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DOTTORATO DI RICERCA IN MATEMATICA
CICLO XXXI

Credit Risk Management and Jump Models

Presentata da: Sidy Diop

Coordinatore dottorato:
Chiar.ma Prof.ssa
Giovanna Citti

Supervisore:
Chiar.mo Prof.
Andrea Pascucci

Co-supervisore:
Chiar.mo Dr.
Marco Di Francesco

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Abstract

Since the breakout of global financial crisis 2007-2009 and the failures of system-relevant financial institutions such as Lehman Brothers, Bear Stearns and AIG, improving credit risk management models and methods is among the top priorities for institutional investors. Indeed, the effectiveness of the traditional quantitative models and methods have been reduced by the new complex credit derivatives introduced by the the financial innovations. The level of implication of the credit default swaps (CDSs) in the crisis makes the CDS market an interesting and active field of research. This doctoral thesis comprises three research papers that seek to improve and create corporate and sovereign credit risk models, to provide an approximate analytic expressions for CDS spreads and a numerical method for partial differential equation arisen from pricing defaultable coupon bond.

First, an extension of Jump to Default Constant Elasticity Variance in more general and realistic framework is provided (see Chapter 3). We incorporate, in the model introduced in [9], a stochastic interest rate with possible negative values. In addition we provide an asymptotic approximation formula for CDS spreads based on perturbation theory. The robustness and efficiency of the method is confirmed by several calibration tests on real market data.

Next, under the model introduced in Chapter 3, we present in Chapter 4 a new numerical method for pricing non callable defaultable bond. we propose appropriate numerical schemes based on a Crank-Nicolson semi-Lagrangian method for time discretization combined with biquadratic Lagrange finite elements for space discretization. Once the numerical solutions of the PDEs are obtained, a post-processing procedure is carried out in order to achieve the value of the bond. This post-processing includes the computation of an integral term which is approximated by using the composite trapezoidal rule. Finally, we present some numerical results for real market bonds issued by different firms in order to illustrate the proper behaviour of the numerical schemes.

Finally, we introduce a hybrid Sovereign credit risk model in which the intensity of default of a sovereign is based on the jump to default extended CEV model (see Chapter 5). The model

captures the interrelationship between creditworthiness of a sovereign, its intensity to default and the correlation with the exchange rate between the bond's currency and the currency in which the CDS spread are quoted. We consider the Sovereign Credit Default Swaps Italy, during and after the financial crisis, as a case of study to show the effectiveness of our model.

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Chapter 1

Introduction

'All of life is the management of risk, not its elimination.'

Walter Wriston, former chairman of Citicorp

Credit risk is most simply defined as the potential loss in the mark-to-market value that may be incurred due to the occurrence of a credit event. A credit event is any sudden and perceptible change in the counterparty's ability to perform its obligations: Bankruptcy, failure to pay, restructuring, repudiation, moratorium, obligation, acceleration and obligation default. Credit default also includes sovereign risk. This happens , for instance, when countries impose foreign-exchange controls that make it impossible for counterparties to respect the terms of a contract. Credit risk management is described in the financial literature as being concerned with identifying and managing a firm's exposure to credit risk.

The financial innovations, after the decision of the US Federal Reserve to lower the interest rate to 1%, followed by the housing bubble, have introduced new complex instruments in the financial market. This led to a rapid growth in the market for credit derivatives, which has become the third-largest derivatives market, after interest rate and foreign exchange derivatives, in terms of gross market value, accounting for USD 1.2 quadrillion as of June 2018. Among the credit derivatives, the credit default swaps (CDS) are the most popular and influential in trading credit risk. However, their level of implication in the recent financial scandals is significant: in the sub-prime crisis in 2007-2008 or the trading losses by the "London Whale" at JP Morgan Chase in 2012. Most of the new credit derivatives have limited historical data and required assumptions regarding risk and correlations with other instruments, reducing the effectiveness and robustness of quantitative risk models prior to the crisis. To deal with these complex financial instruments, the "traditional" models either omitted them and waited until they obtained enough historical data, or created simple approximations or just used proxies. It followed failures of credit risk management. Then, improving models and methods is among the top priorities for institutional investors in the wake up of the recent global financial.

Nowadays the credit risk models can be divided into two primary classes of credit risk modeling approach: the structural approach and the reduced form approach also known as intensity form approach. Structural models, first developed by Black, Scholes and Merton, employ modern option pricing theory in corporate debt valuation. In 1974, Robert Merton, in [34], proposed the first structural model for assessing credit risk of a company by building the company's equity as a call option on its asset, thus allowing the Black-Scholes option pricing method. The structural model has known many improvements due to the simplicity and the several assumptions of its initial version in [34]. Black and Cox [4] introduced the first extension by considering a possible default event before maturity and assumed that default can also occur at the first time the asset price goes below a fixed value. Geske [15] included a coupon bond in the model. Ramaswamy and Sundaresan [40] as well as Kim [25] considered a default event at coupon payment dates and incorporated a stochastic interest rate following a CIR model. In 1995 Longstaff and Schwartz [28] extended the model by assuming that default can happen at anytime, while Zhou [41] modeled the value of the firm as a jump-diffusion process.

The reduced-form models, on the other hand, trace their roots back to the work of Jarrow and Turnbull [17] and subsequently studied by Jarrow and Turnbull [18], Duffie and Singleton [11], and Madan and Unal [31], among others. In reduced form approach models, the set of information requires less detailed knowledge about the firm's assets and liabilities than the structural approach. They are consistent with available market information. The idea is to observe the filtration generated by the default time and the vector of state variables , where the default time is a stopping time generated by a Cox process with an intensity process depending on the state variables which follows a diffusion process. We call the payoff to the firm's debt in the event of default by *recovery rate*, given by a stochastic process. In 2006, Carr and Linetsky introduced the Jump to Default Constant Elasticity Volatility model [9], an improvement of a reduced approach, which unifies credit and equity models into the framework of deterministic and positive interest rates. Assuming that the stock price follows a diffusion process with a possible jump to zero, hazard rate of default is an increasing affine function of the instantaneous variance of returns on the stock price, and stock volatility is defined as in the Constant Elasticity of Variance (CEV) model, the authors developed a model that captures the following three observations:

- credit default swap (CDS) spreads and corporate bond yields are both positively related to implied volatilities of equity options;
- realized volatility of stock is negatively related to its price level;
- equity implied volatilities tend to be decreasing convex functions of option's strike price.

The JDCEV model, thanks to the standard Bessel process with killing, provides an explicit analytical expression for survival probabilities and CDS spreads. However, this approach does not work in the case of a stochastic or negative interest rate. The main purpose of this dissertation is, therefore, to extend the JDCEV models to a more general framework and to provide approximate analytic expressions for CDS spreads for both corporate and sovereign.

In the second chapter, we summarize some preliminaries of mathematical theory (e.g. Itô calculus, Girsanov's theorem etc.) often used in the valuation of options (and other derivatives).

In the third chapter, we propose a new methodology for the calibration of a hybrid credit-equity model to credit default swap (CDS) spreads and survival probabilities. We consider an extended Jump to Default Constant Elasticity of Variance model incorporating stochastic and possibly negative interest rates. Our approach is based on a perturbation technique that provides an explicit asymptotic expansion of the credit default swap spreads. The robustness and efficiency of the method is confirmed by several calibration tests on real market data.

In the fourth chapter, we consider the numerical solution of a two factor-model for the valuation of defaultable bonds which pay coupons at certain given dates. We consider the extended JDCEV model introduced in the previous chapter. From the mathematical point of view, the valuation problem requires the numerical solution of two partial differential equations (PDEs) problems for each coupon and with maturity those coupon payment dates. In order to solve these PDE problems, we propose appropriate numerical schemes based on a Crank-Nicolson semi-Lagrangian method for time discretization combined with biquadratic Lagrange finite elements for space discretization. Once the numerical solutions of the PDEs are obtained, a kind of post-processing is carried out in order to determine the value of the bond. This post-processing includes the computation of an integral term which is approximated by using the composite trapezoidal rule. Finally, we present some numerical results for real market bonds issued by different firms in order to illustrate the proper behavior of the numerical schemes.

The fifth chapter presents a hybrid sovereign risk model in which the intensity of default of the sovereign is based on the jump to default extended CEV model with a deterministic interest rate. The model captures the interrelationship between creditworthiness of a sovereign, its intensity to default and the correlation with the exchange rate between the bond's currency and the currency in which the CDS spread are quoted. We analyze the differences between the default intensity under the domestic and foreign measure and we compute the default-survival probabilities in the bond's currency measure. We also give an approximation formula to sovereign CDS spread obtained by using the same technique as in the Chapter 3. Finally, we test the model on real market data by several calibration experiments to confirm the robustness of our method.

We conclude the thesis and present,in the first section of the appendix, the theory of the asymptotic approximation method used in third and fifth fourth chapters, and introduced in [29]. The second and third sections in the appendix consist of results from calibration tests on corporates and sovereigns credit default swap spreads.

Chapter 2

Preliminaries

In this chapter, we summarize some definitions and results from finance, stochastic calculus and the theory of partial differential equations. We mainly focus on the risk-neutral measure and Girsanov's Theorem, enlargement filtrations and the PDE approach for pricing. The following definitions and results are in major adapted from [21], [39] and [6]. More detailed information can be found in these books.

2.1 Default Times with Stochastic Intensity

2.1.1 Default time

Let $(\Omega, \mathcal{G}, \mathbb{P})$ be a probability space endowed with a filtration $\mathbb{F}$ and let λ be a positive $\mathbb{F}$-adapted process. We assume that there exists, on the space $(\Omega, \mathcal{G}, \mathbb{P})$, a random variable Θ, independent of $\mathcal{F}_\infty$, with exponential law of parameter 1: $\mathbb{P}\left(\Theta \leq t\right) = e^{-t}$. We define the default time τ as the first time when the increasing process $\Lambda_t = \int_0^t \lambda_s \mathrm{d}s$ exceeds the random level Θ, i.e.,

$$\tau = \inf\{t \geq 0 \colon \Lambda_t \geq \Theta\}.$$

In particular, $\{\tau \geq s\} = \{\Lambda_s \leq \Theta\}$. We assume that $\Lambda_t < \infty$, $\forall\, t$, and $\Lambda_\infty = \infty$.

2.1.2 Conditional expectation with respect to $\mathcal{F}_t$

Lemma 2.1.1. *The conditional distribution function of τ given the filtration $\sigma-$algebra $\mathcal{F}_t$ is, for $t \geq s$,*

$$\mathbb{P}\left(\tau > s|\mathcal{F}_t\right) = \exp\left(-\Lambda_s\right).$$

Proof. The proof comes straightforward from the equality $\{\tau > s\}$, the independence assumption and the $\mathcal{F}_t$-measurablity of Λ_s for $s \leq t$

$$\mathbb{P}\left(\tau > s|\mathcal{F}_t\right) = \mathbb{P}\left(\Lambda_s < \Theta|\mathcal{F}_t\right) = \exp\left(-\Lambda_s\right).$$

□

Remark 2.1.2. 1. for $t < s$, we obtain $\mathbb{P}\left(\tau > s|\mathcal{F}_t\right) = \mathbb{E}\left(\exp\left(-\Lambda_s|\mathcal{F}_t\right)\right)$.

2. If the process λ is not positive, the process Λ is not increasing and we obtain, for $s < t$,

$$\mathbb{P}\left(\tau > s|\mathcal{F}_t\right) = \mathbb{P}\left(\sup_{u \leq s} \Lambda_u < \Theta\right) = \exp\left(-\sup_{u \leq s} \Lambda_u\right).$$

2.1.3 Enlargements of Filtrations

The problems of enlargement and immersion of filtration have been first introduced by K. Itô [16] and then later studied in the seventies by Barlow [1], Jeulin and Yor [22]. Let $\mathbb{F}$ and $\mathbb{G}$ be two filtrations. $\mathbb{G}$ is larger than $\mathbb{F}$ does not imply that a $\mathbb{F}$-martingale is a $\mathbb{G}$-martingale.

Definition 2.1.3 (($\mathcal{H}$) hypothesis)**.** The filtration $\mathbb{F}$ is said to be immersed in $\mathbb{G}$ if any square integrable $\mathbb{F}$-martingale is a $\mathbb{G}$-martingale. That is

$$\mathbb{P}\left(\tau \leq t|\mathcal{F}_t\right) = \mathbb{P}\left(\tau \leq t|\mathcal{F}_\infty\right) = \mathbb{P}\left(\tau \leq t|\mathcal{F}_s\right), \ \forall\ t,\ s,\ t \leq s.$$

Proposition 2.1.4. *Hypothesis* $(\mathcal{H})$ *is equivalent to any of the following properties:*

(i) $\forall\ t \geq 0,$ *the* σ*-fields* $\mathcal{F}_\infty$ *and* $\mathcal{G}_t$ *are conditionally independent given* $\mathcal{F}_t$

(ii) $\forall\ t \geq 0,\ \forall A_t \in L^1\left(\mathcal{G}_t\right)\ \mathbb{E}\left(A_t|\mathcal{F}_\infty\right) = \mathbb{E}\left(A_t|\mathcal{F}_t\right)$

(iii) $\forall\ t \geq 0,\ \forall F \in L^1\left(\mathcal{F}_\infty\right),\ \mathbb{E}\left(F|\mathcal{G}_t\right) = \mathbb{E}\left(F|\mathcal{F}_t\right).$

In particular $(\mathcal{H})$ *holds if and only if every* $\mathbb{F}-$*local martingale is* $\mathbb{G}-$*local martingale.*

Proof. For the proof, the reader can refer to the chapter 5 in [20]. □

Consider the default process $D_t = \mathbb{1}_{\{\tau \leq t\}}$ and $\mathcal{D}_t = \sigma\left(D_s \colon s \leq t\right)$. We set the filtration $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{D}_t$, that is, the enlarged filtration generated by the underlying filtration $\mathbb{F}$ and the default time process D. We write $\mathbb{G} = \mathbb{F} \vee \mathbb{D}$, the smallest filtration which contains the filtration $\mathbb{F}$ and such that the default time τ is a $\mathbb{G}$-stopping time. If A_t is an event in the σ-algebra $\mathcal{G}_t$, then there exists $\widehat{A}_t \in \mathcal{F}_t$ such that

$$A_t \cap \{\tau > t\} = \widehat{A}_t \cap \{\tau > t\},$$

It follows that, if $(Y_t)_{t\geq 0}$ is a $\mathbb{G}$-adapted process, there exists an $\mathbb{F}$-adapted process $(\widehat{Y}_t)_{t\geq 0}$ such that

$$Y_t \mathbb{1}_{\{t<\tau\}} = \widehat{Y}_t \mathbb{1}_{\{t<\tau\}}.$$

2.1.4 Conditional expectation with respect to $\mathcal{G}_t$

Lemma 2.1.5. *Let Y be an integrable random variable. Then*

$$\mathbb{1}_{\{\tau>t\}}\mathbb{E}\left(Y|\mathcal{G}_t\right) = \mathbb{1}_{\{\tau>t\}}\frac{\mathbb{E}\left(Y\mathbb{1}_{\{\tau>t\}}|\mathcal{F}_t\right)}{\mathbb{E}\left(\mathbb{1}_{\{\tau>t\}}|\mathcal{F}_t\right)} = \mathbb{1}_{\{\tau>t\}}e^{\Lambda_t}\mathbb{E}\left(Y\mathbb{1}_{\{\tau>t\}}|\mathcal{F}_t\right).$$

Proof. We have, by definition of the conditional expectation, that $Y_t = \mathbb{E}\left(Y|\mathcal{G}_t\right)$ is $\mathcal{G}_t$-measurable. Then

$$\begin{aligned}
&\mathbb{1}_{\{\tau>t\}}Y_t = \mathbb{1}_{\{\tau>t\}}\mathbb{E}\left(Y|\mathcal{G}_t\right)\\
\Rightarrow\ &\mathbb{E}\left(\mathbb{1}_{\{\tau>t\}}Y_t|\mathcal{F}_t\right) = \mathbb{E}\left(\mathbb{1}_{\{\tau>t\}}\mathbb{E}\left(Y|\mathcal{G}_t\right)|\mathcal{F}_t\right)\\
\Rightarrow\ &Y_t\mathbb{E}\left(\mathbb{1}_{\{\tau>t\}}|\mathcal{F}_t\right) = \mathbb{E}\left(\mathbb{1}_{Y\{\tau>t\}}|\mathcal{F}_t\right)\\
&\Rightarrow \mathbb{1}_{\{\tau>t\}}Y_t = \mathbb{1}_{\{\tau>t\}}\frac{\mathbb{E}\left(\mathbb{1}_{Y\{\tau>t\}}|\mathcal{F}_t\right)}{\mathbb{E}\left(\mathbb{1}_{\{\tau>t\}}|\mathcal{F}_t\right)} = \mathbb{1}_{\{\tau>t\}}e^{\Lambda_t}\mathbb{E}\left(Y\mathbb{1}_{\{\tau>t\}}|\mathcal{F}_t\right),
\end{aligned}$$

where the last equality comes from the lemma 2.1.1. □

We denote by F the right-continuous cumulative distribution function of the random variable τ defined as $F_t = \mathbb{P}(\tau \le t)$ and we assume that $F_t < 1$ for any $t \le T$, where T is a finite horizon.

Corollary 2.1.6. *If X is an integrable $\mathcal{F}_T$-measurable random variable, for $t < T$*

$$\mathbb{E}\left(X\mathbb{1}_{\{T<\tau\}}|\mathcal{G}_t\right) = \mathbb{1}_{\{\tau>t\}}e^{\Lambda_t}\mathbb{E}\left(Xe^{-\Lambda_T}|\mathcal{F}_t\right).$$

Proof. Since $\mathbb{E}\left(X\mathbb{1}_{\{T<\tau\}}|\mathcal{G}_t\right)$ is equal to zero on the $\mathcal{G}_t$-measurable set $\{\tau < t\}$, then

$$\begin{aligned}
\mathbb{E}\left(X\mathbb{1}_{\{T<\tau\}}|\mathcal{G}_t\right) &= \mathbb{1}_{\{\tau>t\}}\mathbb{E}\left(X\mathbb{1}_{\{T<\tau\}}|\mathcal{G}_t\right),\\
&= \mathbb{1}_{\{\tau>t\}}e^{\Lambda_t}\mathbb{E}\left(X\mathbb{1}_{\{T<\tau\}}\mathbb{1}_{\{\tau>t\}}|\mathcal{F}_t\right), \quad \text{by lemma 2.1.5}\\
&= \mathbb{1}_{\{\tau>t\}}e^{\Lambda_t}\mathbb{E}\left(\mathbb{E}\left(X\mathbb{1}_{\{T<\tau\}}|\mathcal{F}_T\right)|\mathcal{F}_t\right),
\end{aligned}$$

$$= \mathbb{1}_{\{\tau>t\}} e^{\Lambda_t} \mathbb{E}\left(X \mathbb{E}\left(\mathbb{1}_{\{T<\tau\}} | \mathcal{F}_T\right) | \mathcal{F}_t\right),$$
$$= \mathbb{1}_{\{\tau>t\}} e^{\Lambda_t} \mathbb{E}\left(X e^{-\Lambda_T} | \mathcal{F}_t\right).$$

□

Lemma 2.1.7. *(i) Let h be a (bounded) $\mathbb{F}$-predictable process. Then*

$$\mathbb{E}\left(h_\tau | \mathcal{F}_t\right) = \mathbb{E}\left(\int_0^\infty h_u \lambda_u e^{-\Lambda_u} \mathrm{d}u | \mathcal{F}_t\right) = \mathbb{E}\left(\int_0^\infty h_u \mathrm{d}F_u | \mathcal{F}_t\right)$$

and

$$\mathbb{E}\left(h_\tau | \mathcal{G}_t\right) = e^{\Lambda_t} \mathbb{E}\left(\int_t^\infty h_u \lambda_u \mathrm{d}F_u | \mathcal{F}_t\right) \mathbb{1}_{\{\tau>t\}} + h_\tau \mathbb{1}_{\{\tau\leq t\}}. \tag{2.1.1}$$

In particular

$$\mathbb{E}\left(h_\tau\right) = \mathbb{E}\left(\int_0^\infty h_u \lambda_u e^{-\Lambda_u} \mathrm{d}u\right) = \mathbb{E}\left(\int_0^\infty h_u \mathrm{d}F_u\right).$$

(ii) The process $(D_t - \int_0^{t\wedge\tau} \lambda_s \mathrm{d}s,\ t \geq 0)$ is a $\mathbb{G}$- martingale.

Proof. Let $B_v \in \mathcal{F}_v$ and h the elementary $\mathbb{F}$-predictable process defined as $h_t = \mathbb{1}_{\{t>v\}} B_v$. Then,

$$\mathbb{E}\left(h_\tau | \mathcal{F}_t\right) = \mathbb{E}\left(\mathbb{1}_{\{\tau>v\}} B_v | \mathcal{F}_t\right) = \mathbb{E}\left(\mathbb{E}\left(\mathbb{1}_{\{\tau>v\}} B_v | \mathcal{F}_\infty\right) | \mathcal{F}_t\right)$$
$$= \mathbb{E}\left(B_v \mathbb{P}\left(\tau > v | \mathcal{F}_\infty\right) | \mathcal{F}_t\right) = \mathbb{E}\left(B_v e^{-\Lambda_v} | \mathcal{F}_t\right).$$

It follows that

$$\mathbb{E}\left(h_\tau | \mathcal{F}_t\right) = \mathbb{E}\left(B_v \int_v^\infty \lambda_u e^{-\Lambda_u} \mathrm{d}u | \mathcal{F}_t\right) = \mathbb{E}\left(B_v \int_0^\infty h_u \lambda_u e^{-\Lambda_u} \mathrm{d}u | \mathcal{F}_t\right)$$

and (i) is derived from the monotone class theorem. Equality (2.1.1) follows from the Lemma

2.1.5.

