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Geometric structures underlying mimetic approaches

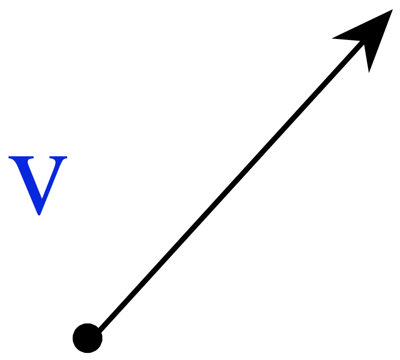
to the discretization of Maxwell's equations

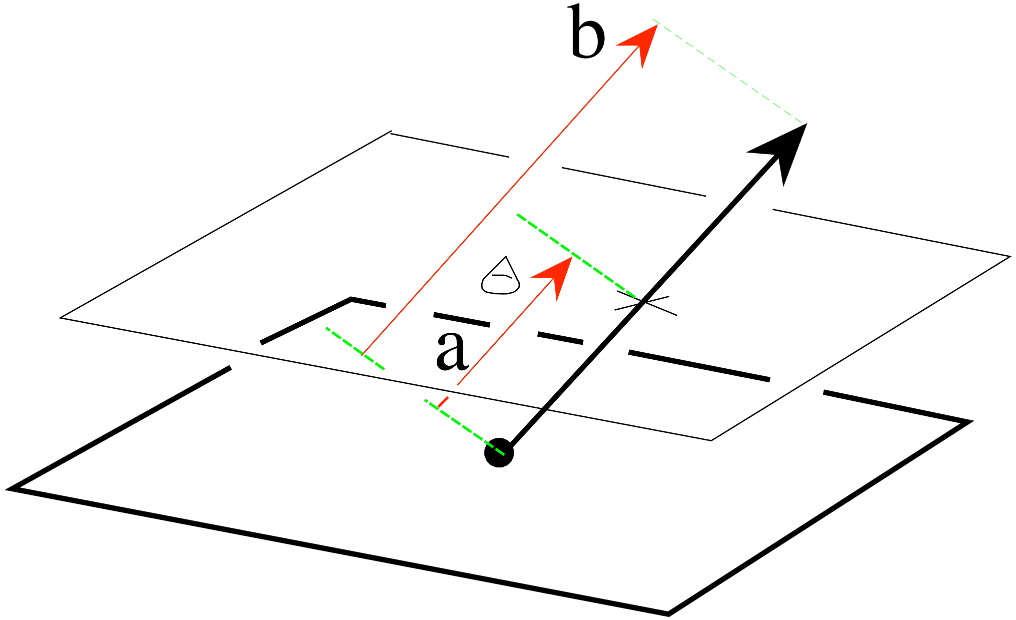
A tour of the workshop

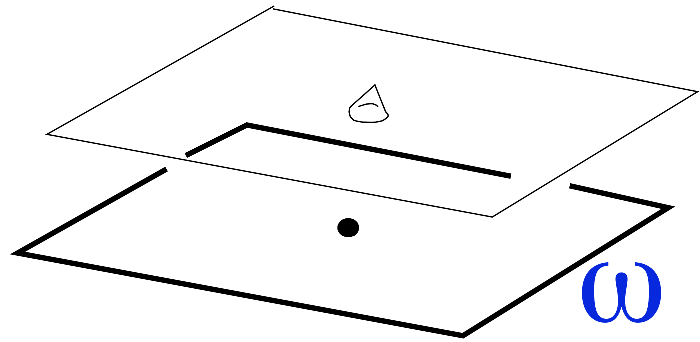
Vector:

Covector:

$$\langle \mathbf{v} \, ; \, \omega \rangle = b/a$$







(virtual) displacement,
velocity, ...
are vectors

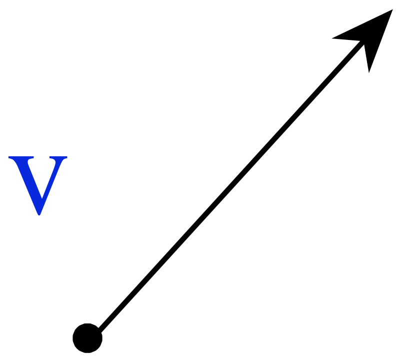
v → <virt. work>
is linear map, i.e.,
a covector, say f.
<virt. work> = ⟨v ; f⟩

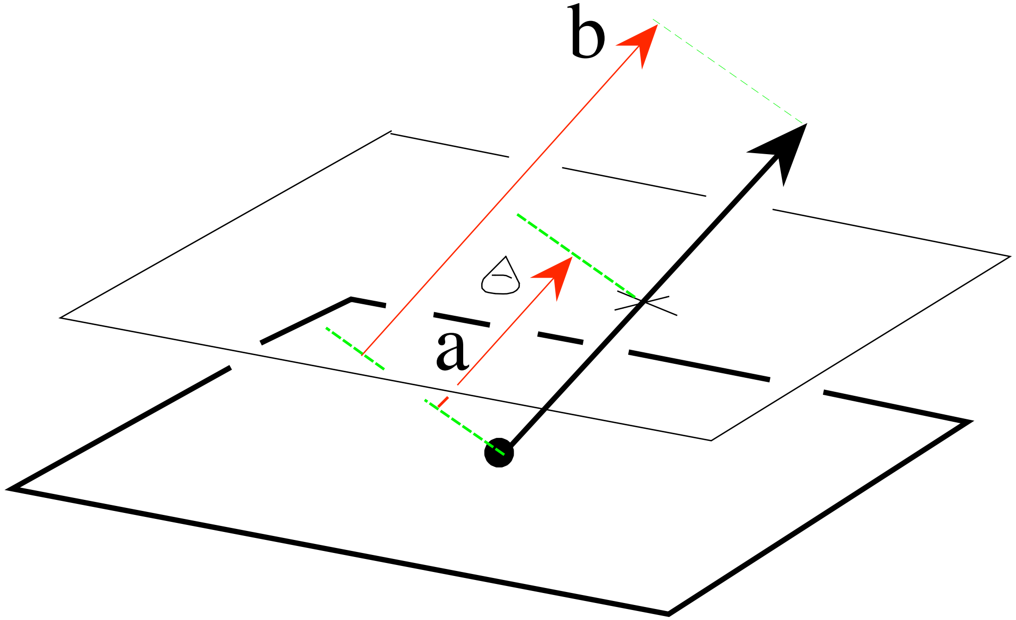
force,
momentum, ...
are covectors

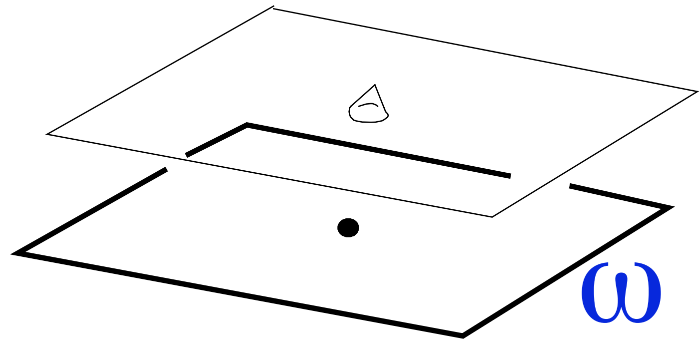
Vector:

Covector:

$$\langle v \,;\, \omega \rangle = b/a$$





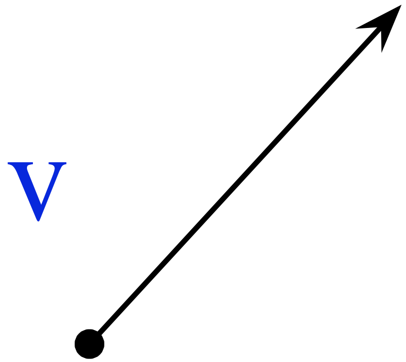


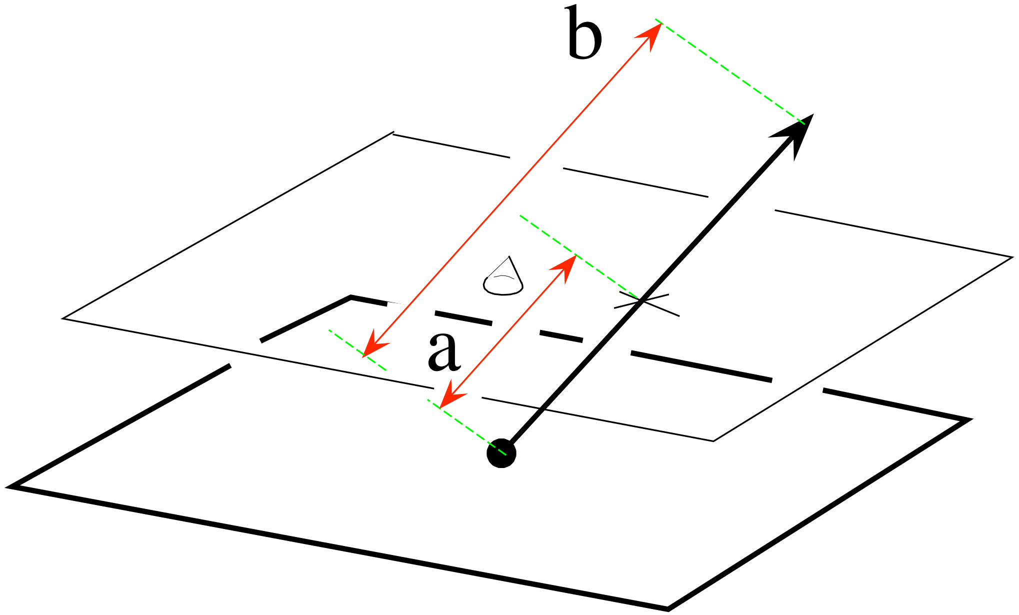
$\langle v \,;\, \omega \rangle = v \cdot \Omega$ – but "proxy vector" Ω depends on (most often, irrelevant) metric of ambient space

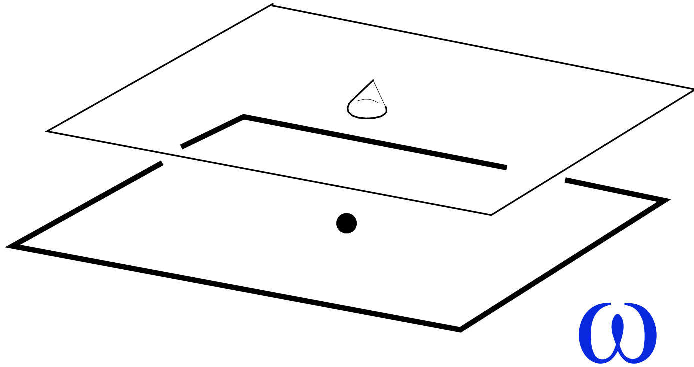
Vector

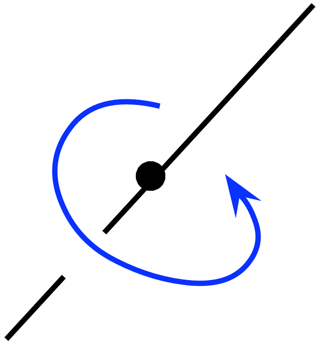
Covector

$$\langle \mathbf{v} ; \omega \rangle = b/a$$



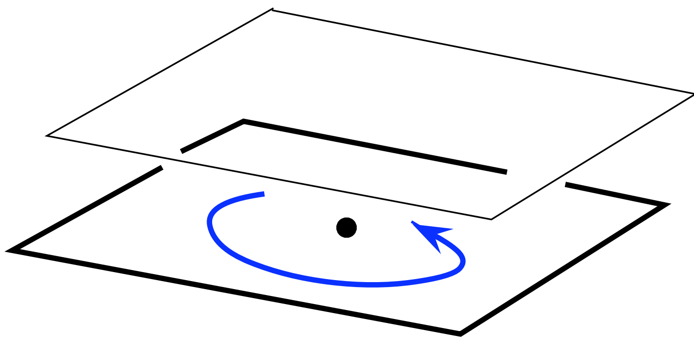






Come also in

"twisted" variety



(also called "axial" vectors or covectors)

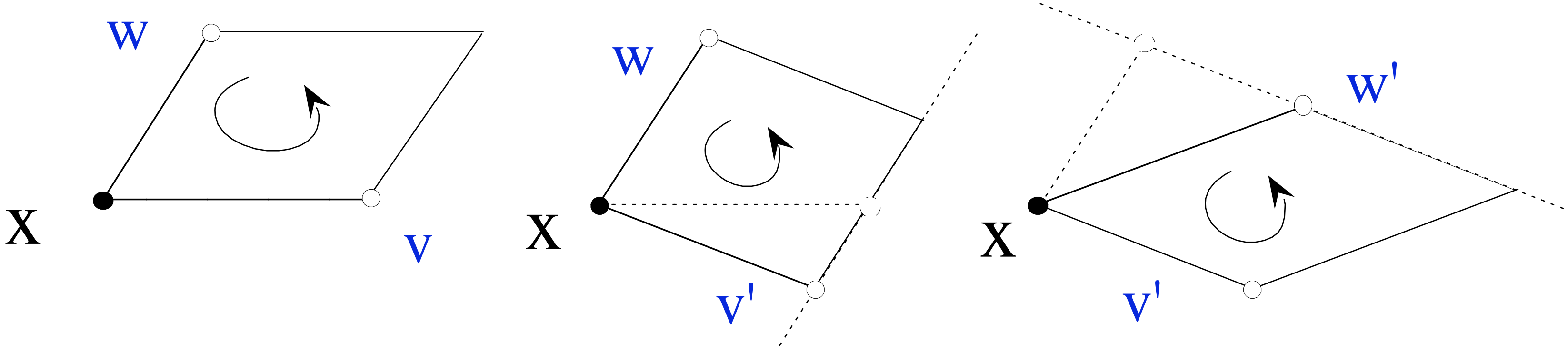
p-vectors:

(Grassmann algebra)

Case p = 2 (bivector)

(denoted $v \wedge w$ or $v \vee w$)

("wedge") ("join")

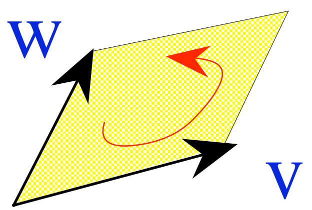


$\{v, w\} \sim \{v', w\} \sim \{v', w'\}$

p-covectors:

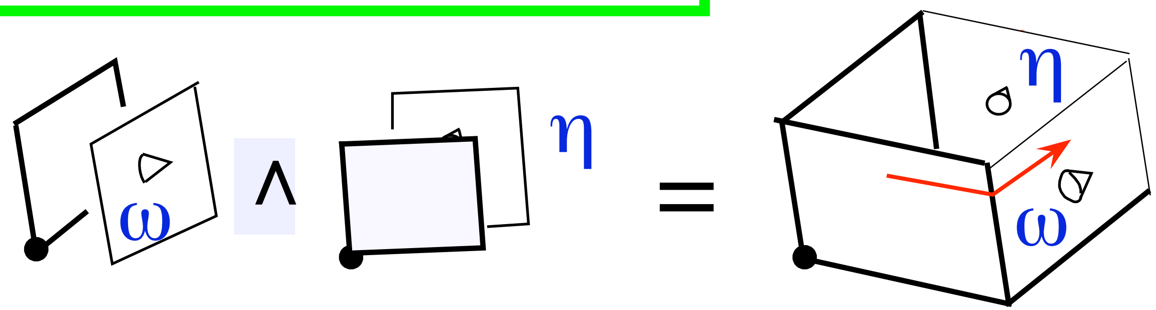
Grassmann algebra
of multi-covectors, too

$v \vee w$



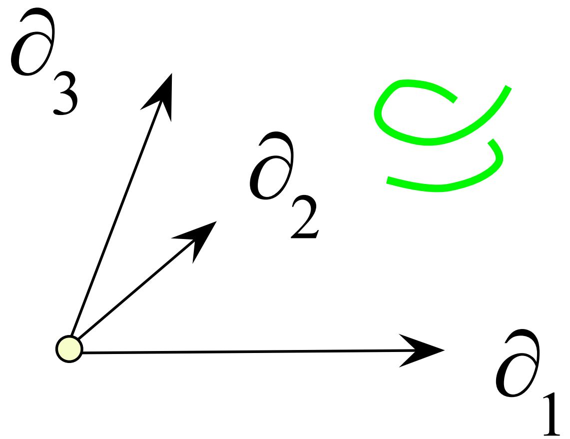
wedge
product

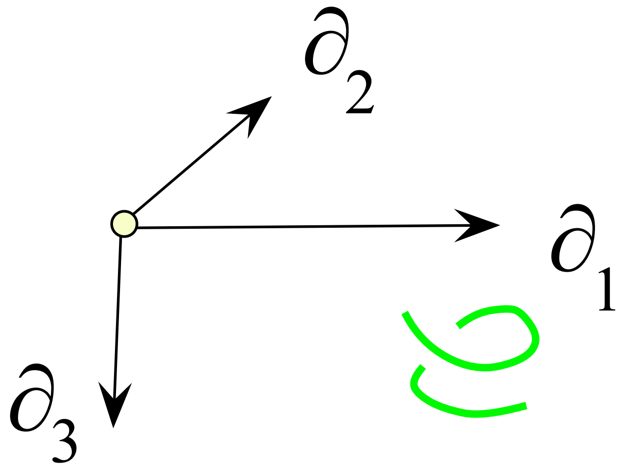
$\omega \wedge \eta$



$$\langle v \vee w \,;\, \omega \wedge \eta \rangle = \langle v \,;\, \omega \rangle \langle w \,;\, \eta \rangle - \langle w \,;\, \omega \rangle \langle v \,;\, \eta \rangle$$

Orientation, twisted objects





$\mathrm{Or} \in \{\text{direct, skew}\}$

$\mathrm{Or} = \circlearrowleft \;\Leftrightarrow\; -\mathrm{Or} = \circlearrowright$,

On the set of pairs $\{\omega, \mathrm{Or}\}$, equivalence relation:

$$\{\omega, \mathrm{Or}\} \sim \{-\omega, -\mathrm{Or}\}$$

Then $\tilde{\omega} \hat{=}$ equivalence class

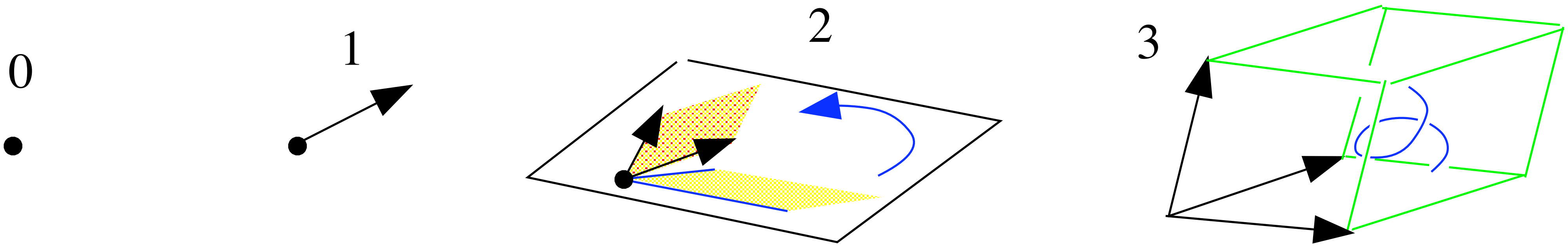
Outer orientation:

- Of vector subspace: an orientation of (one of its) complement(s)
- Of affine subspace: an outer orientation of the vector subspace parallel to it
- Of submanifold: consistent orientations of all its tangent spaces

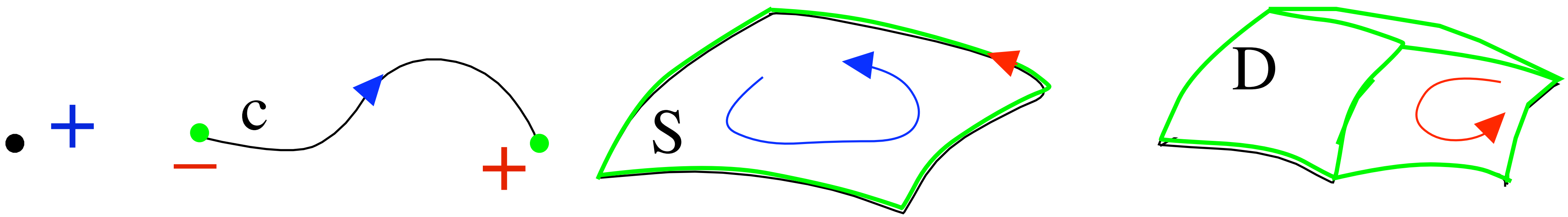
Objects we'll work with – straight

- Affine 3D space, with associated vector space, but no orientation, no metric structure (for a while)

- Points, vectors, multivectors (Grassmann algebra)



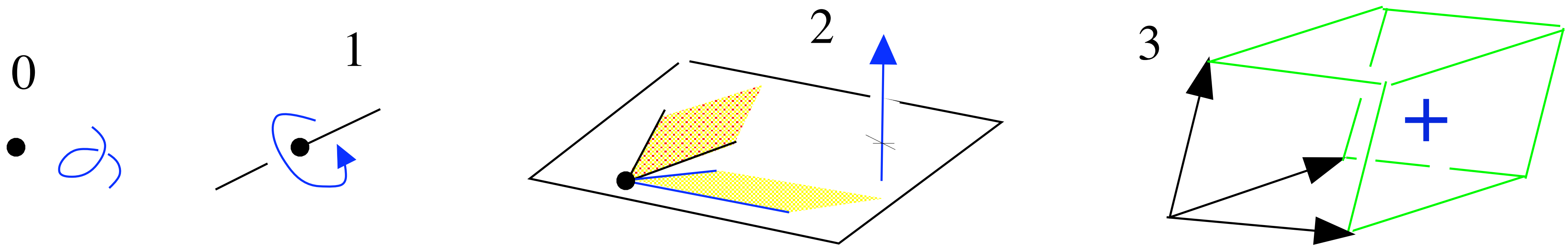
- Smooth sub-manifolds, with own orientation:



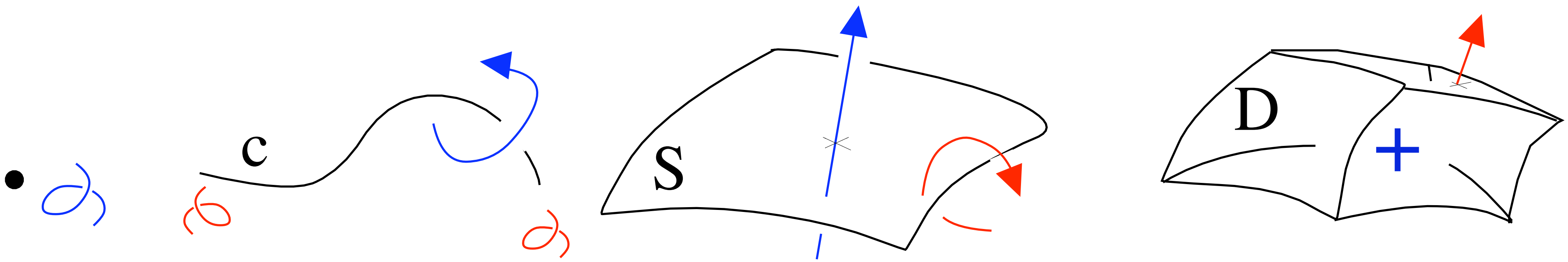
Objects we'll work with – twisted

- Affine 3D space, with associated vector space, but no orientation, no metric structure (for a while)

- Points, vectors, multivectors (Grassmann algebra)



- Sub-manifolds, with own outer orientation:



Mathematical physics → Numerical models

Calculus — Computers — Discrete calculus?

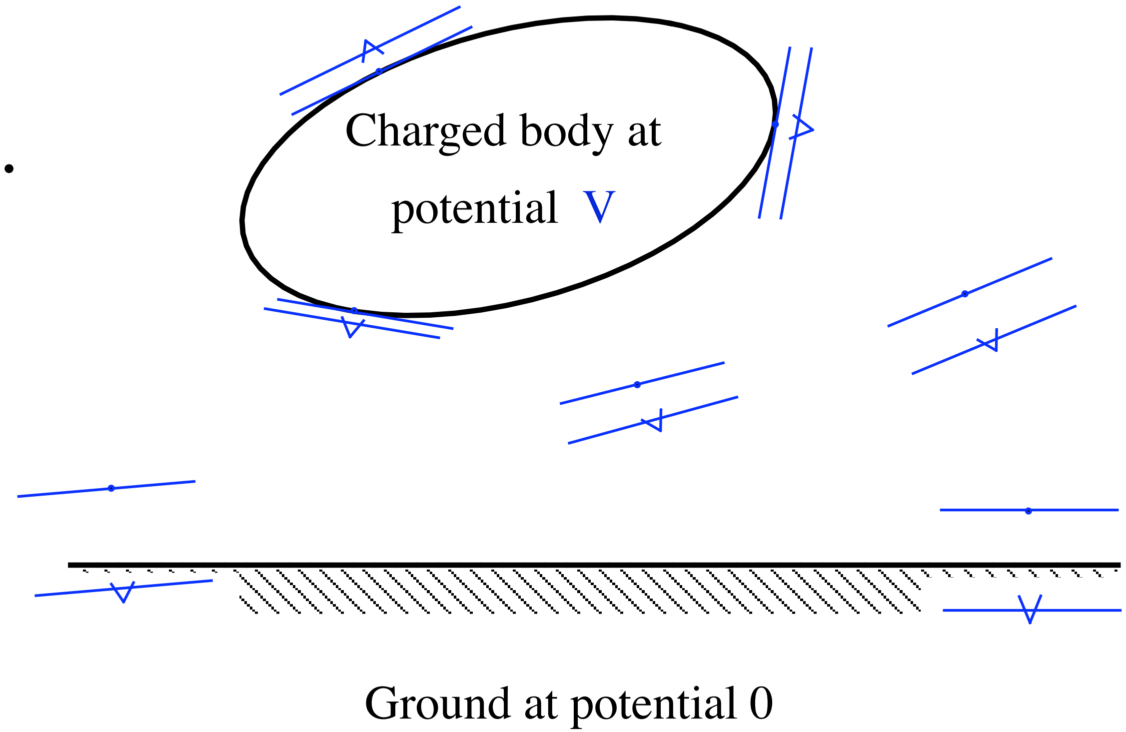
Reformulating theories:

B, H, E, ... are just elements of a mathematical representation of electromagnetic phenomena, and not necessarily the right objects to deal with

Most physical fields are covector-fields rather than vector fields

Ambient electric field ...

a field of covectors...



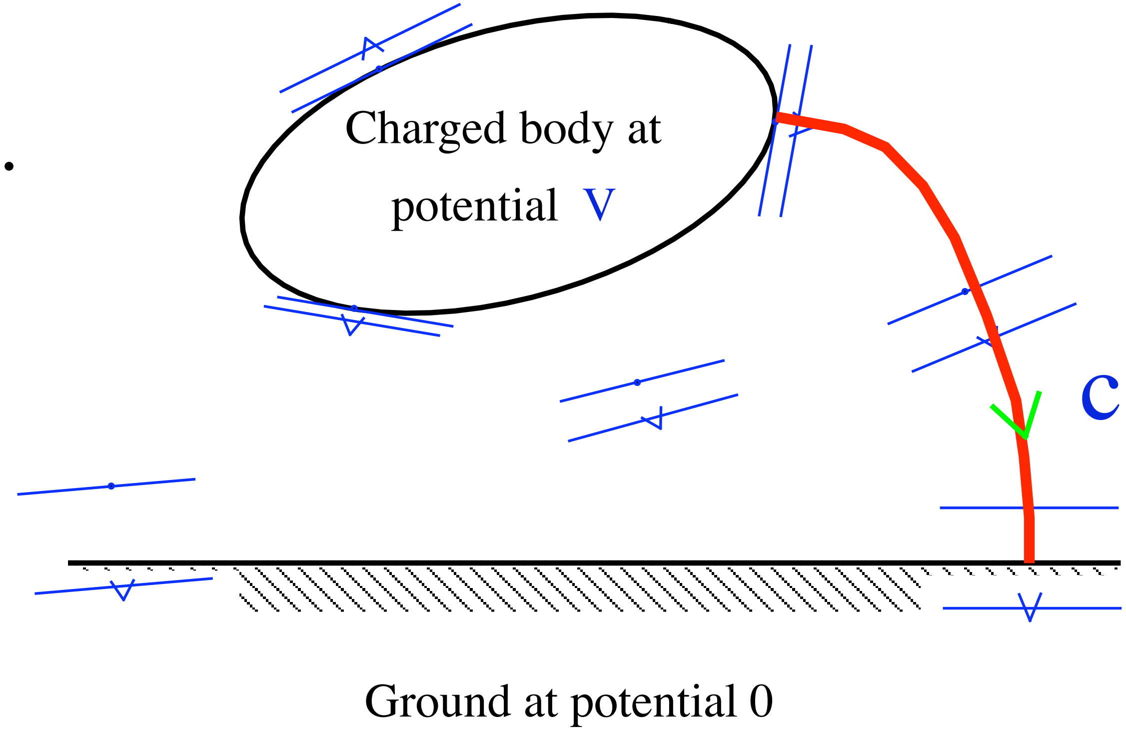
... E = – grad v

... x → e(x), denoted e.

