Fluid-structure interaction problems in the cardiovascular system

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Blood flows and arteries are much more complicated than what will be presented in this talk!
Cardiovascular system modelling ???

- Blood flows and arteries are much more complicated than what will be presented in this talk!

- Nevertheless: even with simplified models, mathematical modelling may help to improve some therapies or medical devices.
Ex 1: Stents and aneurisms

Coils
Ex I: Stents and aneurisms

Coils

Cardiatis *stent*
Ex 1: Stents and aneurisms

Coils

Cardiatis *stent*
Ex 1: Stents and aneurisms

Coils

Cardiatis stent
Ex 1: Stents and aneurisms

Cardiatis stent

Coils

In vivo experiments
(Cardiatis /INRA)
Ex 1: Stents and aneurisms

Coils

Cardiatis stent

In vivo experiments
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« In silico » experiment
(Cardiatis /INRIA)
Ex 1: Stents and aneurisms

Coils

Cardiatis stent

Numerical simulation:
- stent permeability/aneurysm blood flow?
- stent permeability/collateral arteries blood

In vivo experiments (Cardiatis /INRA)

« In silico » experiment (Cardiatis /INRIA)
Ex 1: Stents and aneurisms

Coils

Cardiatis stent

In vivo experiments (Cardiatis /INRA)

Numerical simulation:
- stent permeability/aneurysm blood flow?
- stent permeability/colateral arteries blood

« In silico » experiment (Cardiatis /INRIA)

Without stent
Ex 1: Stents and aneurisms

Numerical simulation:
- stent permeability/aneurysm blood flow?
- stent permeability/collateral arteries blood

In vivo experiments (Cardiatis /INRA)

« In silico » experiment (Cardiatis /INRIA)
Without stent

Coils
Ex 1: Stents and aneurisms

Cardiatis stent

In vivo experiments (Cardiatis /INRA)

Numerical simulation:
- stent permeability/aneurysm blood flow?
- stent permeability/colateral blood flow?

Coils
Ex 2: Abdominal Aortic Aneurism

Endocom project:
wireless pressure monitoring
Ex 2: Abdominal Aortic Aneurism

Numerical simulation:
- optimal position of the sensor

Endocom project:
- wireless pressure monitoring
Focus of this talk: fluid-structure interaction
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Fluid-structure interaction: interaction with vessel, heart muscle, valves, etc.
Focus of this talk: fluid-structure interaction

- **Fluid-structure interaction**: interaction with vessel, heart muscle, valves, etc.
- **Challenging problems** for scientific computing:
  - Efficiency & stability
Focus of this talk: fluid-structure interaction

- Fluid-structure interaction: interaction with vessel, heart muscle, valves, etc.
- Challenging problems for scientific computing:
  - Efficiency & stability
  - Moving domain: geometrical non-linearities, topological change (contact)
Focus of this talk: fluid-structure interaction

- **Fluid-structure interaction**: interaction with vessel, heart muscle, valves, etc.

- **Challenging problems** for scientific computing:
  - Efficiency & stability
  - Moving domain: geometrical non-linearities, topological change (contact)
  - Boundary conditions
Outline

- Fluid-structure interaction in arteries
  - Stability issues - Strongly couple schemes
  - A stable weakly coupled scheme
- Fluid-structure interaction with valves
  - Fictitious domain method
  - FSI and Contact management
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Fluid-structure interaction in arteries
Fluid-structure interaction in arteries

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F. Nicoud, H. Vernhet, M. Dauzat
Fluid-structure interaction in arteries

Wall : Arbitrary Lagrangian Eulerian (ALE)
Fluid-structure interaction in arteries

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The fluid-structure problem:

\[ \Omega^f(t), \] moving fluid domain (filled with a viscous fluid: blood)

\[ \Omega^s(t), \] solid current configuration (artery wall)

\[ \Sigma(t) \] fluid-solid interface, where we enforce

- continuity of velocity
- continuity of stress

Determine \( \Omega^f(t) \), velocity and stress within the fluid and solid
The coupled problem

- Fluid equations:

\[
\rho^f \left( \frac{\partial u}{\partial t} \bigg|_{\tilde{x}} + (u - w) \cdot \nabla u \right) - 2\mu \text{div}\varepsilon(u) + \nabla p = 0, \quad \text{in} \quad \Omega^f(t)
\]

\[
\text{div} u = 0, \quad \text{in} \quad \Omega^f(t)
\]

\[
\sigma(u, p)n = g, \quad \text{on} \quad \Gamma_N^f
\]
The coupled problem

- **Fluid equations:**
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  \rho^f \left( \frac{\partial u}{\partial t} \bigg|_{\hat{x}} + (u - w) \cdot \nabla u \right) - 2\mu \text{div} \varepsilon(u) + \nabla p = 0, \quad \text{in} \quad \Omega^f(t) \\
  \text{div} u = 0, \quad \text{in} \quad \Omega^f(t) \\
  \sigma(u, p)n = g, \quad \text{on} \quad \Gamma^f_N
  \]

- **Solid equations:**
  \[
  \rho^s \frac{\partial^2 d}{\partial t^2} - \text{div}(F(d)S(d)) = 0, \quad \text{in} \quad \hat{\Omega}^s \\
  d = 0, \quad \text{on} \quad \hat{\Gamma}^s_D \\
  F(d)S(d)\hat{n}^s = 0, \quad \text{on} \quad \hat{\Gamma}^s_N
  \]
The coupled problem

- **Fluid equations:**
  \[
  \rho_f \left( \frac{\partial u}{\partial t} |_{\bar{x}} + (u - w) \cdot \nabla u \right) - 2\mu \text{div}\epsilon(u) + \nabla p = 0, \quad \text{in} \quad \Omega^f(t)
  \]
  \[
  \text{div} \ u = 0, \quad \text{in} \quad \Omega^f(t)
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  \]
  \[
  d = 0, \quad \text{on} \quad \hat{\Gamma}_D
  \]
  \[
  F(d)S(d)\hat{n}^s = 0, \quad \text{on} \quad \hat{\Gamma}_N
  \]

- **Coupling conditions:**
  \[
  d^f = \text{Ext}(d|_{\Sigma}), \quad w(d^f) = \frac{\partial d^f}{\partial t}, \quad \text{in} \quad \hat{\Omega}^f, \quad \Omega^f(t) = (I + d^f)(\hat{\Omega}^f), \quad \text{(geometry)}
  \]
  \[
  u = w(d^f), \quad \text{on} \quad \Sigma(t), \quad \text{(velocity)}
  \]
  \[
  F(d)S(d)\hat{n} = J(d^f)\sigma(u, p)F(d^f)^{-T}\hat{n}, \quad \text{on} \quad \hat{\Sigma}, \quad \text{(stress)}
  \]
Remark: alternatives
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- 1D models
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- 3D Interesting simplified approaches have been proposed:
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  - Figueroa, Vignon-Clementel, Jansen, Hughes, Taylor, 2006
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- In those cases, the FSI cost is almost the fluid cost.
Remark: alternatives

- **1D models**

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  - if the stress within the wall is required
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In the sequel, we only consider the cases where a “real” structure problem has to be solved. Useful

if the stress within the wall is required

for valves
Implementation issues

Use independent solvers for fluid and structure:

- Advantage: re-usability of state-of-the-art algorithms
- Difficulties: possible troubles with the coupling algorithms
Implementation issues

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Strong coupling: sub-iterations at each time step

Weak coupling: 1 or 2 iterations per time step
Example: Dirichlet-Neumann

- Fluid sub-problem: \((d^{f,n+1}, u^{n+1}, p^{n+1}) = \mathcal{F}(d^{m+1}_{|\Sigma})\)
Example: Dirichlet-Neumann

- Fluid sub-problem: 
  \[(d_{f,n+1}^n, u_{n+1}^n, p_{n+1}^n) = \mathcal{F}(d_{\Sigma}^{n+1})\]

- Solid sub-problem: 
  \[d_{\Sigma}^{n+1} = S(d_{f,n+1}^n, u_{n+1}^n, p_{n+1}^n)\]
Example: Dirichlet-Neumann

- Fluid sub-problem:
  \[ (d^{f,n+1}, u^{n+1}, p^{n+1}) = \mathcal{F}(d^{n+1}_{\Sigma}) \]