The martingale property (ii) follows from the integration by parts formula. Indeed, let $s < t$. Then, on the one hand from the Lemma 2.1.5

$$\begin{aligned}\mathbb{E}\left(D_t - D_s|\mathcal{G}_s\right) &= \mathbb{P}\left(s < \tau \leq t|\mathcal{G}_t\right) = \mathbb{1}_{\{s<\tau\}}\frac{\mathbb{P}\left(s < \tau \leq t|\mathcal{F}_s\right)}{\mathbb{P}\left(s < \tau|\mathcal{F}_s\right)}\\ &= \mathbb{1}_{\{s<\tau\}}\left(1 - e^{\Lambda_s}\mathbb{E}\left(e^{-\Lambda_t}|\mathcal{F}_s\right)\right).\end{aligned}$$

On the other hand, from part (i), for $s < t$,

$$\begin{aligned}\mathbb{E}\left(\int_{s\wedge\tau}^{t\wedge\tau}\lambda_u \mathrm{d}u|\mathcal{G}_s\right) &= \mathbb{E}\left(\Lambda_{t\wedge\tau} - \Lambda_{s\wedge\tau}|\mathcal{G}_s\right) = \mathbb{E}\left(\psi_\tau|\mathcal{G}_t\right)\\ &= \mathbb{1}_{\{s<\tau\}}e^{\Lambda_s}\mathbb{E}\left(\int_s^\infty \psi_u\lambda_u e^{-\Lambda_u}\mathrm{d}u|\mathcal{F}_s\right)\end{aligned}$$

where $\psi_u = \Lambda_{t\wedge u} - \Lambda_{s\wedge u} = \mathbb{1}_{\{s<u\}}(\Lambda_{t\wedge u} - \Lambda_s)$. Consequently,

$$\begin{aligned}\int_s^\infty \psi_u\lambda_u e^{-\Lambda_u}\mathrm{d}u &= \int_s^t \left(\Lambda_u - \Lambda_s\right)\lambda_u e^{-\Lambda_u}\mathrm{d}u + \left(\Lambda_t - \Lambda_s\right)\int_t^\infty \lambda_u e^{-\Lambda_u}\mathrm{d}u\\ &= \int_s^t \Lambda_u\lambda_u e^{-\Lambda_u}\mathrm{d}u - \Lambda_s\int_s^\infty \lambda_u e^{-\Lambda_u}\mathrm{d}u + \Lambda_t e^{-\Lambda_t}\\ &= \int_s^t \Lambda_u\lambda_u e^{-\Lambda_u}\mathrm{d}u - \Lambda_s e^{-\Lambda_s} + \Lambda_t e^{-\Lambda_t}\\ &= e^{-\Lambda_s} - e^{-\Lambda_t}.\end{aligned}$$

It follows that

$$\mathbb{E}\left(D_t - D_s|\mathcal{G}_s\right) = \mathbb{E}\left(\int_{s\wedge\tau}^{t\wedge\tau}\lambda_y \mathrm{d}u|\mathcal{G}_s\right),$$

hence the martingale property of the process $D_t - \int_0^{t\wedge\tau}\lambda_u \mathrm{d}u$. $\square$

2.1.5 Conditional survival probability

Let G be the survival hazard process, $G_t := \mathbb{P}(\tau > t|\mathcal{F}_t) = 1 - F_t$. Since the default time is constructed with a Cox process model, we can see that

$$G_t = \mathbb{E}\left(\mathbb{1}_{\{\tau>t\}}|\mathcal{F}_t\right) = \exp\left(-\int_0^t \lambda_u \mathrm{d}u\right).$$

It follows that the immersion property holds.

Lemma 2.1.8. *Let X be an $\mathcal{F}_T$-measurable integrable random variable. Then, for $t < T$,*

$$\mathbb{E}\left(X\mathbb{1}_{\{T<\tau\}}|\mathcal{G}_t\right) = \mathbb{1}_{\{\tau>t\}}\left(G_t\right)^{-1}\mathbb{E}\left(XG_T|\mathcal{F}_t\right).$$

Proof. The proof is the same as in Corollary 2.1.6. Indeed

$$\mathbb{1}_{\{\tau>t\}}\mathbb{E}\left(X\mathbb{1}_{\{T<t\tau\}}|\mathcal{G}_t\right) = \mathbb{1}_{\{\tau>t\}}x_t,$$

where x_t is $\mathcal{F}_t$-measurable. Taking conditional expectations w.r.t $\mathcal{F}_t$ of both sides, we deduce

$$x_t = \frac{\mathbb{E}\left(X\mathbb{1}_{\{\tau>T\}}|\mathcal{F}_t\right)}{\mathbb{E}\left(\mathbb{1}_{\{\tau>t\}}|\mathcal{F}_t\right)} = \mathbb{1}_{\{\tau\}}\left(G_t\right)^{-1}\mathbb{E}\left(XG_T|\mathcal{F}_t\right).$$

□

Lemma 2.1.9. *Let h be an $\mathbb{F}$-predictable process. Then*

$$\mathbb{E}\left(h_\tau\mathbb{1}_{\{\tau\le T\}}|\mathcal{G}_t\right) = h_\tau\mathbb{1}_{\{\tau\le t\}} - \mathbb{1}_{\{\tau>t\}}\left(G\right)^{-1}\mathbb{E}\left(\int_t^T h_u \mathrm{d}G_u|\mathcal{F}_t\right)$$

Proof. The proof follows the same line as that of Lemma 2.1.7. □

2.2 Equivalent Probabilities, Radon-Nikodým Densities and Girsanov's Theorem

Let $\mathbb{P}$ and $\mathbb{Q}$ be two probabilities defined on the same measurable space $(\Omega, \mathcal{F})$. The probability $\mathbb{Q}$ is said to be **absolutely continuous** with respect to $\mathbb{P}$, (denoted $\mathbb{Q} \ll \mathbb{P}$) if $\mathbb{P} = 0$ implies $\mathbb{Q}(A) = 0$, for any $A \in \mathcal{F}$. In that case, there exists a positive, $\mathcal{F}$-measurable random variable L, called the **Radon-Nikodým density** of $\mathbb{Q}$ with respect to $\mathbb{P}$, such that

$$\forall A \in \mathcal{F},\ \mathbb{Q}(A) = \mathbb{E}_{\mathbb{P}}(L\mathbb{1}_A).$$

This random variable L satisfies $\mathbb{E}_{\mathbb{P}}(L) = 1$ and for any $\mathbb{Q}$-integrable random variable X, $\mathbb{E}_{\mathbb{Q}}(X) = \mathbb{E}_{\mathbb{P}}(XL)$. The notation $\dfrac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}} = L$ (or $\mathbb{Q}|_{\mathcal{F}} = L\mathbb{P}|_{\mathcal{F}}$) is common use, in particular in the chain of equalities

$$\mathbb{E}_{\mathbb{Q}}(X) = \int X \mathrm{d}\mathbb{Q} = \int X \frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}} \mathrm{d}\mathbb{P} = \int XL\mathrm{d}\mathbb{P} = \mathbb{E}_{\mathbb{P}}(XL).$$

The probabilities $\mathbb{P}$ and $\mathbb{Q}$ are said to be **equivalent**, (this will be denoted $\mathbb{P} \sim \mathbb{Q}$), if they have the same null sets, i.e., if for any $A \in \mathcal{F}$,

$$\mathbb{Q}(A) = 0 \iff \mathbb{P}(A) = 0,$$

or equivalently, if $\mathbb{Q} \ll \mathbb{P}$ and $\mathbb{P} \ll \mathbb{Q}$. In that case, there exists a strictly positive, $\mathcal{F}$-measurable random variable L, such that $\mathbb{Q}(A) = \mathbb{E}_{\mathbb{P}}(L\mathbb{1}_A)$. Note that $\frac{\mathrm{d}\mathbb{P}}{\mathrm{d}\mathbb{Q}} = L^{-1}$ and $\mathbb{P}(A) = \mathbb{E}_{\mathbb{Q}}\left(L^{-1}\mathbb{1}_A\right)$.

Conversely, if L is a strictly positive $\mathcal{F}$-measurable r.v., with expectation 1 under $\mathbb{P}$, then $\mathbb{Q} = L \cdot \mathbb{P}$ defines a probability measure on $\mathcal{F}$, equivalent to $\mathbb{P}$. From the definition of equivalence, if a property holds almost surely (a.s.) with respect to $\mathbb{P}$, it also holds a.s. for any probability

$\mathbb{Q}$ equivalent to $\mathbb{P}$. Two probabilities $\mathbb{P}$ and $\mathbb{Q}$ on the same probability space $(\Omega, \mathcal{F})$ are said to be equivalent if they have the same negligible sets on $\mathcal{F}_t$, for every $t \geq 0$, i.e. ,if $\mathbb{Q}|_{\mathcal{F}_t} \sim \mathbb{P}|_{\mathcal{F}_t}$. In that case, there exists a strictly positive $\mathbb{F}$-adapted process $(L_t)_{t\geq 0}$ such that $\mathbb{Q}|_{\mathcal{F}_t} = L_t \mathbb{P}|_{\mathcal{F}_t}$. Furthermore, if τ is a stopping time , then

$$\mathbb{Q}|_{\mathcal{F}_\tau \cap \{\tau < \infty\}} = L_\tau \cdot \mathbb{P}|_{\mathcal{F}_\tau \cap \{\tau < \infty\}}.$$

Proposition 2.2.1. *(Bayes Formula) Suppose that $\mathbb{Q}$ and $\mathbb{P}$ are equivalent on $\mathcal{F}_T$ with Radon-Nikodým density L. Let X be a $\mathbb{Q}$-integrable $\mathcal{F}_T$-measurable random variable, then, for $t < T$*

$$\mathbb{E}_{\mathbb{Q}}\left(X|\mathcal{F}_t\right) = \frac{\mathbb{E}_{\mathbb{P}}\left(L_T X|\mathcal{F}_t\right)}{L_t}.$$

Proof. The proof follows immediately from the definition of conditional expectation. To check that $\mathcal{F}_t$-measurable r.v. $Z = \frac{\mathbb{E}_{\mathbb{P}}(L_T X|\mathcal{F}_t)}{L_t}$ is the $\mathbb{Q}$-conditional expectation of X, we prove that $\mathbb{E}_{\mathbb{Q}}\left(F_t X\right) = \mathbb{E}_{\mathbb{Q}}\left(F_t Z_t\right)$ for any bounded $\mathcal{F}_t$-measurable r.v. F_t. This follows from the equalities

$$\begin{aligned}\mathbb{E}_{\mathbb{Q}}\left(F_t X\right) &= \mathbb{E}_{\mathbb{P}}\left(L_T F_t X\right) = \mathbb{E}_{\mathbb{P}}\left(F_t \mathbb{E}_{\mathbb{P}}\left(X L_T|\mathcal{F}_t\right)\right) \\ &= \mathbb{E}_{\mathbb{Q}}\left(F_t L_t^{-1} \mathbb{E}_{\mathbb{P}}\left(X L_T|\mathcal{F}_t\right)\right) = \mathbb{E}_{\mathbb{Q}}\left(F_t Z\right).\end{aligned}$$

□

Proposition 2.2.2. *Let $\mathbb{Q}$ and $\mathbb{P}$ be two locally equivalent probability measures with Radon-Nikodým density L. A process M is a $\mathbb{Q}$-martingale if and only if a process LM is a $\mathbb{P}$-martingale. By localization, this result remains true for local martingales.*

Proof. Let M be a $\mathbb{Q}$-martingale. From the Bayes formula, we obtain, for $s \leq t$,

$$M_s = \mathbb{E}_{\mathbb{Q}}\left(M_t|\mathcal{F}_s\right) = \frac{\mathbb{E}_{\mathbb{P}}\left(L_t M_t|\mathcal{F}_s\right)}{L_s}$$

and the result follows. The converse part is now obvious. □

2.2.1 Decomposition of $\mathbb{P}$Martingales as $\mathbb{Q}$semi-martingales

Theorem 2.2.3. *Let $\mathbb{Q}$ and $\mathbb{P}$ be two locally equivalent probability measures with Radon-Nikodým density L. We assume that the process L is continuous. If M is a continuous $\mathbb{P}$-local martingale, then the process $\widetilde{M}$ defined by*

$$\mathrm{d}\widetilde{M} = \mathrm{d}M - \frac{1}{L}\mathrm{d}\langle M, L\rangle$$

is a continuous $\mathbb{Q}$-local martingale. If N is another continuous $\mathbb{P}$-local martingale,

$$\langle M, N\rangle = \langle \widetilde{M}, \widetilde{N}\rangle = \langle M, \widetilde{N}\rangle.$$

Proof. From Proposition 2.2.2, it is enough to check that $\widetilde{M}L$ is a $\mathbb{P}$-local martingale, which follows easily from Itô's calculus. □

Corollary 2.2.4. *Under the hypothesis of Theorem 2.2.3, we may write the process L as a Doleans-Dade martingale: $L_t = \mathcal{E}(\xi)_t$, where ξ is an $\mathbb{F}$-local martingale. The process $\widetilde{M} = M - \langle M, \xi\rangle$ is a $\mathbb{Q}$-local martingale.*

2.2.2 Girsanov's Theorem

Assume that $\mathbb{F}$ is generated by a Brownian motion W and let L be the Radon-Nikodým density of the locally equivalent measures $\mathbb{P}$ and $\mathbb{Q}$. Then the martingale L admits a representation of the form $\mathrm{d}L_t = \psi_t \mathrm{d}W_t$. Since L is strictly positive, this equality takes the form $\mathrm{d}L_t = -\theta_t L_t \mathrm{d}W_t$. where $\theta = -\frac{\psi}{L}$. It follows that

$$L_t = \exp\left(-\int_0^t \theta_s \mathrm{d}W_s - \frac{1}{2}\int_0^t \theta_s^2 \mathrm{d}s\right) = \mathcal{E}(\xi)_t\,,$$

where $\xi_t = -\int_0^t \theta_s \mathrm{d}W_s$.

Proposition 2.2.5. *Let W be an $(\mathbb{P}, \mathbb{F})$-Brownian motion and let θ be an $\mathbb{F}$-adapted process such that the solution of the stochastic differential equation (SDE)*

$$\mathrm{d}L_T = -L_t\theta_t\mathrm{d}W_t, \quad L_0 = 1$$

is a martingale. We set $\mathbb{Q}|_{\mathcal{F}_t} = L_t\mathbb{P}|_{\mathcal{F}_t}$. Then the process W admits a $\mathbb{Q}$-semi-martingale decomposition $\widetilde{W}$ as $W_t = \widetilde{W}_t - \int_0^t \theta_s \mathrm{d}s$ where $\widetilde{W}$ is a $\mathbb{Q}$-Brownian motion.

Proof. From $\mathrm{d}L_t = -L_t\theta_t\mathrm{d}W_t$, using the Girsanov's theorem 2.2.3, we obtain that the decomposition of W under $\mathbb{Q}$ is $\widetilde{W}_t - \int_0^t \theta_s \mathrm{d}s$. The process W is a $\mathbb{Q}$-semi-martingale and its martingale part $\widetilde{W}$ is a Brownian motion. This last fact follows from Levy's theorem, since the bracket of W does not depend on the (equivalent) probability. □

Remark 2.2.6. (Multidimensional case) Let W be an n-dimensional Brownian motion and θ be an n-dimensional adapted process such that $\int_0^t ||\theta_s||^2 \mathrm{d}s < \infty$, a.s.. Define the local martingale L as the solution of

$$\mathrm{d}L_t = L_t\theta_t \cdot \mathrm{d}W_t = L_t \left(\sum_{i=1}^n \theta_t^i \mathrm{d}W_t^i \right),$$

so that

$$L_t = \exp\left(\int_0^t \theta_s \cdot \mathrm{d}W_s - \frac{1}{2} \int_0^t ||\theta_s||^2 \mathrm{d}s \right).$$

If L is a martingale, the n-dimensional process $\left(\widetilde{W}_t = W_t - \int_0^t \theta_s \mathrm{d}s,\ t \geq 0\right)$ is a $\mathbb{Q}$-martingale, where $\mathbb{Q}$ is defined by $\mathbb{Q}|_{\mathcal{F}_t} = L_t\mathbb{P}|_{\mathcal{F}_t}$. Then $\widetilde{W}$ is an n-dimensional Brownian motion (and in particular its components are independent).

If W is a Brownian motion with correlation matrix Λ, then, since the brackets do not depend on the probability, under $\mathbb{Q}$, the process

$$\widetilde{W}_t = W_t - \int_0^t \theta_s \cdot \Lambda \mathrm{d}s$$

is a correlated Brownian motion with the same correlation n matrix Λ.

2.2.3 Risk-neutral measure

The value of a derivative can be calculated as the expectation of the derivative payoff over all possible asset price paths which affect the payoff. The measure under which this expectation is taken is critical, determining whether the pricing of derivatives is in line with the standard no-arbitrage assumptions present in almost all models of derivative pricing. The fundamental theorem of asset pricing tells us that a complete market is arbitrage free if and only if there exists at least one risk-neutral probability measure. Under this measure all assets have an expected return which is equal to the risk-free rate.

The history of the development of risk-neutral pricing is one that spans decades and largely follows the development of quantitative finance. Consider the standard assumption that there is a stock whose price satisfies

$$\mathrm{d}S_t = \mu_t S_t \mathrm{d}t + \sigma_t S_t \mathrm{d}W_t.$$

In addition, suppose that we have an adapted interest rate process $r_t,\ t \geq 0$. The corresponding discount process follows

$$\mathrm{d}D_t = -r_t D_t \mathrm{d}t.$$

The discounted stock price process is given by

$$d\left(D_t S_t\right) = r_t D_t S_t \left(\theta \mathrm{d}t + \mathrm{d}W_t\right),$$

where we define the market price of risk to be

$$\theta_t = \frac{\alpha_t - r_t}{\sigma_t}.$$

We introduce a probability measure $\mathbb{Q}$ defined in Girsanov's theorem, which uses the market price of risk θ_t. In terms of Brownian motion $\widetilde{W}$, we rewrite the discount stock price as

$$\mathrm{d}\left(D_t S_t\right) = \sigma_t D_t S_t \mathrm{d}\widetilde{W}_t$$

We call $\mathbb{Q}$ the *risk-neutral measure.*

2.3 PDE approach for pricing defaultable contingent claim

We assume that $\mathbb{P}$ represents the real-world probability as opposed to the risk neutral probability measure denoted by $\mathbb{Q}$ and chosen by the market. We assume that a defaultable risky asset S and a stochastic interest rate r are the only assets available in the market whose risk-neutral dynamics are as follows:

$$\begin{cases} \mathrm{d}S_t = \mu\left(S_t, r_t\right)\mathrm{d}t + \sigma\left(S_t, r_t\right)\mathrm{d}W_t^1 \\ \mathrm{d}r_t = \alpha\left(S_t, r_t\right)\mathrm{d}t + \beta\left(S_t, r_t\right)\mathrm{d}W_t^2 \\ \mathrm{d}W_t^1 \mathrm{d}W_t^2 = \rho \mathrm{d}t. \end{cases}$$

The filtration $\mathbb{F}$ represents the quantity of information on the assets and the filtration $\mathbb{D}$ is generated by τ which models the time to default of the defaultable asset. We assume that the market is complete and arbitrage-free. Consider a contingent claim H on the defaultable risky asset S that consists of a payment of an amount $H = h\left(S_T, r_T\right)$ at maturity if no default occurs prior to maturity T. The time-t price $V\left(t, H\right)$ of the contingent claim is given, from

Corollary 2.1.6, by

$$V\left(t,H\right)=\mathbb{E}\left(e^{-\int_t^T\left(r_u+\lambda_u\right)\mathrm{d}u}h\left(S_T,r_T\right)|\mathcal{F}_t\right).$$

Let

$$f\left(t,S_t,r_t\right)=V\left(t,H\right)$$

and

$$g\left(S_t,r_t\right)=\mathbb{E}\left(e^{-\int_0^T\left(r_u+\lambda_u\right)\mathrm{d}u}h\left(S_T,r_T\right)|\mathcal{F}_t\right).$$

As soon as g is smooth enough (see [24]), Itô's formula leads to

$$\begin{aligned}\mathrm{d}g\left(S_t,r_t\right)=&\Big(\partial_t g\left(S_t,r_t\right)+\mu\left(S_t,r_t\right)\partial_S g\left(S_t,r_t\right)+\frac{1}{2}\sigma\left(S_t,r_t\right)^2\partial_S^2 g\left(S_t,r_t\right)+\alpha\left(S_t,r_t\right)\partial_r g\left(S_t,r_t\right)\\&+\frac{1}{2}\beta\left(S_t,r_t\right)^2\partial_r^2 g\left(S_t,r_t\right)+\rho\beta\left(S_t,r_t\right)\sigma\left(S_t,r_t\right)\partial_{Sr}g\left(S_t,r_t\right)-\left(r_t-\lambda_t\right)g\left(S_t,r_t\right)\Big)\mathrm{d}t\\&+\sigma\left(S_t,r_t\right)\partial_S g\left(S_t,r_t\right)\mathrm{d}W_t^1+\beta\left(S_t,r_t\right)\partial_r g\left(S_t,r_t\right)\mathrm{d}W_t^2.\end{aligned}$$

Since the process $\left(g\left(S_t,r_t\right),\ t\geq 0\right)$ is a martingale, the dt-term is equal to zero. That is

$$\begin{aligned}0=&\partial_t g\left(S_t,r_t\right)+\mu\left(S_t,r_t\right)\partial_S g\left(S_t,r_t\right)+\frac{1}{2}\sigma\left(S_t,r_t\right)^2\partial_S^2 g\left(S_t,r_t\right)\\&+\rho\beta\left(S_t,r_t\right)\sigma\left(S_t,r_t\right)\partial_{Sr}g\left(S_t,r_t\right)+\alpha\left(S_t,r_t\right)\partial_r g\left(S_t,r_t\right)+\frac{1}{2}\beta\left(S_t,r_t\right)^2\partial_r^2 g\left(S_t,r_t\right)-\left(r_t-\lambda_t\right)g\left(S_t,r_t\right),\end{aligned}$$

and

$$g\left(S_T,r_T\right)=\mathbb{E}\left(e^{-\int_T^T\left(r_u+\lambda-u\right)\mathrm{d}u}h\left(S_T\right)|\mathcal{F}_T\right)=h\left(S_T\right).$$

Proposition 2.3.1. *Let the risk-neutral dynamics of the defaultable risky asset and the stochas-*

tic interest rate be defined as

$$\begin{cases} \mathrm{d}S_t = \mu\left(S_t, r_t\right)\mathrm{d}t + \sigma\left(S_t, r_t\right)\mathrm{d}W_t^1 \\ \mathrm{d}r_t = \alpha\left(S_t, r_t\right)\mathrm{d}t + \beta\left(S_t, r_t\right)\mathrm{d}W_t^2 \\ \mathrm{d}W_t^1\mathrm{d}W_t^2 = \rho\mathrm{d}t. \end{cases}$$

Assume that u solves the Cauchy problem

$$\begin{cases} \left(\partial_t + \mathcal{A}\right)u\left(t, S, r\right) = 0, \quad t \in [0, T),\ S \geq 0,\ r \in \mathbb{R} \\ u\left(T, S, r\right) = h\left(S, r\right), \quad S \geq 0,\ r \in \mathbb{R}, \end{cases}$$

with

$$\mathcal{A} = \partial_t + \mu\left(S_t, r_t\right)\partial_S + \frac{1}{2}\sigma\left(S_t, r_t\right)^2\partial_S^2 + \alpha\left(S_t, r_t\right)\partial_r + \frac{1}{2}\beta\left(S_t, r_t\right)^2\partial_r^2 + \rho\beta\left(S_t, r_t\right)\sigma\left(S_t, r_t\right)\partial_{Sr} - \left(r_t - \lambda_t\right).$$

Then the value at time t of the contingent claim $H = h\left(S_T, r_T\right)$ is equal to $u\left(t, S_t, r_t\right)$.