Most physical fields are covector-fields rather than vector fields

Ambient electric field ...

a field of covectors...



... $E = -\operatorname{grad} v$

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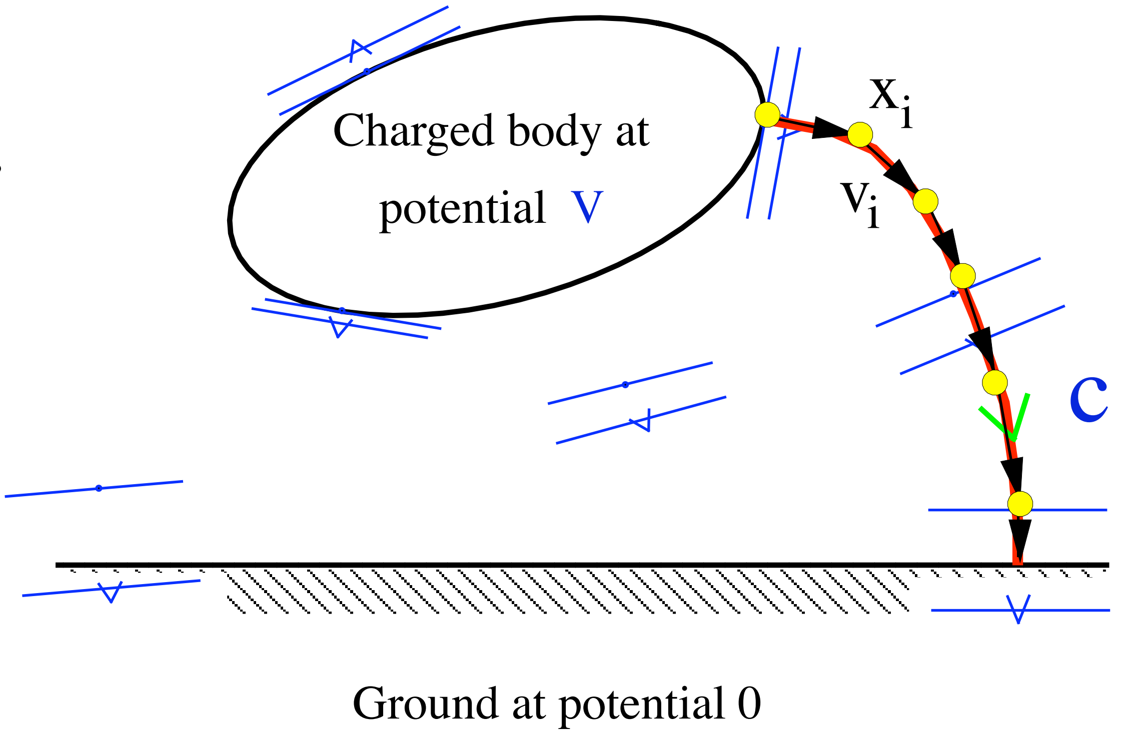


... $E = -\text{grad } v$

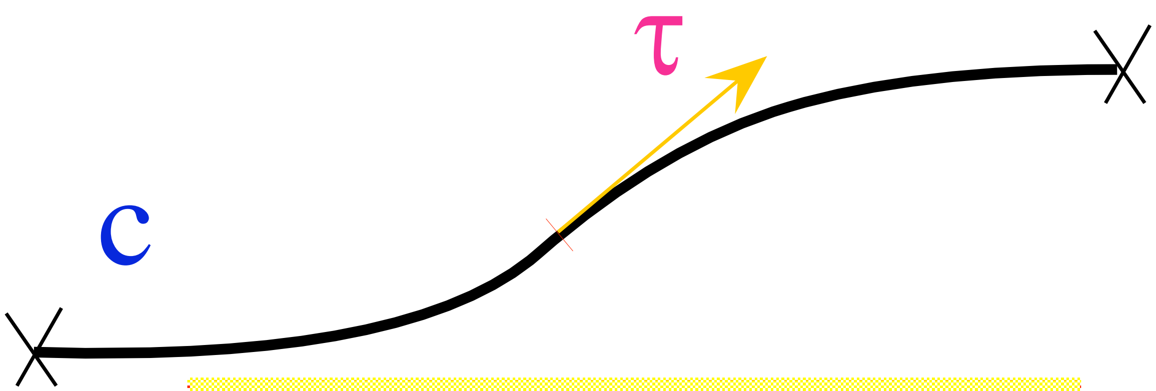
... $x \rightarrow e(x)$, denoted e.

$$V = \lim \sum_i \langle v_i \; ; \; e(x_i) \rangle \equiv \int_c e \equiv \langle c \; ; e \rangle$$

E
(the vector field)

*as a **proxy** for*

e
(the 1-form)



$$\int_c e = \int_c \tau \cdot E$$

Change " • ", change E (and τ), for same e

The observable is not E but e, the form

So what counts is the

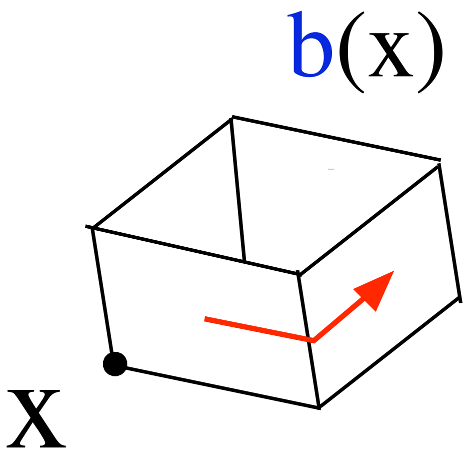
ORIENTED_LINE → *REAL*

map, denoted e here

(later called *cochain*)

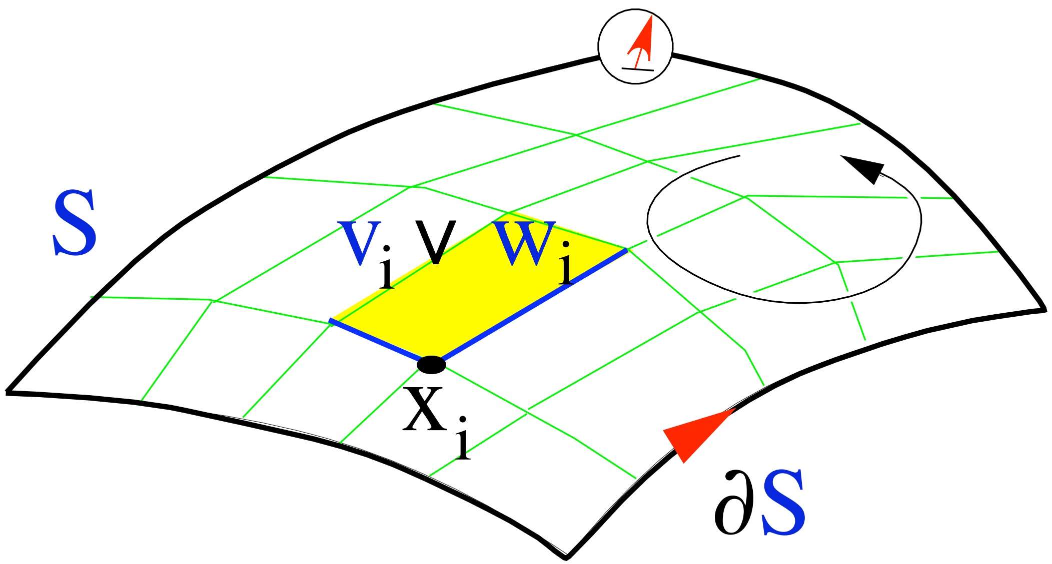
Same about magnetic induction b:

A field of 2-covectors



$$\langle S\,;\,b\rangle = \lim \sum_i \langle v_i \vee w_i\,;\,b(x_i)\rangle$$

$$\equiv \int_S b$$



Same about magnetic induction b:

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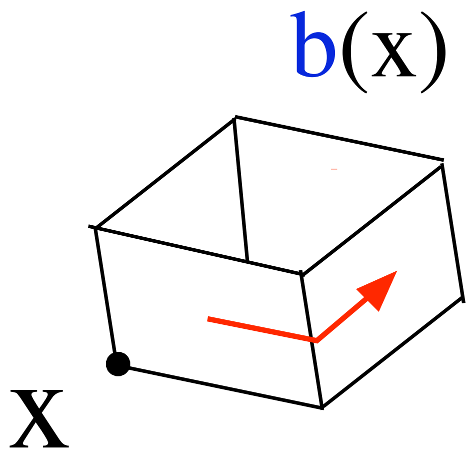
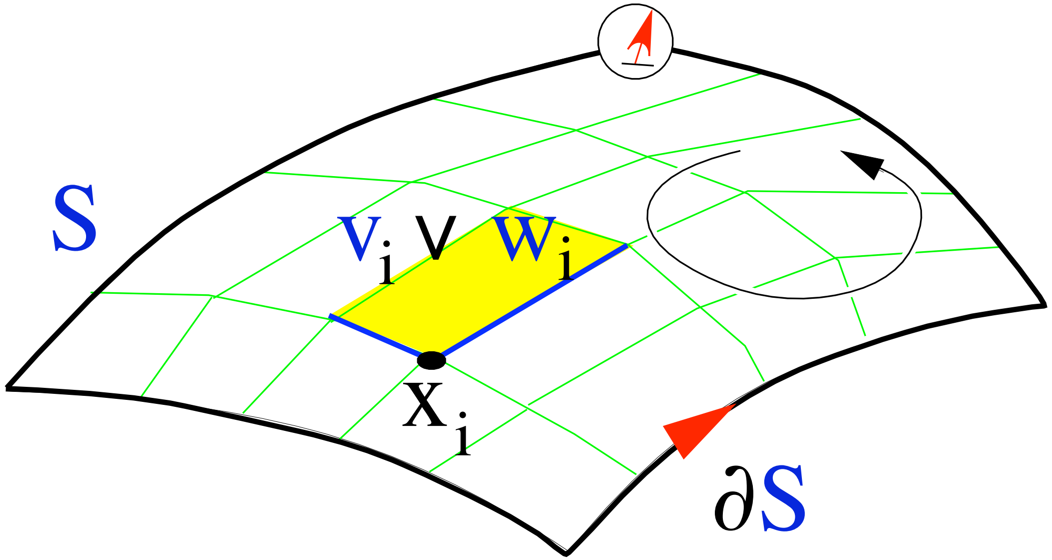


$$\langle S\,;\,b\rangle = \lim \Sigma_i \langle v_i \vee w_i\,;\,b(x_i)\rangle$$

$$\equiv \int_S b$$



Faraday:

$$\forall\, S,\quad \partial_t [\int_S b] + \int_{\partial S} e = 0$$

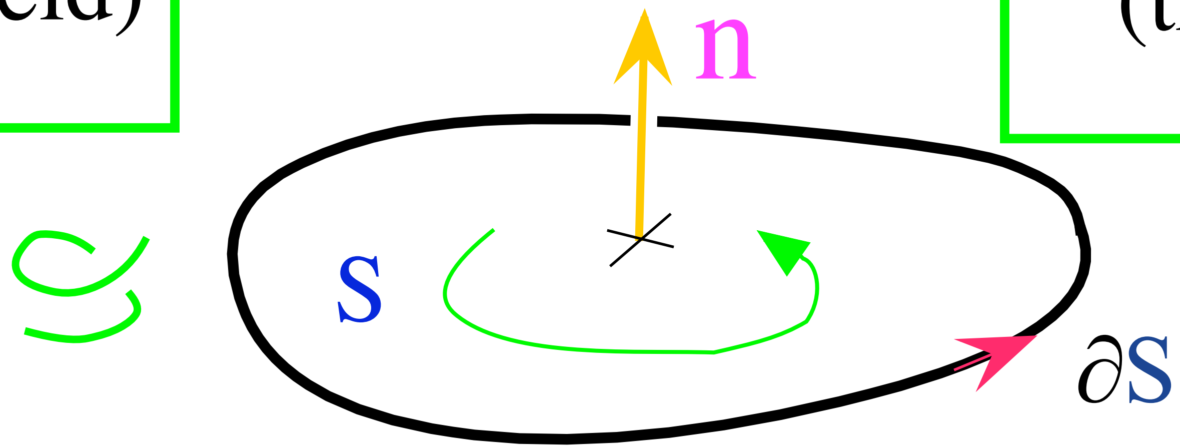
$$\partial_t \langle S\,;\,b\rangle + \langle \partial S\,;\,e\rangle = 0$$

i.e., if one defines d by

$$\langle S\,;\,\mathrm{d}e\rangle = \langle \partial S\,;\,e\rangle,$$

$$\partial_t b + \mathrm{d}e = 0$$

B (the vector field)	*as a proxy for*	b (the 2-form)



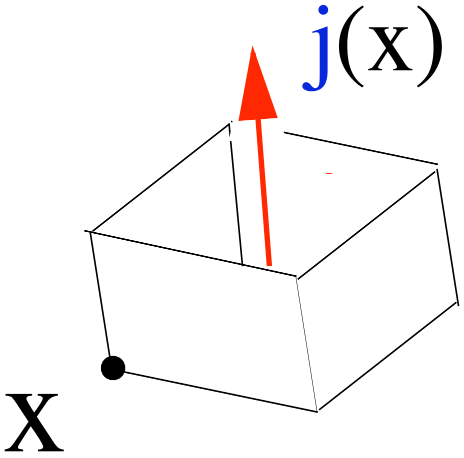
$$\int_S b = \int_S n \cdot B$$

Change "•", change B *(and* n*), for same* b

The observable is not B but b, the 2-form

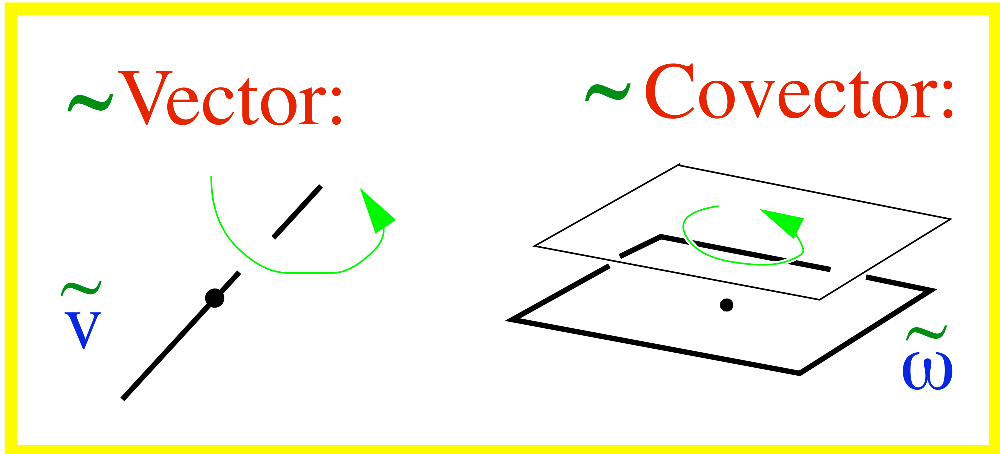
Change to , change B *to* –B, *for same* b

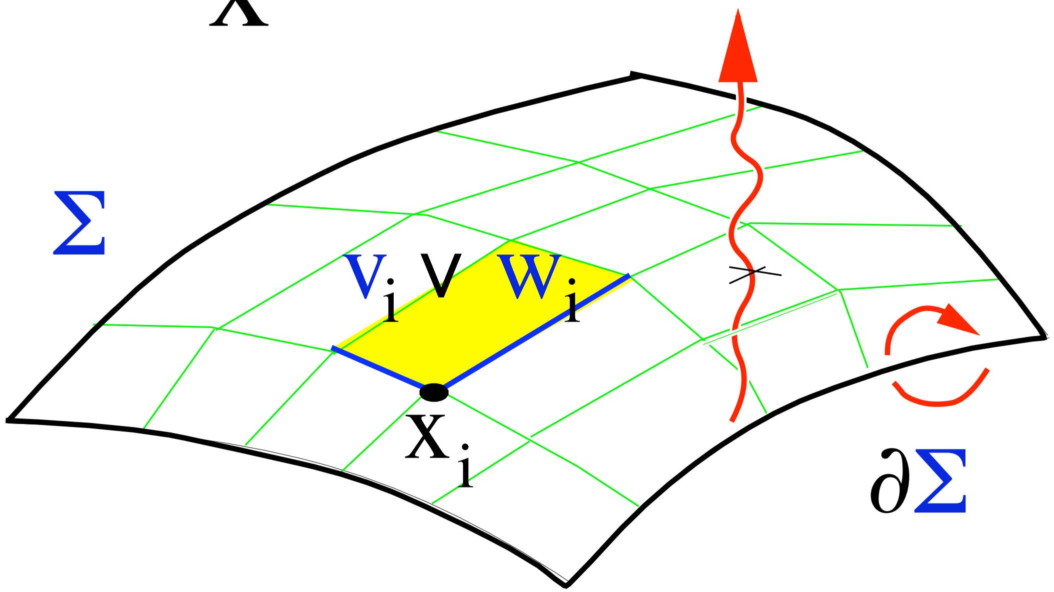
Slightly different for h and j:



j is a field of "twisted" 2-covectors

h a field of twisted covectors



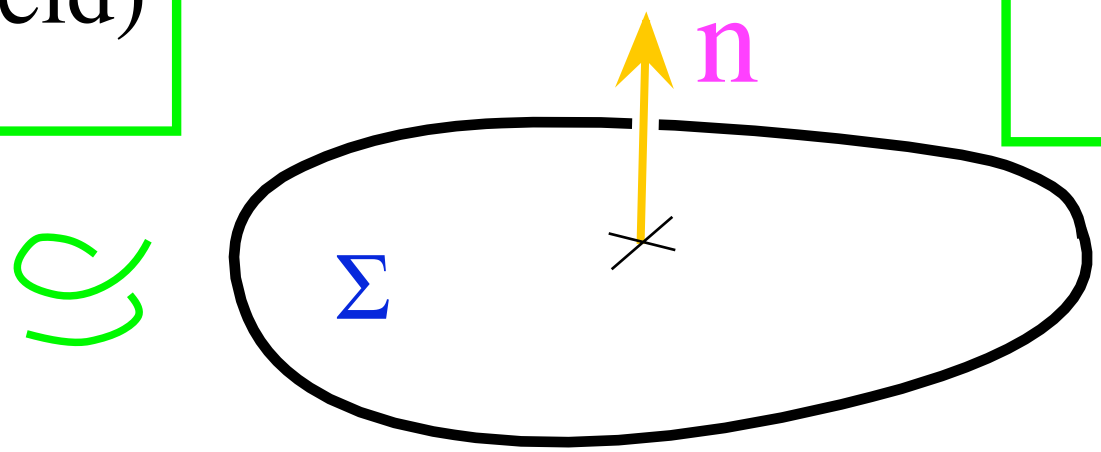


Ampère (in statics): $dh = j$

J
(the vector field)

as a proxy for

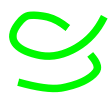
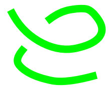
j
(the $\tilde{2}$-form)



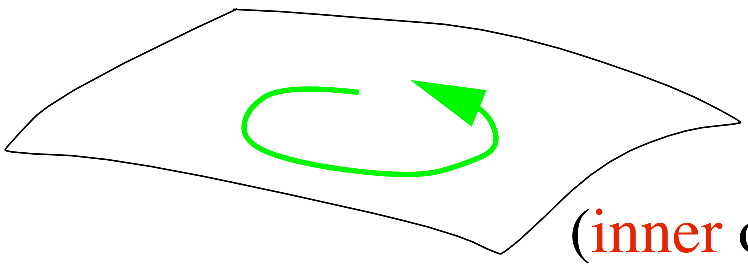
$$\int_S \mathbf{j} = \int_S \mathbf{n} \cdot \mathbf{J}$$

Change " • ", change **J** *(and* **n***), for same* **j**

The observable is not **J** but **j**, the $\tilde{2}$-form

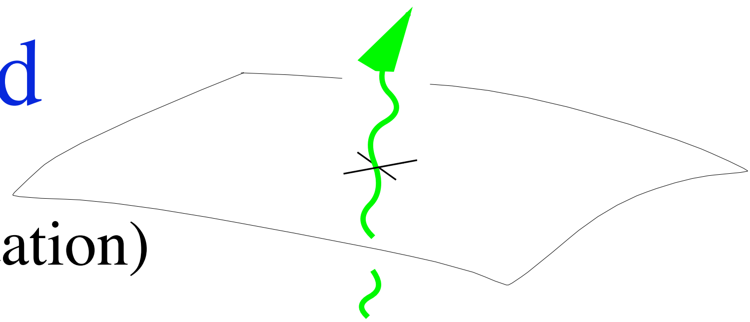
Ambient space orientation,  *or*  *, irrelevant*

- Fields of p-covectors are called p-forms (for "differential forms of degree p")
- Quite often, physical fields are usefully modelled by p-forms
- p-forms, meant to be integrated over p-submanifolds (of space, or spacetime)
- Two kinds of forms, depending on which kind of orientation is conferred to the manifold:



straight (inner orientation)

twisted (outer orientation)

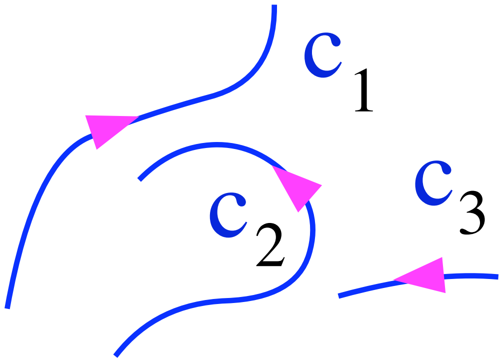


- Highly meaningful distinction in physics: straight [resp. twisted] forms represent intensive [resp. extensive] entities

The concept of chain:

1-chains:

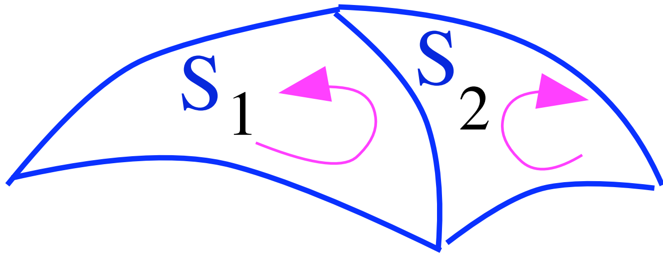
Embed set of curves in vector space of *singular* 1-*chains*



$$c = r^1 c_1 + r^2 c_2 + r^3 c_3$$

2-chains:

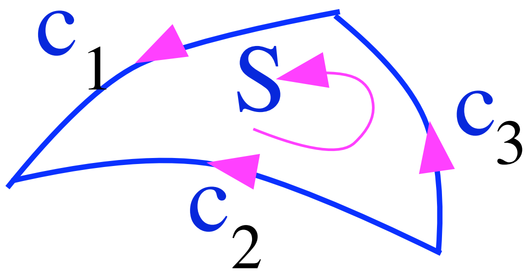
Same with surfaces etc.:



p-*chains*

e.g., $S = S_1 - S_2$

Boundary operator ∂:



$$\partial S = c_1 - c_2 + c_3$$

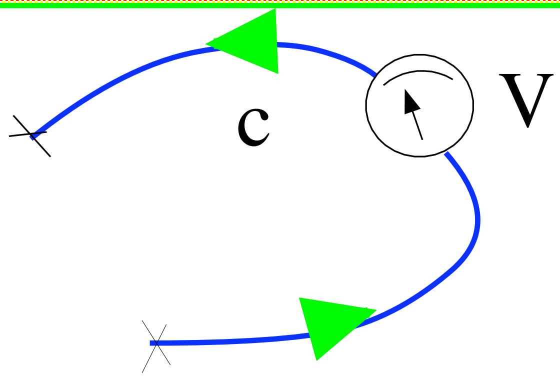
(Linear map: $\partial(S_1 - S_2) = \partial S_1 - \partial S_2$)

What about dual objects (linear functionals), called cochains?

Chains model probes. Cochains model fields.

Voltmeter:

(p = 1)



e.m.f. $V = \int_c e$

Electric field seen as map

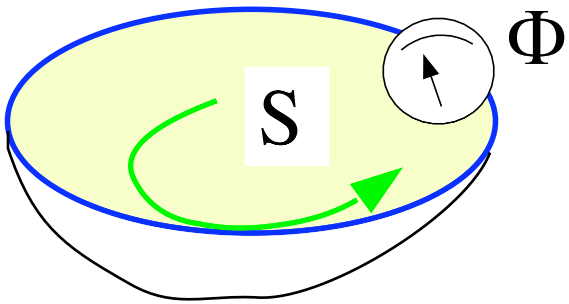
c → <emf along c>,

map here denoted e,

a 1-cochain.

Fluxmeter:

(p = 2)



Magnetic induction as map b,

the 2-cochain

S → <flux embraced by S>.

