coef. from 1 thru k.
- CGLS solution $x^{(k)}$ includes only those coef. we need.





CGLS suppresses noise better than TSVD in this case.

Another Story: CGLS for Tikhonov

Of course, one could also use CGLS to solve the Tikhonov problem in the form

$$\min_x \left\| \begin{pmatrix} A \\ \lambda L \end{pmatrix} - \begin{pmatrix} b \\ 0 \end{pmatrix} \right\|_2^2 .$$

But this approach typically requires that the system is solved many times, for many diffrent values of λ.

Also, preconditioning is often necessary – but it can be difficult to design a good preconditioner for the Tikhonov problem.

We shall not pursue this aspect further in this talk.

Other Krylov Subspace Methods

Sometimes it is impractical to use methods that need A^T, e.g, if A is symmetric or if we have a black-box function that computes $y = A\,x$.

MINRES and GMRES come to mind – these methods are based on the Krylov subspace:

$$\mathcal{K}_k = \mathrm{span}\{b, Ab, A^2b, \ldots, A^{k-1}b\}.$$

Unfortunately it is a bad idea to include the noisy vector b in the subspace.

A better choice is the "shiftet" Krylov subspace:

$$\vec{\mathcal{K}}_k = \mathrm{span}\{Ab, A^2b, \ldots, A^kb\}.$$

The corresponding methods are called MR-II and RRGMRES (they are now included in Regularization Tools).

Example on next slide.

Comparing Krylov Methods

We find:

♥ The absence of b in the Krylov subspace is essential for MR-II.

♠ MR-II computes a filtered SVD solution.

♣ Negative eigenvalues of A do not inhibit the regularizing effect of MR-II, but they can slow down the convergence.

♦ RRGMRES mixes the SVD components in each iteration and $x^{(k)}$ is not a filtered SVD solution.

• RRGMRES works well if the mixing is weak (e.g, if $A \approx A^T$), or if the Krylov basis vectors are well suited for the problem.

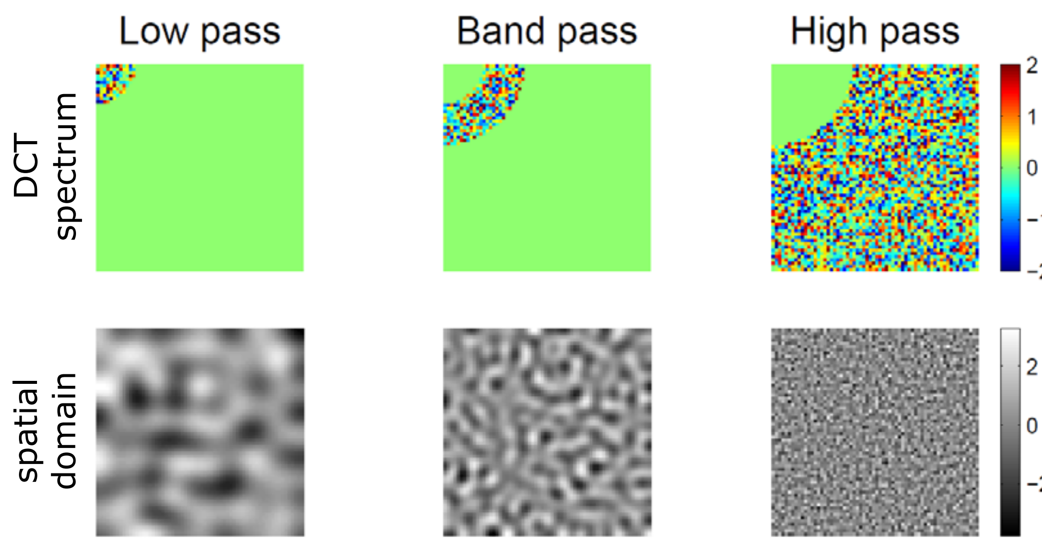
Progress of the Iterations

CGLS:
k = 4, 10
and 25
iterations

Initially, the image gets sharper – then ”freckles” start to appear.

Low frequencies carry the main information.

”Freckles” are band-pass filtered noise.