- Solid sub-problem:
  \[ d^{n+1}_{\Sigma} = S(d^{f,n+1}, u^{n+1}, p^{n+1}) \]

**Fixed-point iterations with acceleration**

(1) Initialization:
\[ \lambda^0 = d^n_{\Sigma} \]

(2) Until convergence \((k \geq 0)\):

(a) Solve fluid and solid:
\[ \tilde{\lambda}^{k+1} = (S \circ \mathcal{F})(\lambda^k) \]

(b) Relaxation:
\[ \lambda^{k+1} = \omega_k \tilde{\lambda}^{k+1} + (1 - \omega_k)\lambda^k, \quad \omega_k \in (0, 1] \]

- **Aitken acceleration** (Wall, Ramm, 2001):
\[ \omega_k = \frac{(\lambda^k - \lambda^{k-1}) \cdot (\lambda^k - \tilde{\lambda}^k - \lambda^{k-1} + \tilde{\lambda}^{k-1})}{|\lambda^k - \tilde{\lambda}^k - \lambda^{k-1} + \tilde{\lambda}^{k-1}|^2} \]
Example: Dirichlet-Neumann

- Fluid sub-problem: 
  \[(d^{f,n+1}, u^{n+1}, p^{n+1}) = F(d_{\Sigma}^{n+1})\]

- Solid sub-problem: 
  \[d_{\Sigma}^{n+1} = S(d^{f,n+1}, u^{n+1}, p^{n+1})\]

### Fixed-point iterations with acceleration

1. Initialization: 
   \[\lambda^0 = d_{\Sigma}^n\]

2. Until convergence \(k \geq 0\):
   a. Solve fluid and solid: 
      \[\tilde{\lambda}^{k+1} = (S \circ F)(\lambda^k)\]
   b. Relaxation: 
      \[\lambda^{k+1} = \omega_k \tilde{\lambda}^{k+1} + (1 - \omega_k) \lambda^k, \quad \omega_k \in (0, 1]\]

   - **Aitken acceleration** *(Wall, Ramm, 2001)*: 
     \[\omega_k = \frac{(\lambda^k - \lambda^{k-1}) \cdot (\lambda^k - \tilde{\lambda}^k - \lambda^{k-1} + \tilde{\lambda}^{k-1})}{|\lambda^k - \tilde{\lambda}^k - \lambda^{k-1} + \tilde{\lambda}^{k-1}|^2}\]

---

Vierendeels, Lanoye, Degroote, Verdonck, 2007
Vierendeels, 2006
Domain decomposition
Option 1: decompose first then linearize

Dirichlet-Neumann
- Newton: *Fernandez-Moubachir* (2003), ...
  *Mischler-van Brummelen-de Borst* (2005), *Vierendeels* (2006), ...

Neumann - Neumann
- *Deparis-Discacciati-Quarteroni* (2005), ...

Robin - Neumann
**Domain decomposition**

**Option 1 : decompose first then linearize**

- Dirichlet-Neumann
  - Newton: *Fernandez-Moubachir* (2003), ...
- Neumann - Neumann
  - *Deparis-Discacciati-Quarteroni* (2005), ...
- Robin - Neumann

**Option 2 : linearize first then decompose**

Domain decomposition

Option 1: decompose first then linearize

- Dirichlet-Neumann
- Neumann - Neumann
  - Deparis-Discacciati-Quarteroni (2005)
- Robin - Neumann
  - Badia-Nobile-Vergara (2007)

Option 2: linearize first then decompose


Common feature:
These algorithms are strongly coupled, for stability.
Number of sub-iterations: between 10 and 100!
Explicit coupling: some observations
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- An explicit algorithm is *a priori* very efficient...

\[ \text{FSI cost} \approx \text{FLUID cost} + \text{SOLID cost} \]
Explicit coupling: some observations

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\[
\text{FSI cost} \approx \text{FLUID cost} + \text{SOLID cost}
\]

- ... but unstable!

![Implicit coupling](image1)

![Explicit coupling](image2)
Explicit coupling: some observations

- An explicit algorithm is *a priori* very efficient...
  
  $$\text{FSI cost} \approx \text{FLUID cost} + \text{SOLID cost}$$

- ... but unstable!

- Energy estimate with an *artificial interface power* term:
  $$\int_{\Sigma^n} \sigma(u^{n+1}, p^{n+1}) n \cdot \left( u^{n+1} - \frac{d^{n+1} - d^n}{\delta t} \right)$$
Explicit coupling: some observations

• An explicit algorithm is *a priori* very efficient...

\[ \text{FSI cost} \approx \text{FLUID cost} + \text{SOLID cost} \]

• *... but unstable!* 

- Implicit coupling
- Explicit coupling

• Energy estimate with an artificial interface power term:

\[ \int_{\Sigma^{n+1}} \sigma(u^{n+1}, p^{n+1}) n \cdot \left( u^{n+1} - \frac{d^{n+1} - d^n}{\delta t} \right) \]

• Explicit coupling is stable and widely used in aeroelasticity!

Ex: Farhat, van der Zee, Geuzaine, 2006
Explicit coupling: some observations

- An explicit algorithm is *a priori* very efficient...

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- *... but unstable!*

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  \]

- Explicit coupling is stable and widely used in aeroelasticity!
  Ex: *Farhat, van der Zee, Geuzaine, 2006*

- What is the source of instabilities of those schemes in blood flows?
A 2D simplified model

(Causin, JFG, Nobile, 2004)
A 2D simplified model

(Causin, JFG, Nobile, 2004)

• **Solid**: string model (*small displacements*)

\[
\rho^s \varepsilon \ddot{d} + Ld = p_{|\Sigma}, \quad \text{in} \quad \Sigma,
\]

with

- \( d \): vertical displacement
- \( \varepsilon \): vessel thickness
- \( L \): linear operator (for instance \( L\eta = a\eta - b \frac{\partial^2 \eta}{\partial x^2} \))
A 2D simplified model

(Causin, JFG, Nobile, 2004)

- **Solid**: string model (infinitesimal displacements)
  \[ \rho^s \varepsilon \ddot{d} + Ld = p_{|\Sigma}, \text{ in } \Sigma, \]

- **Fluid**: fixed fluid domain, no viscous/convective terms
  \[
  \begin{cases}
  \rho^f \frac{\partial u}{\partial t} + \nabla p = 0, & \text{in } \Omega^f \\
  \text{div } u = 0, & \text{in } \Omega^f \\
  u \cdot n = \dot{d}, & \text{on } \Sigma \\
  u \cdot n = 0, & \text{on } \Gamma_1 \\
  p = 0, & \text{on } \Gamma_2
  \end{cases}
  \]
A 2D simplified model

(Causin, JFG, Nobile, 2004)

- **Solid:** string model (infinitesimal displacements)
  \[ \rho^s \ddot{d} + L \dot{d} = p_{\mid \Sigma}, \quad \text{in} \quad \Sigma, \]

- **Fluid:** fixed fluid domain, no viscous/convective terms
  \[
  \begin{align*}
    \rho^f \frac{\partial u}{\partial t} + \nabla p &= 0, \quad \text{in} \quad \Omega^f \\
    \text{div} \, u &= 0, \quad \text{in} \quad \Omega^f \\
    u \cdot n &= \dot{d}, \quad \text{on} \quad \Sigma \\
    u \cdot n &= 0, \quad \text{on} \quad \Gamma_1 \\
    p &= 0, \quad \text{on} \quad \Gamma_2
  \end{align*}
  \]

\[
  \begin{align*}
    -\Delta p &= 0, \quad \text{in} \quad \Omega^f \\
    \frac{\partial p}{\partial n} &= -\rho^f \frac{\partial u}{\partial t} \cdot n = -\rho^f \ddot{d}, \quad \text{on} \quad \Sigma \\
    \frac{\partial p}{\partial n} &= 0, \quad \text{on} \quad \Gamma_1 \\
    p &= 0 \quad \text{on} \quad \Gamma_2
  \end{align*}
\]
A 2D simplified model

(Causin, JFG, Nobile, 2004)

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  p = 0, & \text{on } \Gamma_2 \\
  \end{cases}
  \Rightarrow
  \begin{cases}
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  \end{cases}
  \]