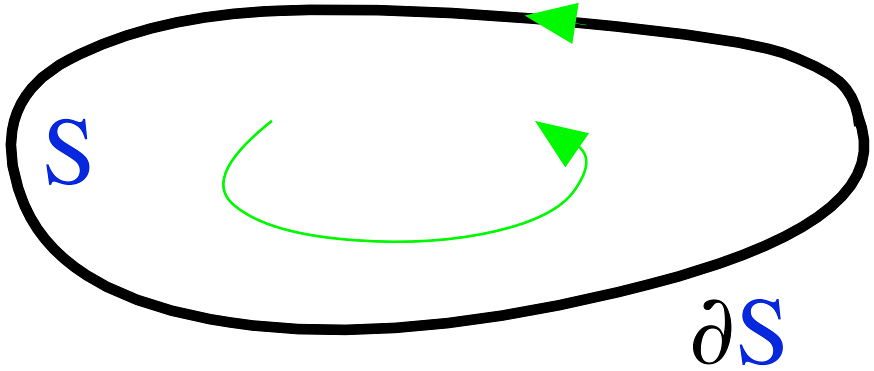
Small probe <—> p-vector

Local field <—> p-covector

Maxwell's Theory

Faraday's law, in terms of cochains:

$\int_{\partial S} e$ — *volts*



$\int_S b$ — *webers*

for all 2-chains S,

$$\frac{d}{dt}\int_S b + \int_{\partial S} e = 0$$

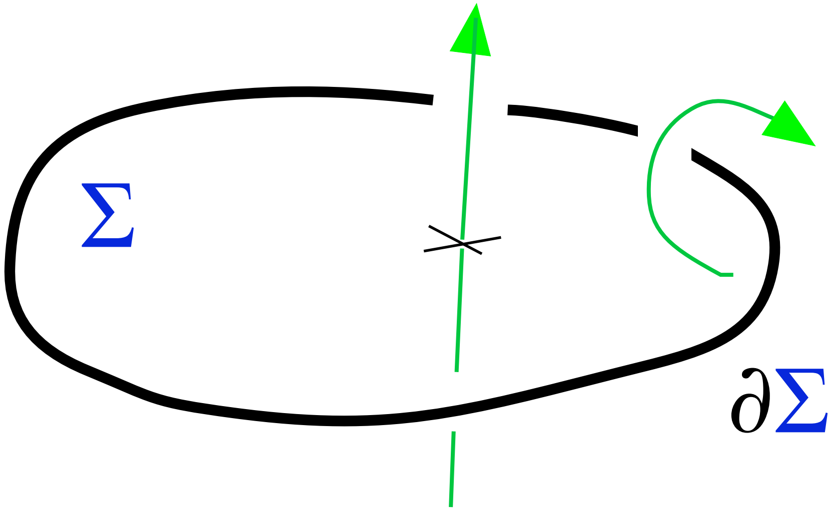
2-cochain (b)   1-cochain (e)

or $\partial_t b + de = 0$, with d defined by $\int_S de = \int_{\partial S} e$

Ampère-Maxwell's law, in terms of cochains:

$$\int_{\partial\Sigma} h$$

ampères



$$\int_{\Sigma} d$$

coulombs

for all $\tilde{2}$-chains Σ,

$$-\frac{d}{dt}\int_{\Sigma} d + \int_{\partial\Sigma} h = \int_{\Sigma} j$$

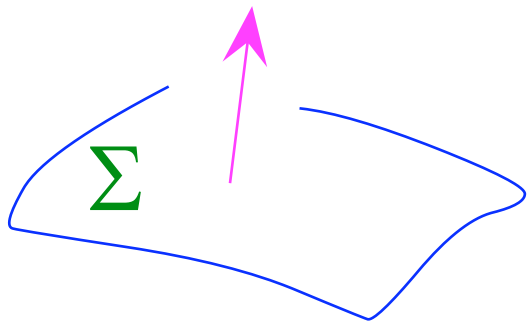
$\tilde{2}$-cochain

$\tilde{1}$-cochain

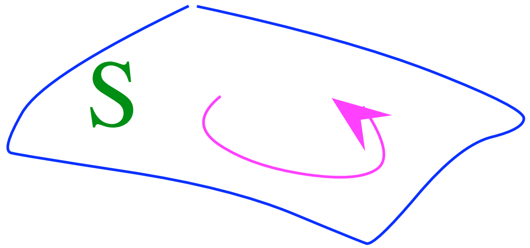
given $\tilde{2}$-cochain

or $-\partial_t d + dh = j$

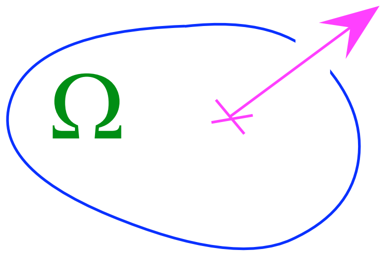
$$-\partial_t \int_\Sigma d + \int_{\partial\Sigma} h = \int_\Sigma j \quad \forall$$



$$\partial_t \int_S b + \int_{\partial S} e = 0 \quad \forall$$

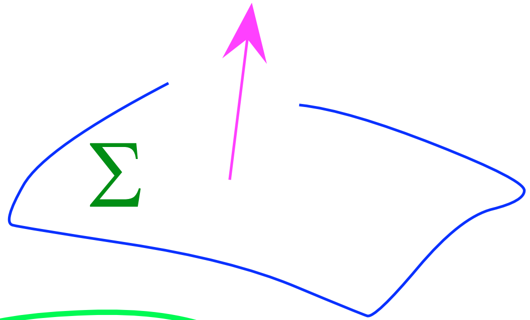


$$\int_\Omega q \mathrel{\widehat{=}} \int_{\partial\Omega} d$$



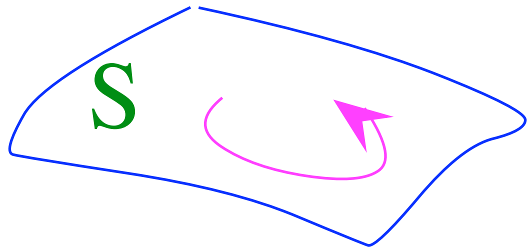
$$\partial_t \int_\Omega q + \int_{\partial\Omega} j = 0$$

$$-\partial_t \int_\Sigma d + \int_{\partial\Sigma} h = \int_\Sigma j \quad \forall$$



$$b = \mu h \qquad ? \qquad d = \varepsilon e$$

$$\partial_t \int_S b + \int_{\partial S} e = 0 \quad \forall$$



$$(-\partial_t D + \mathrm{rot}\, H = J, \quad \partial_t B + \mathrm{rot}\, E = 0)$$

The real nature of μ ("Hodge operator"):

b : a map of type *SURFACE* → *REAL*

("2-cochain")

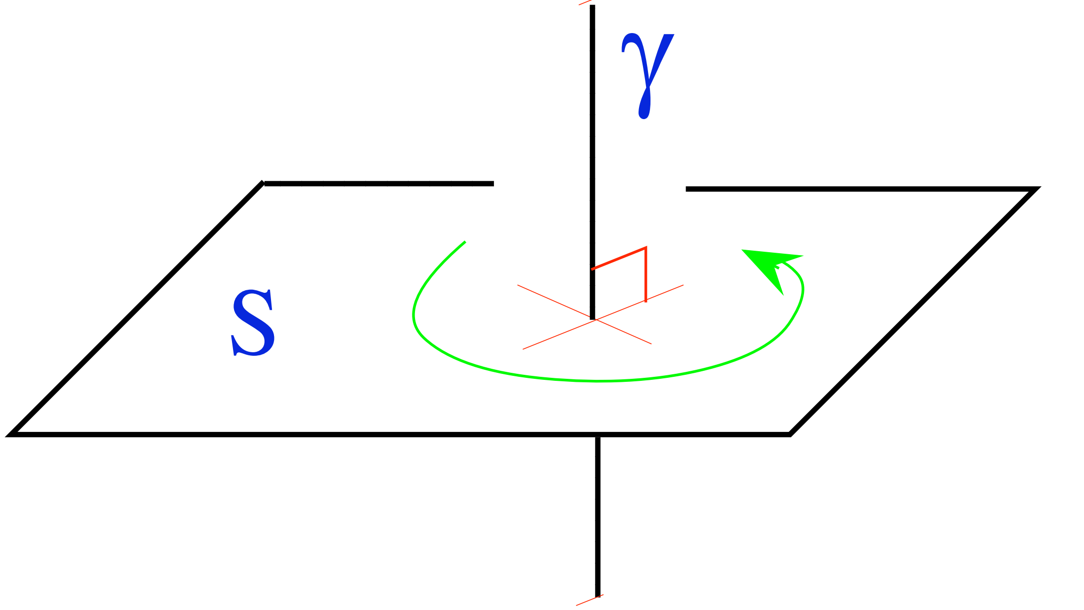
h : a map of type *LINE* → *REAL*

("$\widetilde{1}$-cochain")

$$b = \mu\, h$$

$\vec{S}$ = vectorial area of S

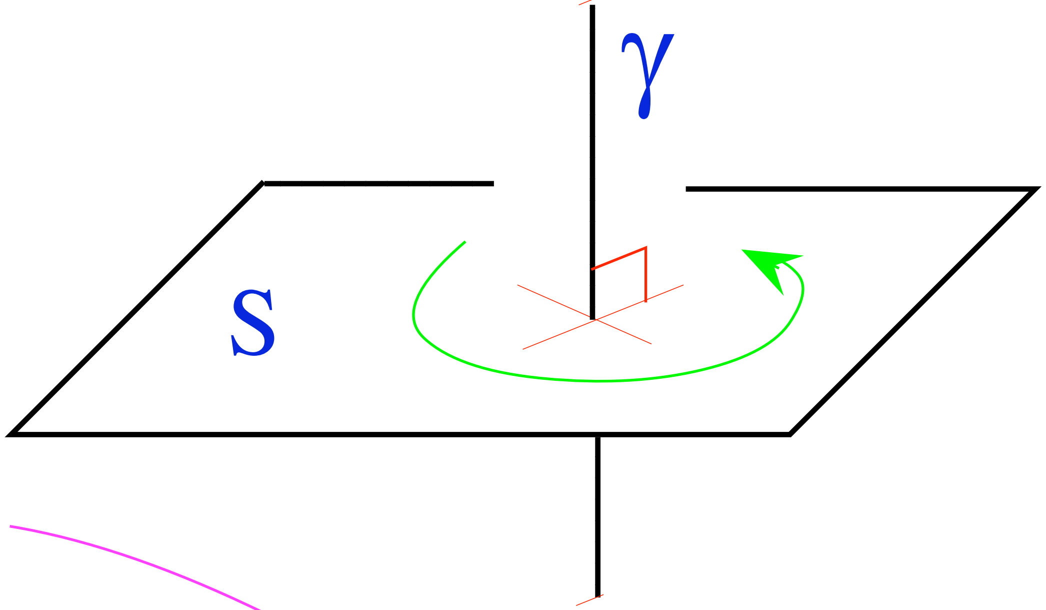
$\vec{\gamma}$ = vector along γ



$$\frac{\vec{S}}{\mathrm{area}(S)} = \frac{\vec{\gamma}}{\mathrm{length}(\gamma)}$$

$\vec{S}$ = vectorial area of S

$\vec{\gamma}$ = vector along γ



=

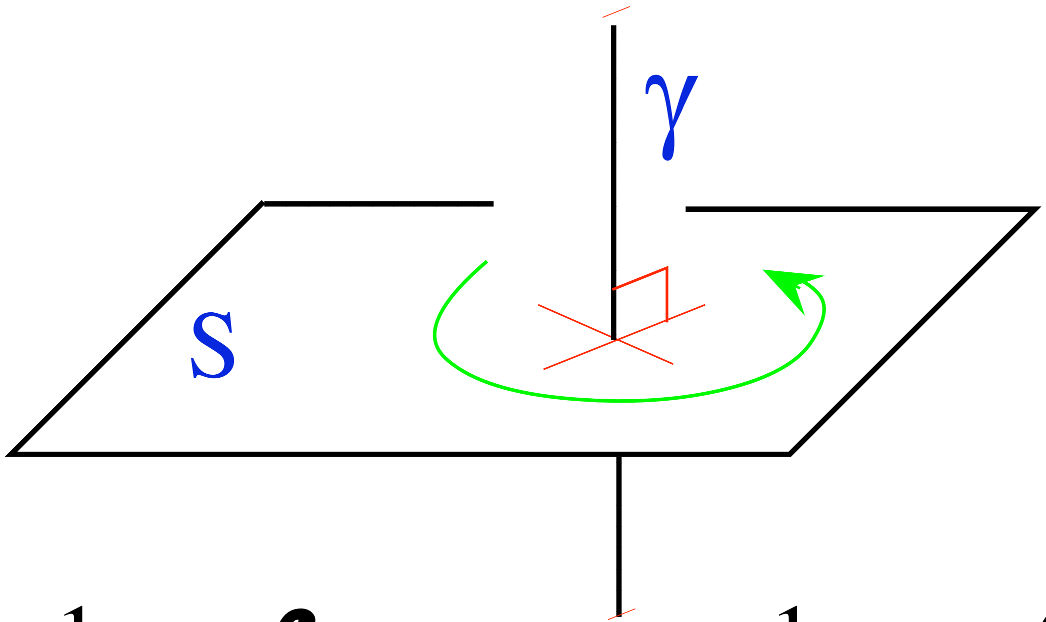
$$B \cdot \frac{\vec{S}}{\text{area}(S)} = \frac{\vec{\gamma}}{\text{length}(\gamma)} \cdot \mu H$$

$$\frac{1}{\text{area}(S)} \int_S b = \mu \frac{1}{\text{lgth}(\gamma)} \int_\gamma h$$

which defines 2-form b knowing

scalar factor μ and 1-form h

The Hodge operator:



$$\frac{1}{\text{area}(S)}\int_S b = \mu \frac{1}{\text{length}(\gamma)}\int_\gamma h$$

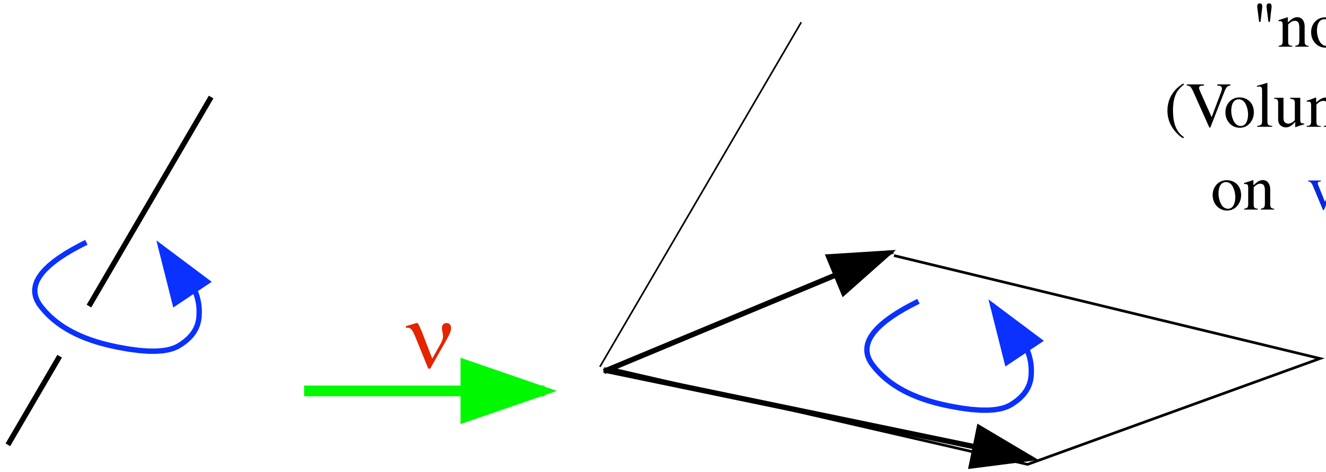
$$b = \mu\ h \quad \Leftrightarrow \quad h = \nu\ b$$

Further structuration of space: the Hodge map

VECTOR ⟶ (n – 1)-*VECTOR*

twisted or straight straight or twisted

Equip space with such a map, ν. (Another one, denoted ε, will be needed.)



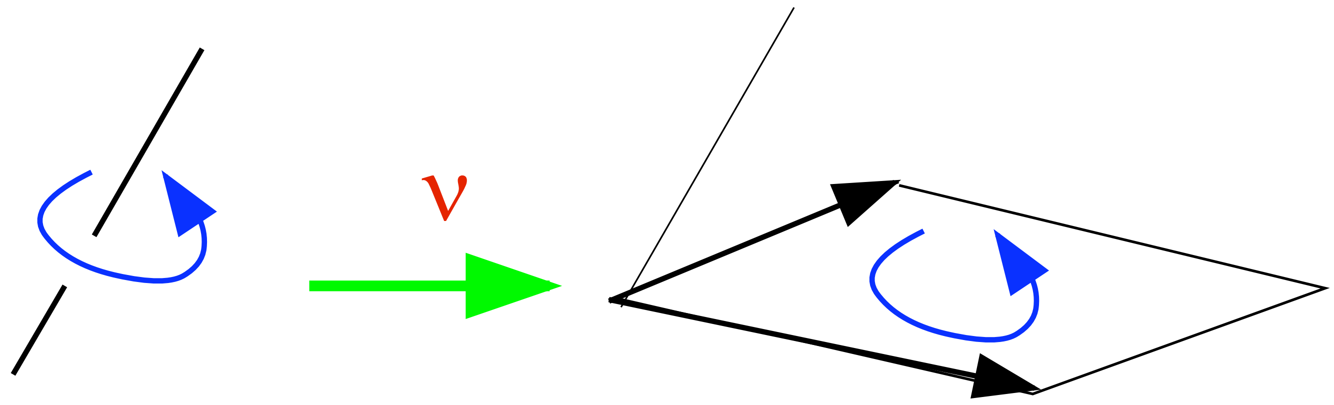
Only requirement, "non-degeneracy". (Volume $v \vee \nu v$, built on v and its image, must be $\neq 0$.)

Determines a metric ("ν-adapted")

(Select reference 3-vector Δ and real λ. Set $\lambda^2 v \vee \nu v = |v|_\lambda^2 \Delta$, hence a norm, scaling as λ. Adjust λ for λ-volume of Δ to be λ^2.)

By duality, yields Hodge map on *co*vectors:

$$\sim 1\text{-}\mathit{VECTOR} \xrightarrow{\nu} 2\text{-}\mathit{VECTOR}$$



$$\underset{\text{1-vector}}{\langle v} \;;\; \overset{\text{1-covector}}{\nu b \rangle} = \underset{\text{2-vector}}{\langle \nu v} \;;\; \overset{\text{2-covector}}{b \rangle}$$

$$\sim 1\text{-}\mathrm{CO}\mathit{VECTOR} \xleftarrow{\nu} 2\text{-}\mathrm{CO}\mathit{VECTOR}$$

Hence relation $h = \nu b$ (and also $d = \varepsilon e$) between cochains, i.e., *fields*

So space geo-*metry* (in the strong sense of assigning *metric* properties—distances, areas, angles, etc.—to the space we inhabit) amounts to specifying *constitutive laws* in electrodynamics.

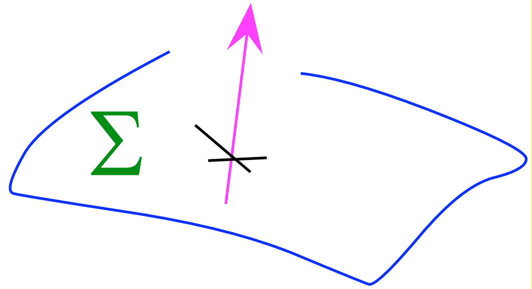
- Should not sound strange: Don't we use *light rays* to measure the Earth?

- Why *two* metrics ($\nu \equiv \mu^{-1}$ and ε)? Because 3D shadows of Minkowski's 4D (pseudo-)metric

- $\varepsilon \neq \varepsilon_0$ and $\mu \neq \mu_0$ when we wish to *ignore* details of microscopic interactions and *geometrize* them wholesale

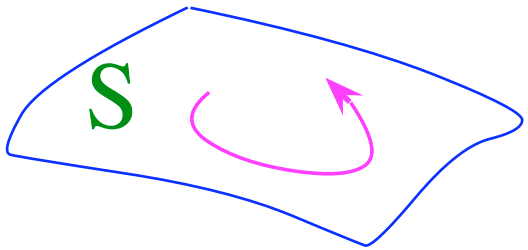
Maxwell, in terms of cochains:

$$-\partial_t \int_\Sigma d + \int_{\partial\Sigma} h = \int_\Sigma j \quad \forall$$



$$d = \varepsilon\, e \qquad\qquad h = \nu\, b$$

$$\partial_t \int_S b + \int_{\partial S} e = 0 \quad \forall$$



$$-\partial_t d + dh = j$$

$$d = \varepsilon\, e$$

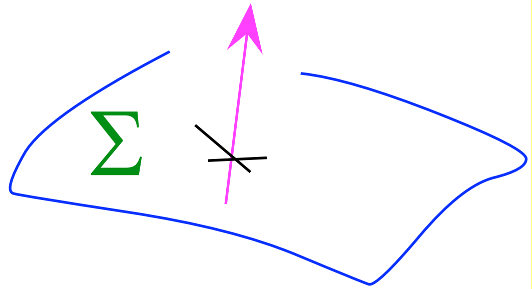
$$h = \nu\, b$$

$$\partial_t b + de = 0$$

	straight	twisted
1	e	h
2	b	d, j

Maxwell, in terms of cochains:

$$-\partial_t \int_\Sigma d + \int_{\partial\Sigma} h = \int_\Sigma j \quad \forall$$



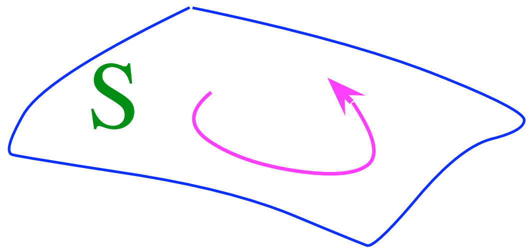
$$-\partial_t d + dh = j$$

$$d = \varepsilon e \qquad\qquad h = \nu b$$

$$d = \varepsilon e$$

$$h = \nu b$$

$$\partial_t \int_S b + \int_{\partial S} e = 0 \quad \forall$$

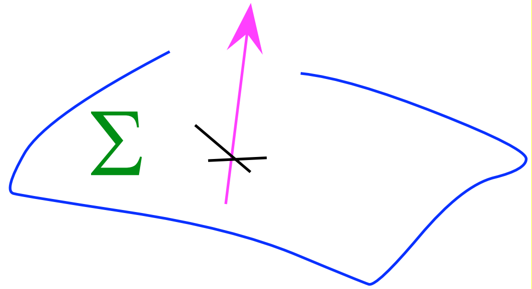


$$\partial_t b + de = 0$$

Discretization strategy: Only enforce these laws for finite system of surfaces S or Σ: those made of faces of a mesh. DoF's are then face-integrals of b, d, and relate to edge-integrals of e, h.

Maxwell, in terms of cochains:

$$-\partial_t \int_\Sigma d + \int_{\partial\Sigma} h = \int_\Sigma j \quad \forall$$



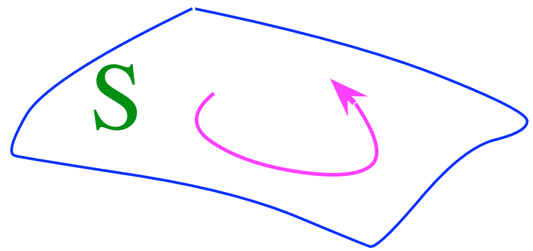
$$-\partial_t d + dh = j$$

$$d = \varepsilon e \qquad\qquad h = \nu b$$

$$d = \varepsilon e$$

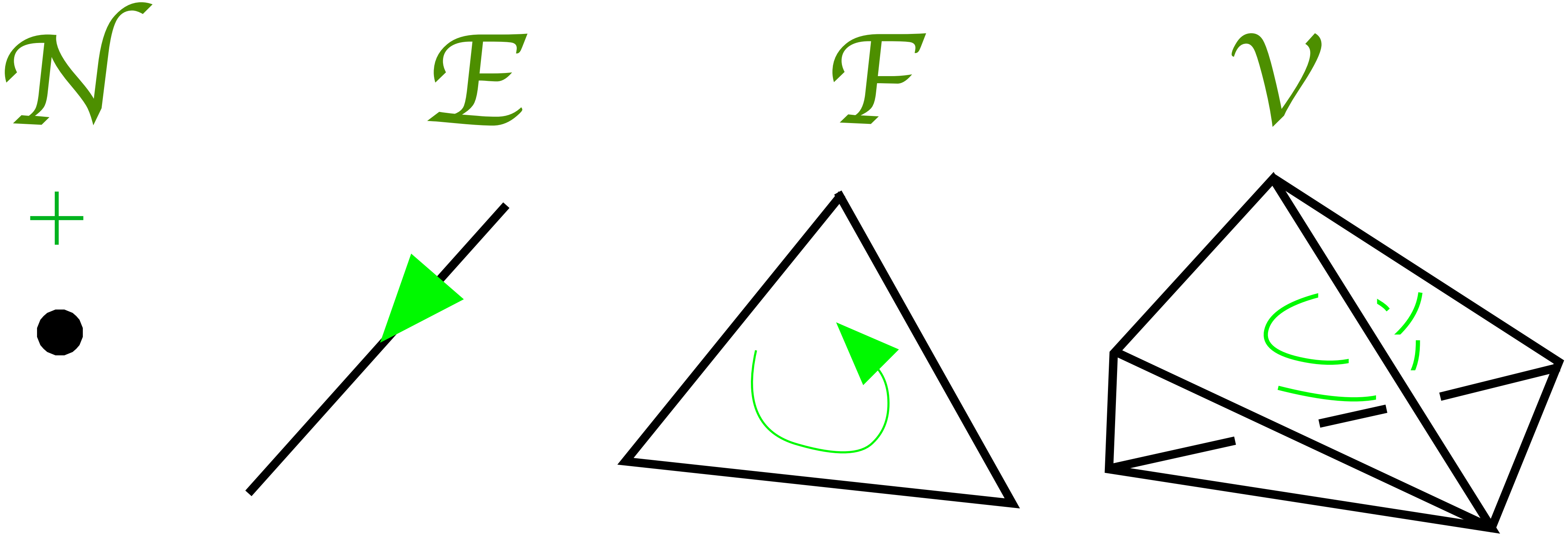
$$h = \nu b$$

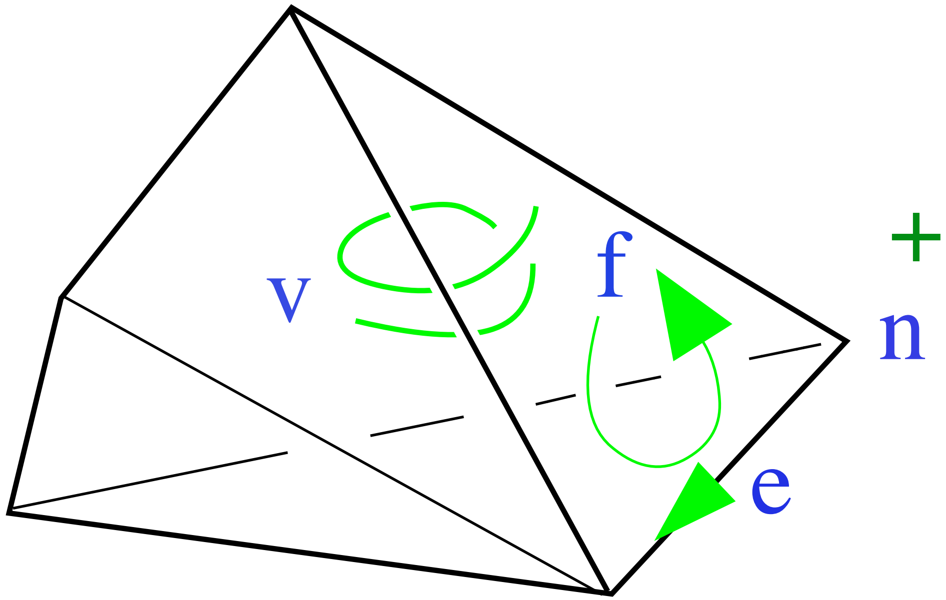
$$\partial_t \int_S b + \int_{\partial S} e = 0 \quad \forall$$



$$\partial_t b + de = 0$$

Problem: Should be same number of DoF's for b and h (resp. for d and e) for discrete versions **ε** and **ν** (matrices) of hodges ε and ν to be square (since they must be invertible).





$$G_{en} = -1$$

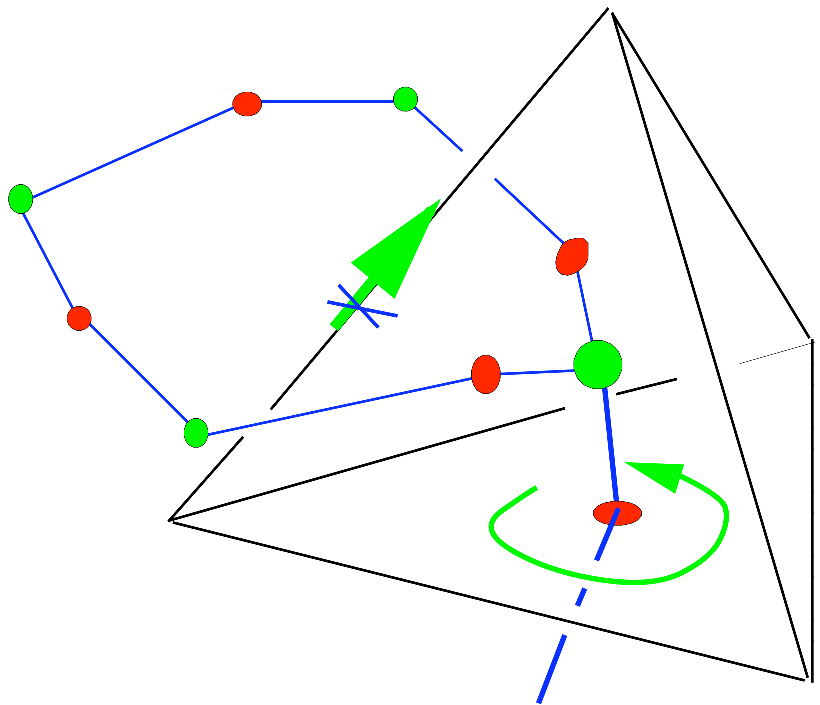
$$R_{fe} = -1$$

$$D_{vf} = 1$$

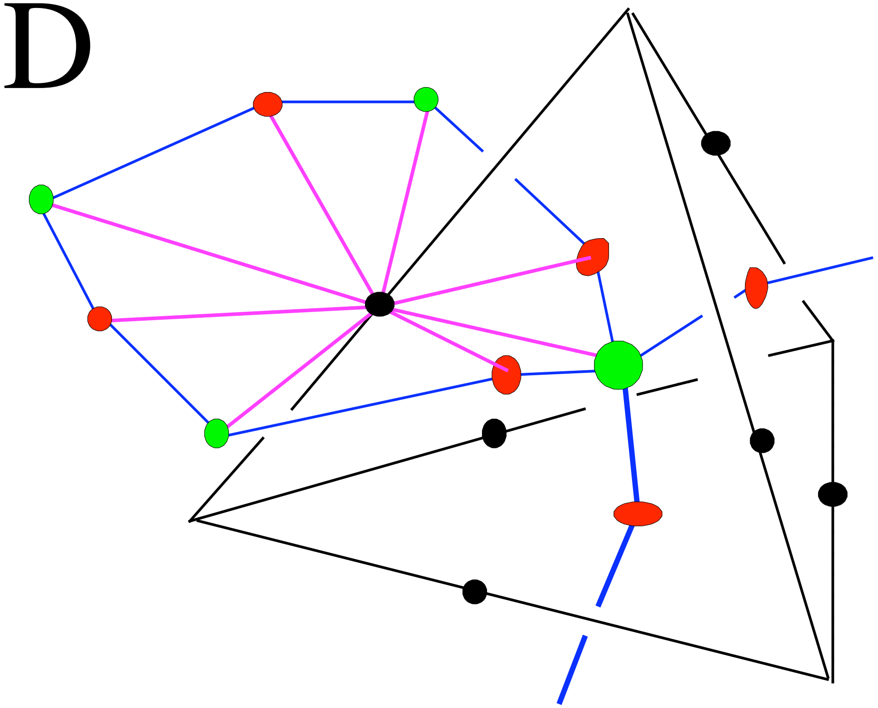
$$DR = 0, \quad RG = 0$$

Select centers inside primal simplexes. Join them to make dual.