Chapter 3

CDS calibration under an extended JDCEV model

3.1 Introduction

The purpose of this chapter is to provide a robust and efficient method to calibrate a hybrid credit-equity model to the CDS market. Credit Default Swaps (CDS) are the most influential and traded credit derivatives. They played an important role in the recent financial scandals: in the sub-prime crisis in 2007-2008 or the trading losses by the "London Whale" at JP Morgan Chase in 2012. On the other hand, large global banks have been successfully exploiting the CDS market in their trading activities: for example, JP Morgan has several trillions of dollars of CDS notional outstanding. In parallel, the academic research on CDS, liabilities and derivatives in general has quickly expanded in the recent years. Among the most important contributions, the Jump to Default Constant Elasticity of Variance (JDCEV) model by Carr and Linetsky [9, 32, 33] is one of the first attempts to unify credit and equity models into the framework of deterministic and positive interest rates. The authors of [9] claim that credit models should not ignore information on stocks and there exists a connection among stock prices, volatilities and default intensities. Indeed, earlier research on credit models (e.g. [10, 18, 19] was more focused on how to palliate the absence of the bankruptcy possibility in classical option pricing theory and take into account that in the real world firms have a positive probability of default in finite time.

Nowadays the restrictive assumption of *positive and deterministic interest rates* of the JDCEV model is not realistic and contradicts market observations. To incorporate stochastic and possibly negative interest rates into the JDCEV model, we propose a fast and efficient technique to compute CDS spreads and default probabilities for calibration purposes. In doing this we employ a recent methodology introduced in [29, 37], which consists of an asymptotic expansion of the solution to the pricing partial differential equation. Our method allows to calibrate the extended JDCEV model to real market data in real time. To assess the robustness of the approximation method and the capability of the model of reproducing price dynamics, we provide several tests on UBS AG and BNP Paribas CDS spreads.

This chapeter is organized as follows. In Section 3.2 we set the notations and review the jump to default diffusion model. In Section 3.3 we introduce an extended JDCEV model with stochastic interest rates and provide explicit approximation formulas for the CDS spreads and the risk-neutral survival probabilities. Section 3.4 contains the numerical tests: we consider both the cases of correlated or uncorrelated spreads and interest rates; we calibrate the model to market data of CDS spreads and compute the risk-neutral survival probabilities: a comparison with standard Monte Carlo methods is provided as well.

3.2 CDS spread and default probability

We consider a probability space $(\Omega, \mathcal{G}, \mathbb{Q})$ carrying a standard Brownian motion W and an exponential random variable $\varepsilon \sim \mathrm{Exp}(1)$ independent of W. We assume, for simplicity, a frictionless market, no arbitrage and take an equivalent martingale measure $\mathbb{Q}$ as given. All stochastic processes defined below live on this probability space and all expectations are taken with respect to $\mathbb{Q}$.

Let $\widetilde{S}$ be the pre-default stock price. We assume that the dynamics of $X = \log \widetilde{S}$ are given by

$$\begin{cases} \mathrm{d}X_t = \left(r_t - \frac{1}{2}\sigma^2\left(t, X_t\right) + \lambda\left(t, X_t\right)\right)\mathrm{d}t + \sigma\left(t, X_t\right)\mathrm{d}W_t^1, \\ \mathrm{d}r_t = \kappa\left(\theta - r_t\right)\mathrm{d}t + \delta \mathrm{d}W_t^2, \\ \mathrm{d}W_t^1 \mathrm{d}W_t^2 = \rho \mathrm{d}t, \end{cases}$$

where the interest rate r_t follows the Vasicek dynamics with parameters $\kappa, \theta, \delta > 0$. The time- and state-dependent stock volatility $\sigma = \sigma(t, X)$ and default intensity $\lambda = \lambda(t, X)$ are assumed to be differentiable with respect to X and uniformly bounded. In general the price can become worthless in two scenarios: either the process e^X hits zero via diffusion or a jump-to-default occurs from a positive value. The default time ζ can be modeled as $\zeta = \zeta_0 \wedge \tilde{\zeta}$, where $\zeta_0 = \inf\{t > 0 \mid \widetilde{S}_t = 0\}$ is the first hitting time of zero for the stock price and $\tilde{\zeta} = \inf\{t \geq 0 \mid \Lambda_t \geq \varepsilon\}$ is the jump-to-default time with intensity λ and hazard rate $\Lambda_t = \int_0^t \lambda(s, X_s)\mathrm{d}s$. In what follows, we denote by $\mathbb{F} = \{\mathcal{F}_t,\ t \geq 0\}$ the filtration generated by the pre-default stock price and by $\mathbb{D} = \{\mathcal{D}_t,\ t \geq 0\}$ the filtration generated by the process $D_t = \mathbb{1}_{\{\zeta \leq t\}}$. Eventually, $\mathbb{G} = \{\mathcal{G}_t,\ t \geq 0\},\ \mathcal{G}_t = \mathcal{F}_t \vee \mathcal{D}_t$ is the enlarged filtration.

A Credit Default Swap (CDS) is one of the most representative financial instruments depending on default probabilities of firms. Designed to protect against default, a CDS with constant rate R and recovery at default $(1 - \eta)$ is a contract between a party A (protection buyer), the buyer of the protection against a reference entity C which defaults at time ξ, and a party B (protection seller). Party A pays a premium R of the CDS notional amount $\mathcal{N}$ to the seller B until $\xi \wedge T$ at some predetermined date $t_i \leq T$ with interval time α, where T is the maturity of the CDS. If the default occurs prior to T, the seller pays, at the default time ξ, $(1 - \eta(\xi))$ of $\mathcal{N}$

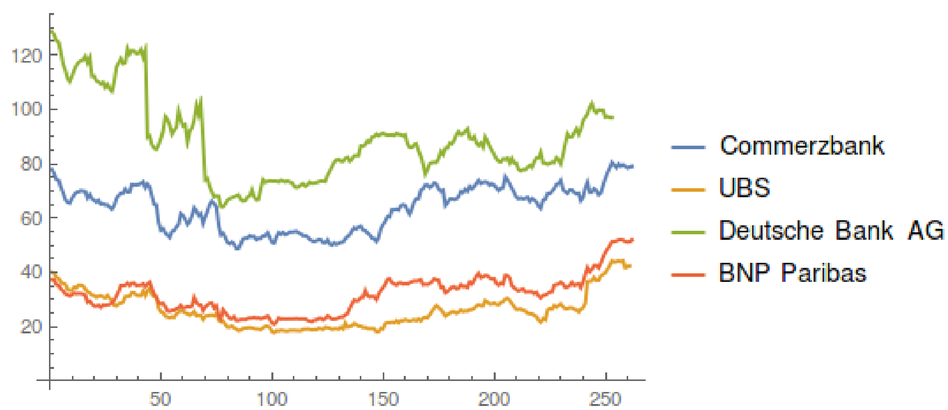
to the buyer.

$$B \rightarrow \text{ protection } (1-\eta(\xi)) \text{ at } \xi \text{, default time of C, } \text{ if } \xi < T \ \rightarrow A$$

$$B \leftarrow \text{ rate } R \text{ at } t_1,\ t_2,\ \cdots,\ \xi \wedge T \ \leftarrow \ A$$

$(1-\eta)$ is called the *loss-given-default*, represents the default protection and R is the CDS rate, also termed *spread, premium* or *annuity* of the CDS. Figure 3.1 represents the evolutions of 5 years maturity CDS spreads of large corporates with respect to the observation date t ($t \mapsto CDS(t, 5Y)$).

Figure 3.1: CDS Spreads with fixed maturity



Evolution of 5 years CDS Spreads in function of observation date of four large corporates in the CDS market.

3.2.1 Valuation of CDS

Consider a CDS contract with rate R, default recovery $(1-\eta)$ and maturity time T. By definition, its market value at time t is given by the expectation of the difference of the discounted payoffs of the protection and premium legs

$$V_t(R) = \mathbb{E}^{\mathbb{Q}}\left(e^{-\int_t^{\xi} r_u \mathrm{d}u}\left(1-\eta(\xi)\right)\mathbb{1}_{\{\xi\le T\}} - \sum_{i=i_t}^{M} e^{-\int_t^{t_i} r_u \mathrm{d}u}\alpha R \mathbb{1}_{\{\xi>t_i\}}|\mathcal{G}_t\right) \tag{3.2.1}$$

where $i_t = \inf\{i \in \{1,\ 2,\ \cdots,\ M\}\colon t \le t_i\}$ with $t_M = T$.

Proposition 3.2.1. *The price of a CDS at time $t \in [0, T]$ is given by*

$$V_t(R) = \mathbb{1}_{\{t<\xi\}}\left(\mathbb{E}^{\mathbb{Q}}\left(\int_t^T e^{-\int_t^s (r_u+\lambda_u)\mathrm{d}u}\left(1-\eta(s)\right)\lambda_s \mathrm{d}s|\mathcal{F}_t\right) - \sum_{i=i_t}^{M}\mathbb{E}^{\mathbb{Q}}\left(\alpha R e^{-\int_t^T (r_u+\lambda_u)\mathrm{d}u}|\mathcal{F}_t\right)\right).$$

Proof. The proof comes straightforward from Corollary 2.1.6 and Lemma 2.1.9 by setting

$$h_s = \left(1-\eta(s)\right)e^{-\int_t^s r_u \mathrm{d}u} \quad \text{and} \quad X^i = e^{-\int_t^{t_i} r_u \mathrm{d}u}.$$

□

3.2.2 Market CDS Spread

Definition 3.2.2. A market CDS spread starting at t is a CDS initiated at time t whose value is equal to zero . A T-maturity market CDS spread at time t is the level of the rate $R = R(t, T)$ that makes a T-maturity CDS starting at t valueless at its inception. A market CDS spread at time t is thus determined by the equation $V_t(R(t, T)) = 0$, where V is defined by (3.2.1). Hence, on $\mathbb{1}_{\{t<\xi\}}$, we have

$$R(t, T) = \frac{\mathbb{E}^{\mathbb{Q}}\left(\int_t^T e^{-\int_t^s (r_u+\lambda_u)\mathrm{d}u}\left(1-\eta(s)\right)\lambda_s \mathrm{d}s|\mathcal{F}_t\right)}{\sum_{i=i_t}^{M}\mathbb{E}^{\mathbb{Q}}\left(\alpha e^{-\int_t^T (r_u+\lambda_u)\mathrm{d}u}|\mathcal{F}_t\right)}. \tag{3.2.2}$$

Proposition 3.2.3. *Consider a CDS contract with constant default recovery $(1-\eta)$, spread R paid at premium payment dates t_i, $i = 1,\ 2, \cdots M$, so that that $\alpha = t_{i+1} - t_i = \frac{T}{M}$. Then, at* *time $t = 0$, the spread $R := R(0,T)$ is given by*

$$R = \frac{(1-\eta)\left(1 - \mathbb{E}\left[e^{-\int_0^T (r_u + \lambda(u,X_u))\mathrm{d}u}\right] - \int_0^T \mathbb{E}\left[e^{-\int_0^s (r_u + \lambda(u,X_u))\mathrm{d}u} r_s\right]\mathrm{d}s\right)}{\frac{T}{M}\sum_{i=1}^{M} \mathbb{E}\left[e^{-\int_0^{t_i} (r_u + \lambda(u,X_u))\mathrm{d}u}\right]}. \tag{3.2.3}$$

Proof. From the definition of equation (3.2.2), the spread at time $t = 0$ is given by

$$R = \frac{(1-\eta)\,\mathbb{E}\left[\int_0^T e^{-\int_0^s (r_u + \lambda(u,X_u))\mathrm{d}u} \lambda(s, X_s)\mathrm{d}s\right]}{\frac{T}{M}\sum_{i=1}^{M} \mathbb{E}\left[e^{-\int_0^{t_i} (r_u + \lambda(u,X_u))\mathrm{d}u}\right]}. \tag{3.2.4}$$

The statement follows by replacing the following identities in (3.2.4):

$$e^{-\int_0^s (r_u + \lambda(u,X_u))\mathrm{d}u} \lambda(s, X_s) = -\frac{\partial}{\partial_s}\left(e^{-\int_0^s (r_u + \lambda(u,X_u))\mathrm{d}u}\right) - r_s e^{-\int_0^s (r_u + \lambda(u,X_u))\mathrm{d}u},$$

and

$$\int_0^T e^{-\int_0^s (r_u + \lambda(u,X_u))\mathrm{d}u} \lambda(s, X_s)\mathrm{d}s = 1 - \left(e^{-\int_0^T (r_u + \lambda(u,X_u))\mathrm{d}u} + \int_0^T e^{-\int_0^s (r_u + \lambda(u,X_u))\mathrm{d}u} r_s \mathrm{d}s\right).$$

□

Remark 3.2.4. The default intensity $\lambda(t, X_t)$ can be considered as the instantaneous probability that the stock will default between t and $t + \mathrm{d}t$, conditioned on the fact that no default has happened before:

$$\lambda(t, X_t)\mathrm{d}t = \mathbb{Q}\left(t \leq \zeta < t + \mathrm{d}t \mid \zeta \geq t\right).$$

The survival probability up to time t is defined as

$$Q(t) := \mathbb{E}\left[e^{-\int_0^t \lambda(u,X_u)\mathrm{d}u}\right]. \tag{3.2.5}$$

3.3 CDS spread approximation under extended JDCEV model

In the JDCEV model the stock volatility is of the form

$$\sigma(t, X) = a(t)e^{(\beta-1)X}$$

where $\beta < 1$ and $a(t) > 0$ are the so-called elasticity parameter and scale function. The default intensity is expressed as a function of the stock volatility and the stock log-price, as follows

$$\lambda(t, X) = b(t) + c\,\sigma(t, X)^2 = b(t) + c\,a(t)^2 e^{2(\beta-1)X}, \tag{3.3.1}$$

where $b(t) \geq 0$ and $c \geq 0$ govern the sensitivity of the default intensity with respect to the volatility. The risk-neutral dynamics of the defaultable stock price $S_t = \{S_t,\ t \geq 0\}$ are then given by

$$\begin{cases} S_t = S_0 e^{X_t} \mathbb{1}_{\{\zeta \geq t\}}, \qquad S_0 > 0, \\ \mathrm{d}X_t = \left(r_t - \frac{1}{2}\sigma^2(t, X_t) + \lambda(t, X_t)\right)\mathrm{d}t + \sigma(t, X_t)\,\mathrm{d}W_t^1, \\ \mathrm{d}r_t = \kappa(\theta - r_t)\,\mathrm{d}t + \delta \mathrm{d}W_t^2, \\ \zeta = \inf\{t \geq 0 \mid \int_0^t \lambda(t, X_t) \geq e\}, \\ \mathrm{d}W_t^1 \mathrm{d}W_t^2 = \rho \mathrm{d}t. \end{cases} \tag{3.3.2}$$

Let us consider a European claim on the defaultable asset, paying $h(X_T)$ at maturity T if no default happens and without recovery in case of default. In case of *constant interest rates*, one deduces the value of the European claim from the following result proved in [9].

Theorem 3.3.1. *Let $r \in \mathbb{R}$ be a non-negative constant and h be a continuous and bounded* *function. Then, for any $0 \leq t \leq T$, we have*

$$\mathbb{E}\left[\exp\left(-c\int_t^T a(u)^2 e^{2(\beta-1)X_u}\mathrm{d}u\right)h\left(X_T\right)|X_t = X_0\right] = \mathbb{E}\left[\left(\frac{Z_{\tau(t)}}{x}\right)^{-\frac{1}{|\beta-1|}} h\left(e^{\int_t^T \alpha(s)\mathrm{d}s}\left(|\beta-1|Z_{\tau(t)}\right)^{\frac{1}{|\beta-1|}}\right)\right], \tag{3.3.3}$$

where $\{Z_t,\ t \geq 0\}$ is a Bessel process starting from x, of index $\nu = \frac{c+1/2}{|\beta-1|}$, and τ is the deterministic time change defined as

$$\tau\left(t\right) = \int_0^t a^2(u)e^{2|\beta-1|\int_0^u \alpha_s \mathrm{d}s}\mathrm{d}u, \qquad \alpha(t) = r + b(t). \tag{3.3.4}$$

By Theorem 3.3.1 and standard results from enlargement filtration theory (cf. [19]), the value of the European claim at time $t < T$ is given by

$$\begin{aligned}\mathbb{E}\left[e^{-\int_t^T r_u \mathrm{d}u}h\left(X_T\right)|\mathcal{G}_t\right] &= \mathbb{1}_{\{\zeta>t\}}\mathbb{E}\left[e^{-\int_t^T (r_u+\lambda(u,X_u))\mathrm{d}u}h\left(X_T\right)|\mathcal{F}_t\right] \\ &= \mathbb{1}_{\{\zeta>t\}}e^{-\int_t^T (r_u+b_u)\mathrm{d}u}\mathbb{E}\left[e^{-c\int_t^T a_u^2 e^{2(\beta-1)X_u}\mathrm{d}u}h\left(X_T\right)|\mathcal{F}_t\right] \\ &= \mathbb{1}_{\{\zeta>t\}}e^{-\int_t^T (r_u+b_u)\mathrm{d}u}\mathbb{E}\left[\left(\frac{Z_{\tau(t)}}{x}\right)^{-\frac{1}{|\beta-1|}} h\left(e^{\int_t^T \alpha_s \mathrm{d}s}\left(|\beta-1|Z_{\tau(t)}\right)^{\frac{1}{|\beta-1|}}\right)\right].\end{aligned} \tag{3.3.5}$$

The validity of the second and third equality above is based on the assumption of deterministic interest rates. In the general case of stochastic rates, the time-change function (3.3.4) is not deterministic anymore and the expectation (3.3.3) cannot be computed analytically. For this reason, to deal with the general case, we adopt a completely different approach and introduce a perturbation technique which provides an explicit asymptotic expansion of the building block (3.3.5). Specifically, we base our analysis on the recent results in [29, 37] on the approximation of solution to parabolic partial differential equations and we derive approximations of the CDS spread (3.2.3) and the risk-neutral survival probability (3.2.5).

We see from (3.2.3) that we have to evaluate expectations of the form

$$u(0, X_0, r_0; T) = \mathbb{E}\left[e^{-\int_0^T (r_u + \lambda(u, X_u))\mathrm{d}u} h(r_T)\right], \tag{3.3.6}$$

with $h(r) = 1$ or $h(r) = r$. By the change of variable $r_t = e^{-\kappa t} y_t$ and from the Feynman-Kac formula (cf., for instance, [39]) it follows that u in (3.3.6) is solution to the Cauchy problem

$$\begin{cases} (\partial_t + \mathcal{A})\, u(t, x, y) = 0, & t < T,\ x, y \in \mathbb{R} \\ u(T, x, y) = h(y), & x, y \in \mathbb{R}, \end{cases} \tag{3.3.7}$$

with

$$\begin{aligned} \partial_t + \mathcal{A} = {} & \partial_t + \frac{1}{2}\sigma^2(t, x)\,\partial_{xx} + \rho\delta\sigma(t, x)e^{\kappa t}\partial_{xy} + \frac{1}{2}\delta^2 e^{2\kappa t}\partial_{yy} \\ & + \left(e^{-\kappa t} y + \lambda(t, x) - \frac{1}{2}\sigma^2(t, x)\right)\partial_x + \kappa\theta e^{\kappa t}\partial_y - \left(e^{-\kappa t} y + \lambda(t, x)\right) \\ = {} & \partial_t + \frac{1}{2}\langle \Sigma\nabla, \nabla\rangle + \langle \mu, \nabla\rangle + \gamma, \end{aligned} \tag{3.3.8}$$

where

$$\Sigma(t, x, y) = \begin{pmatrix} \sigma^2(t, x) & \rho\delta\sigma(t, x)\, e^{\kappa t} \\ \rho\delta\sigma(t, x)\, e^{\kappa t} & \delta^2 \end{pmatrix}, \quad \mu(t, x, y) = \begin{pmatrix} e^{-\kappa t} y - \frac{1}{2}\sigma^2(t, x) + \lambda(t, x) \\ \kappa\theta e^{\kappa t} \end{pmatrix},$$
$$\gamma(t, x) = -\left(e^{-\kappa t} y + \lambda(t, x)\right).$$

Operator $\mathcal{A}$ is *only locally* uniformly parabolic in the sense that, for any ball

$$\mathcal{O}_R := \{(x, r) \in \mathbb{R}^2 \mid |(x, r)| < R\};$$

the coefficients of $\mathcal{A}$ satisfy the following conditions:

Assumption 3.3.2. 1. **(H1)** The matrix $\Sigma(t,x,y)$ is positive definite, uniformly with respect to $(t,x,y) \in (0,T] \times \mathcal{O}_R$.

2. **(H2)** The coefficients Σ, μ, γ are bounded and Hölder-continuous on $(0,T] \times \mathcal{O}_R$.

Under these conditions we can resort to the recent results in [38], Theor. 2.6, or [27], Theor.1.5, about the existence of a *local density* for the process (X, y).

Theorem 3.3.3. *For any $R > 0$, the process (X, y) has a local transition density on $\mathcal{O}_R$, that is a non-negative measurable function $\Gamma = \Gamma(t,x,y;T,z,s)$ defined for any $0 < t < T$, $(x,y) \in \mathbb{R}^2$ and $(z,s) \in \mathcal{O}_R$ such that, for any continuous function $h = h(x,y)$ with compact support in $\mathcal{O}_R$, we have*

$$u(t,x,y) := E_{t,x,y}\left[h(X_T, y_T)\right] = \int_{\mathcal{O}_R} \Gamma(t,x,y;T,z,s) h(z,s) \mathrm{d}z\mathrm{d}s$$

and u satisfies

$$\begin{cases} (\partial_t + \mathcal{A})\, u(t,x,y) = 0, & (t,x,y) \in [0,T) \times \mathcal{O}_R, \\ u(T,x,y) = h(x,y) & (x,y) \in \mathcal{O}_R. \end{cases} \tag{3.3.9}$$

Problem (3.3.9) can be used for numerical approximation purposes. Notice however that (3.3.9) does not posses a unique solution due to the lack of lateral boundary conditions. Nevertheless, numerical schemes can be implemented imposing artificial boundary conditions and the result which guarantees the validity of such approximations is the so-called *principle of not feeling the boundary*. A rigorous statement of this result can be found in [14], Appendix A, or [38], Lemma 4.11. Economically speaking, there exists positive constants $\bar{S}$ and $\bar{r}$ such that for any $t > 0$, we have $|S_t| \leq \bar{S}_t$ and $|r_t| \leq \bar{r}$. In what follows, we assume that the Assumptions 3.3.2 hold.