Interest of this model:
- **Physics**: reproduces propagation phenomena
- **Numerics**: explicit coupling unstable
A 2D simplified model

(Causin, JFG, Nobile, 2004)

- **Solid**: string model (infinitesimal displacements)
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- **Fluid**: fixed fluid domain, no viscous/convective terms
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  u \cdot n = 0, & \text{on} \quad \Gamma_1 \\
  p = 0, & \text{on} \quad \Gamma_2 
  \end{cases}
  \\
  \implies \text{div } u = 0, \quad \text{in} \quad \Omega^f
  \end{aligned}
  \]

  \[
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**Interest of this model:**
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A 2D simplified model

\[ \Sigma \]

\[ \Omega^f \]

\[ \Gamma_1 \]

\[ \Gamma_2 \]

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(Causin, JFG, Nobile, 2004)
The added-mass operator

\[
\begin{aligned}
\text{Fluid:} & \quad \begin{cases}
-\Delta p = 0, & \text{in } \Omega^f \\
\frac{\partial p}{\partial n} = -\rho^f \ddot{d}, & \text{on } \Sigma \\
\frac{\partial p}{\partial n} = 0, & \text{on } \Gamma_1 \\
p = 0 & \text{on } \Gamma_2 
\end{cases} \\
\text{Solid:} & \quad \rho^s \varepsilon \ddot{d} + Ld = p_{|\Sigma}, \quad \text{in } \Sigma,
\end{aligned}
\]
The added-mass operator

\[ \rho_s \ddot{e} + Ld = p|_\Sigma, \quad \text{in} \quad \Sigma, \]

Fluid: \[ \begin{aligned}
-\Delta p &= 0, \quad \text{in} \quad \Omega^f \\
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p &= 0 \quad \text{on} \quad \Gamma_2
\end{aligned} \]

Solid: \[ \rho^s \ddot{e} + Ld = p|_\Sigma, \quad \text{in} \quad \Sigma, \]

**Theorem (Steklov-Poincaré operator)**

The operator \( \mathcal{M}_A : H^{-\frac{1}{2}}(\Sigma) \rightarrow H^{\frac{1}{2}}(\Sigma) \) defined as: for each \( g \in H^{-\frac{1}{2}}(\Sigma) \) we set \( \mathcal{M}_A(g) \overset{\text{def}}{=} q|_{\Gamma^w} \), where \( q \in H^{1}(\Omega^f) \) solves

\[ \begin{aligned}
-\Delta q &= 0, \quad \text{in} \quad \Omega^f \\
\frac{\partial q}{\partial n} &= g, \quad \text{on} \quad \Sigma \\
\frac{\partial q}{\partial n} &= 0, \quad \text{on} \quad \Gamma_1 \\
q &= 0, \quad \text{on} \quad \Gamma_2
\end{aligned} \]

is a linear, compact, positive and self-adjoint operator in \( L^2(\Sigma) \).
The added-mass operator

\[ -\Delta p = 0, \quad \text{in} \quad \Omega^f \]
\[ \frac{\partial p}{\partial n} = -\rho^f \ddot{d}, \quad \text{on} \quad \Sigma \]
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Solid: \( \rho^s \ddot{d} + Ld = p|_\Sigma, \quad \text{in} \quad \Sigma, \)

Theorem (Steklov-Poincaré operator)

The operator \( \mathcal{M}_A : H^{-\frac{1}{2}}(\Sigma) \rightarrow H^{\frac{1}{2}}(\Sigma) \) defined as: for each \( g \in H^{-\frac{1}{2}}(\Sigma) \) we set
\[ \mathcal{M}_A(g) \overset{\text{def}}{=} q|_{\Gamma^w}, \text{where } q \in H^1(\Omega^f) \text{ solves} \]
\[ \begin{cases} -\Delta q = 0, & \text{in } \Omega^f \\ \frac{\partial q}{\partial n} = g, & \text{on } \Sigma \\ \frac{\partial q}{\partial n} = 0, & \text{on } \Gamma_1 \\ q = 0, & \text{on } \Gamma_2 \end{cases} \]
is a linear, compact, positive and self-adjoint operator in \( L^2(\Sigma). \)

✓ From this definition, we have \( p|_\Sigma = \mathcal{M}_A(-\rho^f \ddot{d}) = -\rho^f \mathcal{M}_A \ddot{d} \)
The added-mass effect

\[ \rho^s \ddot{\epsilon} \dd + Ld = p_{|\Sigma}, \quad \text{in} \quad \Sigma, \]

Fluid:
\[
\begin{cases}
-\Delta p = 0, & \text{in} \quad \Omega^f \\
\frac{\partial p}{\partial n} = -\rho^f \ddot{d}, & \text{on} \quad \Sigma \\
\frac{\partial p}{\partial n} = 0, & \text{on} \quad \Gamma_1 \\
p = 0 & \text{on} \quad \Gamma_2
\end{cases}
\]

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\[
p_{|\Sigma} = -\rho^f \mathcal{M}_A \dd
\]
The added-mass effect

Fluid: \[
\begin{aligned}
-\Delta p &= 0, & \text{in } & \Omega^f \\
\frac{\partial p}{\partial n} &= -\rho^f \ddot{d}, & \text{on } & \Sigma \\
\frac{\partial p}{\partial n} &= 0, & \text{on } & \Gamma_1 \\
 p &= 0 & \text{on } & \Gamma_2
\end{aligned}
\]

Solid: \[
\rho^s \varepsilon \ddot{d} + Ld = p|_\Sigma, \quad \text{in } \Sigma,
\]

\[
\begin{aligned}
p|_\Sigma &= -\rho^f M_A \ddot{d}
\end{aligned}
\]

\[
(\rho^s \varepsilon + \rho^f M_A) \ddot{d} + Ld = 0, \quad \text{in } \Sigma
\]
The added-mass effect

Fluid: \[
\begin{align*}
-\Delta p &= 0, \quad \text{in } \Omega_f \\
\frac{\partial p}{\partial n} &= -\rho^f \ddot{d}, \quad \text{on } \Sigma \\
\frac{\partial p}{\partial n} &= 0, \quad \text{on } \Gamma_1 \\
p &= 0 \quad \text{on } \Gamma_2
\end{align*}
\]

Solid: \[
\rho^s \varepsilon \ddot{d} + Ld = p|_{\Sigma}, \quad \text{in } \Sigma,
\]

\[
\begin{aligned}
\dot{p}|_{\Sigma} &= -\rho^f M_A \ddot{d} \\
(\rho^s \varepsilon + \rho^f M_A) \ddot{d} + Ld &= 0, \quad \text{in } \Sigma
\end{aligned}
\]

Remarks:

- This equation looks like a structure equation, except for the extra “mass” term
- The fluid-structure coupling can be condensed into an extra mass action on the structure (hence the terminology “added-mass effect”)
The added-mass effect

\[ \rho_s \varepsilon \ddot{d} + Ld = p|_\Sigma, \quad \text{in} \quad \Sigma, \quad (1) \]

Fluid:
\[
\begin{aligned}
-\Delta p &= 0, & \text{in} & \quad \Omega^f \\
\frac{\partial p}{\partial n} &= -\rho^f \ddot{d}, & \text{on} & \quad \Sigma \\
\frac{\partial p}{\partial n} &= 0, & \text{on} & \quad \Gamma_1 \\
p &= 0 & \text{on} & \quad \Gamma_2
\end{aligned}
\]

Solid:
\[
\rho_s \varepsilon \ddot{d} + Ld = p|_\Sigma, \quad \text{in} \quad \Sigma
\]

\[
(\rho_s \varepsilon + \rho^f M_A) \ddot{d} + Ld = 0, \quad \text{in} \quad \Sigma \quad (2)
\]

Remarks:
- This equation looks like a structure equation, except for the extra “mass” term
- The fluid-structure coupling can be condensed into an extra mass action on the structure (hence the terminology “added-mass effect”)

Question:
What kind of time integration scheme of (2) arises from the explicit coupling of (1)?
Explicit coupling and added-mass