Noise Propagation

CGLS solution can be written in terms of a matrix polynomium:

$$x^{(k)} = \mathcal{P}_k(A^T A)\, A^T b,$$

where $\mathcal{P}_k(\theta) = \big(1 - \mathcal{R}_k(\theta)\big)/\theta$ and $\mathcal{R}_k$ is the Ritz polynomium associated with the Krylov subspade $\mathcal{K}_k$.

Thus $\mathcal{P}_k$ is fixed by A and b, and if $b^{\mathrm{exact}} = b + e$ then

$$x^{(k)} = \mathcal{P}_k(A^T A)\, A^T b^{\mathrm{exact}} + \mathcal{P}_k(A^T A)\, A^T e \equiv x^{(k)}_{b^{\mathrm{exact}}} + x^{(k)}_e .$$

Similarly for the other iterative methods.

Signal component

Noise component

Note that signal component $x^{(k)}_{b^{\mathrm{exact}}}$ depends on the noise e via $\mathcal{P}_k$.

Signal and Noise Components

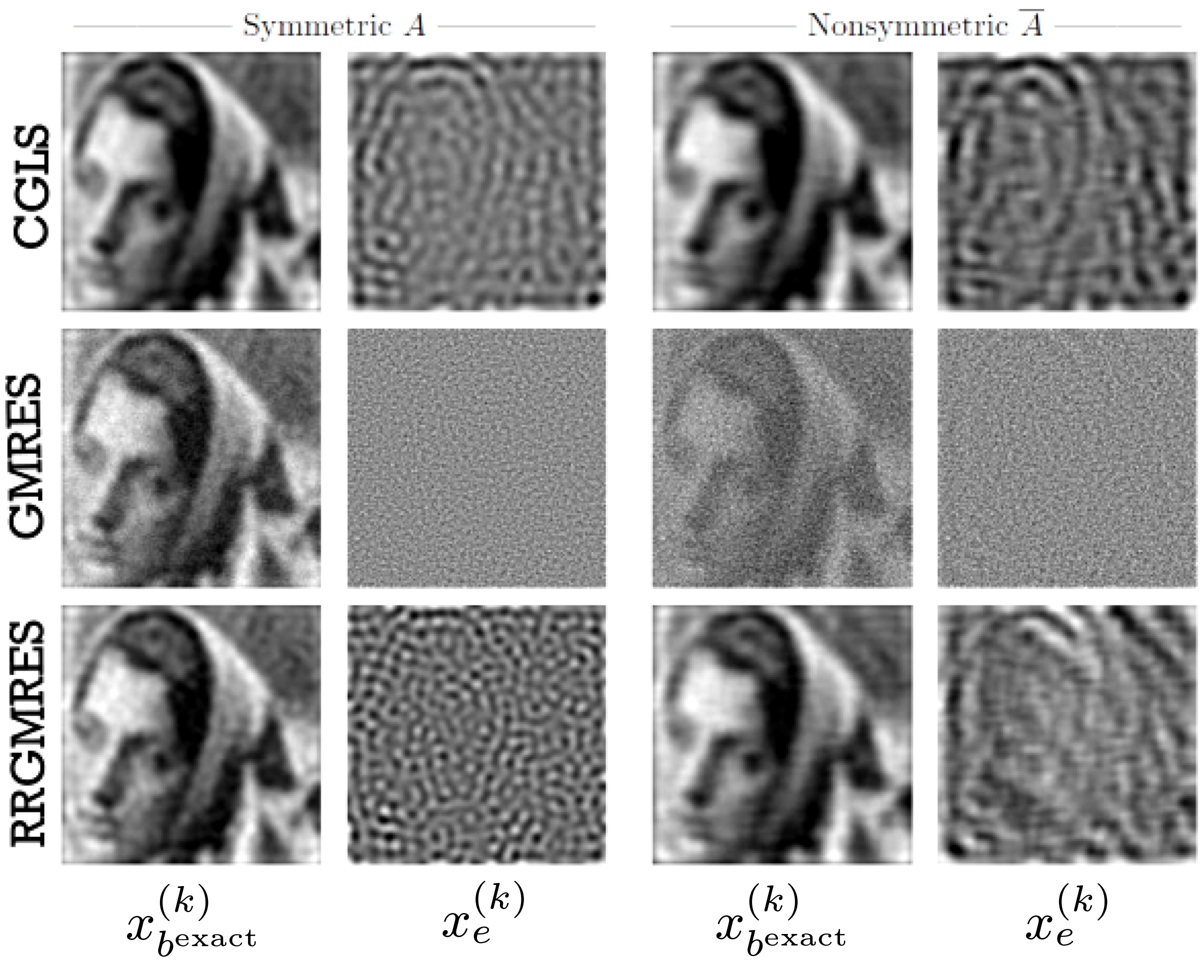
$$x^{(k)} = \mathcal{P}_k(A^T A)\, A^T b = \underbrace{\mathcal{P}_k(A^T A)\, A^T b^{\mathrm{exact}}}_{\text{signal}} + \underbrace{\mathcal{P}_k(A^T A)\, A^T e}_{\text{noise}} \ .$$