Orient all primal cells, independently. Take induced orientation on dual cells:



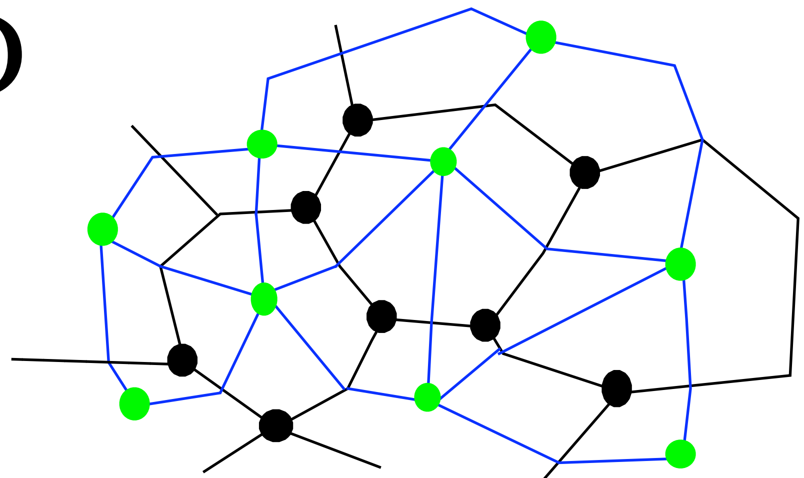
3D



●, —— : primal cells

●, —— : dual cells

2D

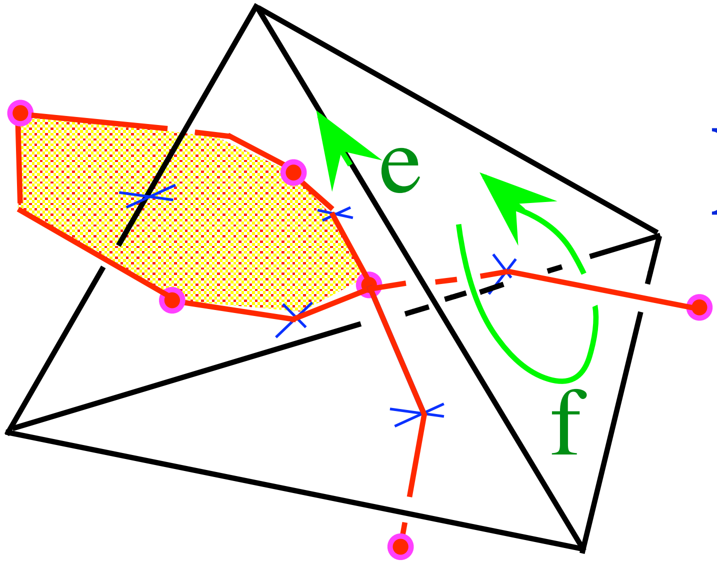


$$\mathcal{N} \xrightarrow[\text{grad}]{\mathbf{G}} \mathcal{E} \xrightarrow[\text{rot}]{\mathbf{R}} \mathcal{F} \xrightarrow[\text{div}]{\mathbf{D}} \mathcal{V}$$

Approximate representation of the field by degrees of freedom assigned to both kinds of cells

b *at faces*

e, a *at edges*



here, $\mathbf{R}_{fe} = -1$

h *at dual edges (i.e., faces)*

d, j *at dual faces*

fluxes

$\mathbf{b} = \{b_f : f \in \mathcal{F}\}$ ——— $\boldsymbol{\nu}$ ———▶ $\mathbf{h} = \{h_f : f \in \mathcal{F}\}$

m.m.f.'s

e.m.f.'s

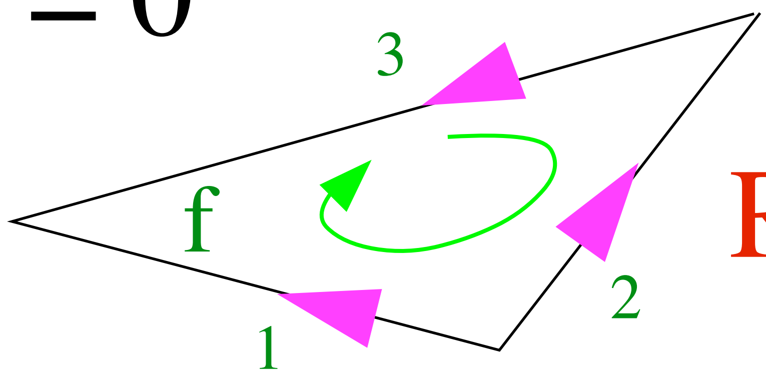
$\mathbf{e} = \{e_e : e \in \mathcal{E}\}$ ——— $\boldsymbol{\varepsilon}$ ———▶ $\mathbf{d} = \{d_e : e \in \mathcal{E}\}$

(cumulated) intensities

Enforce Faraday's law, $\partial_t \int_S b + \int_{\partial S} e = 0$

not for all surfaces S, but for all those made of primal faces. This requires (when $S = f$, a primal face),

$$\partial_t b_f + e_1 - e_2 - e_3 = 0$$



$R_{f2} = -1$

i.e.,

$$\partial_t \mathbf{b} + \mathbf{R}\mathbf{e} = 0$$

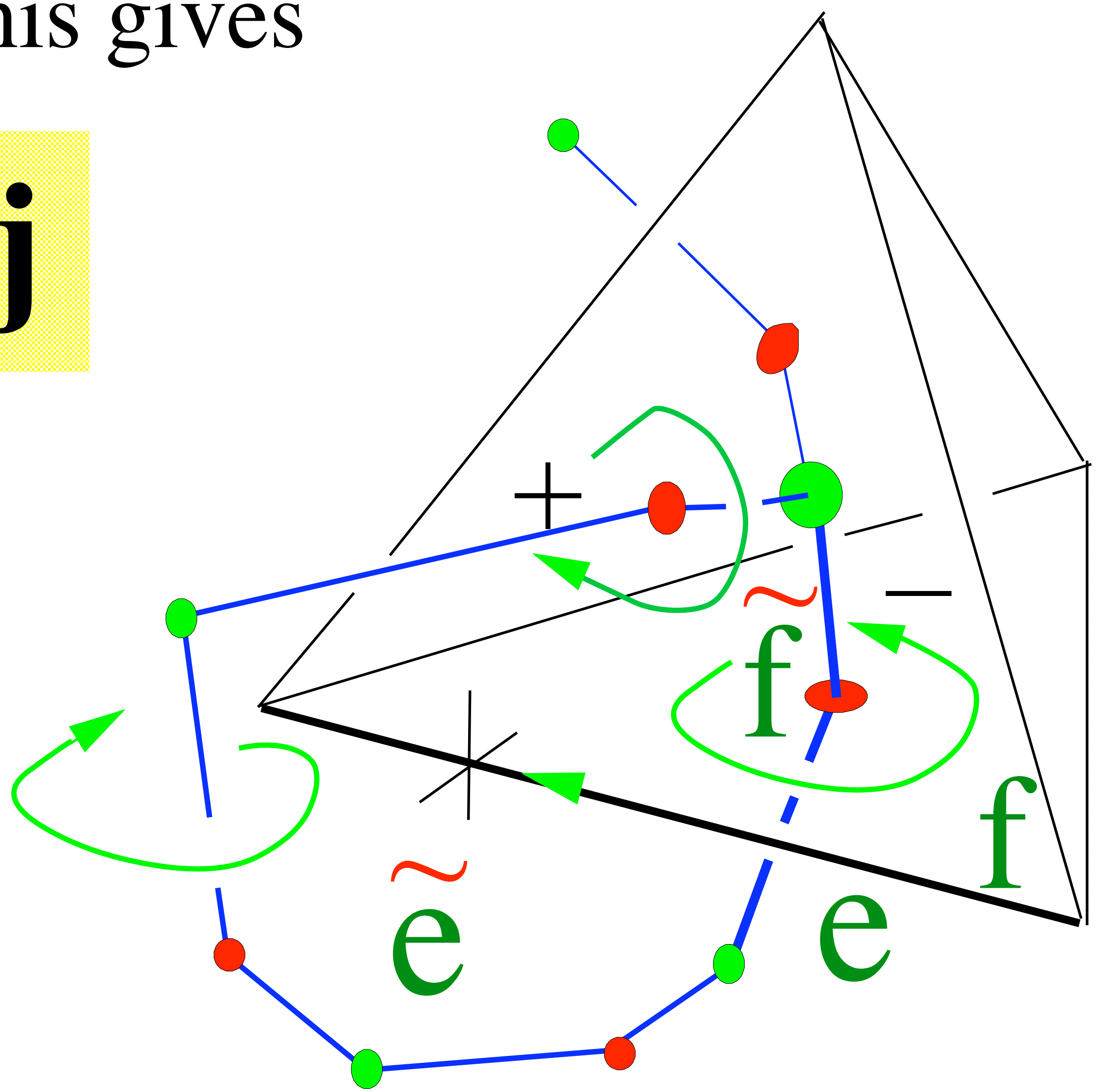
Enforce Ampère's law, $-\partial_t \int_\Sigma d + \int_{\partial\Sigma} h = 0$

not for all surfaces Σ, but for all those made of dual faces such as $\tilde{e}$ here. This gives

$$-\partial_t \mathbf{d} + R^t \mathbf{h} = \mathbf{j}$$

because

$$R_{\tilde{e}\tilde{f}} = R_{fe}$$



The final product:

$$\partial_t \mathbf{b} + \mathbf{R}\mathbf{e} = 0 \qquad -\partial_t \mathbf{d} + \mathbf{R}^t \mathbf{h} = \mathbf{j}$$

$$\mathbf{h} = \boldsymbol{\nu}\, \mathbf{b} \qquad \mathbf{d} = \boldsymbol{\varepsilon}\, \mathbf{e}$$

Leap-frog time discretization gives

$$\frac{\mathbf{b}^{k+1/2} - \mathbf{b}^{k-1/2}}{\delta t} + \mathbf{R}\mathbf{e}^{k} = 0$$

$$-\ \boldsymbol{\varepsilon}\ \frac{\mathrm{e}^{k+1} - \mathrm{e}^{k}}{\delta t} + \mathrm{R}^{t}\, \boldsymbol{\nu}\, \mathrm{b}^{k+1/2} = \mathrm{j}^{k+1/2}$$

"Yee scheme" (1966), aka FDTD

D of this:

$$\partial_t \mathbf{b} + \mathbf{R}\mathbf{e} = 0$$

$$\mathbf{h} = \boldsymbol{\nu}\, \mathbf{b}$$

$\mathbf{G}^t$ of that:

$$-\partial_t \mathbf{d} + \mathbf{R}^t \mathbf{h} = \mathbf{j}$$

$$\mathbf{d} = \boldsymbol{\varepsilon}\, \mathbf{e}$$

Recall that $\partial_t \mathbf{q} - \mathbf{G}^t \mathbf{j} = 0$,

(because $\partial_t q + \operatorname{div} j = 0$, and $-\mathbf{G}^t \sim \operatorname{div}$)

hence $-\mathbf{G}^t \mathbf{d} = \mathbf{q}$

$$\partial_t \mathbf{b} + R\mathbf{e} = 0 \qquad -\partial_t \mathbf{d} + R^t \mathbf{h} = \mathbf{j}$$

$$\mathbf{h} = \nu\, \mathbf{b} \qquad \mathbf{d} = \varepsilon\, \mathbf{e}$$

Use $DR = 0$ and $G^t R^t = 0$ to get

"electric" network		"magnetic" network
$-G^t \mathbf{d} = \mathbf{q}$	← Kirchhoff's node law →	$D\mathbf{b} = 0$
$\mathbf{d} = \varepsilon \mathbf{e}$	$(\partial_t \mathbf{q} - G^t \mathbf{j} = 0)$	$\mathbf{h} = \nu \mathbf{b}$
$R\mathbf{e} = -\partial_t \mathbf{b}$	← Kirchhoff's loop law →	$R^t \mathbf{h} = \mathbf{j} + \partial_t \mathbf{d}$

Two interlocked, cross-talking, networks

If ε *and* ν *diagonal,* ε^{ee} *and* ν^{ff} *can be seen as branch impedances*

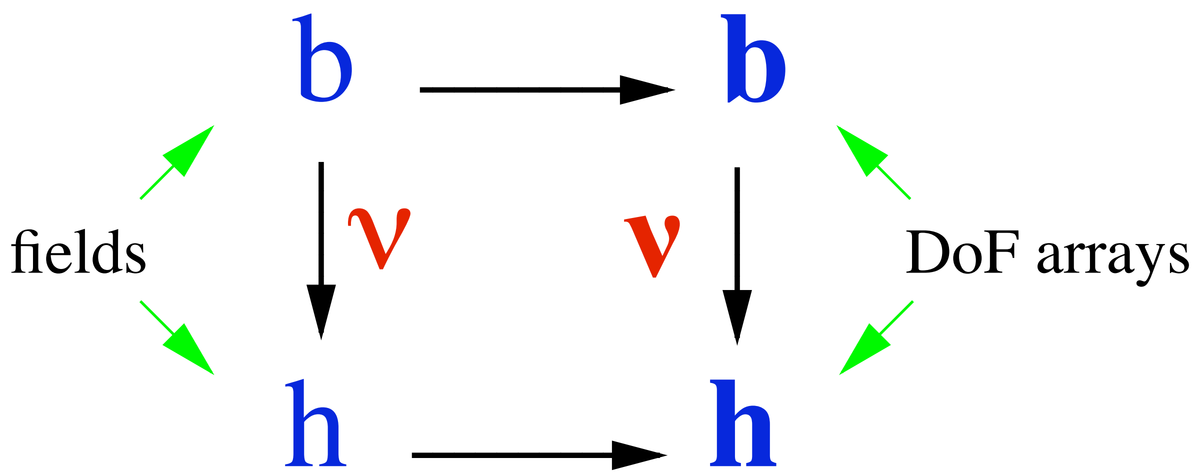
Discrete ("mimetic") structures

Space (comput. domain) → Cell complex

submanifolds (such as S, Σ) → cellular chains

fields (such as b, h, e, d) → cellular cochains

Hodge map(s) → Hodge matrix(es)

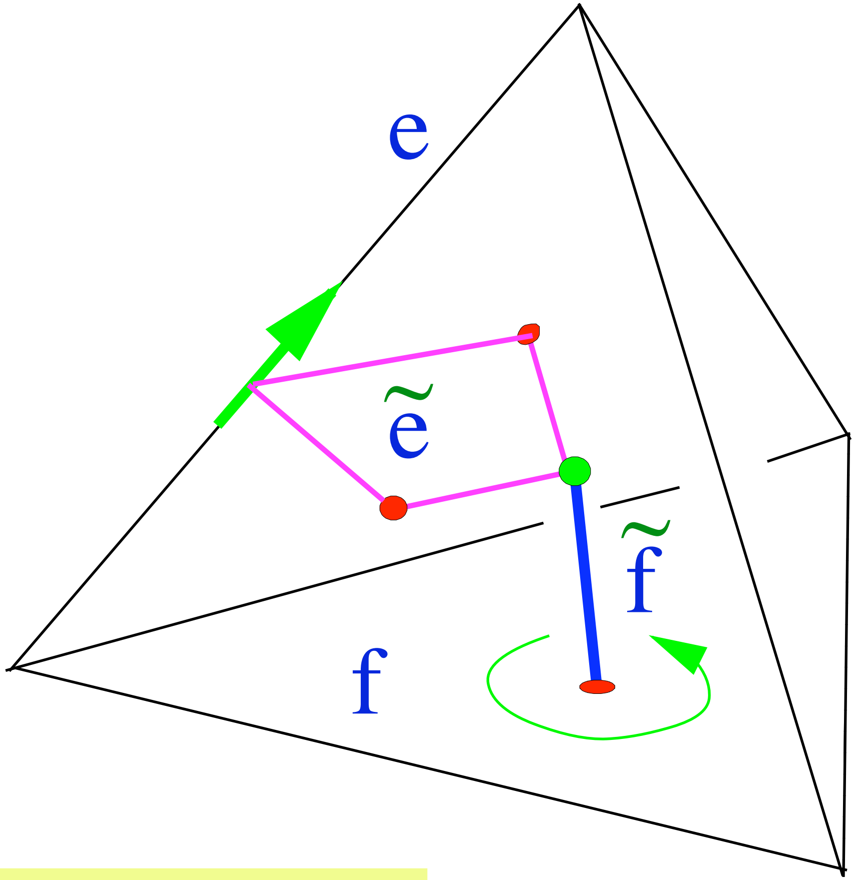


consistency required there, for convergence of numerical schemes

Discrete Hodge map:

$$\tilde{f} \rightarrow \sum_{f' \in \mathcal{F}} \boldsymbol{\nu}^{ff'} f'$$

$\mathcal{F}$: set of mesh faces



Map extends to dual chains (by linearity) and passes (by duality) to cochains

Consistency:

$$\nu[\tilde{1}\text{-vec}(\tilde{f})] = \sum_{f'} \boldsymbol{\nu}^{ff'} [2\text{-vec}(f')]$$

Also needed (for electrostatics and full Maxwell):

$$\tilde{e} \rightarrow \sum_{e' \in \mathcal{E}} \boldsymbol{\varepsilon}^{ee'} e'$$

$\mathcal{E}$: set of mesh edges

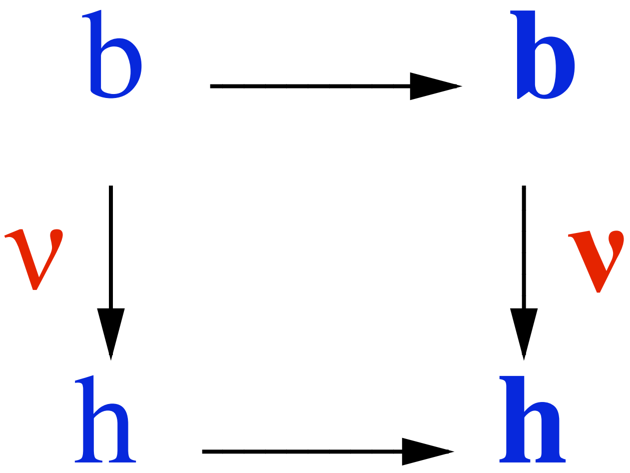
Consistency condition: $\nu[\tilde{1}\text{-vec}(\tilde{f})] = \sum_{f'} \boldsymbol{\nu}^{ff'}[2\text{-vec}(f')]$

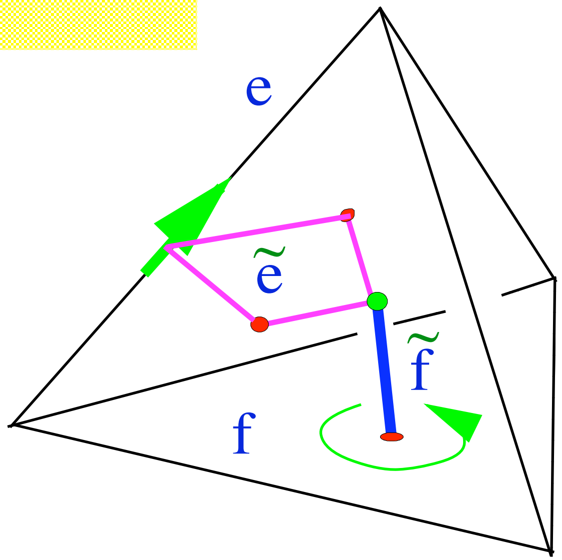
makes commutative

abridged as

$$\nu\vec{\tilde{f}} = \sum_{f'} \boldsymbol{\nu}^{ff'} \vec{f'}$$

the diagram





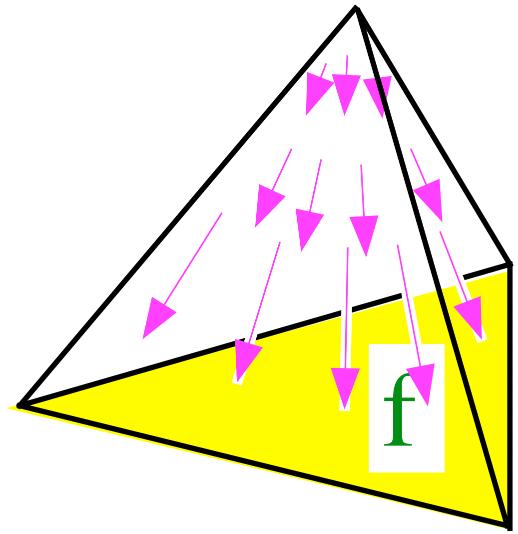
when b and h are piecewise uniform:

$$\mathbf{h}_f = \langle \tilde{f} ; \nu b \rangle \overset{!}{=} \langle \vec{\tilde{f}} ; \nu b \rangle = \langle \nu \vec{\tilde{f}} ; b \rangle = \sum_{f'} \boldsymbol{\nu}^{ff'} \langle \vec{f'} ; b \rangle \overset{!}{=} \sum_{f'} \boldsymbol{\nu}^{ff'} \mathbf{b}_{f'}$$

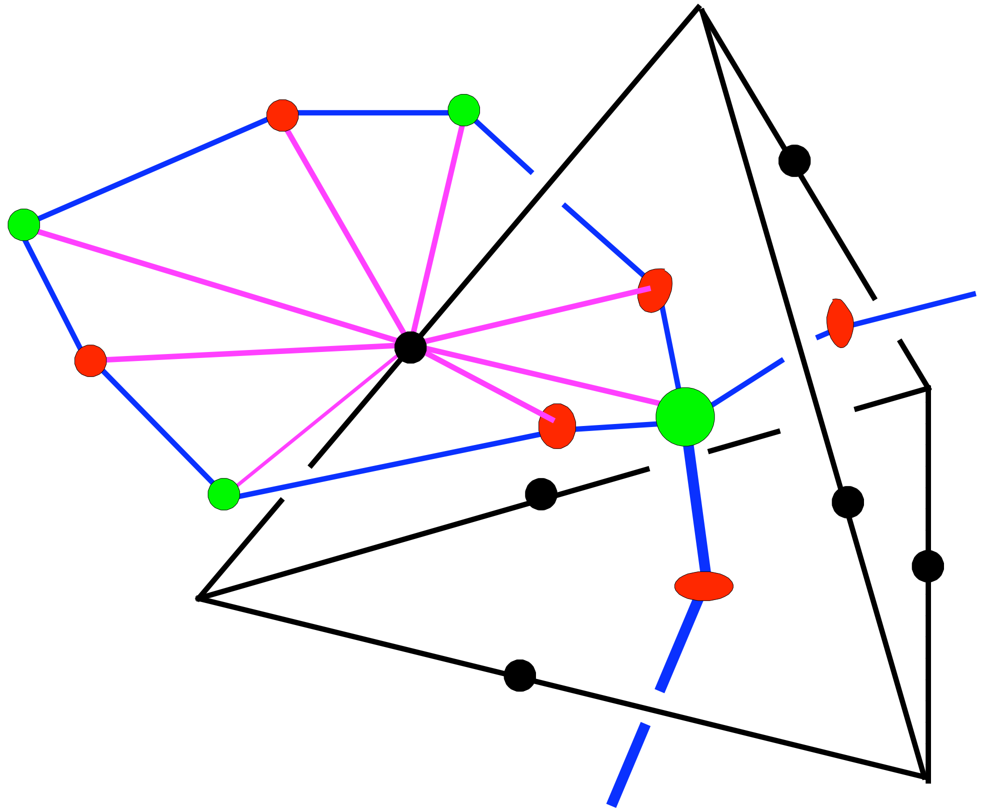
If dual mesh barycentric, criterion met by the "Galerkin Hodge", defined as

$$\nu^{ff'} = \int \nu \, w^{f} \wedge w^{f'}$$

where w^{f} is Whitney form of facet f



Prop. 1: Select centers inside primal simplexes. Join them to make dual. Then unique **ν** conforming to criterion.



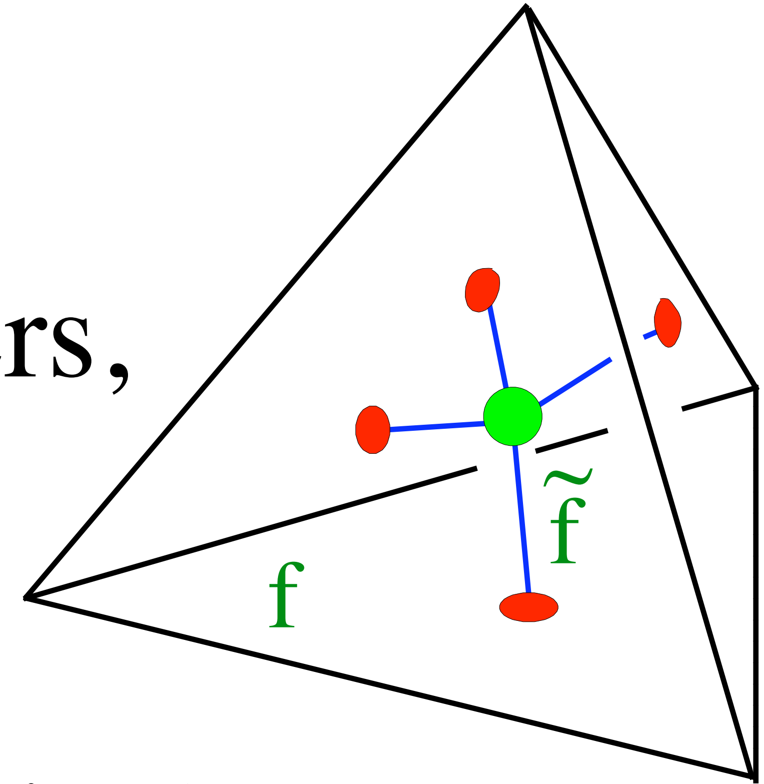
But this **ν** non-symmetric!! (Yet, pos.def.)

Prop. 2: If centers such that

$$\sum_f \text{vec}(f) \times \text{vec}(\tilde{f}) = 0$$

vec(f) = vectorial area of f here
vec($\tilde{f}$) = vector along $\tilde{f}$ (with usual orientation of ambient space)

Then **ν** symmetric.



Corollary: If ● at barycenters, then **ν** symmetric for all positions of ● inside.

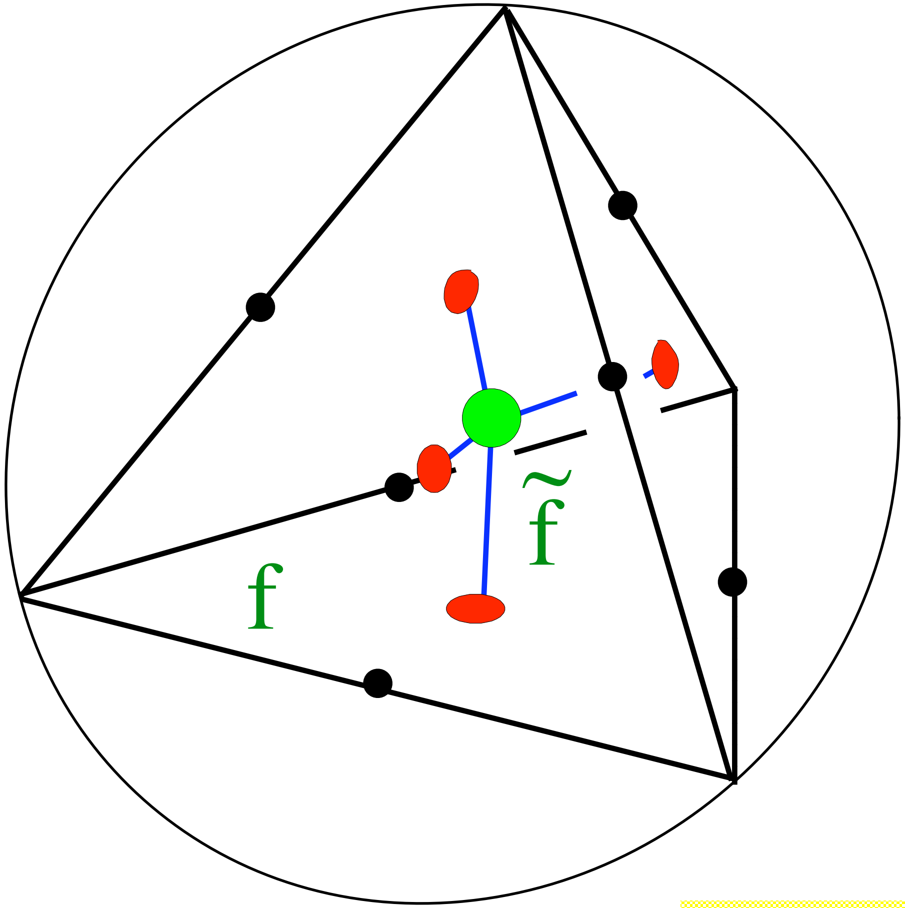
Proof. True if ● at barycenter (Galerkin **ν**). Now,

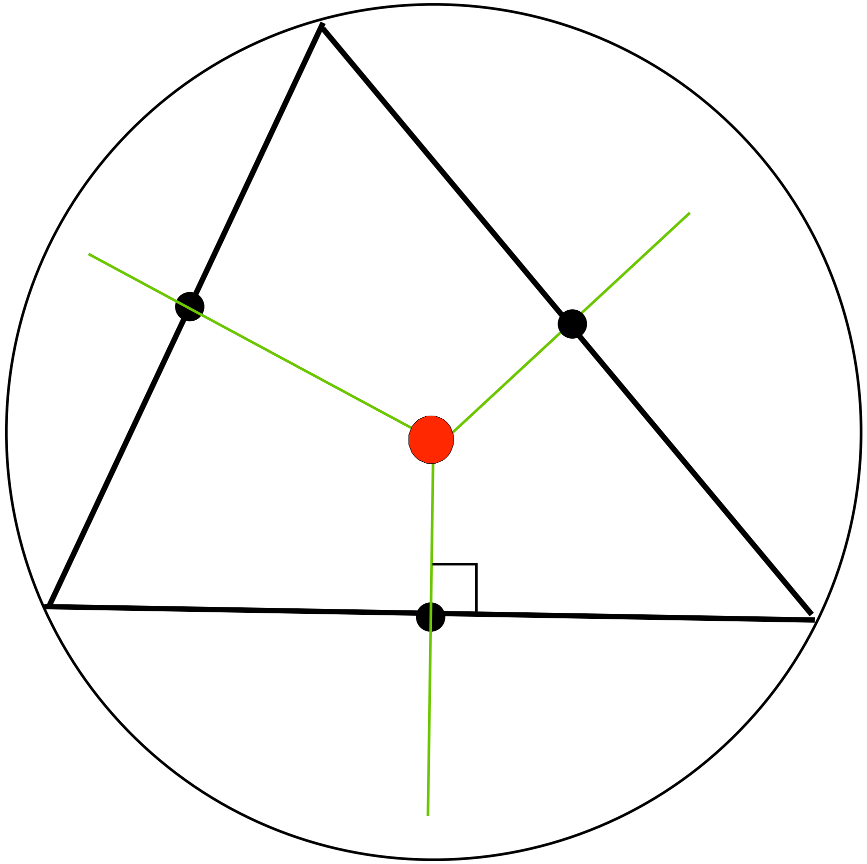
if ● ← ● + v, and because $\sum_f \text{vec}(f) = 0$,

$$\sum_f \text{vec}(f) \times \text{vec}(\tilde{f} + v) = 0 + (\sum_f \text{vec}(f)) \times v = 0. \quad \square$$

An interesting solution (Weiland, Tonti et al., ...)