Fluid:

\[
\begin{aligned}
\rho_f \frac{u^{n+1} - u^n}{\delta t} + \nabla p^{n+1} &= 0 \\
\text{div } u^{n+1} &= 0 \\
u^{n+1} \cdot n &= \frac{d^n - d^{n-1}}{\delta t}
\end{aligned}
\]
Explicit coupling and added-mass

\[
\begin{align*}
\text{Fluid:} & \quad \left\{ \begin{array}{l}
\rho^f \frac{u^{n+1} - u^n}{\delta t} + \nabla p^{n+1} = 0 \\
\text{div } u^{n+1} = 0 \\
u^{n+1} \cdot n = \frac{d^n - d^{n-1}}{\delta t}
\end{array} \right. \\
\Rightarrow & \quad \left\{ \begin{array}{l}
\Delta p^{n+1} = 0 \\
\frac{\partial p^{n+1}}{\partial n} = -\rho^f \frac{d^n - 2d^{n-1} + d^{n-2}}{\delta t^2}
\end{array} \right.
\end{align*}
\]
Explicit coupling and added-mass

**Fluid:**
\[
\begin{align*}
\rho_f \frac{u^{n+1} - u^n}{\delta t} + \nabla p^{n+1} &= 0 \\
\text{div} \ u^{n+1} &= 0 \\
\frac{d^{n+1} - d^{n-1}}{\delta t} 
\end{align*}
\]

\[
\begin{align*}
\text{div} \ u^{n+1} &= 0 \\
\frac{d^{n+1} - d^{n-1}}{\delta t} 
\end{align*}
\]

**Solid:**
\[
\rho_s \varepsilon \frac{d^{n+1} - 2d^n + d^{n-1}}{\delta t^2} + Ld^{n+1} = p^{n+1}_| \Sigma
\]

\[
\begin{align*}
-\Delta p^{n+1} &= 0 \\
\frac{\partial p^{n+1}}{\partial n} &= -\rho_f \frac{d^n - 2d^{n-1} + d^{n-2}}{\delta t^2}
\end{align*}
\]
Explicit coupling and added-mass

\[
\begin{align*}
\text{Fluid:} & \quad \left\{ \begin{array}{l}
\rho^f \frac{u^{n+1} - u^n}{\delta t} + \nabla p^{n+1} = 0 \\
d \cdot u^{n+1} = 0 \\
u^{n+1} \cdot n = \frac{d^n - d^{n-1}}{\delta t}
\end{array} \right. \\
\text{Solid:} & \quad \left\{ \begin{array}{l}
\rho^s \varepsilon \frac{d^{n+1} - 2d^n + d^{n-1}}{\delta t^2} + Ld^{n+1} = p^{n+1}_\Sigma \\
p^{n+1}_\Sigma = -\rho^f M_A \frac{d^n - 2d^{n-1} + d^{n-2}}{\delta t^2}
\end{array} \right.
\end{align*}
\]
Explicit coupling and added-mass

\[
\begin{aligned}
\text{Fluid:} & \quad \left\{ \begin{array}{l}
\rho^f \frac{u^{n+1} - u^n}{\delta t} + \nabla p^{n+1} = 0 \\
\text{div} \, u^{n+1} = 0 \\
u^{n+1} \cdot n = \frac{d^n - d^{n-1}}{\delta t}
\end{array} \right. \\
\Rightarrow & \quad \left\{ \begin{array}{l}
-\Delta p^{n+1} = 0 \\
\frac{\partial p^{n+1}}{\partial n} = -\rho^f \frac{d^n - 2d^{n-1} + d^{n-2}}{\delta t^2}
\end{array} \right.
\end{aligned}
\]

\[
\text{Solid:} \quad \rho^s \varepsilon \frac{d^{n+1} - 2d^n + d^{n-1}}{\delta t^2} + Ld^{n+1} = p^{n+1}_\Sigma
\]

\[
p^{n+1}_\Sigma = -\rho^f \mathcal{M}_A \frac{d^n - 2d^{n-1} + d^{n-2}}{\delta t^2}
\]

Condensed FSI problem:

\[
\rho^s \varepsilon \frac{d^{n+1} - 2d^n + d^{n-1}}{\delta t^2} + \rho^f \mathcal{M}_A \frac{d^n - 2d^{n-1} + d^{n-2}}{\delta t^2} + L\eta^{n+1} = 0
\]
Explicit coupling and added-mass

Fluid:
\[
\begin{align*}
\rho^f \frac{u^{n+1} - u^n}{\delta t} + \nabla p^{n+1} &= 0 \\
\text{div } u^{n+1} &= 0 \\
u^{n+1} \cdot n &= \frac{d^n - d^{n-1}}{\delta t}
\end{align*}
\]

Implict coupling and added-mass
\[
\begin{align*}
-\Delta p^{n+1} &= 0 \\
\frac{\partial p^{n+1}}{\partial n} &= -\rho^f \frac{d^n - 2d^{n-1} + d^{n-2}}{\delta t^2}
\end{align*}
\]

Solid:
\[
\rho^s \varepsilon \frac{d^{n+1} - 2d^n + d^{n-1}}{\delta t^2} + Ld^{n+1} = p_{n+1}^n
\]

Condensed FSI problem:
\[
\begin{align*}
\rho^s \varepsilon \frac{d^{n+1} - 2d^n + d^{n-1}}{\delta t^2} + \rho^f \mathcal{M}_A \frac{d^n - 2d^{n-1} + d^{n-2}}{\delta t^2} + L\eta^{n+1} &= 0
\end{align*}
\]
Explicit coupling and added-mass

\[ \begin{aligned}
\text{Fluid:} & \quad \left\{ \begin{array}{l}
\rho^f \frac{u^{n+1} - u^n}{\delta t} + \nabla p^{n+1} = 0 \\
\text{div} \, u^{n+1} = 0 \\
u^{n+1} \cdot n = \frac{d^n - d^{n-1}}{\delta t}
\end{array} \right. \\
\Rightarrow \quad \left\{ \begin{array}{l}
-\Delta p^{n+1} = 0 \\
\frac{\partial p^{n+1}}{\partial n} = -\rho^f \frac{d^n - 2d^{n-1} + d^{n-2}}{\delta t^2}
\end{array} \right. \\
\text{Solid:} & \quad \rho^s \varepsilon \frac{d^{n+1} - 2d^n + d^{n-1}}{\delta t^2} + Ld^{n+1} = p^{n+1}_\Sigma
\end{aligned} \]

Condensed FSI problem:

\[ \begin{aligned}
\rho^s \varepsilon \frac{d^{n+1} - 2d^n + d^{n-1}}{\delta t^2} + \rho^f M_A \frac{d^n - 2d^{n-1} + d^{n-2}}{\delta t^2} + L\eta^{n+1} = 0
\end{aligned} \]
Explicit coupling and added-mass

Fluid:
\[
\begin{align*}
\rho_f & \frac{u^{n+1} - u^n}{\delta t} + \nabla p^{n+1} = 0 \\
\text{div } u^{n+1} &= 0 \\
\text{div } u^{n+1} &= 0 \\
u^{n+1} \cdot n &= \frac{d^n - d^{n-1}}{\delta t}
\end{align*}
\]

Solid:
\[
\rho_s \varepsilon \frac{d^{n+1} - 2d^n + d^{n-1}}{\delta t^2} + Ld^{n+1} = p_{n+1}^{n+1}
\]

Condensed FSI problem:
\[
\rho_s \varepsilon \frac{d^{n+1} - 2d^n + d^{n-1}}{\delta t^2} + \rho_f \mathcal{M}_A \frac{d^n - 2d^{n-1} + d^{n-2}}{\delta t^2} + L\eta^{n+1} = 0
\]

Observation:
Weak coupling leads to an explicit discretization of the added-mass
An unconditional instability result

Proposition (Causin-JFG-Nobile 04)

Let $\lambda_{\text{max}}$ be the largest eigenvalue of $M_A$ and assume that $L\eta = a\eta$. Then, the previous explicit coupling scheme is unconditionally unstable whenever

$$\frac{\rho^f \lambda_{\text{max}}}{\rho^s \varepsilon} \geq 1.$$  

(1)
An unconditional instability result

Proposition (Causin-JFG-Nobile 04)