Theorem 3.3.4. *Under the assumptions of Proposition 3.2.3 and under the general dynamics (3.3.2), the N-th order approximation of the CDS spread in (3.2.3) is given by*

$$R_N = \frac{(1-L)\left(1 - \sum\limits_{n=0}^{N} \mathcal{L}_n^{(x,y)}(0,T)\, u_0^1(0,x,y;T) - \int_0^T e^{-\kappa s} \sum\limits_{n=0}^{N} \mathcal{L}_n^{(x,y)}(0,s)\, u_0^2(0,x,y;s)\,\mathrm{d}s\right)}{\frac{T}{M} \sum\limits_{i=1}^{M} \sum\limits_{n=0}^{N} \mathcal{L}_n^{(x,y)}(0,t_i)\, u_0^1(0,x,y;t_i)}, \tag{3.3.10}$$

where

$$u_0^1(t,x,y,s) = e^{-\int_t^s \left(e^{-\kappa u} y + \lambda(u,x)\right)\mathrm{d}u},$$

$$u_0^2(t,x,y;s) = e^{-\int_t^s \left(e^{-\kappa u} y + \lambda(u,x)\right)\mathrm{d}u} \left(y + m_2(t,s)\right),$$

$m_2(t,s)$*, the second component of the vector* $m(t,s)$*, and the differential operators* $\mathcal{L}_n^{(x,y)}$ *are respectively defined in equations (7.1.9) and (7.1.11) in the Appendix A.*

Proof. In formula (3.2.3) there appears terms of the form

$$E\left[e^{-\int_0^t (r_u + \lambda(u,X_u))\mathrm{d}u}\right]$$

in the numerator and denominator, that are solutions to problem (3.3.7) with $h(y) = 1$. On the other hand, in (3.2.3) there also appear terms of the form

$$E\left[e^{-\int_0^t (r_u + \lambda(u,X_u))\mathrm{d}u} r_t\right]$$

which are solutions to the same problem with $h(y) = e^{-\kappa t} y$. Theorem 7.1.3 and Theorem 7.1.6 in Appendix A 7.1 yield the approximations

$$E\left[e^{-\int_0^t (r_u + \lambda(u,X_u))\mathrm{d}u}\right] = \sum_{n=0}^{N} \mathcal{L}_n^{(x,y)}(0,t)\, u_0^1(0,x,y;t) + \mathrm{O}\left(t^{\frac{N+2}{2}}\right),$$

$$E\left[e^{-\int_0^t (r_u+\lambda(u,X_u))\mathrm{d}u} r_t\right] = e^{-\kappa t}\sum_{n=0}^{N}\mathcal{L}_n^{(x,y)}\left(0,t\right)u_0^2\left(0,x,y;t\right)+\mathrm{O}\left(t^{\frac{N+2}{2}}\right),$$

as $t \to 0^+$, where

$$\begin{aligned} u_0^1\left(0,x,y,t\right) &= e^{-\int_0^t\left(e^{-\kappa u}y+\lambda(u,x)\right)\mathrm{d}u}\int_{\mathbb{R}^2}\Gamma_0\left(0,x,y;t,\xi_1,\xi_2\right)\mathrm{d}\xi_1\mathrm{d}\xi_2,\\ &= e^{-\int_0^t\left(e^{-\kappa u}y+\lambda(u,x)\right)\mathrm{d}u}, \end{aligned}$$

and

$$\begin{aligned} u_0^2\left(0,x,y;t\right) &= u_0^1\left(0,x,y,t\right)\int_{\mathbb{R}^2}\Gamma_0\left(0,x,y;t,\xi_1,\xi_2\right)h\left(\xi_2\right)\mathrm{d}\xi_1\mathrm{d}\xi_2,\\ &= u_0^1\left(0,x,y,t\right)\int_{\mathbb{R}^2}\Gamma_0\left(0,x,y;t,\xi_1,\xi_2\right)\xi_2\mathrm{d}\xi_1\mathrm{d}\xi_2\\ &= u_0^1\left(0,x,y,t\right)\left(y+m_2(0,t)\right). \end{aligned}$$

□

Remark 3.3.5. We have an analogous approximation result for the survival probability in (3.2.5). Since it can be expressed as the solution to the problem

$$\begin{cases} \left(\partial_t+\mathcal{A}\right)v\left(t,x,y\right)=0, & t<T,\ x,y\in\mathbb{R}\\ v\left(T,x,y\right)=1, & x,y\in\mathbb{R}, \end{cases}$$

then, by Theorem 7.1.3, we have

$$Q\left(t\right)=\sum_{n\geq 0}\mathcal{L}_n^{(x,y)}\left(0,t\right)v_0\left(0,x,y;T\right), \tag{3.3.11}$$

where $v_0(t,x,y;T) = \exp\left(-\int_t^T\lambda(u,x)\mathrm{d}u\right)$ and the operators $\mathcal{L}_n^{(x,y)}(0,t)$ are defined as in (7.1.11) in the Appendix A. As an example, we give here the explicit first-order approximation of the survival probability in the case of constant parameters $a(t) = a$ and $b(t) = b$:

$$
\begin{aligned}
v_0\left(0,x,y;T\right) &= e^{-\left(b+ca^2e^{2(\beta-1)x}\right)T},\\
v_1\left(t,x,y;T\right) &= -ce^{2(\beta-1)x-T\left(b+ca^2e^{(\beta-1)x}\right)}\left(\beta-1\right)a^2\cdot\\
&\quad\cdot\frac{-4y+4\theta+e^{-T\kappa}\left(4y-4\theta+e^{T\kappa}T\kappa\left(4y-4\theta+e^{T\kappa}\left(2\left(b+\theta\right)+\left(-1+2c\right)e^{2(\beta-1)x}a^2\right)\right)\right)}{2\kappa^2}.
\end{aligned}
$$

Implementation technique

These expressions are obtained by using Mathematica symbolic programming. Indeed, by replacing $\mathcal{G}_n^{(x,y)}(0,s)$ and $M^{(x,y)}(0,s)$ by their expressions (7.1.13) and (7.1.14) in (7.1.11), we can write $v_n(0,x,y;T)$ as follow:

$$
v_n\left(0,x,y;T\right) = \sum_{h=1}^{n}\sum_{i=0}^{3n}\sum_{j=0}^{3n-i}\sum_{k=0}^{n}\sum_{l=0}^{n}\left(x-\bar{x}\right)^k\left(y-\bar{y}\right)^l\partial_x^i\partial_y^j v_0\left(0,x,y;T\right)F_{i,j,k,l}^{(n,h)}\left(0,T\right),
$$

with

$$
F_{i,j,k,l}^{(n,h)}\left(0,T\right) = \int_0^T\int_{s_1}^T\cdots\int_{s_{h-1}}^T f_{i,j,k,l}^{(n,h)}\left(0,s_1,\cdots,s_h\right)\mathrm{d}s_1\cdots\mathrm{d}s_h,
$$

and $(\bar{x},\bar{y})$ are chosen and can be time-dependent. The coefficients $f_{i,j,k,l}^{(n,h)}$ have already been computed by symbolic programming with Mathematica and only depend on the coefficients in (3.3.8), $\bar{x}$ and $\bar{y}$. The final expressions are very long but very simple and easy to compute for any $n > 0$. It follows, from integrations of $f_{i,j,k,l}^{(n,h)}$ and partial derivatives of (7.1.8), the expressions of v_n.

3.4 CDS calibration and numerical tests

In this section we apply the method developed in Section 3.2 to calibrate the model to market CDS spreads. We use quotations for different companies (specifically, UBS AG and BNP Paribas) in order to check the robustness of our methodology. The calibration is based on a two-step procedure: we first calibrate separately the interest rate model to daily yields curves for zero-coupon bonds (ZCB), generated using the Libor swap curve. Subsequently, we consider CDS contracts with different maturity dates. We use the approximation formulas (3.3.10) and (3.3.11) for the CDS spreads and survival probabilities, respectively. We use second-order approximations: we have found these to be sufficiently accurate by numerical experiments and theoretical error estimates. The formulas for the second-order approximation are simple, making the method easy to implement.

We distinguish between the uncorrelated and correlated cases: in the first case, i.e. when $\rho = 0$, the survival probability, which is not quoted from the market, can be inferred from the CDS spreads through a bootstrapping formula and therefore it is possible to calibrate directly to the survival probabilities. In the general case when $\rho \neq 0$, we calibrate to the market spreads using formula (3.3.10).

To add more flexibility to the model, we assume that the coefficients $a(t)$ and $b(t)$ in (3.3.1) are linearly dependent on time: more precisely, we assume that

$$a(t) = a_1 t + a_2, \qquad b(t) = b_1 t + b_2,$$

for some constants $a_1,\ a_2,\ b_1$, and b_2.

As defined in (3.3.2), the stochastic interest rate is described by a Vasicek model

$$\mathrm{d}r_t = \kappa\,(\theta - r_t)\,\mathrm{d}t + \delta \mathrm{d}W_t^2.$$

Apart from its simplicity, one of the advantages of this model is that interest rates can take negative values. For the calibration, we use the standard formula for the price $P_t(T)$ of a T-bond, which we recall here for convenience:

$$P_t(T) = A_t(T)\, e^{-B_t(T) r_t},$$

where

$$A_t(T) = e^{\left(\theta - \frac{\delta^2}{2\kappa^2}\right)(B_t(T)-T+t) - \frac{\delta^2}{4\kappa} B_t(T)^2}, \qquad B_t(T) = \frac{1-e^{-\kappa(T-t)}}{\kappa}.$$

The results of the interest rate calibration are given in Table 3.1.

Table 3.1: Calibration to ZCB.

Times to maturity (years)	market ZCB	model ZCB	errors
1.	1.00229	1.00641	-0.410925 %
2	1.00371	1.00844	-0.470717 %
3	1.00333	1.00667	-0.333553 %
4	1.00099	1.00165	-0.0660915 %
5	0.995836	0.993987	0.185689 %
6	0.987867	0.984129	0.378397 %
7	0.977005	0.972476	0.463518 %
8	0.963596	0.959441	0.431229 %
9	0.948371	0.945363	0.317189 %
10	0.933829	0.930451	0.361748 %

Calibration of the term structure formula of the ZCB to the market values of the ZCB. The relative errors between the model and market prices are reported as well. $\kappa = 0.06,\ \theta = 0.09,\ \delta = 0.024,\ r_0 = -0.009$
.

3.4.1 CDS calibration

The problem of calibrating the model (3.3.2) is formulated as an optimization problem. We want to minimize the error between the model CDS spreads and the market CDS spreads. Our approach is to use the square difference between market and model CDS spreads. This leads to the nonlinear least squares method

$$\inf_{\Theta} F(\Theta), \qquad F(\Theta) = \sum_{i=1}^{N} \frac{|R_i - \widehat{R}_i|^2}{\widehat{R}_i^2},$$

where N is the number of spreads used, $\widehat{R}_i$ is the market CDS spreads of the considered reference entity observed at time $t = 0$ and $\Theta = (a_1, a_2, b_1, b_2, \beta, c, \rho)$, with

$$a_2 \geq 0, \quad a_1 \geq -\frac{a_2}{T}, \quad b_2 \geq 0, \quad b_1 \geq -\frac{b_2}{T} \quad c > 0, \quad \beta < 1 \text{ and } -1 < \rho < 1.$$

In order to calibrate our model to data from real market, we received data from Bloomberg for two large credit derivatives dealers: UBS AG and BNP Paribas.

Calibration results

Here, we present the results of calibrating of the model to a set of data covering the period from January, 1st, 2017 to January, 1st , 2023. In the Table 3.2 and the Table 3.4, we present the results of the calibration of the model with $\rho = 0$ to market CDS spreads of UBS and BNP Paribas. The Table 3.3 and the Table 3.5 show the results for the model with correlation ($\rho \neq 0$). In both cases, we can see that the model gives very good fit to the market data, particularly to the most liquid market CDS spreads (2Y, 3Y and 5Y maturities). However we can still observe high relative errors for the BNP Paribas CDS spread with maturity 4 years due to the market incompleteness or the non-liquidity of its 4Y maturity CDS observed at January, 1st, 2017. The interesting fact is that the model gives very good fit to liquid market CDS and this is confirmed, in Appendix 7.2, by more calibration tests on CDS spreads of other different companies.

Table 3.2: Calibration to UBS AG CDS spreads (uncorrelated case).

Time to Maturities	Market CDS spread (bps)	Model CDS spread (bps)	Rel. Errors
1.	25.72	25.914	0.754262 %
1.5	30.2627	30.3312	0.226099 %
2.	35.105	34.7814	-0.921671 %
2.5	39.6922	39.2606	-1.08729 %
3.	43.97	43.7645	-0.467301 %
3.5	48.0616	48.289	0.473158 %
4.	52.3	52.8299	1.01328 %
4.5	56.9646	57.3833	0.73514 %
5.	61.91	61.9452	0.0568152 %
5.5	66.8408	66.5115	-0.492607 %
6.	71.285	71.0784	-0.289867 %

$a_1 = 0.005,\ a_2 = 0.001,\ \beta = 0.91,\ b_1 = 0.003,\ b_2 = 0.003,\ c = 1.4$.

Table 3.3: Calibration to UBS AG CDS spreads (Correlated case).

Time to Maturities	Market CDS spread (bps)	Model CDS spread (bps)	Rel. Errors
1.	25.72	25.9622	0.941777 %
1.5	30.2627	30.3606	0.323276 %
2.	35.105	34.787	-0.905712 %
2.5	39.6922	39.2422	-1.13365 %
3.	43.97	43.7262	-0.554424 %
3.5	48.0616	48.2386	0.368363 %
4.	52.3	52.7785	0.914999 %
4.5	56.9646	57.3445	0.666967 %
5.	61.91	61.9345	0.0396039 %
5.5	66.8408	66.5461	-0.44087 %
6.	71.285	71.1762	-0.152671 %

$a_1 = -0.035,\ a_2 = 0.23,\ \beta = 0.66,\ b_1 = 0.003,\ b_2 = 0.0005,\ c = 0.045,\ \rho = 0.9$.

Table 3.4: Calibration to BNP Paribas CDS spreads (uncorrelated case).

Time to Maturities	Market CDS spread (bps)	Model CDS spread (bps)	Rel. Errors
1.	34.615	34.0535	-1.62217 %
1.5	39.8758	39.6736	-0.50717 %
2.	45.115	45.4635	0.77257 %
2.5	49.9935	51.4145	2.84236 %
3.	56.11	57.517	2.50765 %
3.5	64.5726	63.7612	-1.25665 %
4.	72.59	70.1364	-3.38005 %
4.5	77.6516	76.6317	-1.31333 %
5.	82.27	83.2357	1.17384 %
5.5	89.1455	89.9366	0.887389 %
6.	96.705	96.7222	0.0177715 %

$a_1 = 0.018,\ a_2 = 0.085,\ \beta = 0.88,\ b_1 = 0.002,\ b_2 = 0.0,\ c = 0.53$

Table 3.5: Calibration to BNP CDS spreads (Correlated case).

Time to Maturities	Market CDS spread (bps)	Model CDS spread (bps)	Rel. Errors
1.	34.615	34.0073	-1.7557 %
1.5	39.8758	39.6668	-0.524173 %
2.	45.115	45.4813	0.811846 %
2.5	49.9935	51.4436	2.90047 %
3.	56.11	57.5465	2.56009 %
3.5	64.5726	63.7827	-1.22334 %
4.	72.59	70.1449	-3.36833 %
4.5	77.6516	76.6259	-1.32086 %
5.	82.27	83.2184	1.15283 %
5.5	89.1455	89.9156	0.863835 %
6.	96.705	96.7105	0.00566037 %

$a_1 = 0.023,\ a_2 = 0.08,\ \beta = 0.6,\ b_1 = 0.002,\ b_2 = 0.002,\ c = 0.3,\ \rho = 0.96$

For the calibration, we used a global optimizer, *NMinimize*, from Mathematica's optimization toolbox on a PC with 1× Intel i7-6599U 2.50 GHz CPU and 8GB RAM. We present in Table 3.6 the computational times of the calibration of the model to our two corporates in both uncorrelated and correlated cases. One can conclude that the approximation formula (3.3.10) gives an efficient and fast calibration.

Table 3.6: Computational Times

	Uncorrelated (second)	Correlated (second)
UBS AG	116.856	168.12
BNP Paribas	124.216	161.408

We check the results obtained from the calibration by computing the risk-neutral survival probabilities with the approximation formula (3.3.11). In Tables 3.7 and 3.9, by comparing the real market survival probability (column 2), our method (column 3) and the Monte Carlo (MC) simulation (column 4), we observe that the method provides results as good as the MC. The latter is performed with 100000 iterations and a confident interval of 95%. We do the same test for the correlated case for both corporates and present the results in tables 3.8 and 3.10.

Table 3.7: Risk-neutral UBS AG survival probabilities (uncorrelated case).

Times to maturity	Market probabilities	Model probabilities	Monte Carlo
1.	0.995724	0.99569	$[0.995704, 0.995704]$
2.	0.988367	0.988473	$[0.988531, 0.988531]$
3.	0.978233	0.978335	$[0.978468, 0.978468]$
4.	0.965644	0.965293	$[0.965531, 0.965531]$
5.	0.949437	0.949391	$[0.949765, 0.949766]$
6.	0.93056	0.930706	$[0.931249, 0.93125]$

Table 3.8: Risk-neutral UBS AG survival probabilities (correlated case).

Times to maturity	model probability	Monte Carlo	
1.	0.995724	0.995682	$[0.995695, 0.995697]$
2.	0.988367	0.988471	$[0.988523, 0.988528]$
3.	0.978233	0.978353	$[0.978479, 0.978486]$
4.	0.965644	0.965324	$[0.965553, 0.965563]$
5.	0.949437	0.949399	$[0.949759, 0.94977]$
6.	0.93056	0.930619	$[0.931142, 0.931152]$

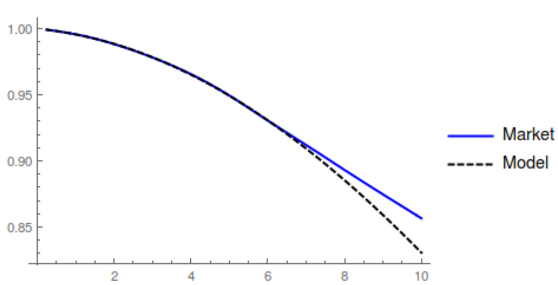
Table 3.9: Risk-neutral BNP Paribas survival probabilities (uncorrelated case).

Times to maturity	Market probabilities	Model probabilities	Monte Carlo
1.	0.994253	0.994339	$[0.994358, 0.994359]$
2.	0.985077	0.984948	$[0.98503, 0.985034]$
3.	0.972302	0.97157	$[0.971779, 0.971788]$
4.	0.952539	0.954016	$[0.954445, 0.954462]$
5.	0.933285	0.932175	$[0.932925, 0.932955]$
6.	0.908874	0.906025	$[0.907231, 0.90728]$

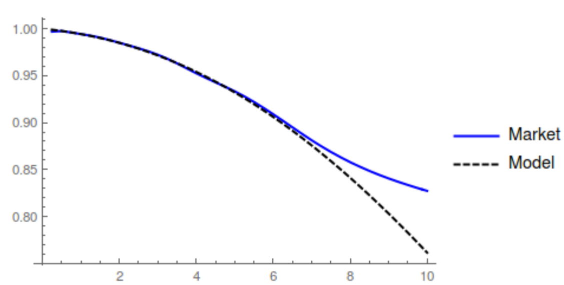
Table 3.10: Risk-neutral BNP Paribas survival probabilities (correlated case).

Times to maturity	market probability	model probability	Monte Carlo
1.	0.994253	0.994347	$[0.994363, 0.994368]$
2.	0.985077	0.98494	$[0.985012, 0.985034]$
3.	0.972302	0.971544	$[0.971742, 0.9718]$
4.	0.952539	0.953977	$[0.95432, 0.954448]$
5.	0.933285	0.932115	$[0.932728, 0.932973]$
6.	0.908874	0.9059	$[0.906818, 0.907245]$

However, as mentioned above in the Appendix (7.1.19), the convergence of the method is in the asymptotic sense; that is it is asymptotically exact as the maturity goes to zero. To show the dependence of the errors on the maturity, we plot the market and model survival probabilities in function of maturity in Figure 3.2, $T \mapsto CDS(0,T)$. The dotted lines correspond to the survival probabilities computed with calibrated parameters and the continuous lines to the real market survival probabilities. We observe that, after 6Y, the errors between the market and model survival probabilities start increasing, as expressed by the error bounds (7.1.19).



(a) UBS AG Survival Probability



(b) BNP Paribas Survival Probability

Figure 3.2: Dependence of the error on the maturity

Influence of the correlation

To see the influence of the correlation in our model, we adopt the test done in [5]. Indeed the authors consider four different payoffs that appear in credit derivatives and compare their present values in very positive and negative correlation cases, i.e. $\rho = 1$ and $\rho = -1$.

$$A = D\left(0,5Y\right)L\left(4Y,5Y\right)\mathbb{1}_{\{\zeta<5Y\}}, \qquad B = D\left(0,5Y\right)\mathbb{1}_{\{\zeta<t\}},$$
$$C = D\left(0,\min\left(\zeta,5Y\right)\right), \quad H = D\left(0,5Y\right)L\left(4Y,5Y\right)\mathbb{1}_{\{\zeta\in[4Y,5Y]\}},$$

where ζ is the time of default and $L(S,T)$ is the market LIBOR rate $T > S$. We consider the UBS AG corporate. First we calibrate the model (3.3.2) to the UBS AG market CDS spreads in both very positive and negative correlation cases. We obtain the following parameters:

$$\rho = 1: \; a_1 = 0.008, \; a_2 = 0.008, \; \beta = 0.5, \; b_1 = 0.003, \; b_2 = 0.003, \; c = 0.68,$$

and

$$\rho = -1: \; a_1 = 0.006, \; a_2 = 0.04, \; \beta = 0.624, \; b_1 = 0.002, \; b_2 = 0.0004, \; c = 1.325.$$

Table 3.11 shows, on one hand, that the correlation has no impact in the payoff of the form B and C. Since the CDS spread and the risk-neutral survival probability expressions are written as function in terms of B and C, the correlation has no influence in the computations of the CDS spreads and the risk-neutral survival probabilities. On the other hand, higher effect can be seen in the values of derivatives including LIBOR rates (A and H). This explains why in both cases (non-correlation and correlation), our model gives a very good fit to the market data. It follows that when we want to use the model for pricing derivatives of types A, H or

Table 3.11: Impact of the correlation

	$\rho = -1$	$\rho = 1$	Rel. Errors	Abs. Errors
A	22.61 bps	90.986 bps	+148.50%	+0.00135
B	505.482 bps	505.058 bps	+0.083%	+0.00004
C	9947.149 bps	9948.279 bps	-0.011%	-0.00011
$\mathbb{D}$	16.244 bps	-0.361 bps	-548.883%	0.00019

pricing in general, it is better and much more accurate to consider the model with correlation.

Chapter 4

Numerical method for evalution of a defaultable coupond bond

4.1 Introduction

The main objective of this chapter is to obtain the price of a defaultable coupon bond under the extended Jump to Default Constant Elasticity of Variance (JDCEV) model proposed in Chapter 3. From the mathematical point of view, the valuation problem of a defaultable coupon bond can be posed in terms of a sequence of partial differential equation (PDE) problems, where the underlying stochastic factors are the interest rates and the stock price. Moreover, the stock price follows a diffusion process interrupted by a possible jump to zero (default), as it is indicated previously. In order to compute the value of the bond we need to solve two partial differential equation problems for each coupon and with maturities those coupon payment dates. Concerning the numerical solution of those PDE problems, after a localization procedure to formulate the problems in a bounded domain and the study of the boundaries where boundary conditions are required following the ideas introduced in [36], we propose appropriate numerical schemes based on a Crank-Nicolson semi-Lagrangian method for time discretization combined with biquadratic Lagrange finite elements for space discretization. The numerical analysis of this Lagrange-Galerkin method has been addressed in [3, 2]. Once the numerical solution of the PDEs is obtained, a kind of post-processing is carried out in order to achieve the value of the bond. This post-processing includes the computation of an integral term which is approximated by using the composite trapezoidal rule.