Centers at circumcenters:

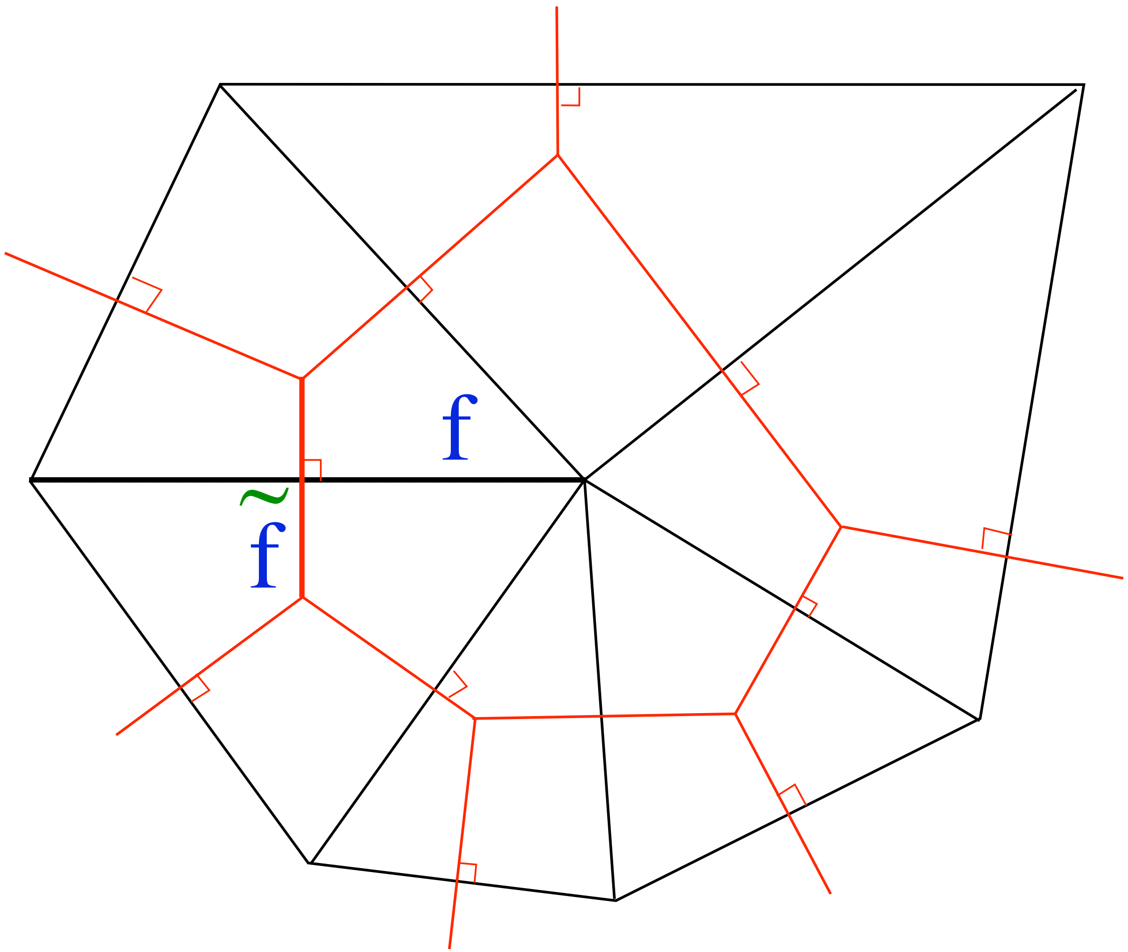


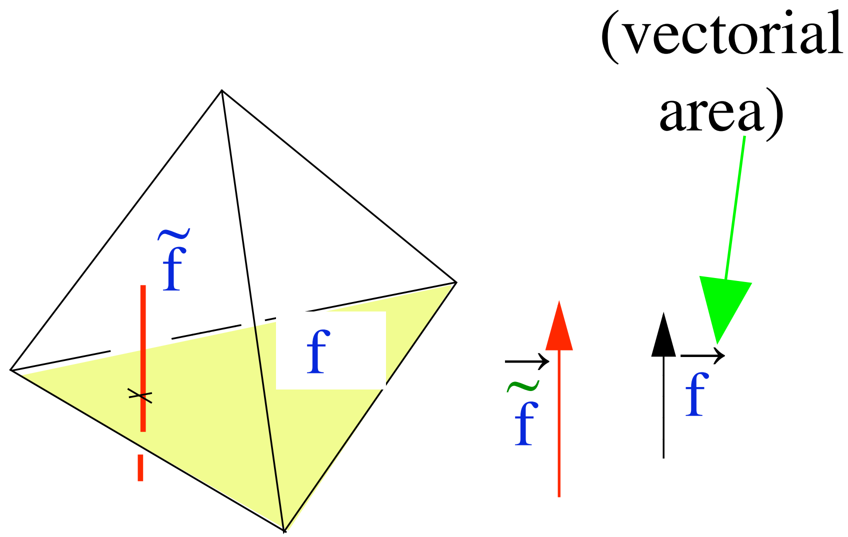


Then, $\vec{f} \,//\, \vec{\tilde{f}}$, so $\boldsymbol{\nu}^{ff} f = \nu \tilde{f}$, other terms 0,

i.e., $\boldsymbol{\nu}^{ff} = \nu \text{ length}(\tilde{f})/\text{area}(f)$

Highly desirable mutual orthogonality of primal and dual meshes

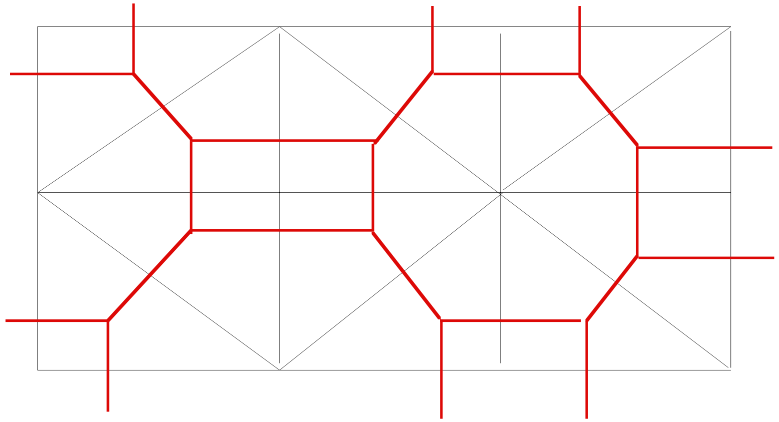


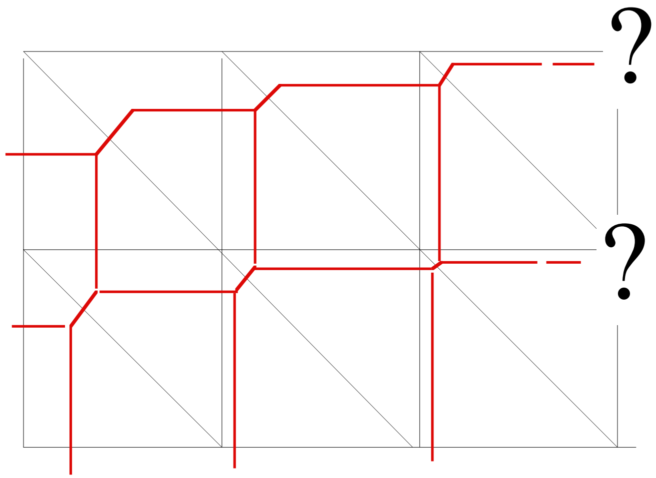


Here, $\vec{\tilde{f}} \,//\, \vec{f}$, and

$$\nu^{ff}\, \vec{f} = \nu\, \vec{\tilde{f}}$$

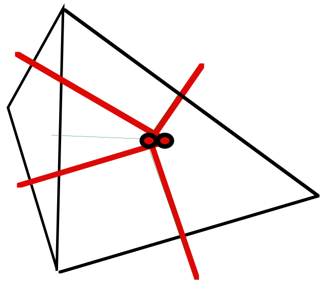
Alas ...





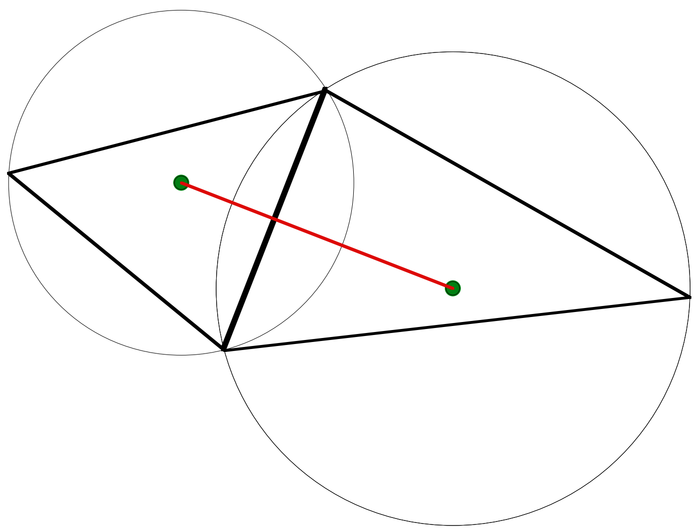
Only specially designed primal meshes will admit an orthogonal dual

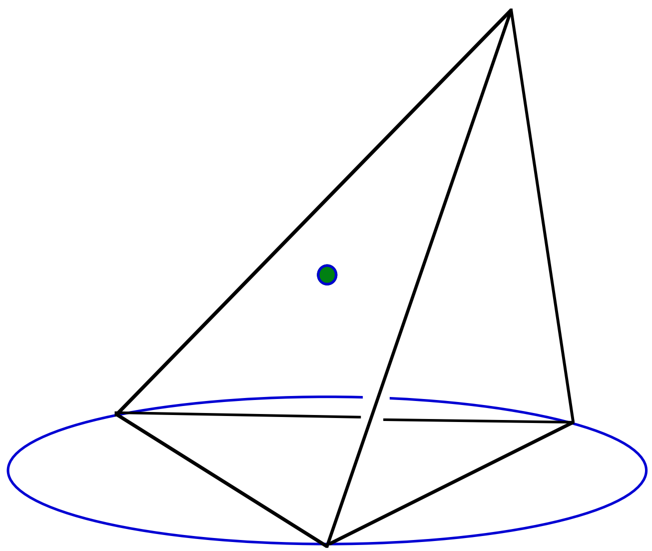
and besides, Delaunay doesn't quite make it:



*A **sufficient** condition:*

The "circumcenter inside" property





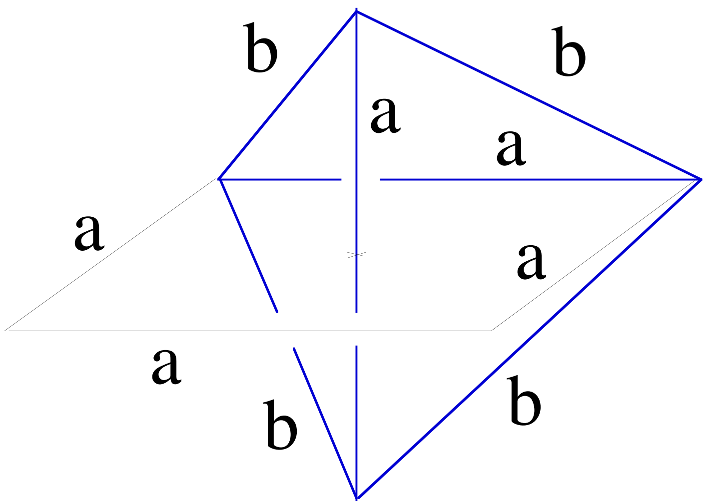
... satisfied by the Sommerville tetrahedron:

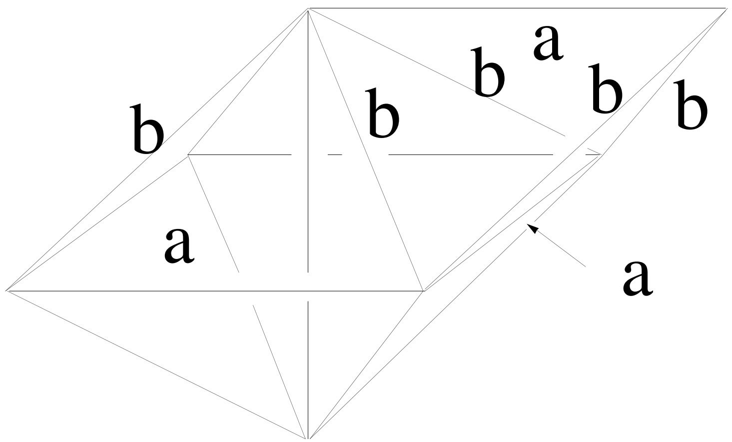
D.M.Y. Sommerville: "Space-filling Tetrahedra in Euclidean Space", **Proc. Edinburgh Math. Soc., 41** (1923), pp. 49-57.

D.M.Y. Sommerville: "Division of Space by Congruent Triangles and Tetrahedra", **Proc. Roy. Soc. Edinburgh, 43** (1923), pp. 85-116.

The Sommerville tetrahedron,

a space-filler:

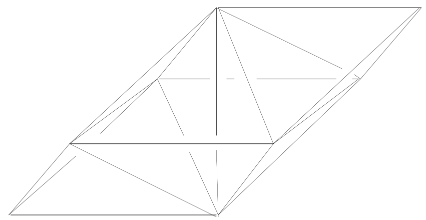




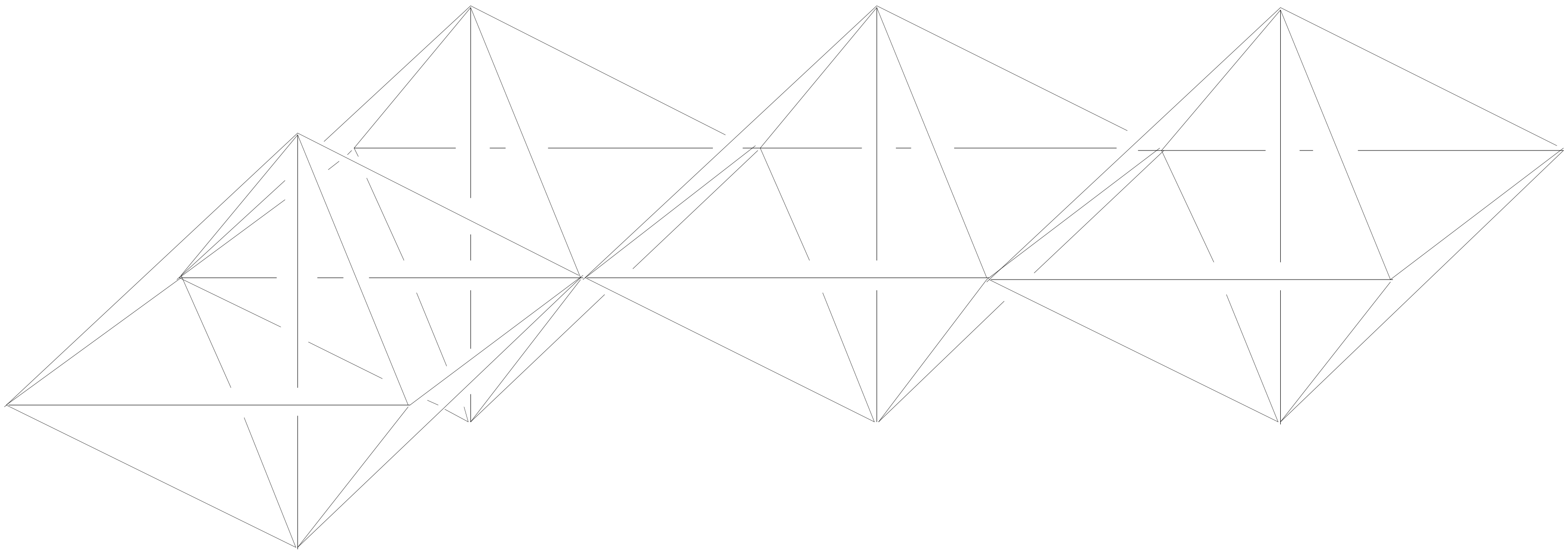
We'll take

a = 2, b = √3

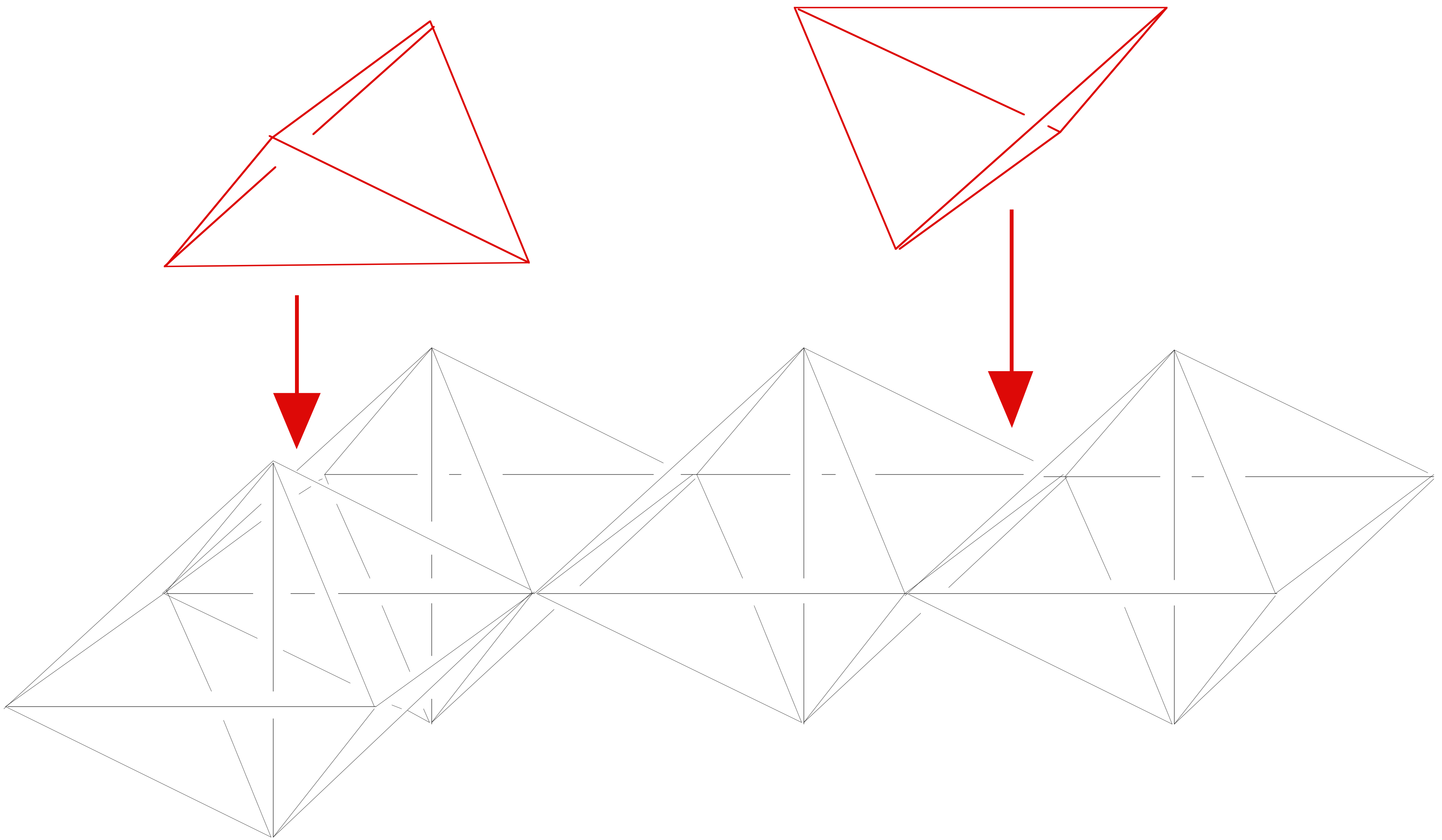
$$3\,a^2 = 4\,b^2$$



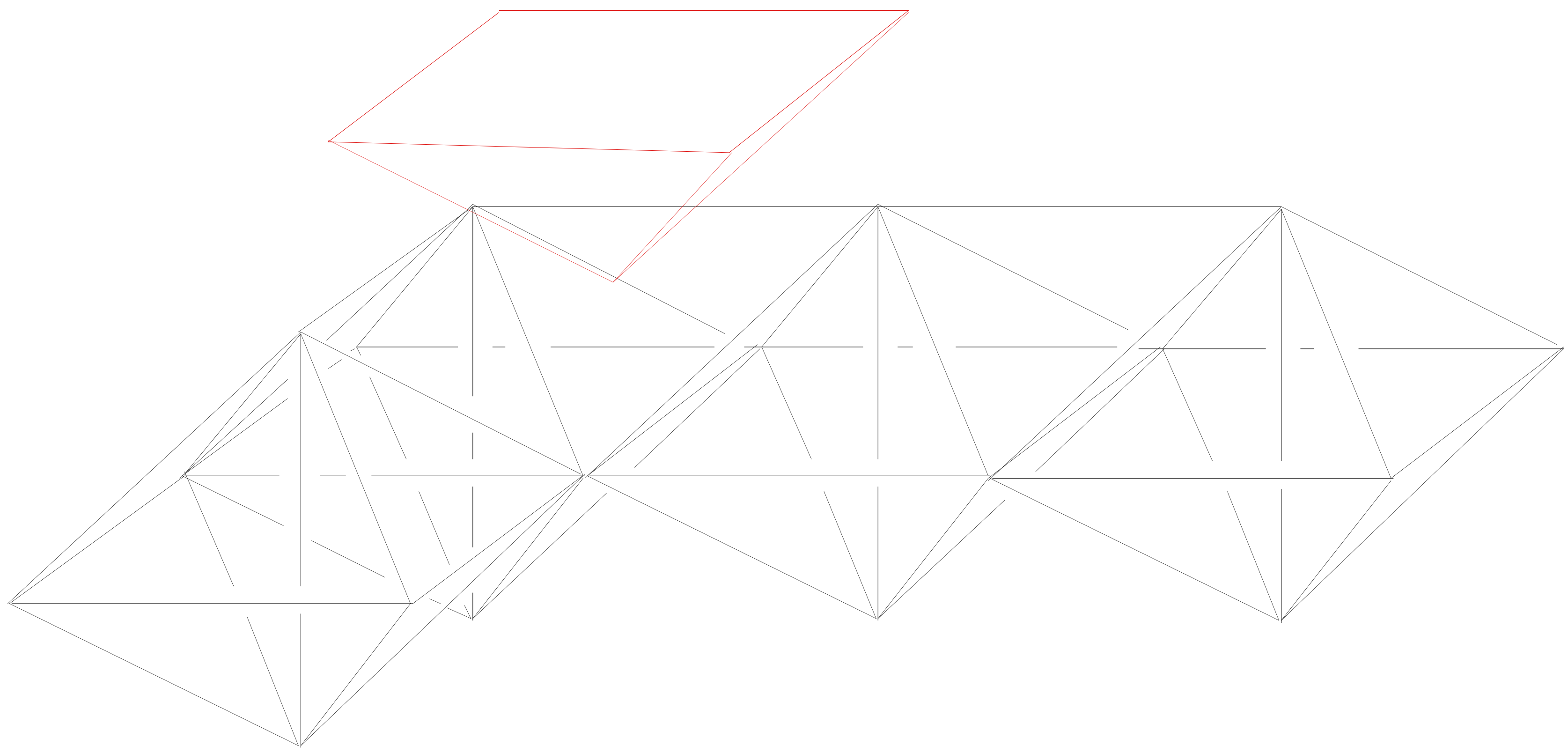
One may now stack the hexahedra thus obtained, which amounts to combine octahedra and tetrahedra in the familiar "octet truss" pattern: First lay the octahedra side by side, like this,



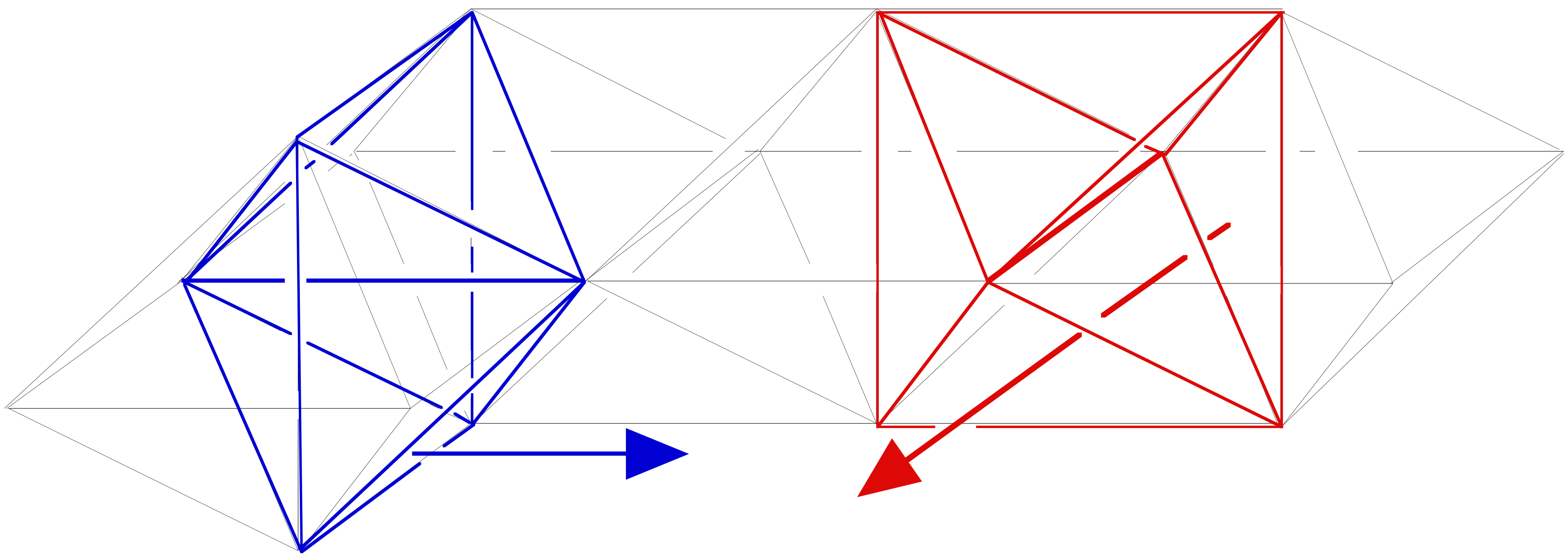
then add S-tetrahedra, two for each octahedron, like this:



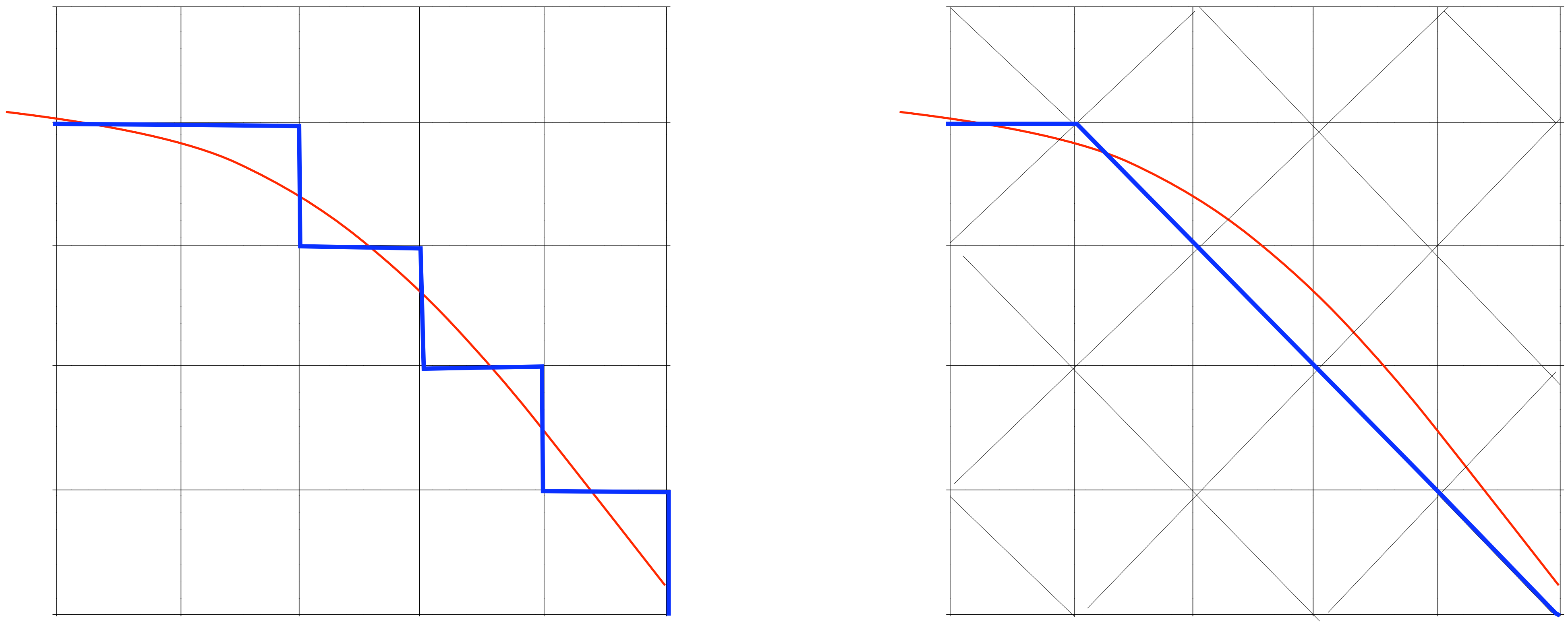
so one is left with a horizontal egg-crate shaped slab, with pyramidal holes, ready to be filled by a similar slab, superposed, thus filling space.