Let $\lambda_{\text{max}}$ be the largest eigenvalue of $M_A$ and assume that $L \eta = a \eta$. Then, the previous explicit coupling scheme is unconditionally unstable whenever

$$\frac{\rho^f \lambda_{\text{max}}}{\rho^s \varepsilon} \geq 1.$$  \hspace{1cm} (1)

Remarks:

- The instability condition does not depend on the time step
- The instability condition confirms two numerical observations:
  - Instabilities might occur when the structure is light, thin and slender
  - In aeroelasticity $\rho^f \ll \rho^s$, hence weak (i.e. explicit) coupling is stable
Implicit / Explicit coupling: summary
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- Implicit coupling stable but too expensive
Implicit / Explicit coupling: summary

- Implicit coupling stable but too expensive
- Explicit coupling cheap but unstable
Implicit / Explicit coupling: summary

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- Other time schemes have been considered by Förster-Wall-Ramm 07 with analogous conclusions
Implicit / Explicit coupling: summary

- Implicit coupling stable but too expensive
- **Explicit** coupling cheap but unstable
- Other time schemes have been considered by *Fürster-Wall-Ramm 07* with analogous conclusions
- Geometrical non-linearities (moving domains), convective and viscous effects do not seem to affect the stability of a coupling algorithm. However, they are implicitly treated in fully implicit schemes (very expensive!)
Implicit / Explicit coupling: summary

• Implicit coupling stable but too expensive

• Explicit coupling cheap but unstable

• Other time schemes have been considered by Förster-Wall-Ramm 07 with analogous conclusions

• Geometrical non-linearities (moving domains), convective and viscous effects do not seem to affect the stability of a coupling algorithm. However, they are implicitly treated in fully implicit schemes (very expensive!)

Three ideas: (Fernandez, JFG, Grandmont, 2006)

➢ Treat implicitly the added-mass effect (incompressibility, pressure stress)
➢ Treat explicitly the fluid domain motion, convective and viscous effects
➢ Perform this using a projection scheme (Chorin-Teman) within the fluid
The Chorin-Teman projection scheme
The Chorin-Teman projection scheme

Main feature:
Incompressibility and viscous/convective effects are decoupled
The Chorin-Teman projection scheme

Main feature:
Incompressibility and viscous/convective effects are decoupled

- Incompressible Navier-Stokes equations:

\[
p^f \left( \frac{\partial u}{\partial t} + u \cdot \nabla u \right) - 2\mu \text{div} \epsilon(u) + \nabla p = 0, \quad \text{in} \quad \Omega^f
\]

\[
\text{div} \, u = 0, \quad \text{in} \quad \Omega^f
\]
The Chorin-Teman projection scheme

**Main feature:**
Incompressibility and viscous/convective effects are decoupled

- **Incompressible Navier-Stokes equations:**
  \[
  \rho^f \left( \frac{\partial u}{\partial t} + u \cdot \nabla u \right) - 2\mu \text{div} \epsilon(u) + \nabla p = 0, \quad \text{in} \quad \Omega^f
  \]
  \[
  \text{div} u = 0, \quad \text{in} \quad \Omega^f
  \]

- **Viscous step:**
  \[
  \begin{cases}
  \rho^f \left( \frac{\tilde{u}^{n+1} - u^n}{\delta t} + \tilde{u}^{n+1} \cdot \nabla \tilde{u}^{n+1} \right) - 2\mu \text{div} \epsilon(\tilde{u}^{n+1}) = 0, \quad \text{in} \quad \Omega \\
  \tilde{u}^{n+1} = 0, \quad \text{on} \quad \partial \Omega
  \end{cases}
  \]

  \[\text{div} u = 0, \quad \text{in} \quad \Omega^f\]
The Chorin-Teman projection scheme

Main feature:
Incompressibility and viscous/convective effects are decoupled

• Incompressible Navier-Stokes equations:
  \[
  \rho^f \left( \frac{\partial u}{\partial t} + u \cdot \nabla u \right) - 2\mu \text{div}\varepsilon(u) + \nabla p = 0, \quad \text{in} \quad \Omega^f
  \]
  \[
  \text{div} u = 0, \quad \text{in} \quad \Omega^f
  \]

• Viscous step:
  \[
  \begin{cases}
  \rho^f \left( \frac{\tilde{u}^{n+1} - u^n}{\delta t} + \tilde{u}^{n+1} \cdot \nabla \tilde{u}^{n+1} \right) - 2\mu \text{div} \varepsilon(\tilde{u}^{n+1}) = 0, \quad \text{in} \quad \Omega \\
  \tilde{u}^{n+1} = 0, \quad \text{on} \quad \partial\Omega
  \end{cases}
  \]

• Projection step:
  \[
  \begin{cases}
  \rho^f \frac{u^{n+1} - \tilde{u}^{n+1}}{\delta t} + \nabla p^{n+1} = 0, \quad \text{in} \quad \Omega \\
  \text{div} u^{n+1} = 0, \quad \text{in} \quad \Omega \\
  u^{n+1} \cdot n = 0, \quad \text{on} \quad \partial\Omega
  \end{cases}
  \]
The Chorin-Teman projection scheme

Main feature:
Incompressibility and viscous/convective effects are decoupled

- Incompressible Navier-Stokes equations:
  \[
  ρ^f \left( \frac{∂u}{∂t} + u \cdot ∇u \right) - 2μ \text{div} ε(u) + ∇p = 0, \quad \text{in} \quad Ω^f
  \]
  \[
  \text{div} u = 0, \quad \text{in} \quad Ω^f
  \]

- Viscous step:
  \[
  \left\{ \begin{aligned}
  ρ^f \left( \frac{u^{n+1} - u^n}{δt} + \tilde{u}^{n+1} \cdot ∇u^{n+1} \right) - 2μ \text{div} ε(\tilde{u}^{n+1}) &= 0, \quad \text{in} \quad Ω \\
  \tilde{u}^{n+1} &= 0, \quad \text{on} \quad ∂Ω
  \end{aligned} \right.
  \]

- Projection step:
  \[
  \left\{ \begin{aligned}
  ρ^f \frac{u^{n+1} - \tilde{u}^{n+1}}{δt} + ∇p^{n+1} &= 0, \quad \text{in} \quad Ω \\
  \text{div} u^{n+1} &= 0, \quad \text{in} \quad Ω \\
  u^{n+1} \cdot n &= 0, \quad \text{on} \quad ∂Ω
  \end{aligned} \right. \quad \Rightarrow \quad \left\{ \begin{aligned}
  -Δp^{n+1} &= -\frac{ρ^f}{δt} \text{div} \tilde{u}^{n+1}, \quad \text{in} \quad Ω \\
  \frac{∂p^{n+1}}{∂n} &= 0, \quad \text{on} \quad ∂Ω
  \end{aligned} \right.
  \]
Semi-implicit coupling: explicit part

- Viscous sub-step:

\[
d^{f,n+1} = \text{Ext}(d^m_{|\Sigma}), \quad w^{n+1} = \frac{d^{f,n+1} - d^n}{\delta t}, \quad \Omega^{f,n+1} = (I + d^{f,n+1})(\hat{\Omega}^f),
\]

\[
\rho^f \left( \frac{\tilde{u}^{n+1} - u^n}{\delta t} + (\tilde{u}^{n+1} - w^{n+1}) \cdot \nabla \tilde{u}^{n+1} \right) - 2\mu \text{div} \epsilon(\tilde{u}^{n+1}) = 0, \quad \text{in} \quad \Omega^{f,n+1}
\]

\[
\tilde{u}^{n+1} = w^{n+1}, \quad \text{on} \quad \Sigma^{n+1}
\]
Semi-implicit coupling: explicit part

- Viscous sub-step:

\[ d^{f,n+1} = \text{Ext}(d^n_{|\Sigma}), \quad w^{n+1} = \frac{d^{f,n+1} - d^n}{\delta t}, \quad \Omega^{f,n+1} = (I + d^{f,n+1})(\omega^f), \]

\[
\rho^f \left( \frac{\tilde{u}^{n+1} - u^n}{\delta t} + (\tilde{u}^{n+1} - w^{n+1}) \cdot \nabla \tilde{u}^{n+1} \right) - 2\mu \text{div} \epsilon(\tilde{u}^{n+1}) = 0, \quad \text{in} \quad \Omega^{f,n+1}
\]

\[
\tilde{u}^{n+1} = w^{n+1}, \quad \text{on} \quad \Sigma^{n+1}
\]