This chapter is organized as follows. In Section 4.2, we present the mathematical modeling with the PDE problem that governs the valuation of non callable defaultable coupon bonds. In Section 4.3, we formulate the pricing problem in a bounded domain after a localization procedure and we impose appropriate boundary conditions. Then, we introduce the discretization in time of the problem by using a Crank-Nicolson characteristic scheme, and we state the variational formulation of the problem in order to apply finite elements. Finally, in Section 4.4 we present some numerical results to illustrate the good performance of the numerical methods and we finish with some conclusions.

4.2 Model and PDE formulation for price of defaultable bond

Consider the following model introduced in Chapter 3

$$
\begin{cases}
S_t = S_0 e^{X_t} \mathbb{1}_{\{\zeta \geq t\}}, \qquad S_0 > 0, \\
\mathrm{d}X_t = \left(r_t + b\,(t) + \left(c - \frac{1}{2}\right) \sigma^2\,(t, X_t)\right) \mathrm{d}t + \sigma\,(t, X_t)\, \mathrm{d}W_t^1, \\
\mathrm{d}r_t = \kappa\,(\theta - r_t)\,\mathrm{d}t + \delta \mathrm{d}W_t^2, \\
\zeta = \inf\{t \geq 0 \mid \int_0^t \lambda\,(t, X_t) \geq e\}, \\
\mathrm{d}W_t^1 \mathrm{d}W_t^2 = \rho \mathrm{d}t,
\end{cases}
\tag{4.2.1}
$$

where S and r are, respectively, the defaultable stock price and the risk free interest rate and $\sigma(t,x) = a(t)\, e^{(\beta-1)x}$, with $a,\ b,\ c$ and β defined as in Chapter 2. Consider a coupon bond with maturity T, coupon rate $(cp_i,\ i = 1, \cdots, M)$ and recovery rate η. It consists of

- A payment a coupon rate cp_i at given dates t_i if no default by t_i, for $i = 1,\ \cdots,\ M$ where M is the number of coupons and $T = t_M$
- A payment of a face value FV at maturity if no default occurs prior to T
- A payment of a recovery rate η, in case of default before the maturity, at the default time ζ.

The value $V(t,S,r;T)$ at time $t > 0$ of this bond is given by

$$V(t,S,r;T) = FV \cdot \mathbb{E}\left(\sum_{i=1}^{M} cp_i e^{-\int_t^{t_i} r_u \mathrm{d}u} \mathbb{1}_{\{\zeta > t_i\}} + \eta e^{-\int_t^{\zeta} r_u \mathrm{d}u} \mathbb{1}_{\{\zeta \le T\}} + e^{-\int_t^{T} r_u \mathrm{d}u} \mathbb{1}_{\{\zeta > T\}} | \mathcal{G}_t\right) \tag{4.2.2}$$

Proposition 4.2.1. *Under the model (4.2.1), the value $V(S,r;T) := V(0,S,r;T)$ at time $t = 0$ of a bond with maturity T, coupon rates $(cp_i,\ i = 1, \cdots, M)$ and constant recovery rate η is given by*

$$\begin{aligned} V(S,r;T) = FV &\left[\sum_{i=1}^{M} cp_i\, \mathbb{E}\left[\exp\left(-\int_0^{t_i} (r_u + \lambda(u,S_u))\, du\right)\right] + \mathbb{E}\left[\exp\left(-\int_0^{T} (r_u + \lambda(u,S_u))\, du\right)\right]\right. \\ &\left. + \eta\left(1 - \mathbb{E}\left[\exp\left(-\int_0^{T} (r_u + \lambda(u,S_u))\, du\right)\right] - \int_0^T \mathbb{E}\left[\exp\left(-\int_0^{\tau_1} (r_u + \lambda(u,S_u))\, du\right) r_{\tau_1}\right] d\tau_1\right)\right]. \end{aligned}$$

Proof. From (4.2.2) and with the Lemma 2.1.8 and 2.1.9, we have on $\{\zeta > t\}$

$$\begin{aligned} V(t,S,r;T) &= FV\left(\sum_{i=1}^{M} cp_i \mathbb{E}\left(e^{-\int_t^{t_i} r_u \mathrm{d}u} \mathbb{1}_{\{\zeta > t_i\}} | \mathcal{G}_t\right) + \eta \mathbb{E}\left(e^{-\int_t^{\zeta} r_u \mathrm{d}u} \mathbb{1}_{\{\zeta \le T\}} | \mathcal{G}_t\right) + \mathbb{E}\left(e^{-\int_t^{T} r_u \mathrm{d}u} \mathbb{1}_{\{\zeta > T\}} | \mathcal{G}_t\right)\right) \\ &= FV\left(\sum_{i=1}^{M} cp_i \mathbb{E}\left(e^{-\int_t^{t_i} (r_u + \lambda_u) \mathrm{d}u} | \mathcal{F}_t\right) + \eta \int_t^T \mathbb{E}\left(e^{-\int_t^{s} (r_u + \lambda_u) \mathrm{d}u} \lambda_s | \mathcal{F}_t\right) \mathrm{d}s + \mathbb{E}\left(e^{-\int_t^{T} (r_u + \lambda_u) \mathrm{d}u} | \mathcal{F}_t\right)\right) \end{aligned}$$

$$\Rightarrow V(S,r;T) = FV\left(\sum_{i=1}^{M} cp_i \mathbb{E}\left(e^{-\int_0^{t_i}(r_u+\lambda_u)\mathrm{d}u}\right) + \eta\int_0^T \mathbb{E}\left(e^{-\int_0^{s}(r_u+\lambda_u)\mathrm{d}u}\lambda_s\right)\mathrm{d}s + \mathbb{E}\left(e^{-\int_0^{T}(r_u+\lambda_u)\mathrm{d}u}\right)\right)$$

The statement follows by replacing the following identities in the equality above:

$$e^{-\int_0^s(r_u+\lambda(u,X_u))\mathrm{d}u}\lambda(s,X_s) = -\frac{\partial}{\partial_s}\left(e^{-\int_0^s(r_u+\lambda(u,X_u))\mathrm{d}u}\right) - r_s e^{-\int_0^s(r_u+\lambda(u,X_u))\mathrm{d}u},$$

and

$$\int_0^T e^{-\int_0^s(r_u+\lambda(u,X_u))\mathrm{d}u}\lambda(s,X_s)\mathrm{d}s = 1 - \left(e^{-\int_0^T(r_u+\lambda(u,X_u))\mathrm{d}u} + \int_0^T e^{-\int_0^s(r_u+\lambda(u,X_u))\mathrm{d}u} r_s \mathrm{d}s\right).$$

□

Next, if we denote by

$$\begin{aligned} u_1(0,S_t,r_t;s) &= \mathbb{E}\left[\exp\left(-\int_0^s (r_u+\lambda(u,S_u))\,du\right)\right], \\ u_2(0,S_t,r_t;\tau_1) &= \mathbb{E}\left[\exp\left(-\int_0^{\tau_1} (r_u+\lambda(u,S_u))\,du\right) r_{\tau_1}\right], \end{aligned}$$

the expression of the bond value (4.2.1) can be written equivalently as

$$\begin{aligned} V(S_t,r_t;T) = FV &\left[\sum_{i=1}^{M} cp_i\, u_1(0,S_t,r_t;t_i) + u_1(0,S_t,r_t;T)\right. \\ &\left. +\eta\left(1 - u_1(0,S_t,r_t;T) - \int_0^T u_2(0,S_t,r_t;\tau_1)\,d\tau_1\right)\right]. \end{aligned} \tag{4.2.3}$$

Moreover, applying the Feynman-Kac formula (see [39], for example) and by using the change of variable $y_t = r_t\exp(\kappa t)$, u_1 and u_2 are solutions of the Cauchy problem

$$\begin{cases} (\partial_t + \mathcal{A})\,u(t,S,y) = 0, & t < T_1,\ (S,y) \in (0,\,\infty)\times(-\infty,\,\infty), \\ u(T_1,S,y) = h(y), & (S,y) \in (0,\,\infty)\times(-\infty,\,\infty), \end{cases} \tag{4.2.4}$$

with $h(y) = 1$ for $u = u_1$ or $h(y) = \exp(-\kappa T_1)y$ for $u = u_2$, respectively. Moreover, the operator $\mathcal{A}$ is defined as follows

$$(\partial_t + \mathcal{A})\, u = \partial_t u + \frac{1}{2}\sigma^2(t,S)S^2\,\partial_{SS}u + \rho\delta\sigma(t,S)\exp(\kappa t)S\,\partial_{Sy}u + \frac{1}{2}\delta^2\exp(2\kappa t)\,\partial_{yy}u$$
$$+ \left(\exp(-\kappa t)y + \lambda(t,S)\right) S\,\partial_S u + \kappa\theta\exp(\kappa t)\,\partial_y u - \left(\exp(-\kappa t)y + \lambda(t,S)\right) u.$$

The existence of the solution to the Cauchy problem (4.2.4) in a bounded domain is ensured and proved by Theorem 3.3.3.

4.3 Numerical methods for picing defaultable bond

In order to obtain a numerical approach of the value of a non callable defaultable coupon bond we need to solve the Cauchy problem (4.2.4) for $u = u_1$ and $u = u_2$ with maturity $T_1 = t_i$ for $i = 1, ..., M$, that is each coupon payment date for both cases. Once these problems are solved, the value of the bond is given by expression (4.2.3) in which appears an integral term. That integral term will be approximated by means of the classical composite trapezoidal rule. For the numerical solution of the PDE problem, we propose a Crank-Nicolson characteristics time discretization scheme combined with a piecewise bi-quadratic Lagrange finite element method. This Lagrange-Galerkin method has been analyzed in [3, 2] for time and space discretization. More recently, it has been applied to the valuation of pension plans without and with early retirement in [8] and [30], respectively. Thus, in order to apply this numerical technique, first a localization procedure is used to cope with the initial formulation in an unbounded domain.

4.3.1 Localization procedure and formulation in a bounded domain

In this section we replace the unbounded domain by a bounded one. To determine the required boundary conditions for the associated PDE problem we follow [36] based on the theory proposed by Fichera in [13]. Let us introduce the following notation:

$$x_0 = t, \quad \tilde{x}_1 = S \quad \text{and} \quad \tilde{x}_2 = y.$$

Let $\tilde{x}_1^\infty$ and $\tilde{x}_2^\infty$ be two large enough real numbers and

$$\Omega^* = (0, x_0^\infty) \times \left(\frac{1}{\tilde{x}_1^\infty}, \tilde{x}_1^\infty\right) \times (-\tilde{x}_2^\infty, \tilde{x}_2^\infty)$$

with $x_0^\infty = T_1$. Additionally, we make the changes of variables $x_1 = \tilde{x}_1 - \frac{1}{\tilde{x}_1^\infty}$, $x_2 = \tilde{x}_2 + \tilde{x}_2^\infty$, $x_1^\infty = \tilde{x}_1^\infty - \frac{1}{\tilde{x}_1^\infty}$ and $x_2^\infty = 2\tilde{x}_2^\infty$. It follows a new bounded domain Ω defined as:

$$\Omega = (0, x_0^\infty) \times (0, x_1^\infty) \times (0, x_2^\infty)$$

We denote the Lipschitz boundary by $\Gamma = \partial\Omega$ such that $\Gamma = \bigcup_{i=0}^{2}(\Gamma_i^- \cup \Gamma_i^+)$, where

$$\Gamma_i^- = \{(x_0, x_1, x_2) \in \Gamma \mid x_i = 0\}, \quad \Gamma_i^+ = \{(x_0, x_1, x_2) \in \Gamma \mid x_i = x_i^\infty\}, \quad i = 0, 1, 2.$$

Then, the operator defined in (4.2) can be written in the form:

$$\mathcal{A}u = \sum_{i,j=0}^{2} b_{ij} \frac{\partial^2 u}{\partial x_i x_j} + \sum_{j=0}^{2} b_j \frac{\partial u}{\partial x_j} + b_0 u,$$

where the involved data are given by

$$\mathbf{B} = (b_{ij}) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{2}a^2(t)\left(x_1 + \frac{1}{\tilde{x}_1^\infty}\right)^{2\beta} & \frac{1}{2}\rho\delta a(t)\left(x_1 + \frac{1}{\tilde{x}_1^\infty}\right)^{\beta} \exp(\kappa t) \\ 0 & \frac{1}{2}\rho\delta a(t)\left(x_1 + \frac{1}{\tilde{x}_1^\infty}\right)^{\beta} \exp(\kappa t) & \frac{1}{2}\delta^2 \exp(2\kappa t) \end{pmatrix},$$

$$\mathbf{b} = (b_j) = \begin{pmatrix} 1 \\ \left(\exp(-\kappa t)(x_2 - \tilde{x}_2^\infty) + \lambda\left(t, x_1 + \frac{1}{\tilde{x}_1^\infty}\right)\right)\left(x_1 + \frac{1}{\tilde{x}_1^\infty}\right) \\ \kappa\theta\exp(\kappa t) \end{pmatrix},$$

$$b_0 = -\left(\exp(-\kappa t)(x_2 - \tilde{x}_2^\infty) + \lambda\left(t, x_1 + \frac{1}{\tilde{x}_1^\infty}\right)\right).$$

Thus, following [36], in terms of the normal vector to the boundary pointing inward Ω, $\mathbf{m} = (m_0, m_1, m_2)$, we introduce the following subsets of Γ:

$$\eth^0 = \left\{ x \in \Gamma / \sum_{i,j=0}^{2} b_{ij} m_i m_j = 0 \right\}, \quad \eth^1 = \Gamma - \eth^0,$$

$$\eth^2 = \left\{ x \in \eth^0 / \sum_{i=0}^{2} \left(b_i - \sum_{j=0}^{2} \frac{\partial b_{ij}}{\partial x_j} \right) m_i < 0 \right\}.$$

As indicated in [36] the boundary conditions at $\eth^1 \bigcup \eth^2$ for the so-called first boundary value problem associated with (4.3.1) are required. Note that $\eth^1 = \{\Gamma_1^-, \Gamma_1^+, \Gamma_2^-, \Gamma_2^+\}$ and $\eth^2 = \{\Gamma_0^+\}$. Therefore, in addition to a final condition (see section 4.2), we need to impose boundary conditions on Γ_1^-, Γ_1^+, Γ_2^- and Γ_2^+. Next, we will impose Dirichlet conditions on Γ_1^-, Γ_2^- and Γ_2^+, whereas on Γ_1^+ we will impose a homogeneous Neumann condition.

Taking into account the previous change of spatial variable and making the change of time variable $\tau = T_1 - t$, we write the equation (4.2.4) in divergence form in the bounded spatial domain $\Omega = (0, x_1^\infty) \times (0, x_2^\infty)$. Thus, the initial boundary value problem (IBVP) takes the following form:

Find $u : [0, T_1] \times \Omega \to \mathbb{R}$ such that

$$\partial_\tau u - Div(A\nabla u) + \mathbf{v} \cdot \nabla u + lu = f \text{ in } (0, T_1) \times \Omega, \qquad (4.3.1)$$

$$u(0,.) = h(x_2 - \tilde{x}_2^\infty) \text{ in } \Omega,$$

$$u = \exp\left(-\int_{T_1-\tau}^{T_1}\left(\exp(-\kappa\tilde{u})(x_2 - \tilde{x}_2^\infty) + \lambda\left(\tilde{u}, x_1 + \frac{1}{\tilde{x}_1^\infty}\right)\right) d\tilde{u}\right) h(x_2 - \tilde{x}_2^\infty) \text{ on } (0, T_1) \times \Gamma_1^-,$$

$$\frac{\partial u}{\partial x_1} = 0 \text{ on } (0, T_1) \times \Gamma_1^+,$$

$$u = \exp\left(-\int_{T_1-\tau}^{T_1}\left(\exp(-\kappa\tilde{u})(x_2 - \tilde{x}_2^\infty) + \lambda\left(\tilde{u}, x_1 + \frac{1}{\tilde{x}_1^\infty}\right)\right) d\tilde{u}\right) h(x_2 - \tilde{x}_2^\infty) \text{ on } (0, T_1) \times \Gamma_2^-,$$

$$u = \exp\left(-\int_{T_1-\tau}^{T_1}\left(\exp(-\kappa\tilde{u})(x_2 - \tilde{x}_2^\infty) + \lambda\left(\tilde{u}, x_1 + \frac{1}{\tilde{x}_1^\infty}\right)\right) d\tilde{u}\right) h(x_2 - \tilde{x}_2^\infty) \text{ on } (0, T_1) \times \Gamma_2^+,$$

where the diffusion matrix $\mathbf{A}$, the velocity field $\mathbf{v}$, the reaction function l and the second member f are defined as follows:

$$\mathbf{A} = \begin{pmatrix} \frac{1}{2}a^2(T_1-\tau)\left(x_1 + \frac{1}{\tilde{x}_1^\infty}\right)^{2\beta} & \frac{1}{2}\rho\delta a(T_1-\tau)\left(x_1 + \frac{1}{\tilde{x}_1^\infty}\right)^{\beta}\exp(\kappa(T_1-\tau)) \\ \frac{1}{2}\rho\delta a(T_1-\tau)\left(x_1 + \frac{1}{\tilde{x}_1^\infty}\right)^{\beta}\exp(\kappa(T_1-\tau)) & \frac{1}{2}\delta^2\exp(2\kappa(T_1-\tau)) \end{pmatrix},$$

$$\mathbf{v} = \begin{pmatrix} \frac{1}{2}a^2(T_1-\tau)(2\beta)\left(x_1 + \frac{1}{\tilde{x}_1^\infty}\right)^{2\beta-1} - \left(\exp(-\kappa(T_1-\tau))(x_2 - \tilde{x}_2^\infty) + \lambda\left(T_1-\tau, x_1 + \frac{1}{\tilde{x}_1^\infty}\right)\right)\left(x_1 + \frac{1}{\tilde{x}_1^\infty}\right) \\ \frac{1}{2}\rho\delta a(T_1-\tau)(\beta)\left(x_1 + \frac{1}{\tilde{x}_1^\infty}\right)^{\beta-1}\exp(\kappa(T_1-\tau)) - \kappa\theta\exp(\kappa(T_1-\tau)) \end{pmatrix},$$

$$l = \exp(-\kappa(T_1-\tau))(x_2 - \tilde{x}_2^\infty) + \lambda\left(T_1-\tau, x_1 + \frac{1}{\tilde{x}_1^\infty}\right),$$

$$f = 0.$$

4.3.2 Time discretization

The method of characteristics is based on a finite differences scheme for the discretization of the material derivative, i.e., the time derivative along the characteristic lines of the convective part of the equation (4.3.1). The material derivative operator is given by

$$\frac{D}{D\tau} = \partial_\tau + \mathbf{v} \cdot \nabla.$$

For a brief description of the method, we first define the characteristics curve through $\mathbf{x} = (x_1, x_2)$ at time $\bar{\tau}$, $X(\mathbf{x}, \bar{\tau}; s)$, which satisfies:

$$\frac{\partial}{\partial s} X(\mathbf{x}, \bar{\tau}; s) = \mathbf{v}(X(\mathbf{x}, \bar{\tau}; s)), \quad X(\mathbf{x}, \bar{\tau}; \bar{\tau}) = \mathbf{x}. \tag{4.3.2}$$

In order to discretize in time the material derivative in the Cauchy problem (4.3.1), let us consider a number of time steps N, the time step $\delta\tau = T/N$ and the time mesh points $\tau^n = n\delta\tau$, $n = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots, N$.

The material derivative approximation by the characteristics method for both problems is given by:

$$\frac{Du}{D\tau} = \frac{u^{n+1} - u^n \circ X^n}{\delta\tau},$$

where $u = u_1, u_2$ and $X^n(\mathbf{x}) = X(\mathbf{x}, \tau^{n+1}; \tau^n)$. In this case, the solution of (4.3.2) is not computed analytically. Instead, we consider numerical ODE solvers to approximate the characteristics curves (see [3], for example). More precisely, we employ the second order Runge-Kutta method.

Next, we consider a Crank-Nicolson scheme around $\big(X(\mathbf{x}, \tau^{n+1}; \tau), \tau\big)$ for $\tau = \tau^{n+\frac{1}{2}}$. So, the time discretized equation for $u = u_1, u_2$ can be written as follows:

Find u^{n+1} such that:

$$\frac{u^{n+1}(\mathbf{x}) - u^n(X^n(\mathbf{x}))}{\delta\tau} - \frac{1}{2}Div(A\nabla u^{n+1})(\mathbf{x}) - \frac{1}{2}Div(A\nabla u^n)(X^n(\mathbf{x})) \\ +\frac{1}{2}(l\, u^{n+1})(\mathbf{x}) + \frac{1}{2}(l\, u^n)(X^n(\mathbf{x})) = 0. \tag{4.3.3}$$

In order to obtain the variational formulation of the semi-discretized problem, we multiply (4.3.3) by a suitable test function, integrate in Ω, use the classical Green formula :

$$\begin{aligned}\int_\Omega Div(\mathbf{A}\nabla u^n)(X^n(\mathbf{x}))\psi(\mathbf{x})d\mathbf{x} = &\int_\Gamma (\nabla X^n)^{-T}(\mathbf{x})\mathbf{n}(x)\cdot(\mathbf{A}\nabla u^n)(X^n(\mathbf{x}))\psi(\mathbf{x})dA_{\mathbf{x}} \\ &- \int_\Omega (\nabla X^n)^{-1}(\mathbf{x})(\mathbf{A}\nabla u^n)(X^n(\mathbf{x}))\cdot\nabla\psi(\mathbf{x})d\mathbf{x} \\ &- \int_\Omega Div((\nabla X^n)^{-T}(\mathbf{x}))\cdot(\mathbf{A}\nabla u^n)(X^n(\mathbf{x}))\psi(\mathbf{x})d\mathbf{x}.\end{aligned} \tag{4.3.4}$$

Note that, due to the characteristics curves can not be obtained analytically, the terms $(\nabla X^n)^{-1}(\mathbf{x})$ and $Div((\nabla X^n)^{-T}(\mathbf{x}))$ in (4.3.4) are replaced by the following approximations (see [3] for more detail):

$$(\nabla X^n)^{-1}(\mathbf{x}) = \mathbf{I}(\mathbf{x}) + \delta\tau\mathbf{L}^n(X^n(\mathbf{x})) + O(\delta\tau^2),$$

$$Div((\nabla X^n)^{-T}(\mathbf{x})) = \delta\tau\,\nabla\, Div\,(\mathbf{v}^n(X^n(\mathbf{x}))) + O(\delta\tau^2),$$

where $\mathbf{L} = \nabla\mathbf{v}$.