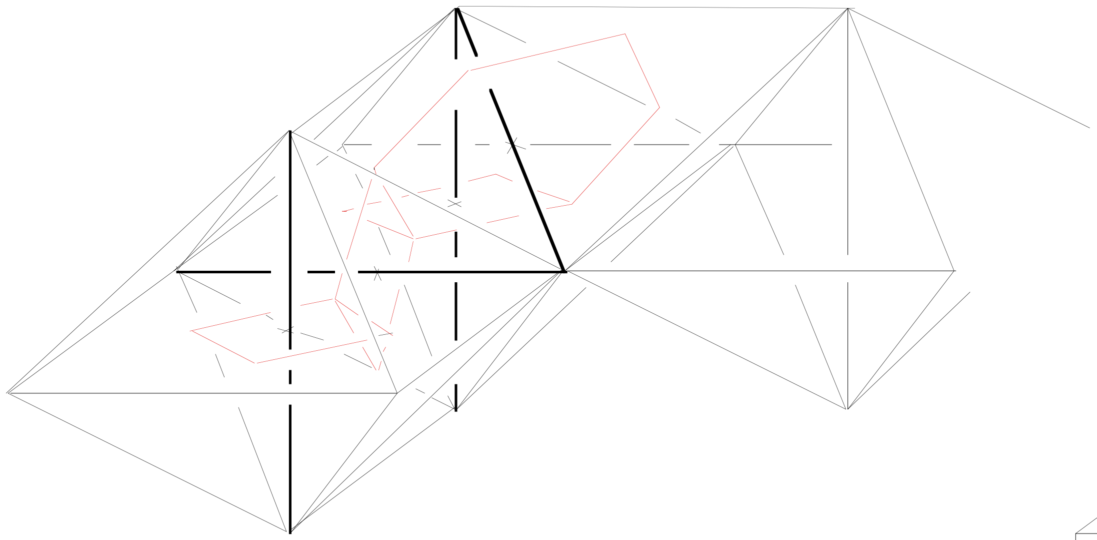
No privileged direction:



Notorious “staircase” problem, alleviated:

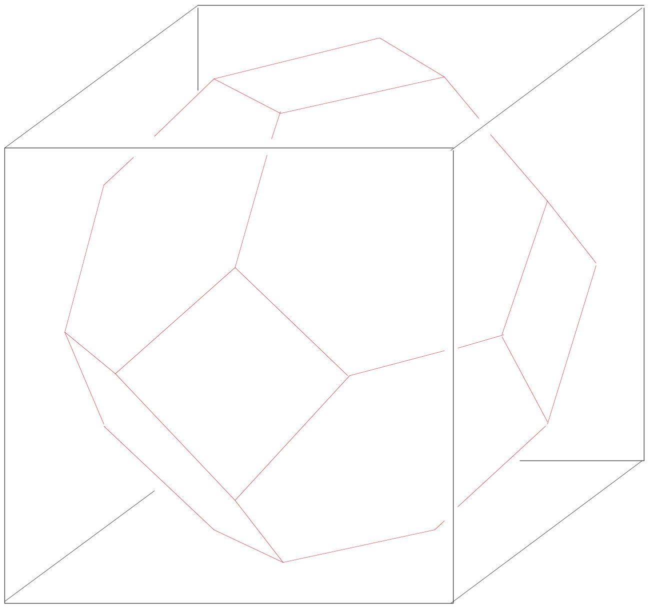


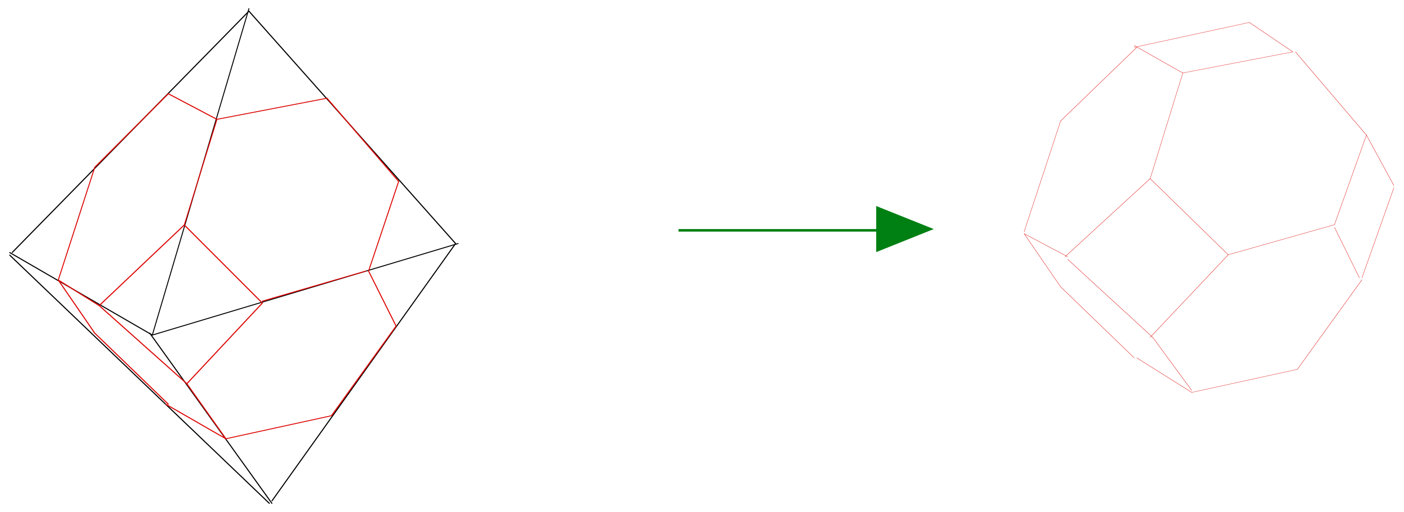
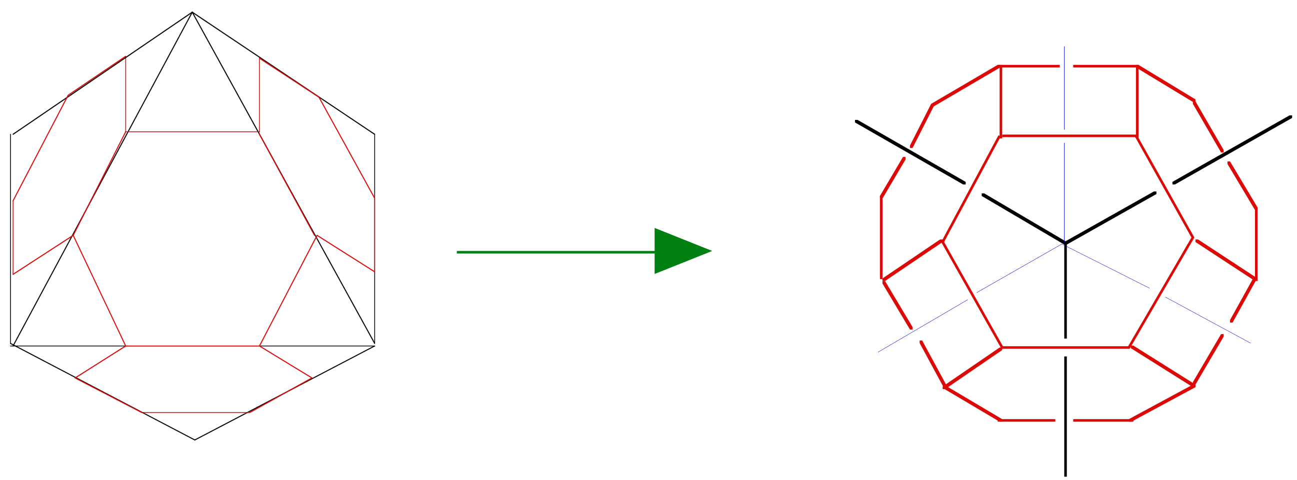
The dual mesh:



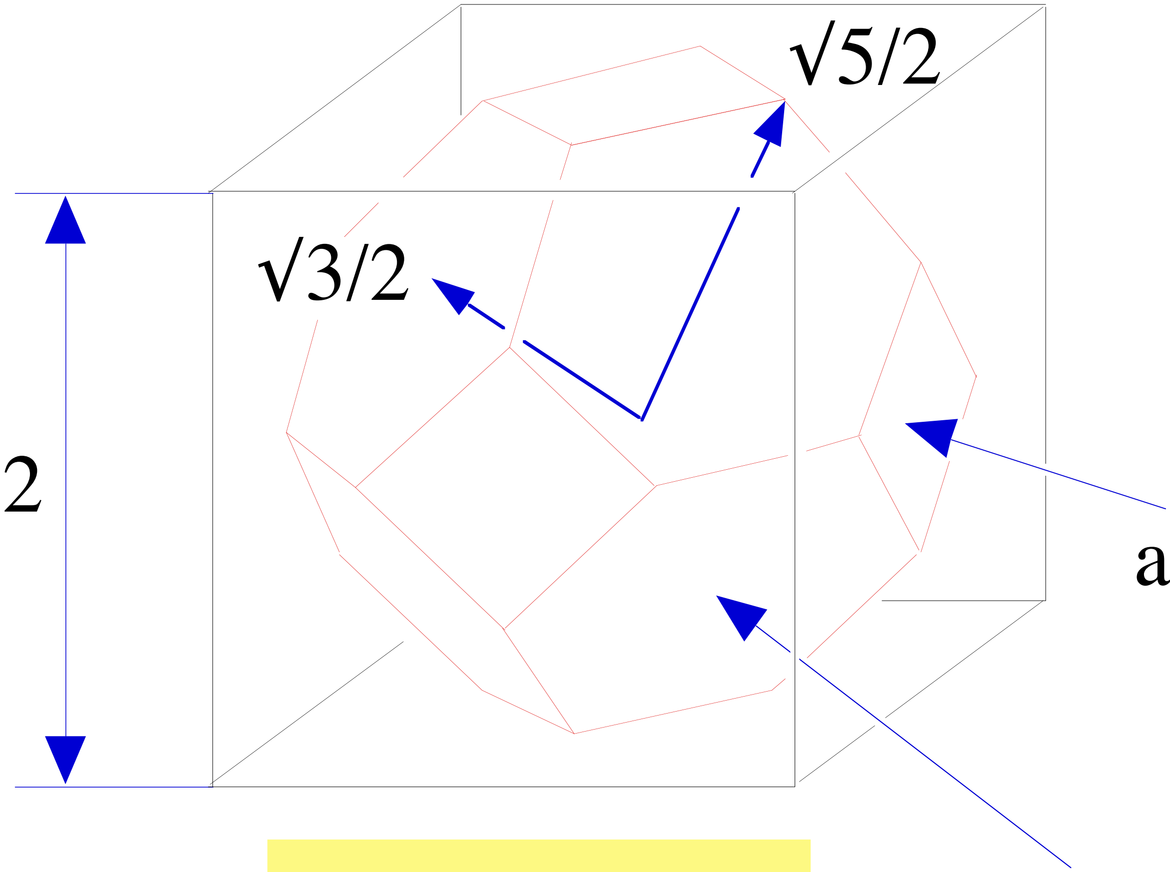
(truncated octahedron, aka

tetrakaidecahedron)



"More isotropic" than the Yee lattice:



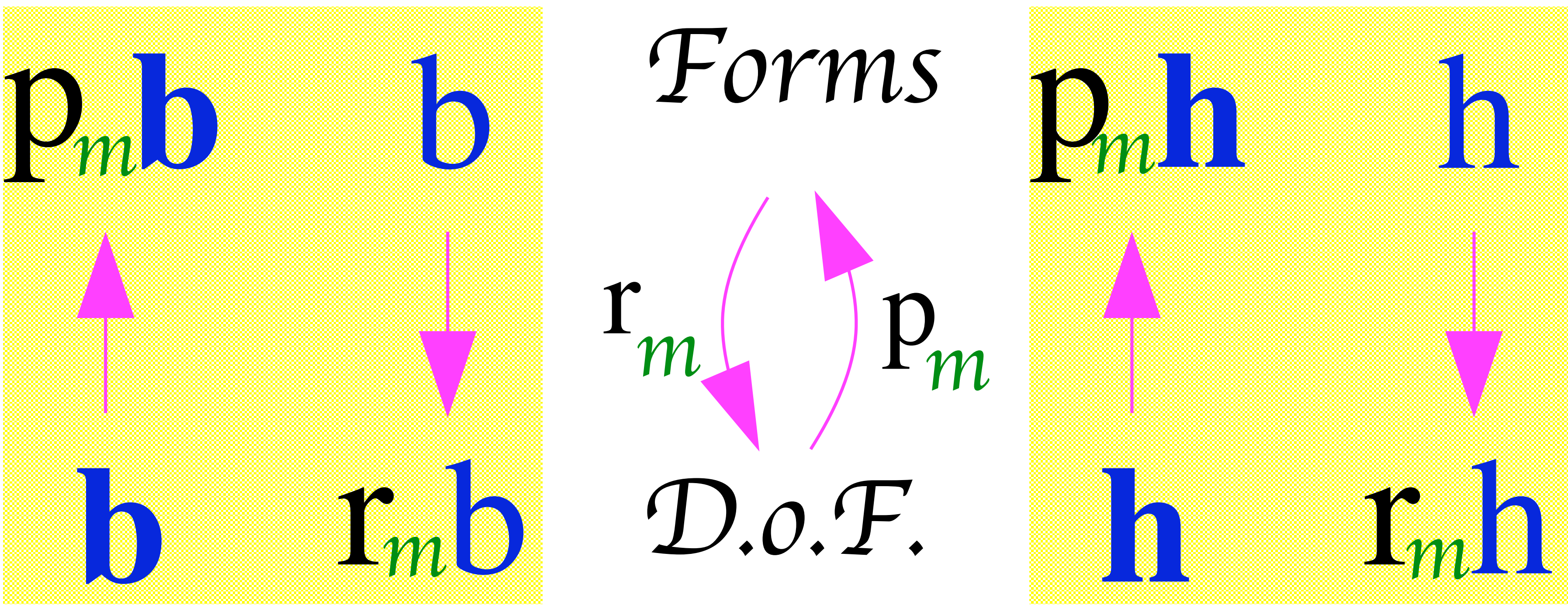
$$\sqrt{5}/\sqrt{3} < \sqrt{3}$$

All dual-edge lengths $1/\sqrt{2}$

$$\frac{\text{area}(f)}{\text{length}(\tilde{f})} = \frac{\sqrt{2}}{1/\sqrt{2}} = 2$$

$$\frac{\text{area}(\tilde{e})}{\text{length}(e)} = \frac{1/2}{2} \quad \text{or} \quad \frac{3\sqrt{3}/4}{\sqrt{3}}$$

Convergence issues



Computed fluxes

$$\mathbf{b} = \{\mathbf{b}_f : f \in \mathcal{F}\}$$

Computed mmf's

$$\mathbf{h} = \{\mathbf{h}_f : f \in \mathcal{F}\}$$

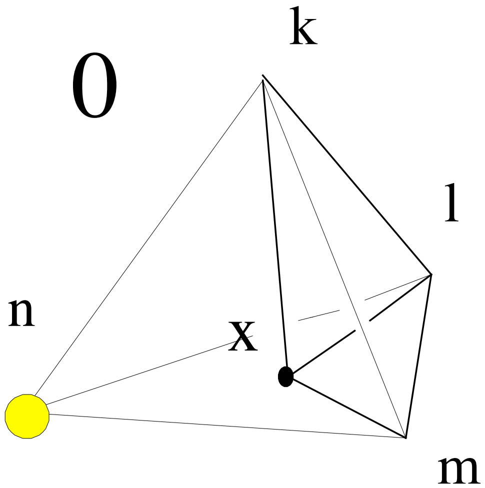
$$(r_m b)_f = \int_f b$$

True ones

$$(r_m h)_f = \int_{\tilde{f}} h$$

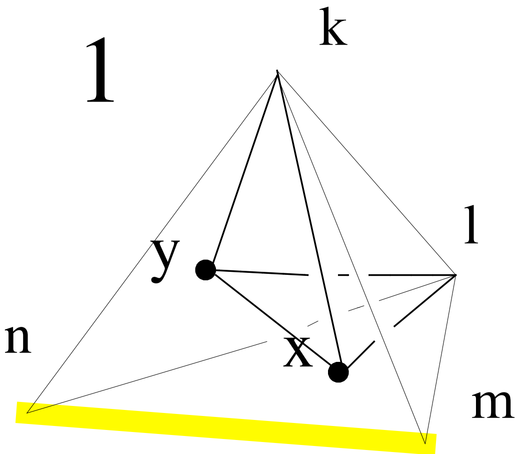
Whitney forms

0



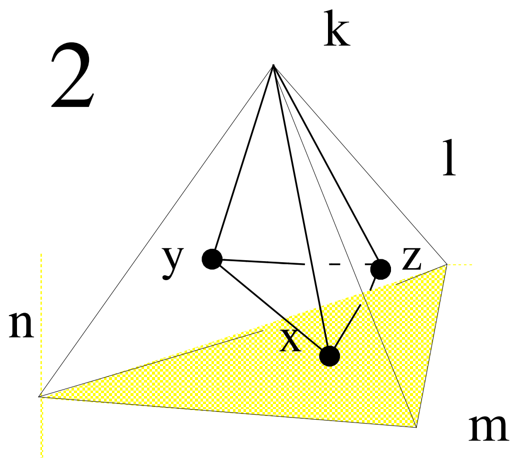
W^{n}

1



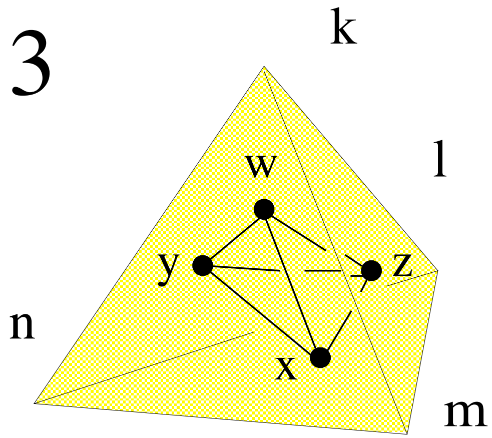
$W^{\{m, n\}}$

2



$W^{\{l, m, n\}}$

3



$W^{\{k, l, m, n\}}$

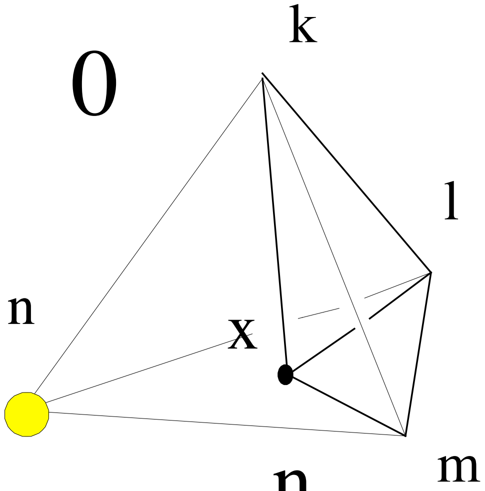
λ^{n}

$\lambda^{n}\, d\lambda^{m} - \lambda^{m}\, d\lambda^{n}$

$2[\, \lambda^{l}\, d\lambda^{m} \wedge d\lambda^{n} + \ldots + \ldots]$

$6\, d\lambda^{k} \wedge d\lambda^{l} \wedge d\lambda^{m}$

0



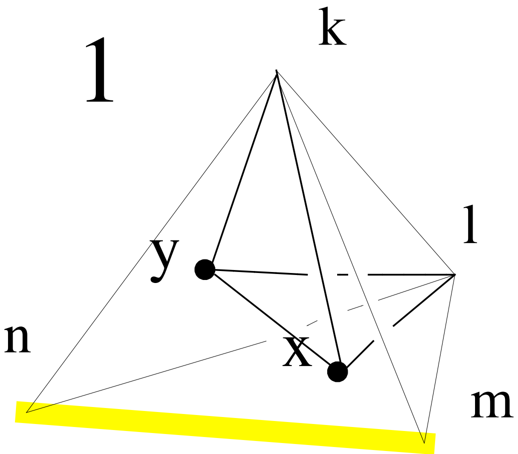
w^{n}

λ^{n}

Mapping points to cellular 0-chains, weights given by Whitney 0-forms:

$$x = \sum_{n \in \mathcal{N}} w^{n}(x)\, n$$

1



$w^{\{m,\,n\}}$

$\lambda^{n}\, d\lambda^{m} - \lambda^{m}\, d\lambda^{n}$

Mapping (bound) vectors to cellular 1-chains, weights given by Whitney 1-forms:

$$v = y - x = \sum_{e \in \mathcal{E}} \langle v \,;\, w^{e}(x) \rangle\, e$$

(last e, by notational abuse, is vec(e), aka $\vec{e}$)

Sketch of convergence proof, in magnetostatics

(easy extension to full Maxwell, by using Laplace transform)

Notation: $\|\mathbf{b}\|_{\boldsymbol{\nu}}^2 = \sum_{f,f'} \boldsymbol{\nu}^{ff'} \mathbf{b}_f \mathbf{b}_{f'}$ ("$\boldsymbol{\nu}$–norm"), $(\mathbf{b}, \mathbf{h}) = \sum_f \mathbf{b}_f \mathbf{h}_f$

$$\mathrm{D}\,\mathbf{b} = 0, \quad \mathbf{h} = \boldsymbol{\nu}\,\mathbf{b}, \quad \mathrm{R}^t\,\mathbf{h} = \mathbf{j}$$

$$\mathrm{D}\,\mathrm{r}_m \mathrm{b} = 0 \qquad \mathrm{R}^t\,\mathrm{r}_m \mathrm{h} = \mathrm{r}_m \mathrm{j}$$

(because $\mathrm{D}\,\mathrm{r}_m = \mathrm{r}_m \mathrm{d}$) (because $\mathrm{R}^t\,\mathrm{r}_m = \mathrm{r}_m \mathrm{d}$)

$$\underbrace{(\mathbf{h} - \mathrm{r}_m \mathrm{h})}_{\in\, \ker(\mathrm{R}^t)} - \boldsymbol{\nu}\underbrace{(\mathbf{b} - \mathrm{r}_m \mathrm{b})}_{\in\, \ker(\mathrm{D})} = (\boldsymbol{\nu}\mathrm{r}_m - \mathrm{r}_m \nu)\mathrm{b}$$

$$\|\mathbf{b} - \mathrm{r}_m \mathrm{b}\|_{\boldsymbol{\nu}}^2 + \|\mathbf{h} - \mathrm{r}_m \mathrm{h}\|_{\boldsymbol{\mu}}^2 = \|(\nu \mathrm{r}_m - \mathrm{r}_m \nu)\mathrm{b}\|_{\boldsymbol{\mu}}^2 \equiv \|(\mu \mathrm{r}_m - \mathrm{r}_m \mu)\mathrm{h}\|_{\boldsymbol{\nu}}^2$$

Consistency

$$\begin{cases} p_m r_m \mathbf{b} \to \mathbf{b} \quad \text{when "} m \to 0 \text{"} \\ \|(\boldsymbol{\nu} r_m - r_m \nu)\mathbf{b}\|_{\boldsymbol{\mu}} \to 0 \end{cases}$$

\+

Stability :

$$\alpha \, \|p_m \mathbf{b}\|_\nu \leq \|\mathbf{b}\|_{\boldsymbol{\nu}}$$

=

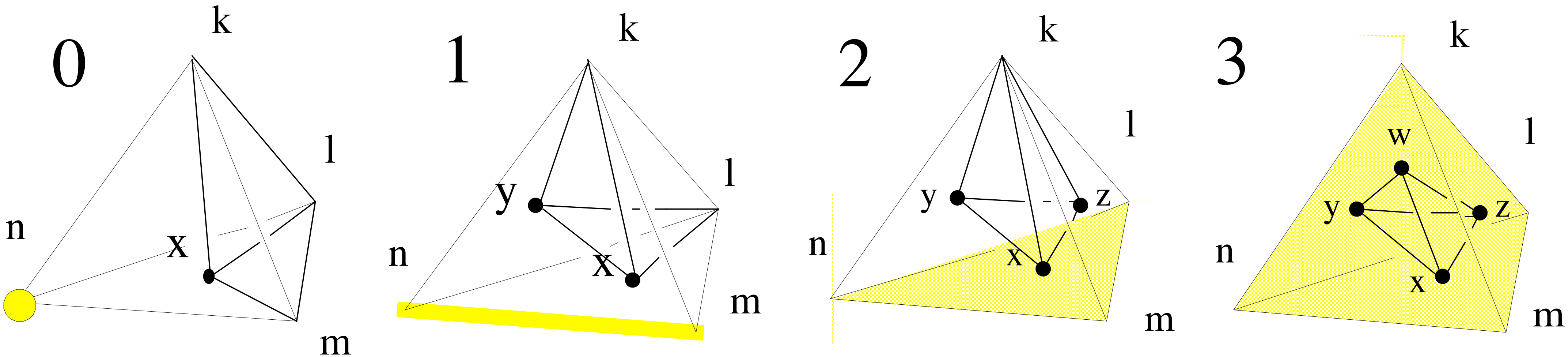
Convergence :

$$\|p_m(\mathbf{b} - r_m \mathbf{b})\|_\nu \leq \frac{1}{\alpha} \|\mathbf{b} - r_m \mathbf{b}\|_{\boldsymbol{\nu}}$$

$$\leq \frac{1}{\alpha} \|(\boldsymbol{\nu} r_m - r_m \nu)\mathbf{b}\|_{\boldsymbol{\mu}} \to 0 \Rightarrow p_m \mathbf{b} \to \mathbf{b}$$

Why Galerkin method fulfills consistency requirement:

Whitney form proxies



$\mathbf{w}^{n}$ $\quad \mathbf{w}^{\{m,\, n\}}$ $\quad \mathbf{w}^{\{l,\, m,\, n\}}$ $\quad \mathbf{w}^{\{k,\, l,\, m,\, n\}}$

λ^{n}

$\lambda^{n}\, \nabla\lambda^{m} - \lambda^{m}\, \nabla\lambda^{n}$

$2[\, \lambda^{l}\, \nabla\lambda^{m} \times \nabla\lambda^{n} + \ldots + \ldots]$

$1/\mathrm{vol}(\{k, l, m, n\})$

Whitney forms as a partition of unity

- $\sum_n w^n(x) = 1 \qquad \forall\, x$

- $\sum_e w^e(x) \otimes e = 1 \qquad \forall\, x$

  i.e., $\sum_e (v \cdot w^e(x))\, e = v \quad \forall\, v$

- $\sum_f w^f(x) \otimes f = 1 \quad \forall\, x$

etc.