**Observation:**
- Fluid domain, viscous and convective effects explicitly treated
Semi-implicit coupling: \textit{implicit part}

- Fluid projection sub-step (in a known domain):

\[
\begin{align*}
\rho^f \frac{u^{n+1} - \tilde{u}^{n+1}}{\delta t} + \nabla p^{n+1} &= 0, \quad \text{in } \Omega^{f,n+1} \\
\text{div} u^{n+1} &= 0, \quad \text{in } \Omega^{f,n+1} \\
\mathbf{u}^{n+1} \cdot \mathbf{n} &= \frac{d^{n+1} - d^n}{\delta t} \cdot \mathbf{n}, \quad \text{on } \Sigma^{n+1}
\end{align*}
\]

- Solid equation:

\[
\begin{align*}
\rho^s \frac{d^{n+1} - 2d^n + d^{n-1}}{\delta t^2} - \text{div} (F(d^{n+1})S(d^{n+1})) &= 0, \quad \text{in } \hat{\Omega}^s \\
F(d^{n+1})S(d^{n+1})\hat{n} &= J(d^{f,n+1})\sigma(\tilde{u}^{n+1}, \rho^{n+1})F(d^{f,n+1})^{-T}\hat{n}, \quad \text{on } \hat{\Sigma}
\end{align*}
\]
Semi-implicit coupling: implicit part

- Fluid projection sub-step (in a known domain):
  \[
  \begin{aligned}
  \rho^f \frac{u^{n+1} - \tilde{u}^{n+1}}{\delta t} + \nabla p^{n+1} &= 0, \quad \text{in} \quad \Omega^{f,n+1} \\
  \text{div} u^{n+1} &= 0, \quad \text{in} \quad \Omega^{f,n+1} \\
  u^{n+1} \cdot n &= \frac{d^{n+1} - d^n}{\delta t} \cdot n, \quad \text{on} \quad \Sigma^{n+1}
  \end{aligned}
  \]

- Solid equation:
  \[
  \begin{aligned}
  \rho^s \frac{d^{n+1} - 2d^n + d^{n-1}}{\delta t^2} - \text{div} \left( F(d^{n+1}) S(d^{n+1}) \right) &= 0, \quad \text{in} \quad \hat{\Omega}^s \\
  F(d^{n+1}) S(d^{n+1}) \hat{n} &= J(d^{f,n+1}) \sigma(\tilde{u}^{n+1}, p^{n+1}) F(d^{f,n+1})^{-T} \hat{n}, \quad \text{on} \quad \hat{\Sigma}
  \end{aligned}
  \]
Semi-implicit coupling: implicit part

- Fluid projection sub-step (in a known domain):

\[
\begin{align*}
\rho^f \frac{u^{n+1} - \tilde{u}^{n+1}}{\delta t} + \nabla p^{n+1} &= 0, \text{ in } \Omega^{f,n+1} \\
\text{div} u^{n+1} &= 0, \text{ in } \Omega^{f,n+1} \\
\frac{u^{n+1} \cdot n}{\delta t} &= \frac{d^{n+1} - d^n}{\delta t} \cdot n, \text{ on } \Sigma^{n+1}
\end{align*}
\]

- Solid equation:

\[
\begin{align*}
\rho^s \frac{d^{n+1} - 2d^n + d^{n-1}}{\delta t^2} - \text{div} \left( F(d^{n+1}) S(d^{n+1}) \right) &= 0, \text{ in } \hat{\Omega}^s \\
F(d^{n+1}) S(d^{n+1}) \hat{n} &= J(d^{f,n+1}) \sigma(\tilde{u}^{n+1}, p^{n+1}) F(d^{f,n+1})^{-T} \hat{n}, \text{ on } \hat{\Sigma}
\end{align*}
\]

Observations:
Semi-implicit coupling: implicit part

- Fluid projection sub-step (in a known domain):

\[
\begin{align*}
\rho^f u^{n+1} - \tilde{u}^{n+1} \over \delta t + \nabla p^{n+1} &= 0, \quad \text{in } \Omega^{f,n+1} \\
\text{div} u^{n+1} &= 0, \quad \text{in } \Omega^{f,n+1} \\
\theta^{n+1} \cdot n &= \frac{d^{n+1} - d^n}{\delta t} \cdot n, \quad \text{on } \Sigma^{n+1}
\end{align*}
\]

- Solid equation:

\[
\begin{align*}
\rho^s \frac{d^{n+1} - 2d^n + d^{n-1}}{\delta t^2} - \text{div} (F(d^{n+1})S(d^{n+1})) &= 0, \quad \text{in } \hat{\Omega}^s \\
F(d^{n+1})S(d^{n+1})\hat{n} &= J(d^{f,n+1})\sigma(\tilde{u}^{n+1}, p^{n+1})F(d^{f,n+1})^{-T}\hat{n}, \quad \text{on } \hat{\Sigma}
\end{align*}
\]

Observations:

- Projection sub-step in a fixed fluid domain (fixed matrix)
Semi-implicit coupling: implicit part

- Fluid projection sub-step (in a known domain):

\[
\begin{align*}
\rho^f \frac{u^{n+1} - \tilde{u}^{n+1}}{\delta t} + \nabla p^{n+1} &= 0, \quad \text{in } \Omega^{f,n+1} \\
\text{div } u^{n+1} &= 0, \quad \text{in } \Omega^{f,n+1} \\
\end{align*}
\]

\[
\frac{\partial p^{n+1}}{\partial n} = -\frac{\rho^f}{\delta t} \text{div } \tilde{u}^{n+1}, \quad \text{in } \Omega^{f,n+1}
\]

\[
\frac{\partial p^{n+1}}{\partial n} = -\rho^f \frac{d^{n+1} - 2d^n + d^{n-1}}{\delta t^2}, \quad \text{on } \Sigma^{n+1}
\]

\[
u^{n+1} \cdot n = \frac{d^{n+1} - d^n}{\delta t} \cdot n, \quad \text{on } \Sigma^{n+1}
\]

- Solid equation:

\[
\begin{align*}
\rho^s \frac{d^{n+1} - 2d^n + d^{n-1}}{\delta t^2} - \text{div } (F(d^{n+1})S(d^{n+1})) &= 0, \quad \text{in } \hat{\Omega}^s \\
F(d^{n+1})S(d^{n+1})\hat{n} = J(d^{f,n+1})\sigma(\tilde{u}^{n+1}, p^{n+1})F(d^{f,n+1})^{-T}\hat{n}, \quad \text{on } \hat{\Sigma}
\end{align*}
\]

Observations:

- Projection sub-step in a fixed fluid domain (fixed matrix)
- Implicit part solved with cheaper (inner) iterations
A stability result (linear case)

Theorem: (Fernandez-JFG-Grandmont 2006)

Assume the interface matching operator to be $L^2$-stable. Then, under condition

$$\rho^s \geq C \left( \rho^f \frac{h}{H^\alpha} + 2 \frac{\mu \delta t}{h H^\alpha} \right),$$

with $\alpha \overset{\text{def}}{=} \begin{cases} 0, & \text{if } \Omega^s = \Sigma, \\ 1, & \text{if } \Omega^s \neq \Sigma, \end{cases}$

the following discrete energy inequality holds:

$$\frac{1}{\delta t} \left[ \frac{\rho^f}{2} \left\| u^h_{n+1} \right\|_{0, \Omega^f}^2 - \frac{\rho^f}{2} \left\| u^n_h \right\|_{0, \Omega^f}^2 + \frac{\rho^s}{2} \left\| d^{n+1}_H - d^n_H \right\|_{0, \Omega^f}^2 - \frac{\rho^s}{2} \left\| d^n_H - d^{n-1}_H \right\|_{0, \Omega^f}^2 \right]$$

$$+ \frac{1}{2 \delta t} \left[ a^s(d^{n+1}_H, d^{n+1}_H) - a^s(d^n_H, d^n_H) \right] + \mu \left\| \epsilon(u^h_{n+1}) \right\|_{0, \Omega^f}^2 \leq 0$$
A stability result (linear case)

Theorem: (Fernandez-JFG-Grandmont 2006)

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$$+ \frac{1}{2\delta t} \left[ a^s(d_H^{n+1}, d_H^{n+1}) - a^s(d_H^n, d_H^n) \right] + \mu \|\epsilon(\bar{u}_h^{n+1})\|_{0,\Omega^f}^2 \leq 0$$

Therefore, the semi-implicit coupling scheme is conditionally stable in the energy norm.
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the following discrete energy inequality holds:

$$\frac{1}{\delta t} \left[ \frac{\rho^f}{2} \| u^{n+1}_h \|_{0,\Omega^f}^2 - \frac{\rho^f}{2} \| u^n_h \|_{0,\Omega^f}^2 + \frac{\rho^s}{2} \left\| \frac{d^{n+1}_H - d^n_H}{\delta t} \right\|_{0,\Omega^f}^2 \right] \leq \frac{1}{2\delta t} \left[ a^s(d^{n+1}_H, d^{n+1}_H) - a^s(d^n_H, d^n_H) \right] + \frac{\mu}{\delta t} \epsilon(\tilde{u}^{n+1}_h)_{0,\Omega^f}^2 \leq 0$$

Therefore, the semi-implicit coupling scheme is conditionally stable in the energy norm.