After the previous steps, we can write a variational formulation for the time discretized problem as follows:

Find $u^{n+1} \in H^1(\Omega)$ for all $\psi \in H^1(\Omega)$ such that $\psi = 0$ on Γ_1^-, Γ_2^- and Γ_2^+ :

$$\begin{aligned}
\frac{1}{\delta\tau}\int_\Omega u^{n+1}(\mathbf{x})\psi(\mathbf{x})d\mathbf{x} + \frac{1}{2}\int_\Omega(\mathbf{A}\nabla u^{n+1})(\mathbf{x})\cdot\nabla\psi(\mathbf{x})d\mathbf{x} + \frac{1}{2}\int_\Omega(lu^{n+1})(\mathbf{x})\psi(\mathbf{x})d\mathbf{x} \\
= \frac{1}{\delta\tau}\int_\Omega u^n(X^n(\mathbf{x}))\psi(\mathbf{x})d\mathbf{x} - \frac{1}{2}\int_\Omega(\mathbf{A}\nabla u^n)(X^n(\mathbf{x}))\cdot\nabla\psi(\mathbf{x})d\mathbf{x} \\
-\frac{\delta\tau}{2}\int_\Omega \mathbf{L}^n(X^n(\mathbf{x}))(\mathbf{A}\nabla u^n)(X^n(\mathbf{x}))\cdot\nabla\psi(\mathbf{x})d\mathbf{x} - \frac{1}{2}\int_\Omega(lu^n)(X^n(\mathbf{x}))\psi(\mathbf{x})d\mathbf{x} \\
-\frac{\delta\tau}{2}\int_\Omega \nabla\, Div\,(\mathbf{v}^n(X^n(\mathbf{x})))\cdot(\mathbf{A}\nabla u^n)(X^n(\mathbf{x}))\psi(\mathbf{x})d\mathbf{x} \\
+\frac{1}{2}\int_\Gamma(\mathbf{I}(\mathbf{x}) + \delta\tau\mathbf{L}^n(X^n(\mathbf{x})))^T\mathbf{n}(x)\cdot(\mathbf{A}\nabla u^n)(X^n(\mathbf{x}))\psi(\mathbf{x})dA_\mathbf{x} + \frac{1}{2}\int_{\Gamma_1^+} a_{12}(\mathbf{x})\frac{\partial u}{\partial x_2}(\mathbf{x})\psi(\mathbf{x})dA_\mathbf{x},
\end{aligned}$$

where a_{12} is the corresponding coefficient of the diffusion matrix $\mathbf{A}$.

4.3.3 Finite elements discretization

For the spatial discretization, we consider $\{\tau_h\}$ a quadratic mesh of the domain Ω. Let $(T_2, \mathcal{Q}_2, \sigma_{T_2})$ be a family of piecewise quadratic Lagrangian finite elements, where $\mathcal{Q}_2$ denotes the space of polynomials defined in $T_2 \in \tau_h$ with degree less or equal than two in each spatial variable and σ_{T_2} the subset of nodes of the element T_2. More precisely, let us define the finite elements space u_h by

$$u_h = \{\phi_h \in \mathcal{C}^0(\Omega) : \phi_{h_{T_2}} \in \mathcal{Q}_2, \quad \forall T_2 \in \tau_h\},$$

where $\mathcal{C}^0(\Omega)$ is the space of piecewise continuous functions on Ω.

4.3.4 Composite trapezoidal rule

In order to obtain the value of the bond at origination by means of expression (4.2.3) the computation of an integral term is required. The approximation of this integral is carried out by using a suitable numerical integration procedure. More precisely, we employ the classical composite trapezoidal rule with $M+1$ points, where M is the number of coupons, in the following way:

$$\int_0^T u_2(0,S_t,r_t;\tau_1)d\tau_1 \approx \frac{h}{2}\left[u_2(0,S_t,r_t;0) + 2\sum_{j=1}^{M-1} u_2(0,S_t,r_t;k_j) + u_2(0,S_t,r_t;T)\right]$$

where $h = \frac{T}{M}$, $k_j = jh$ for $j = 1,...,M-1$ and $u_2(0,S_t,r_t;0) = r_0$.

4.4 Empirical Results

In order to show the good performance of the numerical methods explained in Section 4.3, we present some numerical results. In the following examples, the value of some of the parameters involved in the underlying factors are obtained by applying the method introduced in Chapter 2. More precisely, first the interest rate model is calibrated to zero-coupon bonds (ZCB) and next the model is calibrated to CDS spreads whose price is obtained by means of an asymptotic expansion method.

In both examples, the number of elements and nodes of the finite element meshes employed in the numerical solution of the problems are shown in Table 4.1.

4.4.1 Example 1

First, we consider the simple case of the valuation of default-free zero-coupon bonds with different maturities. In this setting, the valuation problem is reduced to a one-factor model. The purpose of this example is to compare the value of the bonds we obtain with the market zero-coupon curve. In order to obtain the value of the bonds, we solve the IBVP (4.3.1) with initial condition $h = 1$ and only taking into account as underlying factor the interest rate. The value of the parameters involved in the interest rate model are the ones collected in Table 4.2. The values of the zero-coupon curve and the approximated ones obtained by solving the here proposed model are presented in Table 4.3. For the numerical solution we consider Mesh 64 and the time step $\delta\tau = \frac{1}{360}$ (one day).

	Number of elements	Number of nodes
Mesh 4	16	81
Mesh 8	64	289
Mesh 16	256	1089
Mesh 32	1024	4225
Mesh 64	4096	16641

Table 4.1: Different finite element meshes (number of elements and nodes).

Parameters of the defaultable stock price model
$a_1 = 0.0338$ $a_2 = 0.0524$ $b_1 = 0.0027$ $b_2 = 0.0028$ $c = 0.0436$ $\beta = 0.731504,$
Parameters of the interest rate model
$\kappa = 0.0452$ $\delta = 0.0215$ $\theta = 0.1033$
Correlation coefficient
$\rho = 0.0$
Initial conditions
$S_0 = 1.0$ $r_0 = -0.0092$

Table 4.2: Parameters of the model for the UBS bond.

Maturity (years)	Market ZCB	Model ZCB
1	1.00229	1.006751
2	1.00372	1.009062
3	1.00333	1.007495
4	1.00099	1.002601
5	0.995825	0.994902
6	0.987805	0.984889
7	0.976833	0.973024
8	0.963223	0.959738
9	0.947687	0.945429
10	0.932845	0.930463

Table 4.3: Market and model values of the ZCB.

4.4.2 Example 2

Next, we consider the valuation of two real defaultable bonds traded in the market and issued by different firms. On one hand, we take into account the pricing of a UBS bond quoted in

Euro with maturity 5 years and a face value of 100. The bond pays annually coupon rates of 1.25 basis points and the recovery rate at the event of default is 40%. The model parameter values for this example are collected in Table 4.2. In this case the correlation coefficient ρ is assumed to be zero. Next, in Table 4.4 we present the value of the bond for different meshes and time steps. In this case, we can appreciate that the price of the bond converges to 102.62.

On the other hand, we present a correlated case. More precisely, we address the valuation of a JP Morgan(JPM) bond quoted in US Dollar with maturity 5 years and a face value of 100. The bond pays annually coupon rates of 3.25 basis points and the recovery rate at the event of default is again 40%. For this second example the parameter values of the model are the ones which appear in Table 4.5. As we have just pointed out the correlation coefficient ρ is different from zero. Finally, the value of this bond is shown in Table 4.6. In this case, the value of the bond converges to 103.57.

In both cases, in order to obtain the value of the bond for we need to solve for each coupon payment date the IBVP (4.3.1) with maturity those dates and with initial condition $h = 1$ or $h = \exp(-\kappa T)(x_2 - \tilde{x}_2^{\infty})$. More precisely, in both examples the maturity of the bond is 5 years and the frequency of coupon payments is annually, thus to obtain the value of the bond we need to solve the problem (4.3.1) 10 times, i.e. 5 times with one initial condition to obtain the value of u_1 and 5 times with the other initial condition to obtain the value of u_2.

Table 4.7 presents the a comparison of our pricing method to Monte Carlo Simulation in both non callable defaultable coupon bonds. The Monte Carlo simulation is performed with 100000 iterations and a confident interval of 95%. Next, in Figures 4.1 and 4.2 we show the mesh value of the UBS bond and the JP Morgan bond, respectively. Both figures are obtained with the finer mesh and with time step $\delta\tau = \frac{1}{360}$ (one day).

Time steps per year	Mesh 4	Mesh 8	Mesh 16	Mesh 32	Mesh 64
90	102.499496	102.603837	102.616859	102.619028	102.618959
180	102.499599	102.605953	102.617976	102.619681	102.619602
360	102.500454	102.606351	102.618421	102.620069	102.619925
720	102.500663	102.606518	102.618587	102.620248	102.620094

Table 4.4: Value of the UBS bond for different meshes and time steps.

Parameters of the defaultable stock price model
$a_1 = 0.0313$ $a_2 = 0.0357$ $b_1 = 0.0004$ $b_2 = 0.0017$ $c = 0.3466$ $\beta = 0.777$
Parameters of the interest rate model
$\kappa = 0.1449$ $\delta = 0.0133$ $\theta = 0.0347$
Correlation coefficient
$\rho = 0.4971$

Initial conditions
$S_0 = 1.0$ $r_0 = 0.0147$

Table 4.5: Parameters of the model for the JP Morgan bond.

Time steps per year	Mesh 4	Mesh 8	Mesh 16	Mesh 32	Mesh 64
90	103.725041	103.596891	103.572191	103.570225	103.570524
180	103.841153	103.605155	103.575270	103.572747	103.573144
360	103.794567	103.602389	103.576483	103.574147	103.574509
720	103.794298	103.602461	103.577110	103.574772	103.575161

Table 4.6: Value of the JP Morgan bond for different meshes and time steps.

	Step = 720 and Mesh = 64	Monte Carlo
UBS	102.620	[102.525, 102.727]
JPM	103.575	[103.556, 103.657]

Table 4.7: New method vs Monte Carlo simulation

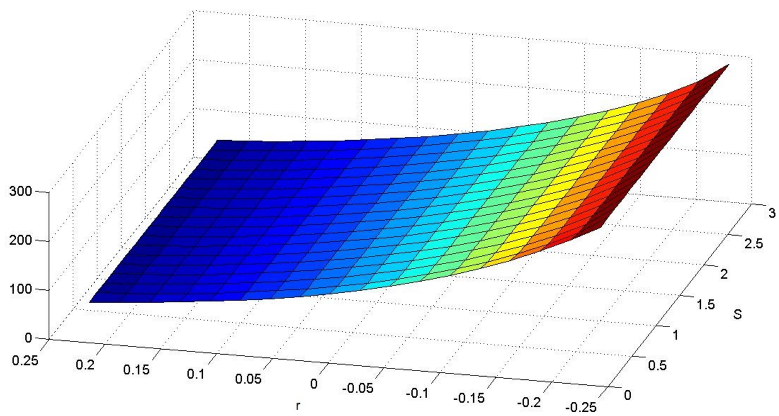


Figure 4.1: Value of the UBS bond

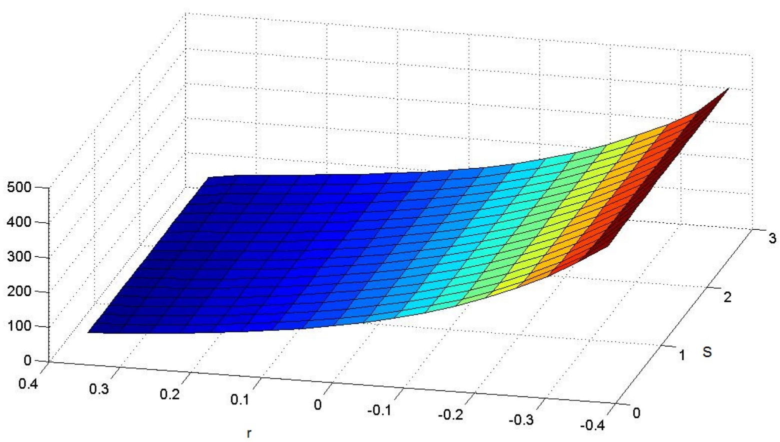


Figure 4.2: Value of the JP Morgan bond

Chapter 5

Sovereign CDS calibration under a hybrid Sovereign Risk Model

5.1 Introduction

Recent dynamics of sovereign credit risk in Europe have determined some significant doubts on the paradigm considering a Euro area government bond as a risk free investment. Consequently for investors the identification and pricing of sovereign bonds becomes a crucial issue. Main factors determining this structural change are the following:

- lack of a common economic and financial policy, with investors' perception that the economic and political convergence of the Euro area still require a long time;
- target to stabilize government deficits constantly disregarded by governments with the impossibility of financing infrastructural investments and difficulties in reforming the social security system;

- slowdown in economic growth and interdependence between financial sector crisis and sovereign risk for some countries;
- contagion effect triggered by the PSI in Greece and hence extended to the entire Euro system (aggravated by the downgrading of rating agencies) with a consequent increase in the risk premium requested by investors.

In the second half of 2011, following the escalation of the sovereign debt crisis in the Euro area and the contagion of tensions from the peripheral countries (Greece, Ireland and Portugal) to the core countries of the Euro area, foreign demand for Eurozone debt has suffered a major collapse involving the liquidation of outstanding positions in particular by institutional investors. The central element that led to this substantial change in terms of asset allocation was the perception that only a few Eurozone countries could be considered risk free; in addition, there was the growing fear of the Euro break-up which helped to stimulate the dynamics of cross-border capital outflow. During the period July - October 2011, foreign investors sold Eurozone fixed income instruments for around 88 bln Euro against a 320 bln Euro inflow in the first half of 2011. Japanese investors sold almost 98% of the Greek bonds and 61% of the Portuguese bonds; in the same period the sale of Italian bonds was almost 10.5%. Starting from the Lehman default event (September 2008), government bonds spreads have suffered a dramatic widening phase both in countries with a weak public sector finance and in countries considered to be safer. The volatility of the government bonds spread seems to reflect not only the perceived default risk of the issuers but also some other new relevant factors:

- Aggregate risk (change in monetary policy, global uncertainty, risk aversion);
- Liquidity risk;
- Country specific risk;

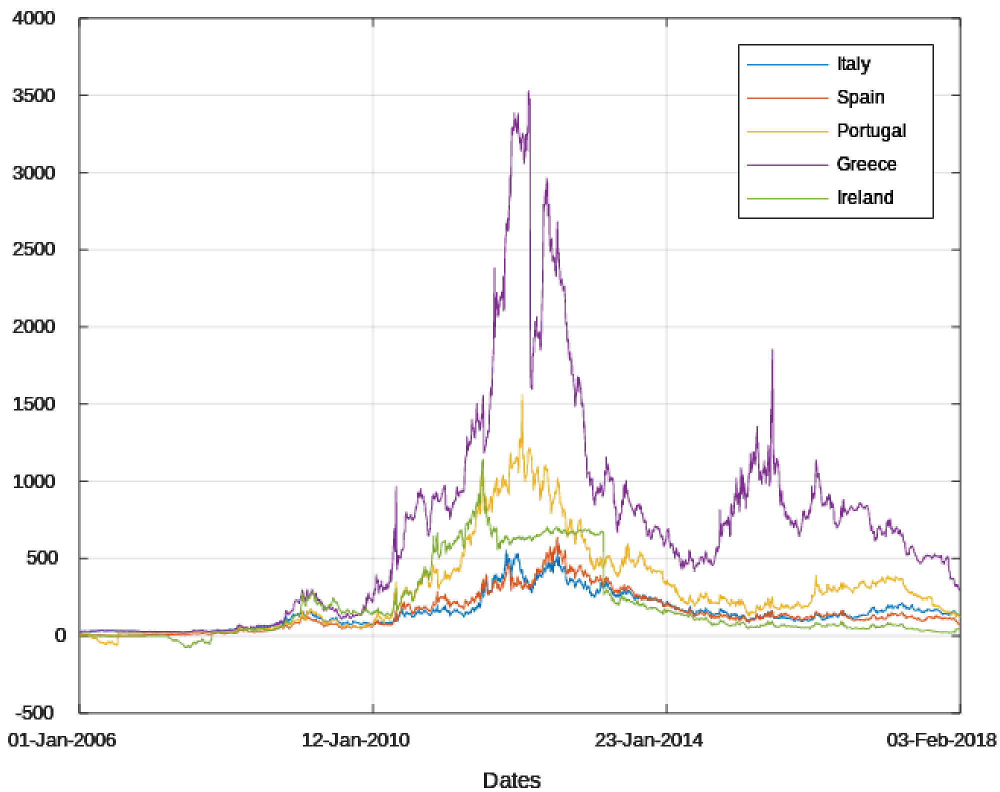


Figure 5.1: Government Bonds Spread versus Bund

- Contagion and systematic risk;
- Exchange rate risk.

The lack of models for the assessment of the component represented in the spread risk (eg Break up Euro scenario) and the lack of measures deemed sufficiently robust to quantify sovereign risk have led most investors to a hyper-prudent assessment of the situation, based on worst case hypotheses, negatively distorting the dynamics of spreads.

Default statistics currently used to calibrate corporate credit ratings are not applicable to sovereign risk. Furthermore, as reported in Moody's Investor Service Sovereign Default and Recovery Rates study [35], there is a very limited number of developed countries default events in the last 30 years (Greece in March 2012 and December 2012, Cyprus in July 2013) and consequently it is not possible to infer a consistent rating migration rates matrix for those countries. Also statistics on recovery rates available on defaulted sovereign bonds are estimated mainly with reference to emerging countries; the average recovery rates reported by Moody's in the sovereign default study is higher than the recovery rates for the two defaults of Greece in 2012 and for the one of Cyprus in 2013.

The need for banks and financial institutions to assess the risk associated with government bonds exposures has posed the problem for asset managers, traders and risk managers to determine how to assess sovereign default risk. There is no a specific standard in models used to assess the sovereign default risk and practitioners make use of consolidated models developed for corporate bonds. The two main families of models used to price and assess the risk of corporate and sovereign bonds are reduced-form models and structural models. Whereas reduced-form models are based on the specification of the risk-neutral default intensity and the fractional loss model, the structural models focus on the behavior of the assets of the issuer and the relative volatility compared to the value of the liabilities. Structural models have varied widely in their implementation, starting from the original models developed by Black and Scholes (1973) and

Merton (1974) and moving to more complex specifications making assumptions concerning the capital structures of the issuers and including different types of debts and other form of liabilities. While in structural models the default time is usually a predictable stopping time, defined as the first hitting time to a certain barrier by the asset process, in the reduce form the default time is a totally unpredictable stopping time modeled as the first jump of a Poisson process with stochastic intensity.

In the reduced form models, thanks to one of the fundamental properties of jumps in Poisson processes, the survival probabilities can be computed as a discount factors, and so it is a common market practice to compute these probabilities from credit default swap market information instead of the bond market. Moreover, the market of sovereign credit default swaps (SCDS) contracts has grown very fast in the last decade and has become very liquid, clean and standardized. So, the market of SCDS offers a consistent data framework set to estimate the default-survival probabilities. Furthermore estimates retrieved from CDS market prices allow practitioners to exclude the issue to represent the liquidity component of bonds spreads.

In this chapter we consider fixed Loss Given Default, that is a standard practice in the market and supported by historical observation. Unlike corporate CDS contracts, SCDS are usually denominated in a difference currency than the currency of the underling bonds. This is due to avoiding the risk of depreciation of the bond's currency in case of a credit event. In fact, if SCDS were denominated in the same currency as the bond, the recovery value would be significantly distorted by exchange rate fluctuation. So, for example, the market convention is to trade Euro CDS in US dollar and US CDS in Euro. The different currency between SCDS and bonds market makes it impossible to use the usual bootstrap technique to compute the default-survival probabilities in the bond's currency measure as for a corporate firm. Moreover, the assumption that the foreign and domestic hazard rate are identical is not realistic and contradicts market observations. So, the joint evolution of the domestic hazard rates and the FX rate between the two currencies must be modeled.

One of the motivations of this work has been to better understand the interrelationship between the creditworthiness of a sovereign, its intensity to default and the exchange rate between its bond's currency and the currency in which SCDS contracts are quoted. We analyze the differences between the default intensity under the domestic and foreign measure and we compute the default-survival probabilities in the bond's currency measure. Finally, we test our calibration to the valuation of sovereign bonds even during the period of sovereign crisis.

We start by providing a robust and efficient method to calibrate a hybrid sovereign risk model to SCDS market. We first present a model for the intensity of default of a sovereign government based on the jump to default extended CEV (Constant Elasticity Variance) model introduced in [9] in 2006 by establishing the link with the exchange rate. Then we give an approximation formula to the SCDS spread obtained from perturbation theory.

Our approach is similar to [7] where the authors presented a model that captures the link between the sovereign default intensity and the foreign exchange rate by adding a constant in case of credit event to this exchange rate process. As shown in [12], the introduction of a jump in the dynamics of the FX rate is necessary since a purely diffusion-based correlation between the exchange rate the hazard rate is not able to explain market observations. The default intensity is described by the exponential of some Ornstein-Uhlenbeck processes. Our work differs from [7] in several aspects: first we provide a hybrid model that captures the default intensity of the sovereign. Second, to approximate the SCDS spread, we employ a recent methodology introduced in [29, 37], which consists in an asymptotic expansion of the solution to the pricing partial differential equation. This approach of describing the sovereign default intensity with a hybrid model has been introduced in [23]. The authors are also inspired by the JDCEV model [9] which has been originally proposed for assessing corporate credit risks.

This chapter is organized as follows. In Section 5.2 we set the notations and introduce the model. In Section 5.3 we recall the definitions, properties on SCDS spreads and provide an explicit approximation formula. Section 5.4 contains the numerical test: we calibrate the model to Italian USD-quoted CDS contracts assessing in two different periods: at the outbreak of the government crisis at the end of 2011, in which the Italian CDS spreads reached the maximum, and at the present date. In Appendix 7.3, to show the robustness and the accuracy of our method, we present other several calibration tests, at the same dates as for Italian USD-quoted CDS spreads, for other European sovereign CDS spreads (France, Spain, Portugal).

5.2 Model and Set-up

In this section, we follow the approach in [23] to capture the dynamics of the default intensity by considering a hybrid model. This approach is inspired by the work [9] introduced in 2006 and establishes the dependency of the default intensity of the sovereign to its solvency. This latter is an indicator taking into account macro-economical factors like the public debt of the GDP (Gross Domestic Product) ratio, the surplus to GDP, interest rate on the sovereign bonds, GDP growth rate, etc... In what follows we model this solvency by a continuous-time process S. Consider the filtered probability space $(\Omega, \mathcal{G}, \mathbb{G}, \mathbb{Q})$ with finite time horizon $T < \infty$. The filtration $\mathbb{G} = (\mathcal{G}_t)_{t\in[0,T]}$ is assumed to satisfy the usual conditions, $\mathcal{G}_T = \mathcal{G}$ and is generated by the Brownian motions W_t^1 and W_t^2 and some discontinuous stochastic process D_t. Let ε be an exponentially distributed random variable independent of the Brownian motions W_t^1 and W_t^2 with parameter 1 (i.e. $\varepsilon \sim \mathrm{Exp}(1)$).