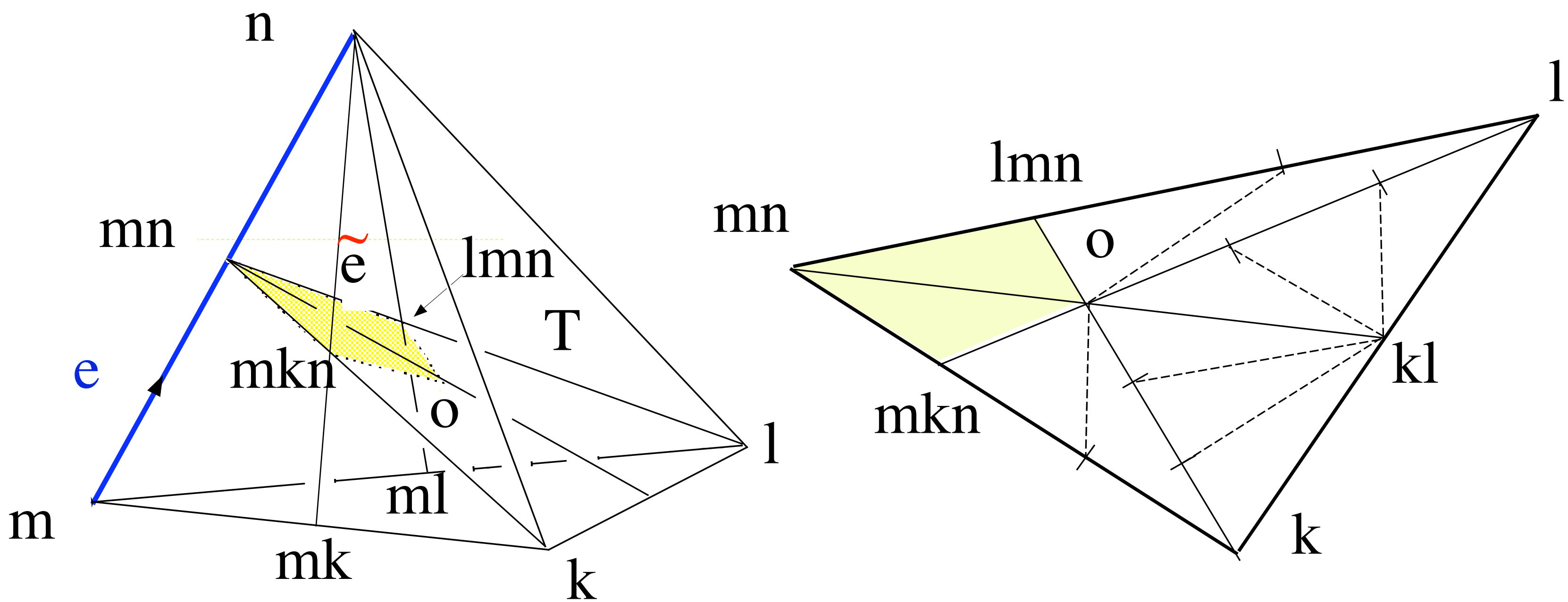
Consequence: The "mass matrix" $\boldsymbol{\varepsilon}$ *of edge elements ...*

$$\sum_{e'} (\varepsilon w^{e}(x) \cdot w^{e'}(x))\, e' = \varepsilon w^{e}(x)$$

$$\sum_{e'} \int_{D} (\varepsilon\, w^{e}(x) \cdot w^{e'}(x))\, e' = \int_{D} \varepsilon w^{e}(x)$$

$$\sum_{e'} \boldsymbol{\varepsilon}^{ee'}\, e' = \int_{D} \varepsilon w^{e}(x) = \widetilde{\varepsilon e} \quad (!)$$

... satisfies the consistency requirement



$$\int_T \nabla w^n = \{k, l, m\}/3$$

$$\int_T w^m \nabla w^n - w^m \nabla w^n =$$

$$(\{k, l, m\}/3 + \{k, l, n\}/3)/4 = \tilde{e}$$

So Galerkin *is* a mimetic method too!

But non-diagonal $\boldsymbol{\varepsilon}$,
making Yee scheme
*im*plicit, thus expensive

Diagonal lumping at the rescue

There is a unique diagonal matrix ε_{diag}, *indexed over edges, such that* $G^t(\varepsilon_{diag} - \varepsilon_{Gal})G = 0$. *Its entries are*

$$\varepsilon^{ee}_{diag} = -(G^t \varepsilon_{Gal} G)^{mn}$$

for each edge e *going from node* m *to node* n. *If* $\varepsilon^{ee}_{diag} > 0$ (plus mild stability assumptions), *the Yee schemes with* $\varepsilon = \varepsilon_{diag}$ *and* $\varepsilon = \varepsilon_{Gal}$ *have the same limit when* $"m \to 0"$

But note that $\varepsilon^{ee}_{diag} > 0$ requires acute dihedral angle at e!

A.B. and L. Kettunen, paper #128 at http://butler.cc.tut.fi/~bossavit/Papers.html

Which primal mesh, which discrete Hodge?

- Galerkin works on all simplicial meshes

  But *non-diagonal* **ε** and **ν**. Diagonal lumping?

  Yes, for **ε** (not for **ν**) if *acute* dihedral angles

- FIT/CM make diagonal hodges

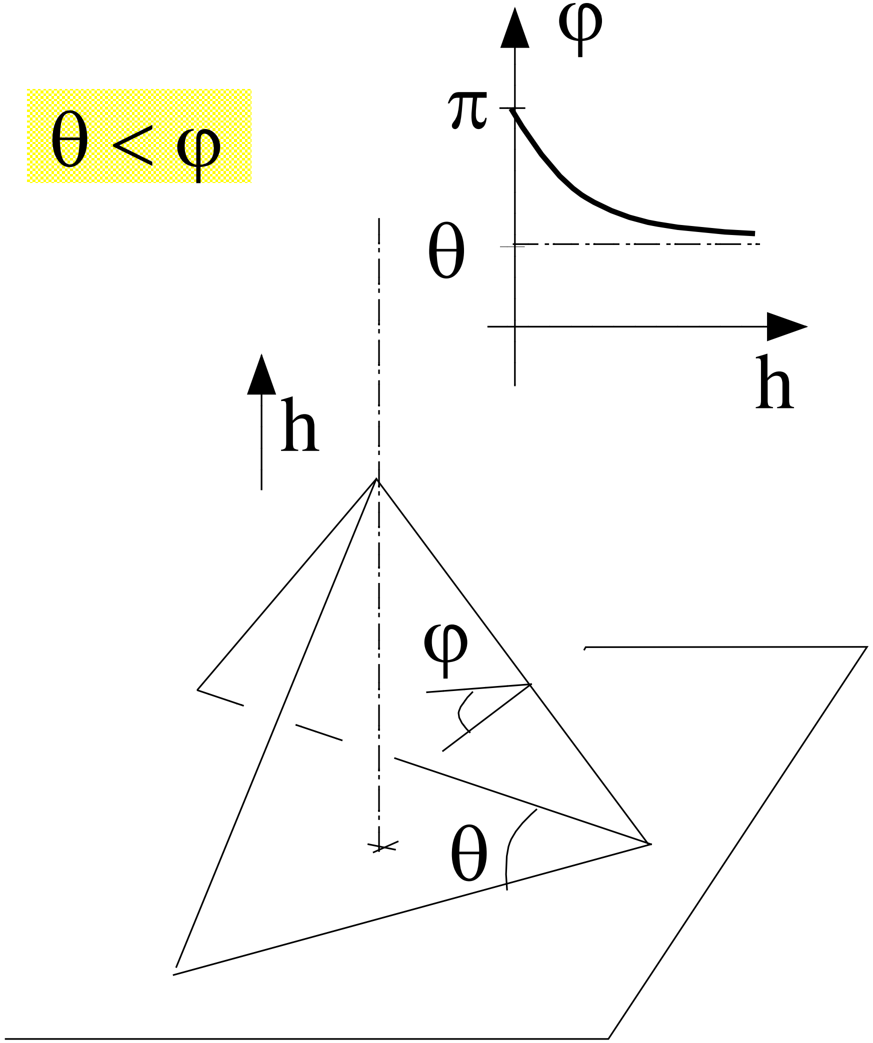
  but require *mutual orthogonality* of primal/dual cell pairs.

Definition. *Acute* n-simplex: Dihedral angles (i.e., angles between hyperplanes subtending (n – 1)-faces) all < 90°.

Proposition. Faces of an acute n-simplex are acute.

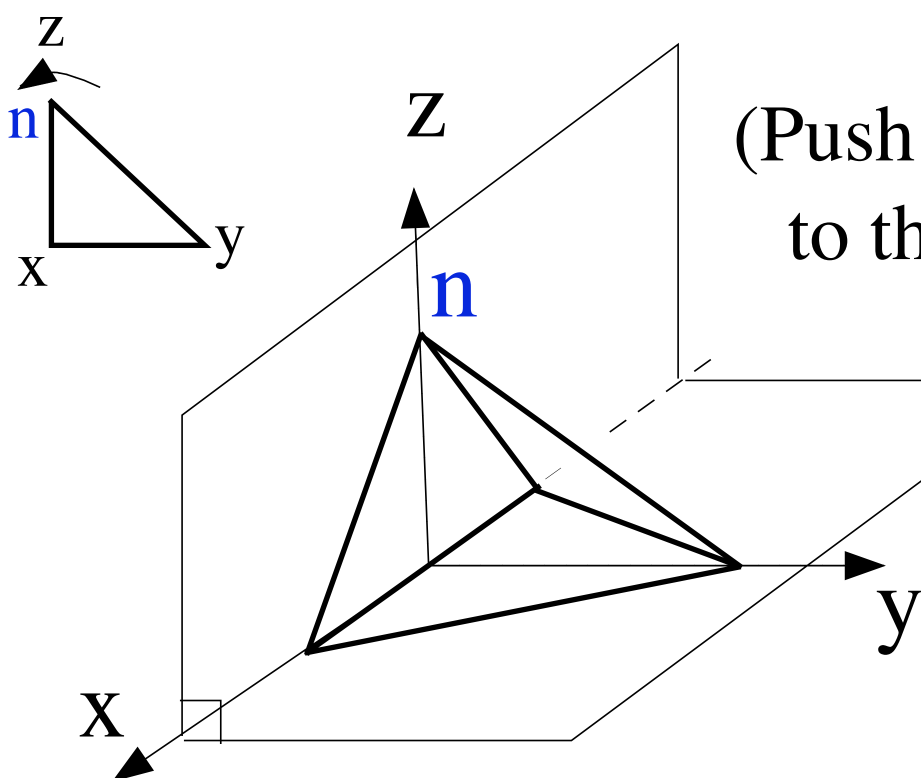
Proof:

$\theta < \varphi$



Converse not true:

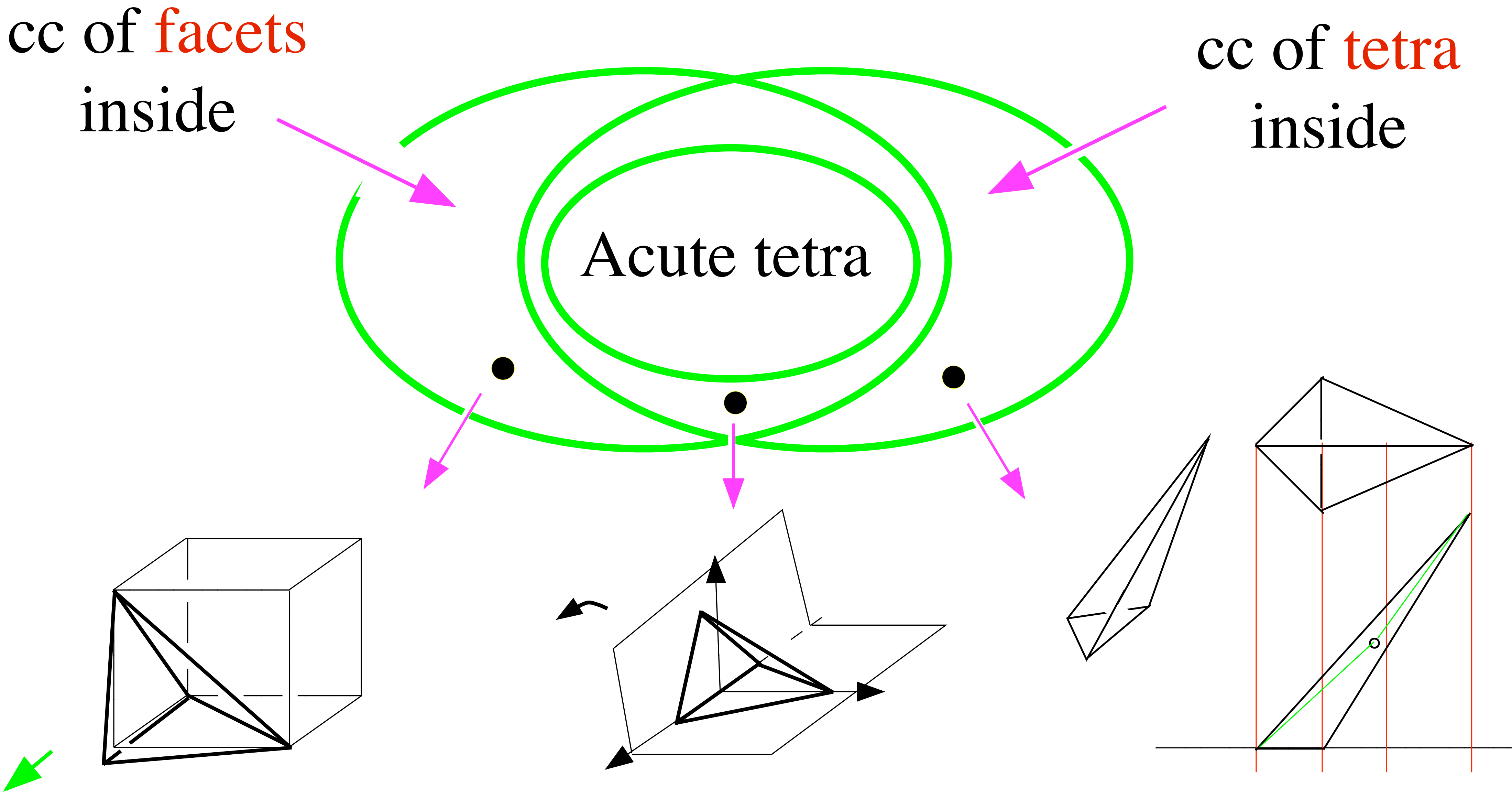
A non-acute tetrahedron with acute facets:



(Push n a bit to the left)

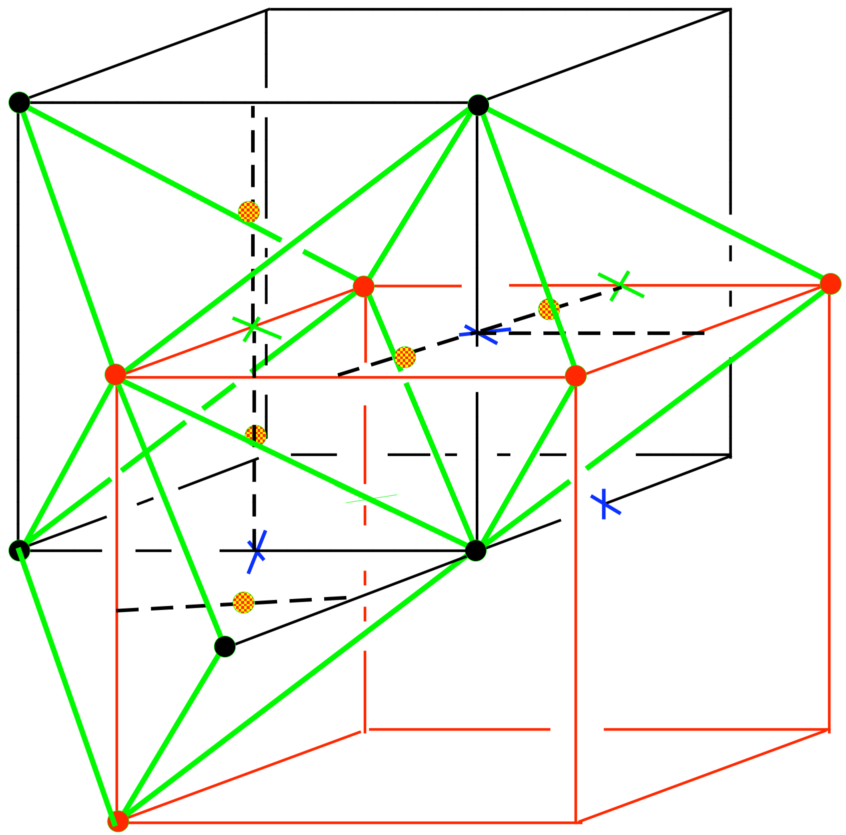
Couldn't acute tetrahedra be preferable?

A Venn diagram:

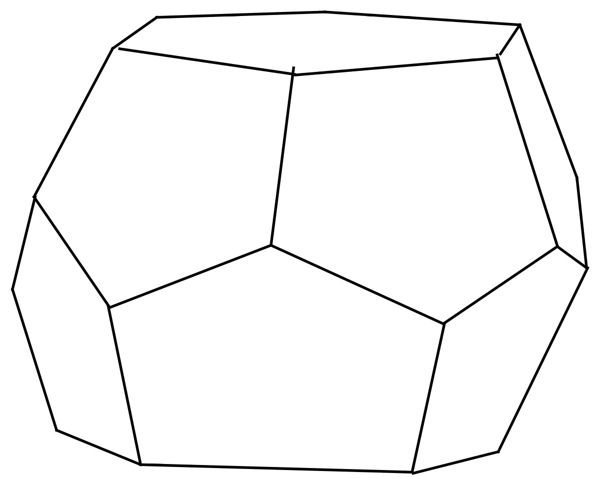


*The A15 acute tiling of space**

To nodes of Sommerville mesh, add centers of one S. tetra out of two...



... build Voronoi cells of lattice thus obtained, then take Delaunay tetras of this.



* D. Eppstein, J.M. Sullivan, A. Üngör: "Tiling space and slabs with acute tetrahedra", **arXiv:**cs.CG/0302027 v1 (19 Feb. 2003).

The tools in the box:

Surfaces, curves, etc. → Cell chains

Fields b, h, ... → Cell cochains (DoF arrays) $\mathbf{b}$, $\mathbf{h}$, ...

Constitutive laws → "Discrete hodges", $\boldsymbol{\varepsilon}$, $\boldsymbol{\nu}$, $\boldsymbol{\sigma}$ ...

grad, rot, div → G, R, D (primal side), $-D^t$, R^t, $-G^t$ (dual side)

products, $E \times H$, $J \cdot E$ → "wedge" product, $e \wedge h$, $j \wedge e$

$$-\partial_t D + \text{rot}\, H = J, \quad D = \varepsilon E \qquad -\partial_t \mathbf{d} + R^t \mathbf{h} = \mathbf{j}, \quad \mathbf{d} = \boldsymbol{\varepsilon} \mathbf{e}$$

$$\partial_t B + \text{rot}\, E = 0, \quad H = \nu B \qquad \partial_t \mathbf{b} + R\, \mathbf{e} = 0, \quad \mathbf{h} = \boldsymbol{\nu} \mathbf{b}$$

$$\text{div}\, D = Q, \quad \text{div}\, B = 0 \qquad -G^t \mathbf{d} = \mathbf{q}, \quad D\mathbf{b} = 0$$

$$E = -\,\text{grad}\, \varphi - \partial_t A \qquad \mathbf{e} = -G\, \boldsymbol{\varphi} - \partial_t \mathbf{a}$$

etc.

Good, but not enough:

What about "force related" entities, like

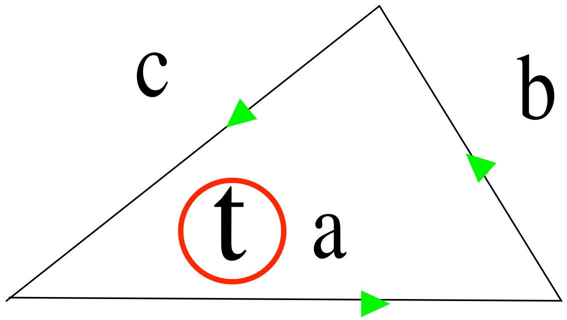
- $E \times H$ (Poynting) ?
- $Q(E + v \times B)$ (Lorentz) ?
- $J \times B$ (Laplace) ?
- $B \otimes H$ (Maxwell) ?

Heuristic hint: force is a covector, cf. $v \rightarrow \langle v\,;\, f \rangle$

Flux of Poynting "vector"

Computing $\int_t e \wedge h$, for primal triangle t,

knowing DoF-arrays $\mathbf{e}$, $\mathbf{h}$, would be simple:



$$\int_t e \wedge h = \frac{1}{6}[\mathbf{e}_a \mathbf{h}_b + \mathbf{e}_b \mathbf{h}_c + \mathbf{e}_c \mathbf{h}_a - \mathbf{h}_a \mathbf{e}_b - \mathbf{h}_b \mathbf{e}_c - \mathbf{h}_c \mathbf{e}_a]$$

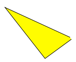
(get e and h from $\mathbf{e}$ and $\mathbf{h}$ using 2D Whitney 1-forms and develop)

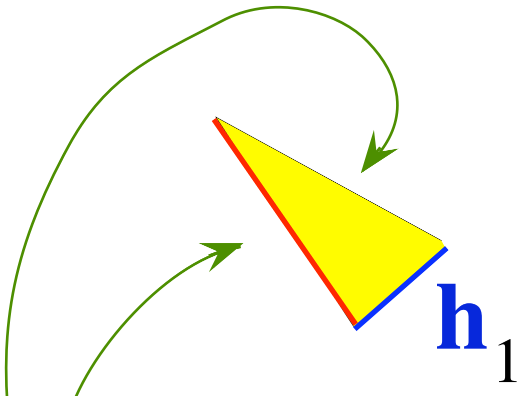
But ...

Flux of Poynting "vector"

... we want $\int_{\Sigma} e \wedge h$ with Σ a dual 2-chain,

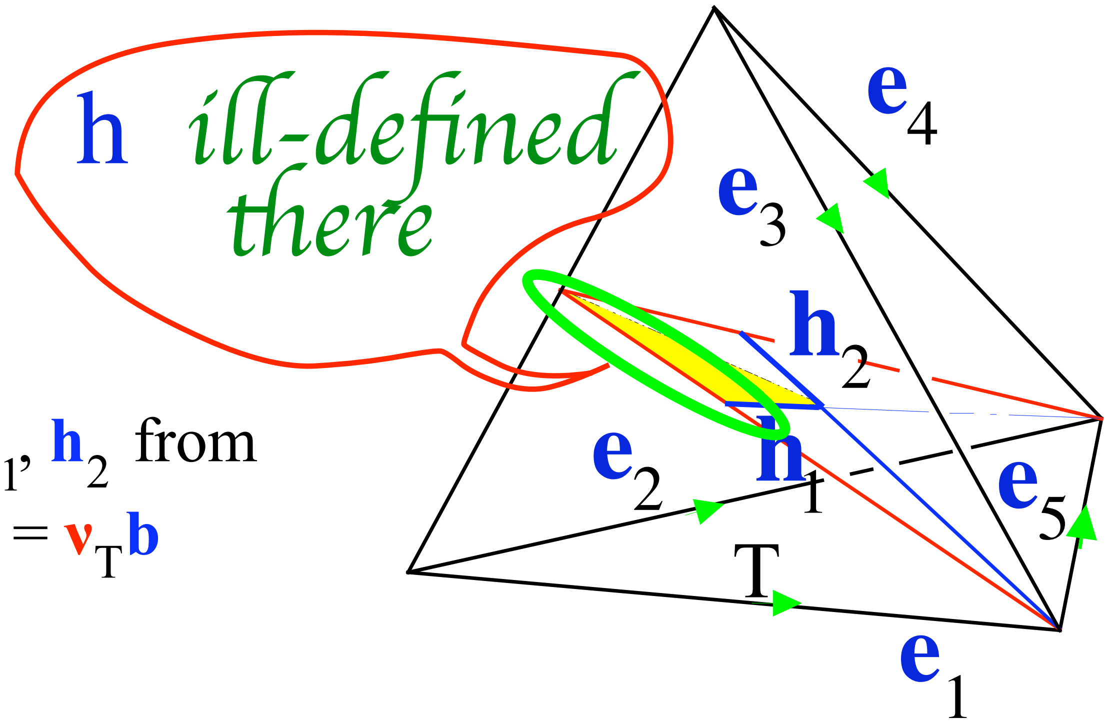
i.e., a sum of integrals

like $\int$  $e \wedge h$ here:



Get $\mathbf{h}_1$, $\mathbf{h}_2$ from

$\mathbf{h} = \boldsymbol{\nu}_T \mathbf{b}$

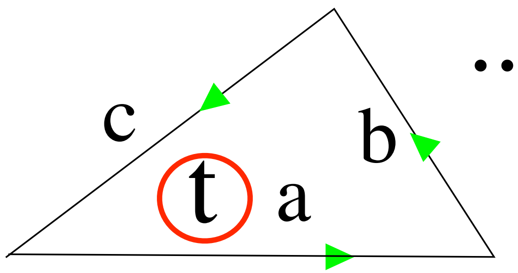


and needed edge values of h not available. Reconstruct them from $\mathbf{h}_1$, $\mathbf{h}_2$ shown here, thanks to the fact that $h = \nu b = \nu da$ (only way to obtain h) is uniform in the tetrahedron

Flux of Poynting "vector"

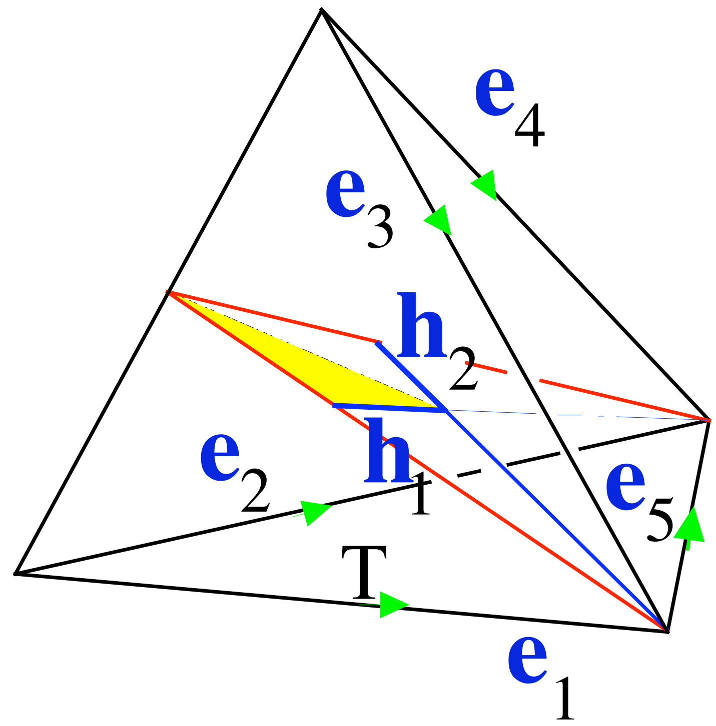
Final recipe for $\int_{\triangle} e \wedge h$:

$$\int_t e \wedge h = \frac{1}{6}\,[\mathbf{e}_a \mathbf{h}_b + \mathbf{e}_b \mathbf{h}_c + \mathbf{e}_c \mathbf{h}_a \; \dots \; - \mathbf{h}_a \mathbf{e}_b - \mathbf{h}_b \mathbf{e}_c - \mathbf{h}_c \mathbf{e}_a]$$

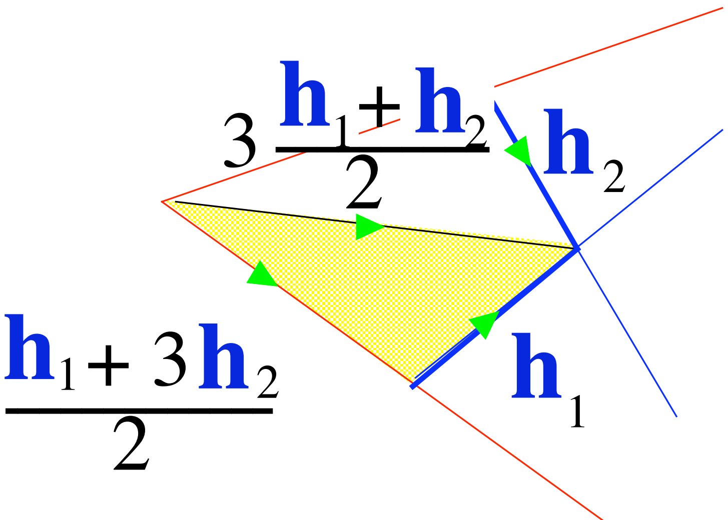


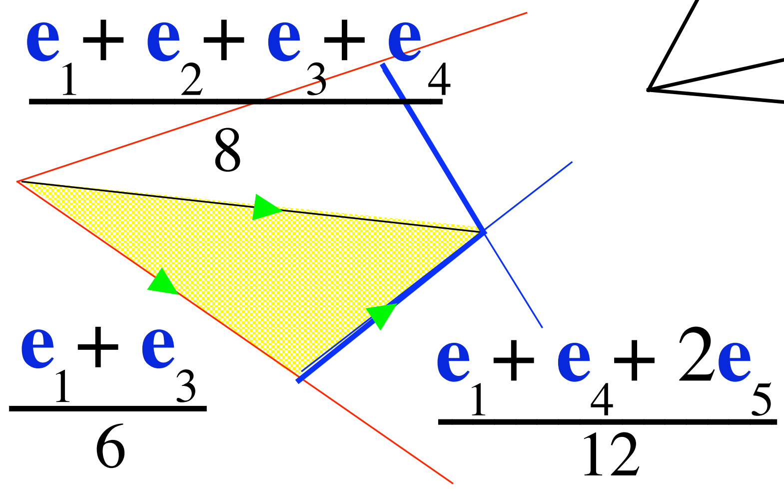
Get $\mathbf{h}_1$, $\mathbf{h}_2$ from

$\mathbf{h} = \boldsymbol{\nu}_T \mathbf{b}$



with these values and orientations:

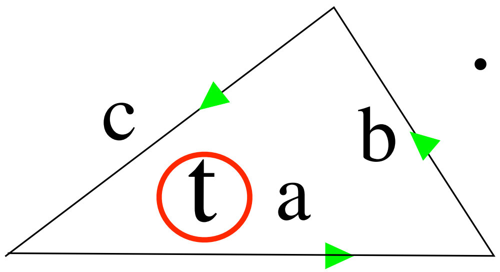




Flux of Poynting "vector"

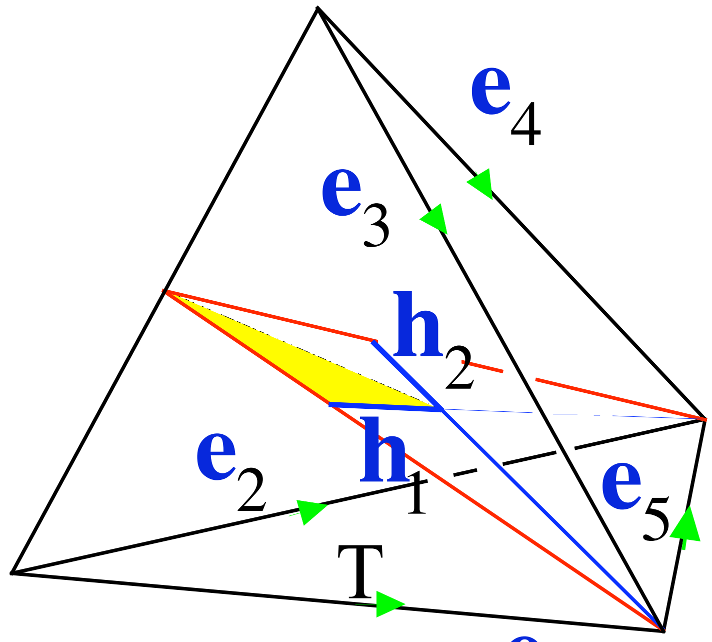
Final recipe for $\int_{\triangle} e \wedge h$:

$$\int_t e \wedge h = \frac{1}{6}\,[\mathbf{e}_a \mathbf{h}_b + \mathbf{e}_b \mathbf{h}_c + \mathbf{e}_c \mathbf{h}_a \;\ldots - \mathbf{h}_a \mathbf{e}_b - \mathbf{h}_b \mathbf{e}_c - \mathbf{h}_c \mathbf{e}_a]$$

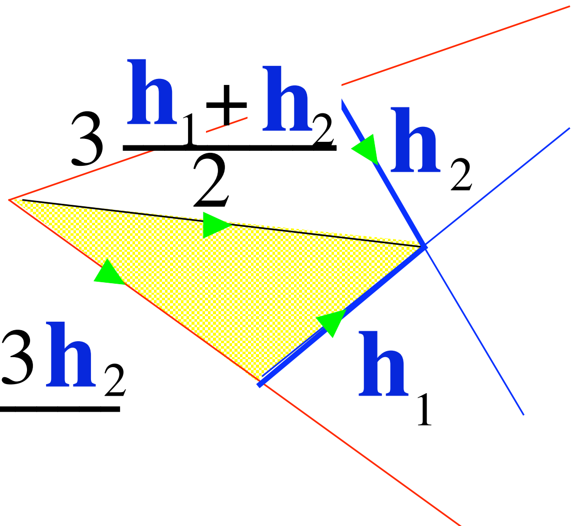


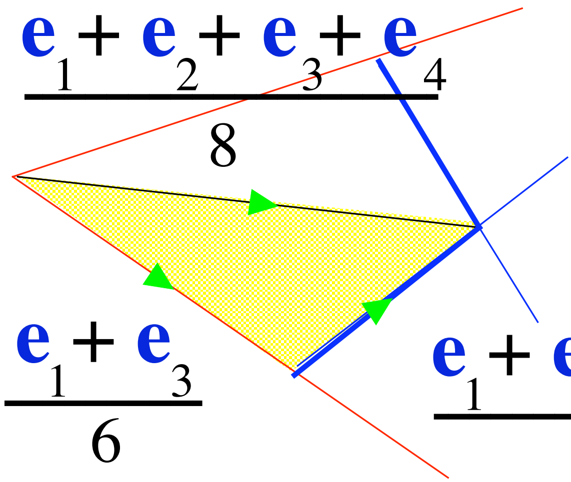
Get $\mathbf{h}_1$, $\mathbf{h}_2$ from

$\mathbf{h} = \boldsymbol{\nu}_T \mathbf{b}$



with these values and orientations:





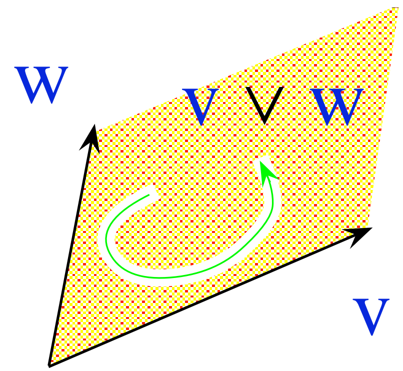
The Lorentz force

Force $\quad F = E + v \times B \quad$ on unit charge

B proxy for b: $\langle v \vee w \,;\, b\rangle = B \cdot (v \times w) \equiv -(v \times B) \cdot w$

Define $i_v b$ as the covector $w \to \langle v \vee w \,;\, b\rangle$

called interior product of b and v



$v \times B$	proxy for	$-\,i_v b$
E	proxy for	e

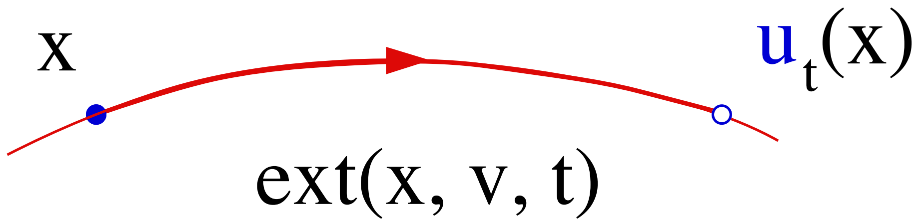
Lorentz force on unit charge passing through point x with velocity v is the covector $e - i_v b$ at point x

So how to "mimic" the inner product

$i_V\, b$?

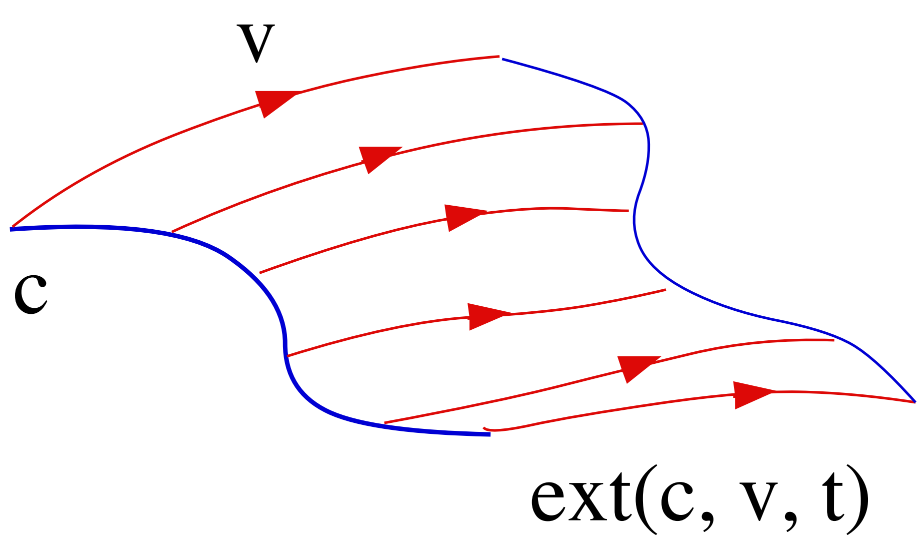
Extrusion *(by the flow of a vector field* v):

- of a point:



$$\left|\begin{array}{l} d_t u_t(x) = v(u_t(x)) \\ u_0(x) = x \end{array}\right.$$

- of a p-manifold:



Inner product:

$$\int_c i_v b = \lim_{t \to 0} \frac{1}{t} \int_{ext(c,\, v,\, t)} b$$

The Lorentz force

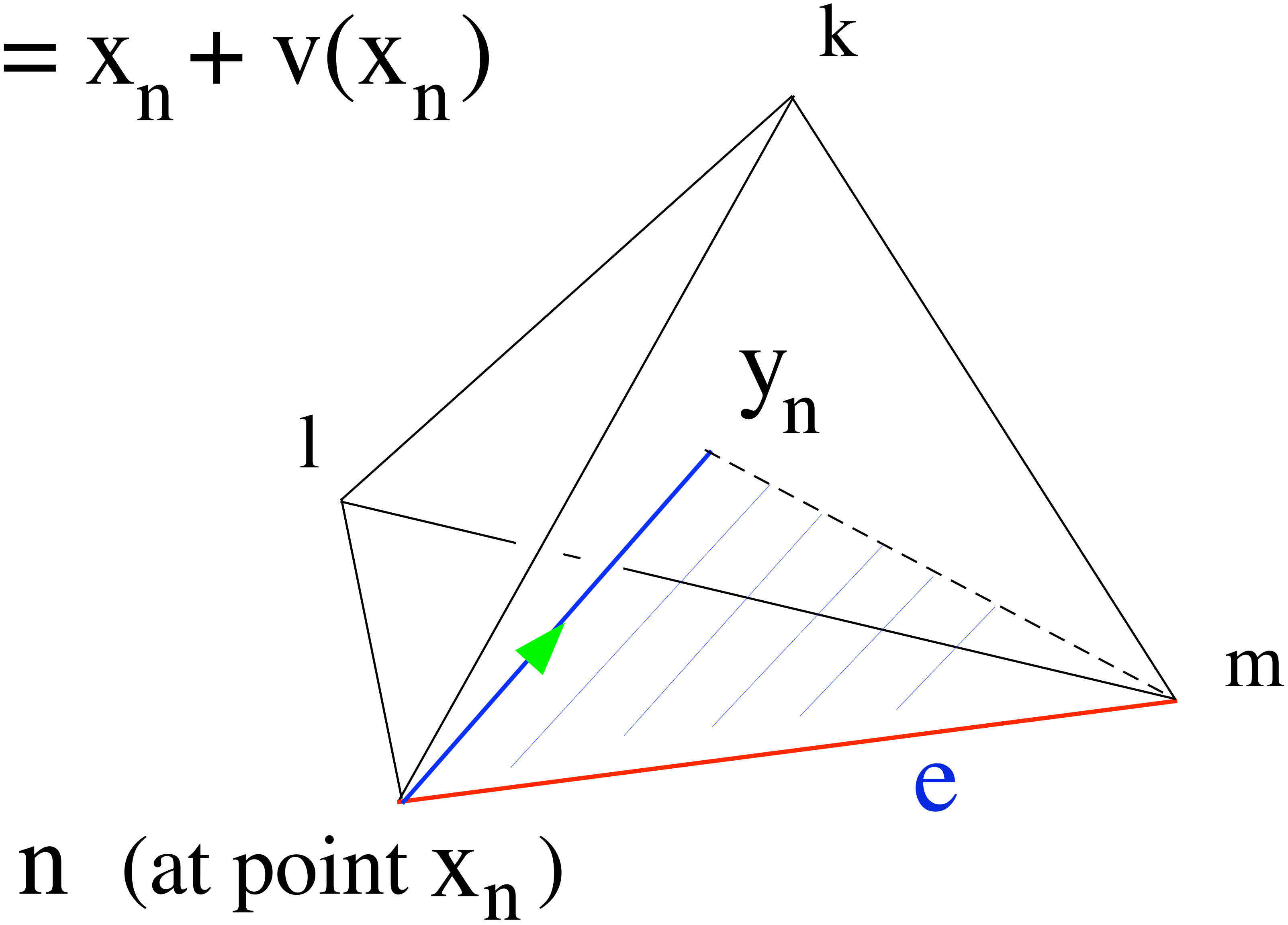
$v \times B$ proxy for $-i_v b$

(vector fields) (1-cochain)

$$\int_e i_v b \sim \int_{ext(e,\,v)} b$$

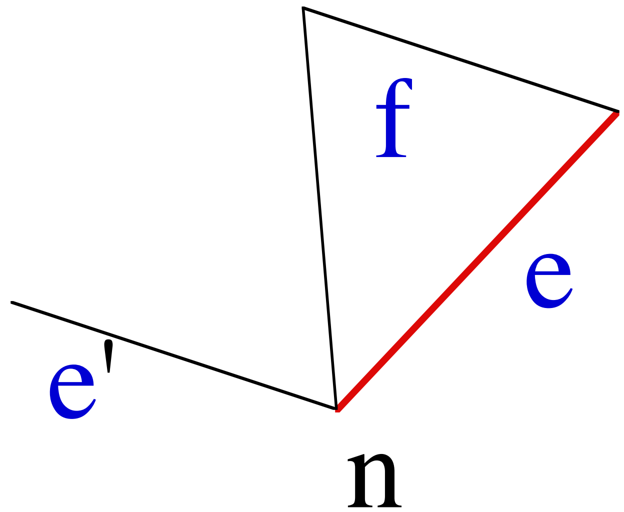
Extrusion of an edge, as a chain of facets?

$$y_n = x_n + v(x_n)$$



$$\mathrm{ext}(e, v) \approx \lambda^k(y_n)\,\mathrm{nmk} + \lambda^l(y_n)\,\mathrm{nml}$$

$I(e, e', f)$ = weight of facet f in extrusion of edge e by the field $\lambda^n e'$



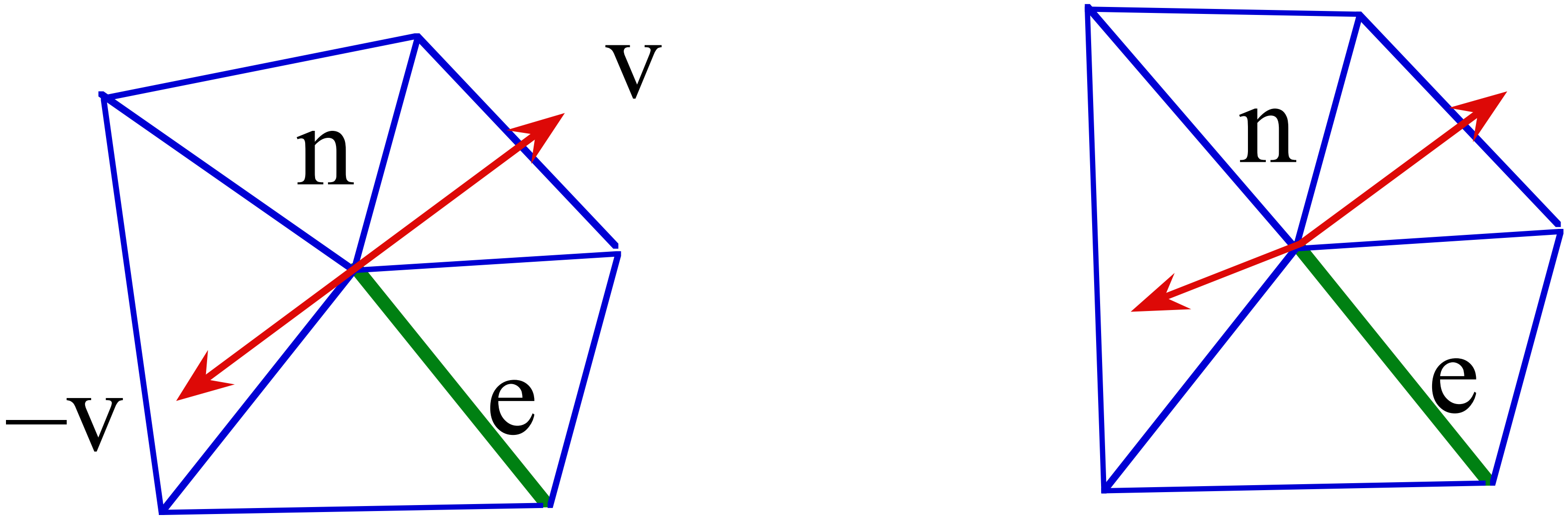
$$v \approx \sum_n \lambda^n(x)\, \mathbf{v}_n = \sum_{n,\, e'} \lambda^n(x)\, v_n^{e'}\, e'$$

$$b = \sum_f \mathbf{b}_f\, w^f$$

$$(i_v b)_e = \sum_{e',\, f} I(e, e', f)\, \mathbf{b}_f\, v_n^{e'}$$

Well and good. But is it true that

$$(i_{-v}\mathbf{b})_e = -\,(i_v\,\mathbf{b})_e \;?$$



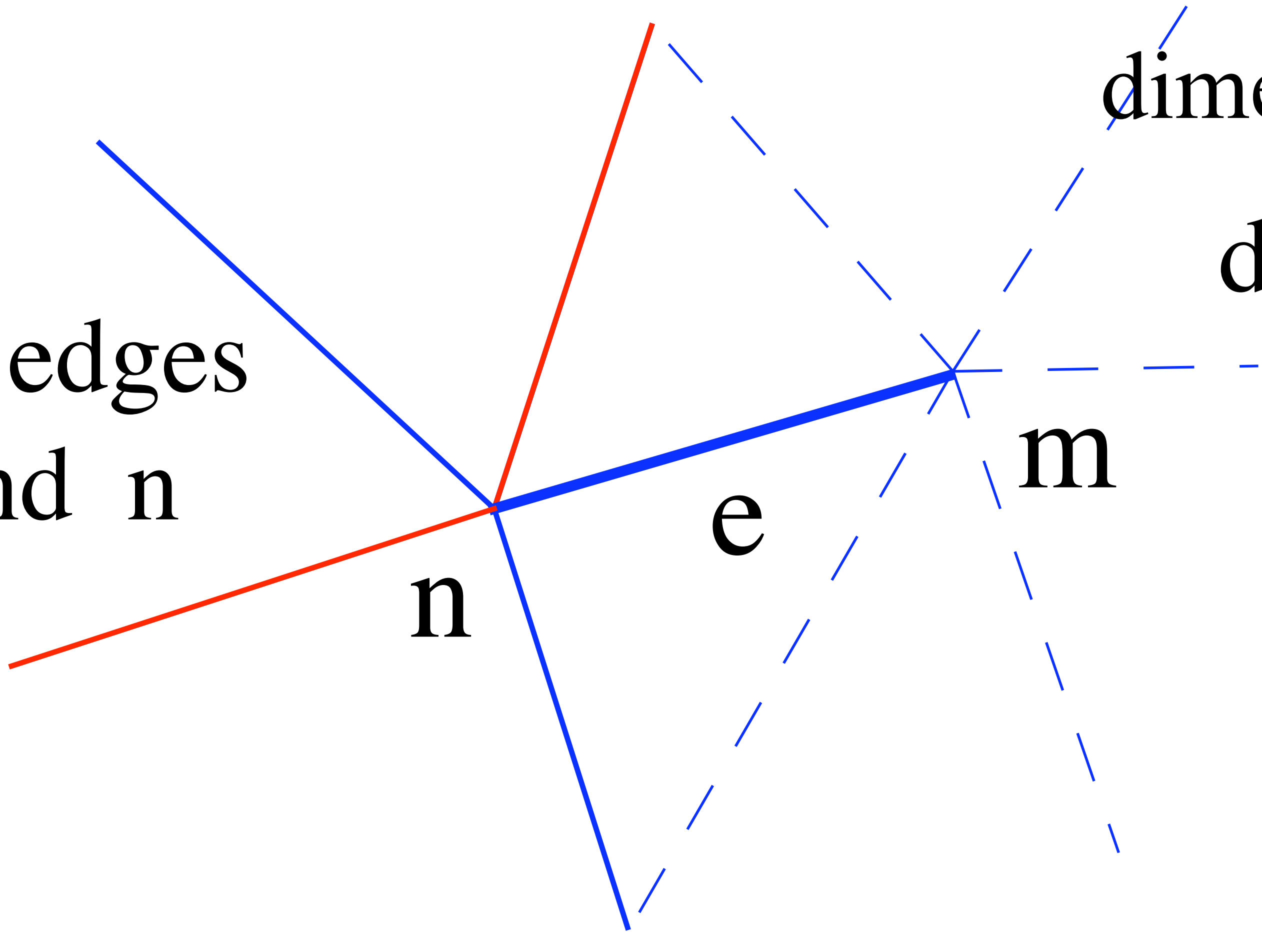
Needed: a discrete notion of "tangent plane at n", or local affine structure

But there is a hitch: Missing the notion of tangent space at a node, we miss the linearity of inner product (and hence, of Lie derivative) w.r.t. flow vector field

This structural element must be specified apart (just as discrete Hodge needed to be)

Local affine structure:

d(n) edges around n



dimension D (2 here)

d(n) – D relations

of the form

$$\Sigma_e \; a_{ne} \, e = 0$$

Now, one can assign a map from T_n to T_m to edge e:
Parallel transport from n to m, connection, etc.

The Laplace force

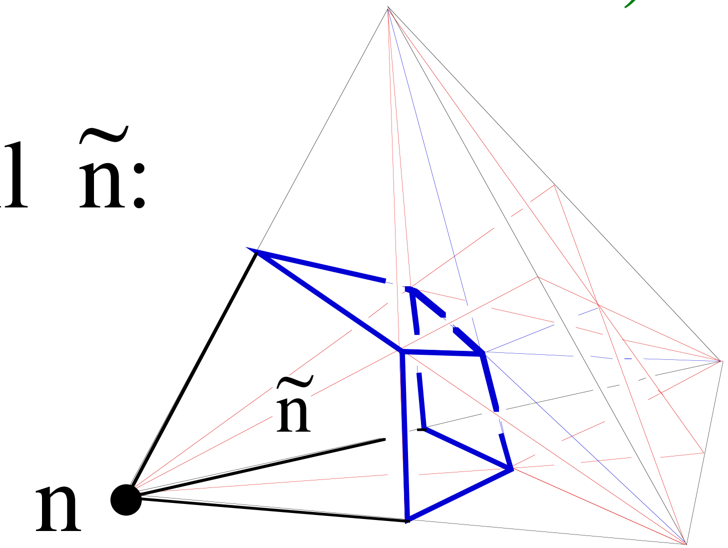
$J \times B$ proxy for $v \to i_v b \wedge j$

(vector field) (covector-valued twisted 3-form)

To be integrated over dual 3-cell $\tilde{n}$:

Similar to $\int e \wedge h$, but now

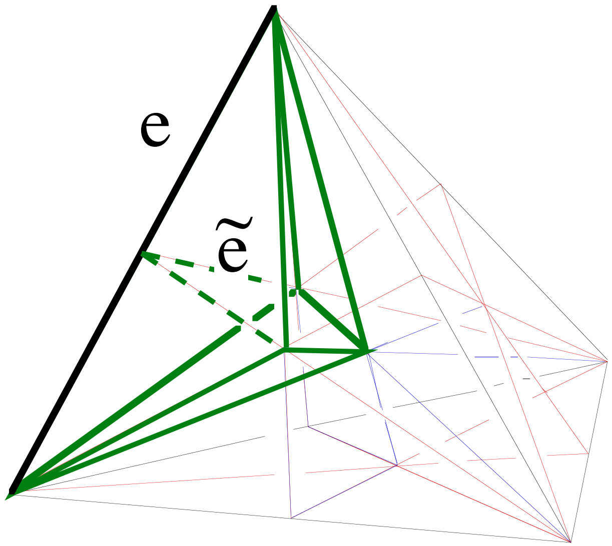
$1 \wedge \tilde{2}$ instead of $1 \wedge \tilde{1}$

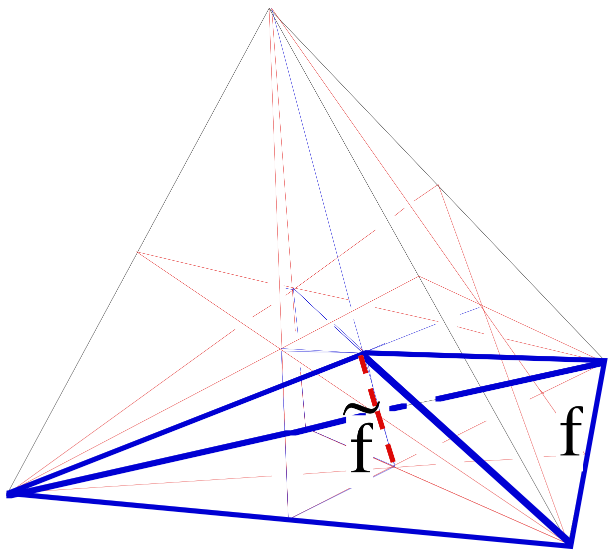


Then, covector $v \to \int_{\tilde{n}} i_v b \wedge j$ is force exerted on $\tilde{n}$

Electric energy, $\int_{\tilde{n}} e \wedge d$, treated like $\int_{\tilde{n}} i_v b \wedge j$

Energy





$$\sum_{e \in \mathcal{E}} e_e\, d_e$$

(electric)

$$\sum_{f \in \mathcal{F}} h_f\, b_f$$

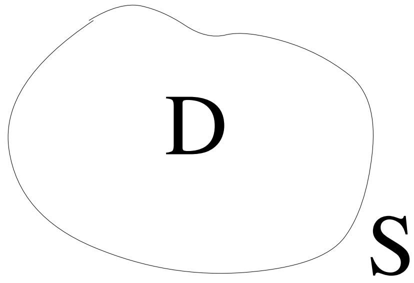
(magnetic)

The Maxwell "tensor"

Start from wedge multiply by

$$-\partial_t \mathbf{d} + \mathrm{d}\mathbf{h} = \mathbf{j} \qquad \wedge\ i_v \mathbf{b}$$

$$\partial_t \mathbf{b} + \mathrm{d}\mathbf{e} = 0 \qquad \wedge\ i_v \mathbf{d}$$



add, integrate over D, use $\mathbf{q} = \mathrm{d}\mathbf{d}$, set

$$\mathbf{f} = v \rightarrow (i_v \mathbf{q} \wedge \mathbf{e} + i_v \mathbf{b} \wedge \mathbf{j})$$

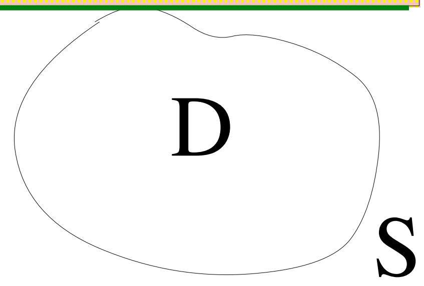
(force *density*, covector-valued twisted 3-form)

find eventually that $\int_D \mathbf{f}$ is equal to

$$\partial_t [\int_D i_v \mathbf{d} \wedge \mathbf{b}] + \int_S [i_v \mathbf{h} \wedge \mathbf{b} + i_v \mathbf{e} \wedge \mathbf{d} - \frac{1}{2}\, i_v(\mathbf{h} \wedge \mathbf{b} + \mathbf{e} \wedge \mathbf{d})]$$

momentum Maxwell (covector-valued, twisted) 2-form

The Maxwell "tensor"



$$\int_D f =$$

$$\partial_t [\int_D i_v d \wedge b] + \int_S [i_v h \wedge b + i_v e \wedge d - \frac{1}{2} i_v (h \wedge b + e \wedge d)]$$

momentum Maxwell (covector-valued, twisted) 2-form

$$\int_S [i_v h \wedge b - \frac{1}{2} i_v (h \wedge b)] = \int_S [i_v b \wedge h + \frac{1}{2} i_v (h \wedge b)]$$

treat like $e \wedge h$

extrude dual faces by v, use result about $h \wedge b$

Conclusion

- Object-oriented programming agenda
- Specific difficulty: infinite dimensional entities (fields) vs finite data structures
- Candidates to "object" status (mesh-related things) have been identified,
- and procedures that apply to them, described
- Discrete avatars of *geometrical* objects, for which traditional vector fields are only *proxies*

Thanks