Remark: this semi-implicit algorithm has been extended by Badia, Quarteroni, Quaini, 2007 to other projection schemes.
3D Navier-Sokes / Non-linear Shell coupling

- Straight cylinder: 50 time steps of length $\delta t = 2 \times 10^{-4}s$

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<th>CPU time</th>
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2001

2003

2006
3D Navier-Stokes / Non-linear Shell coupling

- Straight cylinder: 50 time steps of length $\delta t = 2 \times 10^{-4} s$

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- Cerebral aneurysm: 20 time steps of length $\delta t = 10^{-3} s$

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</table>
Abdominal aortic aneurysm (in-vitro model): 2 cardiac cycles, 1000 times steps

- $\delta t = 1.68 \times 10^{-3} \text{s}$

- Fluid: 26950 Hexahedra ($Q_1/Q_1$ FE)

- Solid: 2240 Quadrilaterals (MITC4 FE)

- Parameters: $\mu = 0.035 \text{ poise}$, $\rho^f = 1 \text{ g/cm}^3$, $\rho^s = 1.2 \text{ g/cm}^3$, $E = 610^6 \text{ dynes/cm}^2$, $\nu = 0.3$

<table>
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</table>

Dimensionless CPU time
• Carotid artery (in-vivo model): 9 cardiac cycles, 4500 times steps
  • $\delta t = 1.68 \times 10^{-3} \text{s}$
• Fluid: 70047 Tetrahedra ($P_1/P_1$ FE)
• Solid: 8103 Quadrilaterals (MITC4 FE)
• Parameters: $\mu = 0.035 \text{ poise}$, $\rho_f = 1 \text{ g/cm}^3$, $\rho_s = 1.2 \text{ g/cm}^3$, $E = 6 \times 10^6 \text{ dynes/cm}^2$, $\nu = 0.3$.

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Dimensionless CPU time
### 3D Navier-Stokes / Non-linear Shell coupling

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**Dimensionless CPU time**

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<td>1.0</td>
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![Graph showing out-flow](image)
Remarks on boundary conditions

Spurious reflexion of pressure wave: 3D-1D coupling
Remarks on boundary conditions

- Spurious reflexion of pressure wave: 3D-1D coupling
Remarks on boundary conditions

Spurious reflexion of pressure wave: 3D-1D coupling

Formaggia, JFG, Quarteroni, Nobile, 2001
Formaggia, Moura, Nobile, 2007
Formaggia, Veneziani, Vergara, 2006
Remarks on boundary conditions

- Spurious reflexion of pressure wave: 3D-1D coupling

- But pressure wave reflexion is not the only issue

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Remarks on boundary conditions

- Spurious reflexion of pressure wave: 3D-1D coupling

- But pressure wave reflexion is not the only issue

The best non-reflecting outlet boundary condition cannot prevent the global (non physiological) bending!
Remarks on boundary conditions
Remarks on boundary conditions

- Surrounding tissues play a key role
Remarks on boundary conditions

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- Typical b.c. on the external part of the vessel:

\[ F(d) \ S(d) \ \hat{n} = p_0 \]
Remarks on boundary conditions

- Surrounding tissues play a key role

- Typical b.c. on the external part of the vessel:
  \[ F(d) \ S(d) \ \hat{n} = p_0 \]

- A simple and affordable way to model the surrounding tissues ("silent blocks"):
  \[ F(d) \ S(d) \ \hat{n} = -k_s d - c_s \frac{\partial d}{\partial t} \]
Remarks on boundary conditions

\[ F(d) \ S(d) \ \hat{n} = -k_s d - c_s \frac{\partial d}{\partial t} \] ("silent blocks")

\[ F(d) \ S(d) \ \hat{n} = p_0 \]
Remarks on boundary conditions

\[ F(d) \ S(d) \ \hat{n} = -k_s d - c_s \frac{\partial d}{\partial t} \]  

(“silent blocks”)

\[ F(d) \ S(d) \ \hat{n} = p_0 \]

Astorino, JFG
Outline

- Fluid-structure interaction in arteries
  - Stability issues - Strongly coupled schemes
  - A stable weakly coupled scheme
- Fluid-structure interaction with valves
  - Fictitious domain method
  - FSI and Contact management
FSI with valves
FSI with valves

Valve: "Fictitious Domains" (FD)
"Fictitious Domain" for valves

Basic idea: impose the kinematic constraint in a weak form

$$\langle \mu, u_f \rangle_\Sigma = \langle \mu, u_s \rangle_\Sigma, \forall \mu \in \Lambda$$
“Fictitious Domain” for valves

Basic idea: impose the kinematic constraint in a weak form

$$\langle \mu, u_f \rangle_\Sigma = \langle \mu, u_s \rangle_\Sigma, \forall \mu \in \Lambda$$

A saddle point problem has to be solved in the fluid

$$a_f(u_f, v_f) + \langle \lambda, v_f \rangle_\Sigma = \int_{\Omega_f(t)} f_f \cdot v_f, \forall v_f \in X_f$$

$$\langle \mu, u_f \rangle_\Sigma - \langle \mu, u_s \rangle_\Sigma = 0, \forall \mu \in \Lambda$$

$$\hat{a}_s(\hat{u}_s, \hat{v}_s) - \langle \lambda, v_s \rangle_\Sigma = \int_{\hat{\Omega}_s} \hat{f}_s \cdot \hat{v}_s, \forall \hat{v}_s \in \hat{X}_s$$
“Fictitious Domain” for valves

- Basic idea: impose the kinematic constraint in a weak form
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  \begin{align*}
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  \end{align*}
  \]

- Lagrange multiplier space:
  \[ \Lambda_h = \{ \mu_h \text{ measure on } \Sigma, \mu_h = \sum_{i=1}^{N_\Sigma} \mu_i \delta(x_i^{n+1}), \mu_i \in \mathbb{R}^n \} \]
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Basic idea: impose the kinematic constraint in a weak form

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\begin{align*}
\begin{cases}
a_f(u_f, v_f) + \langle \lambda, v_f \rangle_\Sigma & = \int_{\Omega_f(t)} f_f \cdot v_f, \quad \forall v_f \in X_f \\
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\hat{a}_s(\hat{u}_s, \hat{v}_s) - \langle \lambda, v_s \rangle_\Sigma & = \int_{\hat{\Omega}_s} \hat{f}_s \cdot \hat{v}_s \quad \forall \hat{v}_s \in \hat{X}_s
\end{cases}
\end{align*}
\]

Lagrange multiplier space:

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Other Lagrange multiplier spaces are possible

Baaijens, 2001, de Hart et al. 2003
Comparison:

Time = 2,900
Comparison:

Time = 2,900

FD:

ALE:

Valve displacement
Comparison:

Time = 2.900

Mix ALE + FD:

N. Diniz dos Santos
FSI and kinematic constraints
Valves are submitted to various kinematic constraints:
Valves are submitted to various kinematic constraints:
- contact between leaflets
Valves are submitted to various kinematic constraints:

- contact between leaflets
- *chordae tendinae* (mitral valve)
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Valves are submitted to various kinematic constraints:  
- contact between leaflets  
- *chordae tendinae* (mitral valve)