Let X be a stochastic process defined as

$$\mathrm{d}X_t = \left(r_d\left(t\right) - \frac{1}{2}\sigma^2\left(t, X_t\right) + \lambda\left(t, X_t\right)\right)\mathrm{d}t + \sigma\left(t, X_t\right)\mathrm{d}W_t^1,$$

where r_d is deterministic taking values. We assume that the time- and state-dependent functions $\sigma = \sigma(t, X)$ and $\lambda = \lambda(t, X)$ are positive, differentiable with respect to X and uniformly bounded. Let L be a real positive constant with $L < e^{X_0}$. Let ζ be defined as

$$\zeta = \inf\{t > 0 \mid e^{X_t} \leq L\} \wedge \inf\{t \geq 0 \mid \int_0^t \lambda(s, X_s)\mathrm{d}s \geq \varepsilon\} \tag{5.2.1}$$

By definition, ζ is a $\mathbb{G}$-stopping time.

Assumption 5.2.1. 1. The market is modelled by the filtered probability space $(\Omega, \mathcal{G}, \mathbb{G}, \mathbb{Q}_d)$ defined above where $\mathbb{Q} := \mathbb{Q}_d$ is a domestic spot risk-neutral martingale measure and $\mathbb{G}$ represents the quantity of information of the market and to which all processes are adapted.

2. The time to default of the sovereign is the stopping time ζ defined in (5.2.1) and we define the solvency S of the sovereign as follows:

$$S_t = S_0 e^{X_t} 1_{\{\zeta > t\}}, \quad S_0 > 0.$$

Default happens when the solvency becomes worthless in one of these two ways. Either the process e^X falls below L via diffusion or a jump-to-default occurs from a value greater than L, where L represents a threshold of the sovereign debt crisis. In what follows, we denote by $\mathbb{F} = \{\mathcal{F}_t,\ t \geq 0\}$ the filtration generated by the sovereign solvency and by $\mathbb{D} = \{\mathcal{D}_t,\ t \geq 0\}$ the filtration generated by the process $D_t = \mathbb{1}_{\{\zeta \leq t\}}$. Eventually, $\mathbb{G} = \{\mathcal{G}_t,\ t \geq 0\}$, $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{D}_t$ is the enlarged filtration.

3. The rate of exchange between foreign currency c_f and domestic currency c_d is denoted by $Z_t \geq 0$, r_i are the short-term interest rates and $B_i(t) = e^{\int_0^t r_i(u)\mathrm{d}u}$ the instantaneous bank accounts in the respective currencies $c_i,\ i = d,\ f$. For simplicity, we assume that the domestic and foreign interest rates are deterministic and are function of time t.

Remark 5.2.2. One can notice that the definition of the solvency of the sovereign is similar to the one of the defaultable stock prices of a corporate in the previous chapter. Indeed solvency can be seen as the "stock price" of a sovereign.

We assume that the rate of exchange Z, defining the value of unit of the foreign currency c_f in the domestic currency c_d, satisfies a SDE of the form

$$\mathrm{d}Z_t = \mu_t^Z Z_{t^-}\mathrm{d}t + \delta\ Z_{t^-}\mathrm{d}W_t^2 + \gamma\ Z_{t^-}\mathrm{d}D_t, \quad \text{with } \mathrm{d}W_t^1\mathrm{d}W_t^2 = \rho\mathrm{d}t, \tag{5.2.2}$$

where $\delta > 0$ and $\gamma \in [0,\ 1]$ is the devaluation/revaluation rate of the FX process. The dynamics (5.2.2) captures the dependency between the sovereign default risk and the rate of exchange, first through the correlation ρ between the Brownian motion W^1 and W^2 and then via the coefficient of devaluation/evaluation γ. Indeed, there is a jump on the rate of exchange at the time of default ζ by

$$\Delta Z_\zeta = \gamma Z_{\zeta^-}$$

That is at ζ, the foreign currency c_f is evaluated/devaluated with respect to the domestic currency c_d in a jump fraction γ of the pre-default value of Z. Therefore the price in c_d of the foreign instantaneous bank account at time t is $B_f(t)\,Z_t$. By Itô formula and (5.2.2)

$$\begin{aligned}
\mathrm{d}\left(B_f(t)\,Z_t\right) &= B_f(t)\,\mathrm{d}Z_t + r_f(t)\,B_f(t)\,Z_t\mathrm{d}t \qquad (5.2.3)\\
&= r_f(t)\,B_f(t)\,Z_t\mathrm{d}t + \mu_t^Z B_f(t)\,Z_t\mathrm{d}t + \delta B_f(t)\,Z_t\mathrm{d}W_t^2 + \gamma B_f(t)\,Z_t\mathrm{d}D_t\\
&= B_f(t)\,Z_t\left(\left(r_f(t) + \mu_t^Z + \gamma\left(1-D_t\right)\lambda\left(t, X_t\right)\right)\mathrm{d}t + \delta\mathrm{d}W_t^2 + \gamma\mathrm{d}M_t\right)
\end{aligned}$$

where the process $\mathrm{d}M_t = \mathrm{d}D_t - \mathrm{d}A_t$ is a martingale with $A_t = \int_0^t (1-D_s)\lambda_s\mathrm{d}s$ the compensator of D_t.

Proposition 5.2.3. *If the rate of exchange between the foreign and domestic currencies obeys a stochastic differential equation* (5.2.2)*, and if the riskless short-term rates of return in the* *domestic and foreign currencies are respectively r_i, $i = d,\ f$, then under $\mathbb{Q}_d$*

$$\mu_t^Z = r_d(t) - r_f(t) - \gamma(1 - D_t)\lambda(t, X_t).$$

Therefore, the exchange rate is given by

$$Z_t = Z_0 \exp\left(\int_0^t (r_d(s) - r_f(s) - \gamma(1 - D_s)\lambda(s, X_s))\,\mathrm{d}s - \frac{1}{2}\delta^2 t + \delta W_t^2 + \gamma D_t\right), \tag{5.2.4}$$

Proof. Under $\mathbb{Q}_d$, the discounted value in c_d of the foreign bank account must be a martingale. But the dynamics of the discounted value $\frac{B_f(t)}{B_d(t)}Z_t$ at time t is , by equation (5.2.3)

$$\begin{aligned}
\mathrm{d}\left(\frac{B_f(t)}{B_d(t)}Z_t\right) &= -r_d(t)\frac{B_f(t)}{B_d(t)}Z_t\mathrm{d}t + \frac{1}{B_d(t)}\mathrm{d}(B_f(t)Z_t) \\
&= \frac{B_f(t)}{B_d(t)}Z_t\left(\left(r_f(t) - r_d(t) + \mu_t^Z + \gamma(1 - D_t)\lambda(t, X_t)\right)\mathrm{d}t + \left(\mathrm{d}W_t^2 + \gamma\mathrm{d}M_t\right)\right)
\end{aligned}$$

Since the term $\left(\delta\mathrm{d}W_t^2 + \gamma\mathrm{d}M_t\right)$ is a martingale, then we must have

$$r_f(t) - r_d(t) + \mu_t^Z + \gamma(1 - D_t)\lambda(t, X_t) = 0.$$

$\square$

Proposition 5.2.4. *Let $\mathbb{Q}_f$ be the risk-neutral foreign martingale measure. Then $\mathbb{Q}_d$ and $\mathbb{Q}_f$ are mutually absolutely continuous; that is they are related by the likelihood ratio*

$$\left(\frac{\mathrm{d}\mathbb{Q}_f}{\mathrm{d}\mathbb{Q}_d}\right)_{\mathcal{F}_T} = \exp\left(\delta W_T^2 + \gamma M_T - \frac{1}{2}\delta^2 T\right)$$

Proof. Consider a contingent claim whose value at time t in c_f is V_t. The price V_0 of the claim at time $t = 0$ in c_f is the discounted expected value of its price, in c_f, at time T, where the expectation is computed under $\mathbb{Q}_f$, the risk-neutral foreign martingale measure:

$$V_0 = e^{-\int_0^T r_f(s)\mathrm{d}s}\mathbb{E}^f\left[V_T\right]. \tag{5.2.5}$$

Let U_t be the time-t price of the claim in c_d. Then $U_t = V_t Z_t$, where Z_t is the rate of exchange between c_f and c_d. Since the claim is a tradable asset, its price in c_d must be a martingale under $\mathbb{Q}_d$. In particular the time-zero price is the discounted expected value of the time-T price:

$$\begin{aligned} U_0 &= e^{-\int_0^T r_d(s)\mathrm{d}s}\mathbb{E}^d\left[U_T\right] \\ V_0 Z_0 &= e^{-\int_0^T r_d(s)\mathrm{d}s}\mathbb{E}^d\left[V_T Z_T\right] \\ V_0 &= e^{-\int_0^T r_f(s)\mathrm{d}s}\mathbb{E}^d\left[V_T \frac{Z_T}{Z_0} e^{-\int_0^T \left(r_f(s)-r_d(s)\right)\mathrm{d}s}\right]. \end{aligned}$$

Comparing equations (5.2.5) and (5.2.6) shows that

$$\mathbb{E}^f\left(V_T\right) = \mathbb{E}^d\left(V_T \frac{Z_T}{Z_0} e^{-\int_0^T \left(r_f(s)-r_d(s)\right)\mathrm{d}s}\right).$$

Since it holds for any non-negative $\mathcal{G}_T$-measurable random variable V_T, it follows from (5.2.4) that

$$\left(\frac{\mathrm{d}\mathbb{Q}_f}{\mathrm{d}\mathbb{Q}_d}\right)_{\mathcal{F}_T} = \frac{Z_T}{Z_0} e^{-\int_0^T r_f(s)-r_d(s)\mathrm{d}s} = \exp\left(\delta W_T^2 + \gamma M_T - \frac{1}{2}\delta^2 T\right).$$

$\square$

Set $Y_t = \log\left(Z_t\right)$. By Itô formula, the dynamics of Y is given by

$$\begin{aligned} \mathrm{d}Y_t &= \left(\mu_t^Z - \frac{1}{2}\delta^2\right)\mathrm{d}t + \delta\,\mathrm{d}W_t^2 + \gamma\,\mathrm{d}D_t \\ &= \left(r_d\left(t\right) - r_f\left(t\right) - \frac{1}{2}\delta^2 - \gamma\left(1-D_t\right)\lambda\left(t, X_t\right)\right)\mathrm{d}t + \delta\,\mathrm{d}W_t^2 + \gamma\,\mathrm{d}D_t. \end{aligned}$$

oAs in [9], we set the volatility of the solvency and the intensity of default as

$$\sigma(t, X) = a(t)e^{(\beta-1)X} \tag{5.2.6}$$

and

$$\lambda(t, X) = b(t) + c\,\sigma(t, X)^2 = b(t) + c\,a(t)^2 e^{2(\beta-1)X} \tag{5.2.7}$$

where $\beta < 1$ and $a(t) > 0$ are the so-called elasticity parameter and scale function, while $b(t) \geq 0$ and $c \geq 0$ govern the sensitivity of the default intensity with respect to the solvency. Under $\mathbb{Q}_d$, it follows that the risk-neutral dynamics of the solvency $S_t = \{S_t,\ t \geq 0\}$ is then given by

$$\begin{cases} S_t = S_0 e^{X_t}\left(1 - D_t\right), \qquad S_0 > 0, \\ \mathrm{d}X_t = \left(r_d\left(t\right) - \frac{1}{2}\sigma^2\left(t, X_t\right) + \lambda\left(t, X_t\right)\right)\mathrm{d}t + \sigma\left(t, X_t\right)\mathrm{d}W_t^1, \\ \mathrm{d}Y_t = \left(r_d\left(t\right) - r_f\left(t\right) - \frac{1}{2}\delta^2 - \gamma\left(1 - D_t\right)\lambda\left(t, X_t\right)\right)\mathrm{d}t + \delta\ \mathrm{d}W_t^2 + \gamma\ \mathrm{d}D_t, \\ \mathrm{d}D_t = \gamma\left(1 - D_t\right)\lambda\left(t, X_t\right)\mathrm{d}t + \mathrm{d}M_t, \qquad \text{with } M \text{ a martingale.} \\ \mathrm{d}W_t^1 \mathrm{d}W_t^2 = \rho \mathrm{d}t, \\ \zeta = \inf\{t > 0 \mid e^{X_t} \leq L\} \wedge \inf\{t \geq 0 \mid \int_0^t \lambda\left(t, X_t\right) \geq \varepsilon\}. \end{cases} \tag{5.2.8}$$

we adopt a martingale modeling approach, in some sense similarly to what is usually done in the theory of interest rate models. Since our market model is incomplete, we directly consider a martingale dynamics for S. Even if our framework is completely different, the martingale approach is also reminiscent of Carr&Linetsky's model [9] where S represents the price of a traded asset.

5.3 Sovereign Credit Default Swap spread

As presented in chaper 3, a CDS is an agreement between two parties, called the protection buyer and the protection seller, typically designed to transfer to the protection seller the financial loss that the protection buyer would suffer if a particular default event happened to a third party, called the reference entity. The protection seller delivers a protection payment to the protection buyer at the time of the default event. In exchange the protection buyer makes periodic premium payments at time intervals α at the credit default swap rate up to the default event or the expiry maturity, whichever comes first. The protection payment is the specified percentage $(1-\eta)$ of the CDS notional amount $\mathcal{N}$ (=1 by assumption), called *default recovery or loss-given-default*, paid at default time. The valuation problem is to determine the arbitrage-free CDS rate R that makes the present value of the CDS contract equal to zero. This rate equates the present value of the protection payoff to the present value of all the premium payments.

By *Sovereign*, we understand, from the definition given by the International Swaps and Derivatives Association (ISDA), *"any state, potential subdivision or government, or any agency, instrumentality, ministry, department or other authority (including ... central bank) thereof"*. Here, for simplicity, we consider sovereign governments. Hence a sovereign Credit Default Swap is a CDS where the reference entity is a government. e.g Eurozone States Members. From ISDA, a credit event in sovereign CDS contracts is induced among others by

- *Failure to pay*: a sovereign fails to make a payment on its obligations (principle, coupons, etc..) in an amount at least as large as the payment requirement beyond the period allowed;
- *Restructuring*: a sovereign alters the principle amount, coupon, currency, maturity or the ranking in priority of repayment of an obligation;

- *Repudiation/moratorium*: a sovereign refuses to honor its obligation and declares a moratorium.

Proposition 5.3.1. *Consider a sovereign CDS contact settled in foreign currency with constant default recovery* $(1-\eta)$*, spread* R *paid at premium dates* t_i*,* $i = 1, 2, \cdots, M$*, so that* $\alpha = t_{i-1} - t_i = \dfrac{T}{M}$*. Then, at time* $t = 0$*, the spread* $R \equiv R(0,T)$ *is given by*

$$R = \frac{(1-\eta)\int_0^T \mathbb{E}^d\left(e^{Y_s - Y_0 - \int_0^s (r_d(u)+\lambda(u,X_u))\mathrm{d}u}\lambda(s,X_s)\right)\mathrm{d}s}{\frac{T}{M}\sum_{i=1}^M \mathbb{E}^d\left(e^{Y_{t_i} - Y_0 - \int_0^{t_i}(r_d(u)+\lambda(u,X_u))\mathrm{d}u}\right)}. \tag{5.3.1}$$

Proof. By Lemma 2.1.8 and 2.1.9, the price of the SCDS at time t is given by

$$\begin{aligned}
V_t(R) &= \mathbb{E}^f\left(e^{-\int_t^\zeta r_f(u)\mathrm{d}u}(1-\eta)\mathbb{1}_{\{\zeta\le T\}} - \sum_{i=1}^M e^{-\int_t^{t_i} r_f(u)\mathrm{d}u}\tfrac{T}{M}R\mathbb{1}_{\{\zeta>t_i\}}|\mathcal{G}_t\right)\\
&= \mathbb{E}^d\left(e^{-\int_t^\zeta r_f(u)\mathrm{d}u}(1-\eta)L_\xi\mathbb{1}_{\{\zeta\le T\}} - \sum_{i=1}^M e^{-\int_t^{t_i} r_f(u)\mathrm{d}u}\tfrac{T}{M}RL_{t_i}\mathbb{1}_{\{\zeta>t_i\}}|\mathcal{G}_t\right) \quad \text{(by change of measure)}\\
&= (1-\eta)\mathbb{E}^d\left(\frac{B_f(t)}{B_f(\zeta)}\frac{B_f(\zeta)Z_\zeta B_d(t)}{B_f(t)Z_tB_d(\zeta)}\mathbb{1}_{\{\zeta\le T\}}|\mathcal{G}_t\right) - \tfrac{T}{M}\sum_{i=1}^M \mathbb{E}^d\left(\frac{B_f(t)}{B_f(t_i)}\frac{B_f(t_i)Z_{t_i}B_d(t)}{B_f(t)Z_tB_d(t_i)}R\mathbb{1}_{\{\zeta>t_i\}}|\mathcal{G}_t\right)\\
&= (1-\eta)\mathbb{E}^d\left(\frac{Z_\zeta B_d(t)}{Z_tB_d(\zeta)}\mathbb{1}_{\{\zeta\le T\}}|\mathcal{G}_t\right) - \tfrac{T}{M}\sum_{i=1}^M \mathbb{E}^d\left(\frac{Z_{t_i}B_d(t)}{Z_tB_d(t_i)}R\mathbb{1}_{\{\zeta>t_i\}}|\mathcal{G}_t\right)\\
&= \mathbb{1}_{\{\zeta>t\}}(1-\eta)\int_t^T \mathbb{E}^d\left(e^{Y_s-Y_t-\int_t^s(r_d(u)+\lambda(u,X_u))\mathrm{d}u}\lambda(s,X_s)|\mathcal{F}_t\right)\mathrm{d}s\\
&\qquad - \mathbb{1}_{\{\zeta>t\}}\tfrac{T}{M}R\sum_{i=1}^M \mathbb{E}^d\left(e^{Y_{t_i}-Y_t-\int_t^{t_i}(r_d(u)+\lambda(u,X_u))\mathrm{d}u}|\mathcal{F}_t\right).
\end{aligned}$$

By definition, the market CDS spread $R(t,T)$ at time t is determined by the equation $V_t(R(t,T)) = 0$. Hence on $\mathbb{1}_{\{\zeta>t\}}$, we have

$$R(t,T) = \frac{(1-\eta)\int_t^T \mathbb{E}^d\left(e^{Y_s-Y_t-\int_t^s(r_d(u)+\lambda(u,X_u))\mathrm{d}u}\lambda(s,X_s)|\mathcal{F}_t\right)\mathrm{d}s}{\frac{T}{M}\sum_{i=1}^M \mathbb{E}^d\left(e^{Y_{t_i}-Y_t-\int_t^{t_i}(r_d(u)+\lambda(u,X_u))\mathrm{d}u}|\mathcal{F}_t\right)}.$$

It follows the equality (5.3.1). ∎

We aim to give an explicit approximation formula to SCDS spread (5.3.1) based on an asymptotic expansion technique introduced in [29, 39]. From Propositon 2.3.1, computing the expectations in (5.3.1) equals to solving general backward Cauchy problems

$$\begin{cases} (\partial_t + \mathcal{A})\, u(t,z) = 0, & t < T,\ z \in \mathbb{R}^d, \\ u(T,z) = h(z), & z \in \mathbb{R}^d, \end{cases}$$

where $\mathcal{A} = \mathcal{A}(t,z)$ is a (locally) parabolic differential operator of the form

$$\mathcal{A}(t,z) = \sum_{|\alpha| \le 2} a_\alpha(t,z)\, D_z^\alpha, \qquad t \in \mathbb{R}^+,\ z \in \mathbb{R}^d, \tag{5.3.2}$$

with

$$\alpha = (\alpha_1, \ldots, \alpha_d), \qquad |\alpha| = \sum_{i=1}^{d} \alpha_i,\ D_z^\alpha = \partial_{z_1}^{\alpha_1} \ldots \partial_{z_d}^{\alpha_d}.$$

In our specific setting, we will consider $\mathcal{A}$ to be the infinitesimal generator of the stochastic processes (X, Y) in (5.2.8), whose precise expression in given in formula (5.3.2).

This leads to an explicit approximation formula for the SCDS spread (5.3.1).