Note that the noise components (the freckles) are *correlated* with structures in the image!

Tends to mask the appearance of the noise!!

Two different matrices *A*

Same Behavior in All Methods



The noise components are always correlated with the image!

General-Form Tikhonov Regularization

CGLS is linked to the SVD of A and thru the Krylov subspace, the Ritz polynomium, and the convergence of the Ritz values.

Thus CGLS is also related to Tikhonov regularization in standard form

$$\min_x \left\{ \|A\,x - b\|_2^2 + \lambda^2\,\|x\|_2^2 \right\}$$

But occationally we prefer the *general* formulation

$$\min_x \left\{ \|A\,x - b\|_2^2 + \lambda^2\,\|L\,x\|_2^2 \right\}, \qquad L \neq I.$$

How do we modify CGLS such that it can incorporate the matrix L?

Use the <u>standard-form transformation</u>

$$\min_{\bar{x}} \|\overline{A}\,\bar{x} - b\|_2^2 + \lambda^2 \|\bar{x}\|_2^2 \quad \text{with} \quad \overline{A} = A\,L^{\#} \quad \text{and} \quad x = L^{\#}\bar{x} + x_{\mathcal{N}},$$

where $L^{\#}$ = oblique pseudoinverse of L and $x_{\mathcal{N}} \in \mathcal{N}(L)$.

Subspace Preconditioning

If we apply CGLS to the standard-form problem

$$\min_{\bar{x}} \|\overline{A}\,\bar{x} - b\|_2^2 + \lambda^2 \|\bar{x}\|_2^2,$$

then the iterates, when transformed back via $L^{\#}$, lie in the affine space

$$\mathrm{span}\{M A^T b, (M A^T A)\, M A^T b, (M A^T A)^2 M A^T b, \ldots\} + x_{\mathcal{N}} \;,$$

where $M = L^{\#}(L^{\#})^T$.

Hence L is a preconditioner for CGLS that provides a better suited subspace.