We propose a framework to deal with these constraints in partitionned FSI algorithms.
Solid-wall contact

\( M \) a solid with energy \( J \)

\( \mathcal{T}_h \) a mesh of \( M \)

\[ X_h = \{ \varphi_h \in C^0(M; \mathbb{R}^3), \varphi_h|_T \in P_1, \forall T \in \mathcal{T}_h \} \]

Minimization with convex constraint:

\[ \inf_{\varphi_h \in \mathcal{U}_h} J(\varphi_h) \]
Solid-wall contact

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Minimization with convex constraint:
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\inf_{\varphi_h \in \mathcal{U}_h} J(\varphi_h)
\]

With \( \mathcal{U}_h = \{ \varphi_h \in X_h, F_{x_i}(\varphi_h) \leq 0, \forall x_i \in M \} \)

\[
F_{x_i}(\varphi_h) = \varepsilon - n \cdot \varphi_h(x_i) - c
\]
Solid-wall contact

$M$ a solid with energy $J$

$\mathcal{T}_h$ a mesh of $M$

$X_h = \{ \varphi_h \in C^0(M; \mathbb{R}^3), \varphi_h|_T \in P_1, \forall T \in \mathcal{T}_h \}$

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$F_{x_i}(\varphi_h) = \varepsilon - \mathbf{n} \cdot \varphi_h(x_i) - c$

van Loon, Anderson, van de Vosse, 2006
Dual approach

\[ G(\mu) = \inf_{\varphi \in X_n} \left[ J(\varphi) + \sum_{i=1}^{N_\Sigma} \mu_i F_{x_i}(\varphi) \right] \]

\[ G(\lambda_c) = \max_{\mu_i \geq 0} G(\mu) \]

\[ \lambda_c : \text{contact pressure} \]
**Dual approach**

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\( \lambda_c \) : contact pressure

The constraint is now simple to enforce
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\[ \lambda_c \quad : \text{contact pressure} \]

The constraint is now simple to enforce.

**Gradient method with projection**

1. \[ \langle J'(\varphi^k), \xi \rangle = -\sum_{i=1}^{N_{\Sigma}} \lambda_{c,i}^k \langle F'_{x_i}(\varphi^k), \xi \rangle = \sum_{i=1}^{N_{\Sigma}} \lambda_{c,i}^k n \cdot \xi(x_i) \]

2. \[ \lambda_{c,i}^{k+1} = P_{\mathbb{R}^+} \left( \lambda_{c,i}^k + \alpha^k \nabla G(\lambda_c^k)_i \right) = P_{\mathbb{R}^+} \left( \lambda_{c,i}^k + \alpha^k F_{x_i}(\varphi^k) \right) \]

3. Iterate on \( k \)
Dual approach

\[ G(\mu) = \inf_{\varphi \in X_h} \left[ J(\varphi) + \sum_{i=1}^{N_\Sigma} \mu_i F_{x_i}(\varphi) \right] \]

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Gradient method with projection

1. \( \langle J'(\varphi^k), \xi \rangle = -\sum_{i=1}^{N_\Sigma} \lambda_{c,i}^k \langle F'_{x_i}(\varphi^k), \xi \rangle = \sum_{i=1}^{N_\Sigma} \lambda_{c,i}^k n \cdot \xi(x_i) \)

2. \( \lambda_{c,i}^{k+1} = P_{\mathbb{R}^+}(\lambda_{c,i}^k + \alpha^k \nabla G(\lambda_c^k)_i) = P_{\mathbb{R}^+}(\lambda_{c,i}^k + \alpha^k F_{x_i}(\varphi^k)) \)

3. Iterate on \( k \)

The contact force is added to the hydrodynamics force
Implementation

FSI Master

Fluid

Struct Master

Struct 1

$\sigma_f$

$u$

$\sigma_f$

$\sigma_f + \sigma_c$

$u$

$u$
Diniz dos Santos, JFG, Bourgat 2007
Solid-solid contact

\[ M = (M_1, M_2, \ldots) \]  a family of solid with energy \( J \)

\[ T_h \]  a mesh of \( M \)

\[ X_h = \{ \varphi_h \in C^0(M; \mathbb{R}^3), \varphi_h|_T \in P_1, \forall T \in T_h \} \]

Minimization with \textit{non convex} constraints:

\[ \inf_{\varphi_h \in U_h} J(\varphi_h) \]
$M = (M_1, M_2, \ldots)$ a family of solid with energy $J$

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Minimization with non convex constraints:

$$\inf_{\varphi_h \in U_h} J(\varphi_h)$$

with

$$U_h = \{\varphi_h \in X_h, dist(\varphi_h(T_1), \varphi_h(T_2)) \geq \varepsilon, \forall T_1, T_2 \in T_h\}$$
Solid-solid contact

Optimization algorithm (O. Pantz, 2007):

- replace a problem with nonconvex constraints with a sequence of problems with convex constraints.
- Solve
  \[ J(\varphi_h^{k+1}) = \inf_{\psi_h \in T(\varphi_h^k)} J(\psi_h) \]
- where \( T(\varphi_h^k) \) is a convex neighborhood of \( \varphi_h^k \)
- Iterate on \( k \) until convergence
Solid-solid contact

\[
T(\psi_h) = \left\{ \varphi_h \in X_h, \min_{x_e \in e} n_{e,x}(\psi_h) \cdot (\varphi_h(x_e) - \varphi_h(x)) \geq \varepsilon \right. \\
\text{for all edges } e \text{ and all node } x \notin e \right\}
\]
At convergence, $\varphi_h$ does not satisfy \textit{a priori} the optimality conditions of the original problem
At convergence, $\varphi_h$ does not satisfy \textit{a priori} the optimality conditions of the original problem.

... but the error is $O(h)$ (O. Pantz, 2007).
At convergence, \( \varphi_h \) does not satisfy \textit{a priori} the optimality conditions of the original problem.

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Same kinds of constraint as for the solid-wall case:
At convergence, $\varphi_h$ does not satisfy \textit{a priori} the optimality conditions of the original problem

... but the error is $O(h)$ (O. Pantz, 2007)

Same kinds of constraint as for the solid-wall case:

$$T(\psi_h) = \left\{ \varphi_h \in X_h, F^0_{e,x}(\varphi_h) \leq 0, F^1_{e,x}(\varphi_h) \leq 0, \right.\
\left. \text{for all edges } e \text{ and all node } x \notin e \right\}$$

$$F^j_{e,x}(\varphi_h) = \varepsilon - n_{e,x}(\psi_h) \cdot (\varphi_h(e_j) - \varphi_h(x))$$
Implementation

FSI Master

 Fluid

 u

 Struct Master

 Struct 1

 u

 Struct 2

 u

\[ \sigma_f \]

\[ \sigma_f + \sigma_c \]

\[ \sigma_f + \sigma_c \]
FSI & Solid-solid contact

Diniz dos Santos, JFG, 2007
FSI & Solid-solid contact

Diniz dos Santos, JFG, 2007
Astorino, JFG, Pantz, Traoré, 2008
Conclusion
Conclusion
Conclusion

Fluid-Structure coupling algorithm
Conclusion

Fluid-Structure coupling algorithm

Tremendous progress in the last few years, but a “totally” weak-coupling is still a open problem
Conclusion

Fluid-Structure coupling algorithm

- Tremendous progress in the last few years, but a “totally” weak-coupling is still an open problem

- Interesting new ideas with Nitsche BC (Burman, Fernandez, 2007)
Conclusion

Fluid-Structure coupling algorithm

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- FSI and Contact management: still a lot of works to do to improve efficiency and maybe modelling (adhesion, friction ?)
Conclusion

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- Boundary conditions for FSI is still mainly an open problem
Conclusion

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  - Tremendous progress in the last few years, but a “totally” weak-coupling is still an open problem
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- Boundary conditions for FSI is still mainly an open problem

- Grand challenge: coupling numerical models with medical data.
Collaborators

- M. Astorino (INRIA)
- N. Diniz dos Santos (INRIA)
- M.A. Fernandez (INRIA)
- C. Grandmont (INRIA)
- V. Martin (UTC)
- O. Pantz (Ecole Polytechnique)