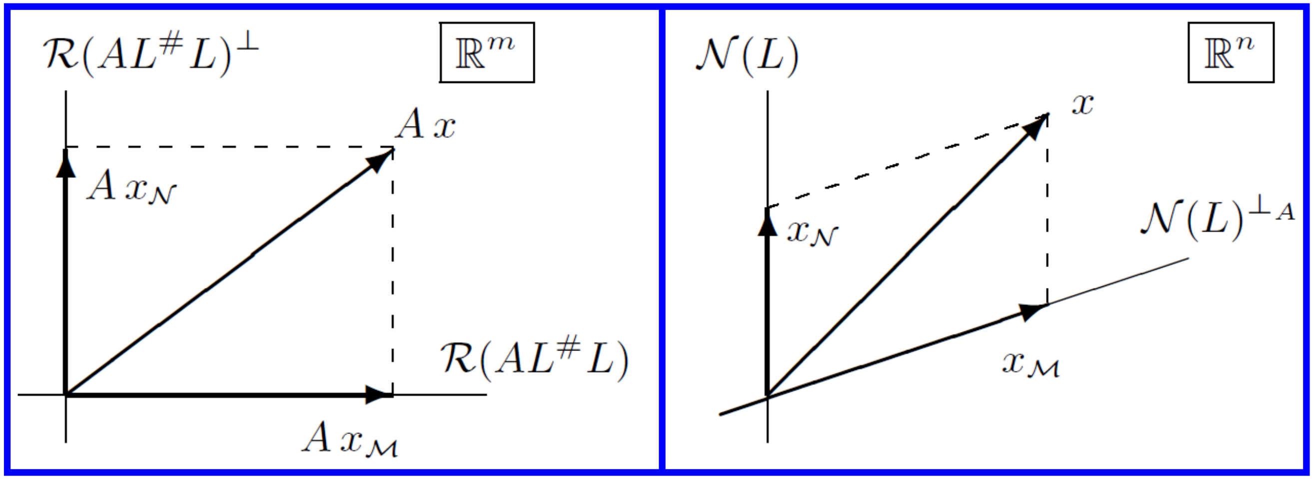
The Krylov subspace methods are implemented such that $\overline{A}$ is never formed.

How is the oblique pseudoinverse $L^{\#}$ defined? And why this particular matrix?

Outside scope
of this talk.

Next slide please ...

Splitting!



Write $x = x_{\mathcal{M}} + x_{\mathcal{N}}$ with $x_{\mathcal{N}} \in \mathcal{N}(L)$ and $x_{\mathcal{M}}$ being $A^T A$–orthogonal to $x_{\mathcal{N}}$.

This corresponds to an *oblique* splitting of the subspace $\mathbb{R}^n$.

Then the vector $A\,x = A\,x_{\mathcal{M}} + A\,x_{\mathcal{N}}$ splits into two *orthogonal* components.

The Tikhonov problem reduces to two independent problems for $x_{\mathcal{M}}$ and $x_{\mathcal{N}}$:

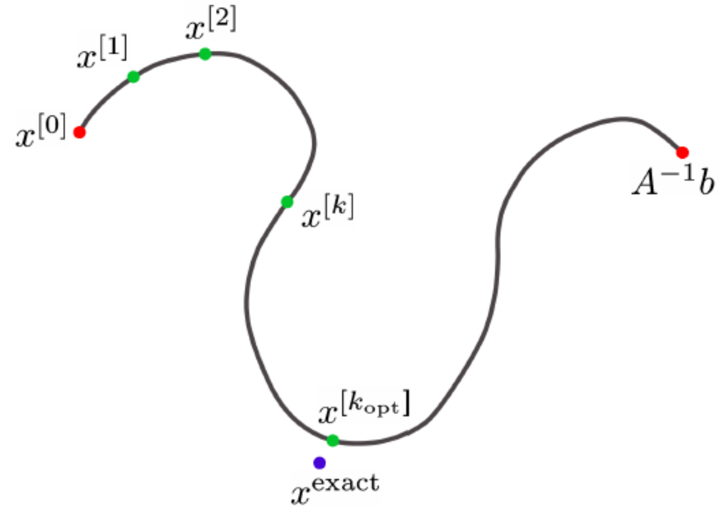
$$\min \|A\,x_{\mathcal{M}} - b\|_2^2 + \lambda^2 \|x_{\mathcal{M}}\|_2^2 \qquad \text{and} \qquad \min \|A\,x_{\mathcal{N}} - b\|_2^2.$$

Since $x_{\mathcal{M}} = L^{\#} L\,x$ we get $A\,x_{\mathcal{M}} = (A\,L^{\#})\,(L\,x) \rightarrow$ the standard-form problem.

Conclusion

An important area with plenty of

- theoretical aspects,
- computational challenges, and
- important applications.



More stuff not covered here:

- Boundary conditions
- Stopping criteria
- Hybrid methods: projection + regularization
- Nonnegativity constraints
- Other iterative methods: ART, SIRT, Richardson-Lucy, ...
- Blind deconvolution
- Applications in astronomy, biometrics, computer-vision, ...