Theorem 5.3.2. *Let T be the expiry date of the SCDS contract settled in foreign currency with constant default recovery $(1-\eta)$, spread R paid at premium dates $t_i,\ i = 1,\ 2,\ \cdots,\ M$, so that $\alpha = t_{i-1} - t_i = \frac{T}{M}$. Under the general dynamics (5.2.8), there exist sequences of differential operators $\left(\mathcal{L}_n^{1,(x,y)}\right)_{n\ge0}$ and $\left(\mathcal{L}_n^{2,(x,y)}\right)_{n\ge0}$ of the form of (7.1.11) and acting on (x,y) such that the N-th order approximation of the SCDS spread in (5.3.1) is given by*

$$R_N = \frac{(1-\eta)\int_0^T \sum\limits_{n=0}^{N} \mathcal{L}_n^{1,(x,y)}(0,s)\, u_0(0,x,y;s)\, \mathrm{d}s}{\frac{T}{K}\sum\limits_{i=1}^{K}\sum\limits_{n=0}^{N} \mathcal{L}_n^{2,(x,y)}(0,t_i)\, v_0(0,x,y;t_i)}, \tag{5.3.3}$$

where

$$u_0\left(0,x,y,s\right)=e^{-\int_0^s\left(r_f(u)+\frac{1}{2}\delta^2+\lambda(u,x)\right)\mathrm{d}u}$$
$$\cdot\left(b\left(s\right)+c\ a\left(s\right)^2\exp\left(2\left(\beta-1\right)\left(x+\int_0^s\left(r_d\left(u\right)+\left(\beta-\frac{3}{2}\right)\sigma\left(u,x\right)^2+\lambda\left(u,x\right)\right)\mathrm{d}u\right)\right)\right)$$
$$v_0\left(0,x,y;s\right)=e^{-\int_0^s\left(r_f(u)+\frac{1}{2}\delta^2+2\lambda(u,x)\right)\mathrm{d}u}.$$

Proof. We see from (5.3.1) that we have to evaluate expectations of the form

$$\mathbb{E}^d\left(e^{Y_T-Y_t-\int_t^T(r_d(u)+\lambda(u,X_u))\mathrm{d}u}h\left(X_T,Y_T\right)|\mathcal{F}_t\right).$$

They are functions of t, X_t, Y_t and D_t. Let us denote its value at time t for $X_t=x,\ Y_t=y$ and $D_t=d$ as $f\left(t,x,y,d\right)$. Set $g\left(t,x,y,d\right)$ the value at time $t=0$ of

$$\mathbb{E}^d\left(e^{Y_T-\int_0^T(r_d(u)+\lambda(u,X_u))\mathrm{d}u}h\left(X_T,Y_T\right)\right).$$

By Itô formula and its martingale property, one can see that

$$\partial_t f\left(t,x,y,d\right)+\left(r_d\left(t\right)-\frac{1}{2}\sigma\left(t,x\right)^2+\lambda\left(t,x\right)\right)\partial_x f\left(t,x,y,d\right)+\frac{1}{2}\sigma\left(t,x\right)^2\partial_x^2 f\left(t,x,y,d\right)$$
$$+\left(r_d\left(t\right)-r_f\left(t\right)-\frac{1}{2}\delta^2+\gamma\left(1-d\right)\lambda\left(t,x\right)\right)\partial_y f\left(t,x,y,d\right)+\frac{1}{2}\delta^2\partial_y^2 f\left(t,x,y,d\right)+\rho\delta\sigma\left(t,x\right)\partial_{xy} f\left(t,x,y,d\right)$$
$$+\left(1-d\right)\lambda\left(t,x\right)\left(f\left(t,x,y,1\right)-f\left(t,x,y,0\right)\right)+\left(-r_f\left(t\right)-\frac{1}{2}\delta^2-\lambda\left(t,x\right)\right)f\left(t,x,y,d\right)=0.$$

Set

$$u\left(t,x,y\right)=f\left(t,x,y,1\right)\ \text{ and }\ v\left(t,x,y\right)=f\left(t,x,y,0\right)$$
$$\Rightarrow f\left(t,x,y,d\right)=1_{\{d=1\}}u\left(t,x,y\right)+1_{\{d=0\}}v\left(t,x,y\right)$$

For the premium leg, the final conditions of u and v are

$$u(T,x,y) = f(T,x,y,1) = 0,$$
$$v(T,x,y) = f(T,x,y,0) = h(x,y) = 1.$$

u is solution to the PDE

$$\left(\partial_t + \left(r_d(t) - \frac{1}{2}\sigma(t,x)^2 + \lambda(t,x)\right)\partial_x + \frac{1}{2}\sigma(t,x)^2\partial_x^2 + \left(r_d(t) - r_f(t) - \frac{1}{2}\delta^2\right)\partial_y + \right.$$
$$\left. + \frac{1}{2}\delta^2\partial_y^2 + \rho\delta\sigma(t,x)\partial_{xy} + \left(-r_f(t) - \frac{1}{2}\delta^2 - \lambda(t,x)\right)\right)u = 0, \quad \text{for } t < T \text{ and } x, y \in \mathbb{R}^2$$
$$u(T,x,y) = 0 \quad x, y \in \mathbb{R}^2.$$

It follows that $u \equiv 0$. Therefore , for computing the premium leg, one only need to solve directly the PDE for v

$$\left(\partial_t + \left(r_d(t) - \frac{1}{2}\sigma(t,x)^2 + \lambda(t,x)\right)\partial_x + \frac{1}{2}\sigma(t,x)^2\partial_x^2 + \left(r_d(t) - r_f(t) - \frac{1}{2}\delta^2 + \gamma\lambda(t,x)\right)\partial_y + \right.$$
$$\left. + \frac{1}{2}\delta^2\partial_y^2 + \rho\delta\sigma(t,x)\partial_{xy} + \left(-r_f(t) - \frac{1}{2}\delta^2 - 2\lambda(t,x)\right)\right)v = 0, \quad \text{for } t < T \text{ and } x, y \in \mathbb{R}^2$$
$$v(T,x,y) = 1 \quad x, y \in \mathbb{R}^2.$$

Analogously , to compute the protection leg, one solve the following PDE for u

$$\left(\partial_t + \left(r_d(t) - \frac{1}{2}\sigma(t,x)^2 + \lambda(t,x)\right)\partial_x + \frac{1}{2}\sigma(t,x)^2\partial_x^2 + \left(r_d(t) - r_f(t) - \frac{1}{2}\delta^2\right)\partial_y + \right.$$
$$\left. + \frac{1}{2}\delta^2\partial_y^2 + \rho\delta\sigma(t,x)\partial_{xy} + \left(-r_f(t) - \frac{1}{2}\delta^2 - \lambda(t,x)\right)\right)u = 0, \quad \text{for } t < T \text{ and } x, y \in \mathbb{R}^2$$
$$u(T,x,y) = \lambda(T,x) \quad x, y \in \mathbb{R}^2.$$

Hence to approximate the Quanto CDS spread, we must deal with two Cauchy problems with different operators $\mathcal{A}^1$ and $\mathcal{A}^2$ and different terminal conditions:

$$\begin{cases} \left(\partial_t + \mathcal{A}^1\right) u\left(t, x, y\right) = 0, & t < T,\ x, y \in \mathbb{R} \\ u\left(T, x, y\right) = \lambda\left(T, x\right), & x, y \in \mathbb{R}, \end{cases} \tag{5.3.4}$$

and

$$\begin{cases} \left(\partial_t + \mathcal{A}^2\right) v\left(t, x, y\right) = 0, & t < T,\ x, y \in \mathbb{R} \\ v\left(T, x, y\right) = 1, & x, y \in \mathbb{R}, \end{cases} \tag{5.3.5}$$

where

$$\begin{aligned} \mathcal{A}^1 = {} & \frac{1}{2}\sigma\left(t, x\right)^2 \partial_x^2 + \rho\delta\sigma\left(t, x\right)\partial_{xy} + \frac{1}{2}\delta^2\partial_y^2 + \left(r_d\left(t\right) - \frac{1}{2}\sigma\left(t, x\right)^2 + \lambda\left(t, x\right)\right)\partial_x \\ & + \left(r_d\left(t\right) - r_f\left(t\right) - \frac{1}{2}\delta^2\right)\partial_y - \left(r_f\left(t\right) + \frac{1}{2}\delta^2 + \lambda\left(t, x\right)\right) \end{aligned}$$

and

$$\begin{aligned} \mathcal{A}^2 = {} & \frac{1}{2}\sigma\left(t, x\right)^2 \partial_x^2 + \rho\delta\sigma\left(t, x\right)\partial_{xy} + \frac{1}{2}\delta^2\partial_y^2 + \left(r_d\left(t\right) - \frac{1}{2}\sigma\left(t, x\right)^2 + \lambda\left(t, x\right)\right)\partial_x \\ & + \left(r_d\left(t\right) - r_f\left(t\right) - \frac{1}{2}\delta^2 + \gamma\lambda\left(t, x\right)\right)\partial_y - \left(r_f\left(t\right) + \frac{1}{2}\delta^2 + 2\lambda\left(t, x\right)\right). \end{aligned}$$

Hence by Theorem 7.1.3, there exists a sequence $(\mathcal{L}_n^{1,(x,y)})_{b\geq n}$ of differential operators such that the N-th approximation of the solution u of (5.3.4) is given by

$$u\left(0, x, y; s\right) = \sum_{n=0}^{N} \mathcal{L}_n^{1,(x,y)}\left(0, s\right) u_0\left(0, x, y; s\right),$$

where

$$u_0\left(0, x, y; s\right) = e^{\int_0^s \chi(u,x,y)\mathrm{d}u} \int_{\mathbb{R}^2} \Gamma_0\left(0, x, y; s, \xi_1, \xi_2\right)\lambda\left(s, \xi_1\right)\mathrm{d}\xi_1\mathrm{d}\xi_2,$$

with

$$\chi(u,x,y) = -\left(r_f(u) + \frac{1}{2}\delta^2 + \lambda(u,x)\right)$$

and Γ_0 is the probability density of the 2-dimensional Gaussian random variable (Ξ_1, Ξ_2) with covariance and mean respectively given by

$$C(0,x,y;s) = \begin{pmatrix} \int_0^s \sigma(u,x)^2\,\mathrm{d}u & \rho\delta\int_0^s \sigma(u,x)\,\mathrm{d}u \\ \rho\delta\int_0^s \sigma(u,x)\,\mathrm{d}u & \delta^2 * s \end{pmatrix}$$

and

$$m(0,x,y;s) = \left(x + \int_0^s \left(r_d(u) - \tfrac{1}{2}\sigma(u,x)^2 + \lambda(u,x)\right)\mathrm{d}u, \quad y + \int_0^s \left(r_d(u) - r_f(u) - \tfrac{1}{2}\delta^2\right)\mathrm{d}u\right),$$

$$\int_{\mathbb{R}^2} \Gamma_0(0,x,y;s,\xi_1,\xi_2)\,\lambda(s,\xi_1)\,\mathrm{d}\xi_1\mathrm{d}\xi_2 = \int_{\mathbb{R}} \bar{\Gamma}_0(0,x;s,\xi_1)\,\lambda(s,\xi_1)\,\mathrm{d}\xi_1,$$

where $\bar{\Gamma}_0$ is the marginal probability density function of the random Gaussian random variable Ξ_1 with variance and mean

$$\overline{C}(0,x,y;s) = \int_0^s \sigma(u,x)^2\,\mathrm{d}u$$

and

$$\overline{m}(0,x,y;s) = x + \int_0^s \left(r_d(u) - \frac{1}{2}\sigma(u,x)^2 + \lambda(u,x)\right)\mathrm{d}u.$$

It follows

$$\begin{aligned}\int_{\mathbb{R}} \bar{\Gamma}_0(0,x;s,\xi_1)\,\lambda(s,\xi_1)\,\mathrm{d}\xi_1 &= b(s) + c\,a(s)\int_{\mathbb{R}} \bar{\Gamma}_0(0,x;s,\xi_1)\,e^{2(\beta-1)\xi_1}\mathrm{d}\xi_1 \\ &= b(s) + c\,a(s)\,\mathbb{E}^d[\Pi],\end{aligned}$$

where $\Pi = e^{2(\beta-1)\Xi_1}$ is a log-normal distributed random variable with mean $\mathbb{E}^d[\Pi]$ equals to

$$\begin{aligned}
\mathbb{E}^d(\Pi) &= \exp\left(2(\beta-1)\overline{m}(0,x,y;s) + 4(\beta-1)^2\frac{\overline{C}(0,x,y;s)}{2}\right)\\
&= \exp\left(2(\beta-1)\left(\overline{m}(0,x,y;s) + (\beta-1)\overline{C}(0,x,y;s)\right)\right)\\
&= \exp\left(2(\beta-1)\left(x + \int_0^s\left(r_d(u) - \frac{1}{2}\sigma(u,x)^2 + \lambda(u,x)\right)\mathrm{d}u + (\beta-1)\int_0^s \sigma(u,x)^2\,\mathrm{d}u\right)\right)\\
&= \exp\left(2(\beta-1)\left(x + \int_0^s\left(r_d(u) + \left(\beta-\frac{3}{2}\right)\sigma(u,x)^2 + \lambda(u,x)\right)\mathrm{d}u\right)\right).
\end{aligned}$$

Hence

$$\begin{aligned}
u_0(0,x,y,s) &= e^{-\int_0^s\left(r_f(u)+\frac{1}{2}\delta^2+\lambda(u,x)\right)\mathrm{d}u}\times\\
&\quad\times\left(b(s) + c\;a(s)^2\exp\left(2(\beta-1)\left(x+\int_0^s\left(r_d(u)+\left(\beta-\frac{3}{2}\right)\sigma(u,x)^2+\lambda(u,x)\right)\mathrm{d}u\right)\right)\right).
\end{aligned}$$

Analogously, we see from theorem 7.1.3 that there exists a sequence $(\mathcal{L}_n^{2,(x,y)})_{b\geq n}$ of differential operators such that the N-th approximation of the solution v of (5.3.5) is given by

$$v(0,x,y;s) = \sum_{n=0}^{N}\mathcal{L}_n^{2,(x,y)}(0,s)\,v_0(0,x,y;s),$$

where

$$v_0(0,x,y;s) = e^{\int_0^s \chi(u,x,y)\mathrm{d}u}\int_{\mathbb{R}^2}\Gamma_0(0,x,y;s,\xi_1,\xi_2)\,\mathrm{d}\xi_1\mathrm{d}\xi_2 = e^{\int_0^s \chi(u,x,y)\mathrm{d}u}$$

with

$$\chi(u,x,y) = -\left(r_f(u) + \frac{1}{2}\delta^2 + 2\lambda(u,x)\right).$$

□

5.4 Sovereign CDS calibration and empirical test

In this section we apply the method developed in Section 5.3 to calibrate the model (5.2.8) to the sovereign CDS spreads quoted by the market. We use quotations for Italian USD-quoted CDS provided by Bloomberg database on the date November, 15^{th} 2011 in order to check the robustness of our methodology. We use the second-order approximation formula (5.3.3) for the SCDS. We consider SCDS contracts with maturity from one up to four years and paid quarterly with recovery rate 40% at the event of default.

Since the formula (5.3.3) gives the approximation of the SCDS spread in the domestic currency c_d (EUR) and the market SCDS is in the foreign currency c_f (USD), we consider the following formula introduced in [7]:

$$R = \frac{\widetilde{R}}{1+\gamma},$$

where $\widetilde{R}$ is SCDS spread in the USD. This shows the link between the SCDS spreads in two currencies. To add more flexibility to the model, we assume that the coefficients $a(t)$ and $b(t)$ in (5.2.7) are linearly dependent on time: more precisely, we assume that

$$a(t) = a_1 t + a_2, \qquad b(t) = b_1 t + b_2,$$

for some constants $a_1,\ a_2,\ b_1$, and b_2. The problem of calibrating the model (5.2.8) is formulated as an optimization problem. We want to minimize the error between the model CDS spread and the market CDS spreads. Our approach is to use the square difference between market and model CDS spreads. This leads to the nonlinear least squares method

$$\inf_{\Theta} F\left(\Theta\right), \qquad F\left(\Theta\right) = \sum_{i=1}^{N} \frac{|R_i - \widehat{R}_i|^2}{\widehat{R}_i^2},$$

where N is the number of spreads used, $\widehat{R}_i$ is the market CDS spreads of the considered reference entity observed at time $t = 0$ and $\Theta = (a_1, a_2, b_1, b_2, \beta, c, \delta, \rho, \gamma)$, with

$$a_2 \geq 0,\ a_1 \geq -\frac{a_2}{T},\ b_2 \geq 0,\ b_1 \geq -\frac{b_2}{T},\ c \geq 0,\ \delta \geq 0,\ \beta < 1,\ -1 < \gamma < 1 \text{ and } -1 < \rho < 1.$$

For the calibration, we use a global optimizer, *NMinimize*, from Mathematica's optimization toolbox on a PC with 1× Intel i7-6599U 2.50 GHz CPU and 8GB RAM. First we calibrate the model to real market SCDS spreads taken at the outbreak of the government crisis at the end of 2011. Table 5.1 shows the results of the calibration to Italian SCDS contracts settled in USD and we can observe that the model gives very good fit to the real market data with a computational time equals to 45.808 seconds. We calibrate our model to market data quoted at November, 15th, 2011, when the Italian CDS spreads reached their maximum value.

Table 5.1: Calibration to Italy USD CDS quoted as COB November, 15th, 2011

Times to maturity (Year)	Market spreads (bps)	Model spreads (bps)	Rel. errors
1.25	639.604	641.307	0.266268 %
1.5	634.042	627.823	-0.98089 %
1.75	617.96	615.888	-0.335402 %
2.	601.52	605.829	0.71629 %
2.25	592.68	597.979	0.894177 %
2.5	590.589	592.642	0.347765 %
2.75	592.193	590.018	-0.367292 %
3.	594.44	590.072	-0.734818 %
3.25	594.935	592.299	-0.442919 %
3.5	593.917	595.317	0.235591 %
3.75	592.288	596.172	0.655759 %
4.	590.945	589.227	-0.290722 %

$a_1 = 0.5,\ a_2 = 0.2,\ \beta = -0.77,\ b_1 = -0.014,\ b_2 = 0.056,\ c = 0.015,\ \delta = 0.94\ \rho = -0.41,\ \gamma = 1.0,\ 45.808\ \textit{seconds}$

We follow the same process as above but this time we calibrate to Italian SCDS spreads quoted on May, 05th 2017. Table 5.2 shows that the method still provides very good fit to real market data.

Table 5.2: Calibration to Italy USD CDS quoted as COB May, 30th 2017

Times to maturity (Year)	Market spreads (bps)	Model spreads (bps)	Rel. errors
1.25	77.3576	78.0834	0.938368 %
1.5	88.3681	87.4887	-0.995104 %
1.75	97.1146	96.3335	-0.804301 %
2.	104.695	104.63	-0.0616158 %
2.25	112.006	112.395	0.346509 %
2.5	119.142	119.642	0.419618 %
2.75	125.994	126.39	0.314503 %
3.	132.455	132.658	0.15294 %
3.25	138.435	138.46	0.0180013 %
3.5	143.918	143.811	-0.0747112 %
3.75	148.906	148.714	-0.128773 %
4.	153.4	153.161	-0.15609 %

$a_1 = 0.2,\ a_2 = 0.03,\ \beta = 0.63,\ b_1 = 0.01,\ b_2 = 0.002,\ c = 0.005,\ \delta = 1.068,\ \rho = 0.24,\ \gamma = 0.54,\ 41.992\ \textit{seconds}$

At time $t = 0$, the foreign survival probability of the SCDS is given by

$$p_0^f(T) = \mathbb{E}^f\left(e^{-\int_0^T \lambda^f(u,X_u)\mathrm{d}u}\right) = \mathbb{E}^f\left(e^{-(1+\gamma)\int_0^T \lambda(u,X_u)\mathrm{d}u}\right),$$

where λ^f is the default intensity in the foreign economy and is linked to the domestic default intensity by the relation $\lambda^f(t,X_t) = (1+\gamma)\lambda(t,X_t)$. The dynamics of the underlying process X in the foreign risk-neutral measure $\mathbb{Q}^f$ is

$$\mathrm{d}X_t = \left(r_d(t) - \frac{1}{2}\sigma^2(t,X_t) + \lambda(t,X_t) - \rho\eta\sigma(t,X_t)\right)\mathrm{d}t + \sigma(t,X_t)\,\mathrm{d}\widehat{W}_t^1,$$

where $\widehat{W}^1$ is given by

$$\mathrm{d}\widehat{W}_t^1 = \mathrm{d}W_t^1 - \frac{\mathrm{d}\langle W^1, Z\rangle_t}{Z_t} = \mathrm{d}W_t^1 - \rho\eta\mathrm{d}t.$$

By Feynman-Kac representation formula, $p_0^f(T) = u(0,x;T)$, where u is solution to the Cauchy problem

$$\begin{cases} (\partial_t + \mathcal{A})\, u\,(t,x) = 0, & t < T,\ x \in \mathbb{R}, \\ u\,(T,x) = 1, & x \in \mathbb{R}, \end{cases}$$

with

$$\mathcal{A} = \frac{1}{2}\sigma\,(t,x)^2\,\partial_x^2 + \left(r_d\,(t) - \frac{1}{2}\sigma\,(t,x)^2 + \lambda\,(t,x) - \rho\eta\sigma\,(t,x)\right)\partial_x - \lambda\,(t,x)\,.$$

By Theorem 7.1.3, there exists a sequence of operator $(\mathcal{L}_n^x)_{n\geq 0}$, acting on the variable x, such that

$$p_0^f\,(T) = u\,(0,x) \approx u_N\,(0,x;T) = \sum_{n=0}^{N} \mathcal{L}_n^x\,(0,T)\,u_0\,(0,x;T)\,, \tag{5.4.1}$$

where u_0 is given by

$$u_0\,(t,x;T) = e^{-(1+\gamma)\int_t^T \lambda(s,x)\mathrm{d}s}.$$

In Figures 5.3 and 5.2 we present a comparison between the expansion approximation method and Monte Carlo simulation by computing the foreign survival probabilities (5.4.1) of Italy USD CDS quoted as COB November, 15th, 2011 (Table 5.1) and Italy USD CDS quoted as COB May, 30th 2017 (Table 5.2). The Monte Carlo is performed with 100000 iterations and a confident interval of 95%. As mentioned in the Appendix (7.1.19), the convergence of the method is in the asymptotic sense. Up to four years maturity, the method coincides with the Monte Carlo simulation. After then, we can see that the curves of the survival probabilities (dashed line) start moving away from the Monte Carlo confidence intervals (blue line).

Figure 5.2: SCDS foreign Survival Probabilities of Italy USD CDS quoted as COB November, 15th, 2011

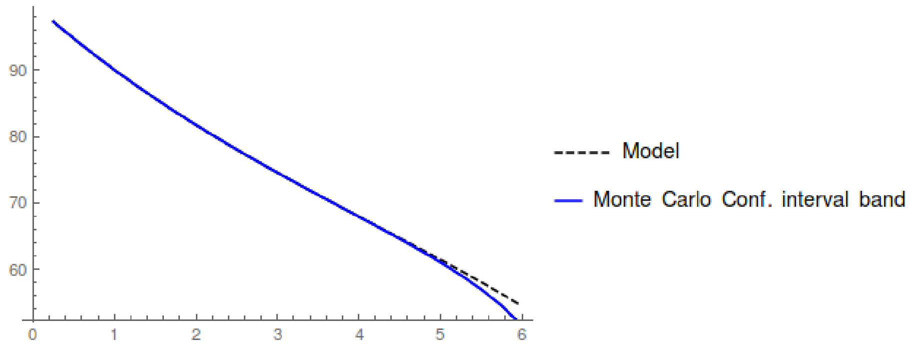
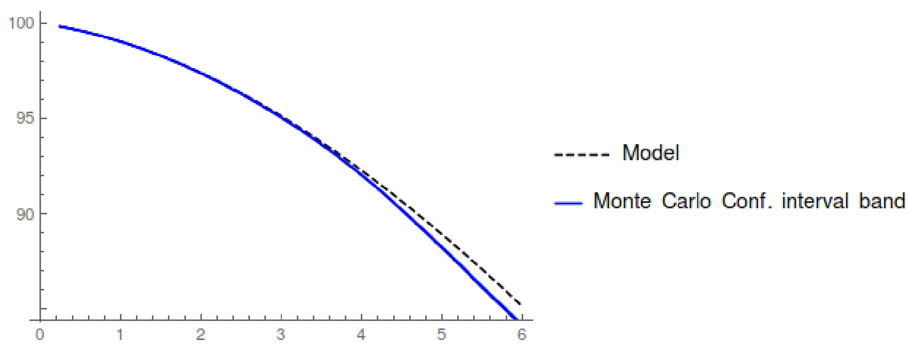


Figure 5.3: SCDS foreign Survival Probabilities of Italy USD CDS quoted as COB May, 30th 2017



To show the accuracy of the method, we present, in the Appendix 7.3, further calibration tests of the model on SCDS of sovereigns belonging to Eurozone (cf. 7.3). In particular, we consider the same dates used for the calibration tests to Italian CDS spreads, and we calibrate our model to French, Spanish and Portuguese USD-quoted CDS spreads.

Chapter 6

Conclusion

The main objectives of this dissertation are to enrich existing models and to provide new calibration and pricing techniques in order to give more realistic descriptions of the financial market in the wake up of the global crisis 2007/2009. With respect to the research questions outlined in the introduction, the following summary gives an overview of the main findings.

In Chapter 3, we have proposed a new methodology for the calibration of a hybrid credit-equity model to credit default swap (CDS) spreads and survival probabilities. We have considered the Jump to Default Constant Elasticity Variance model in a more general framework. More precisely, we assume that the interest rate is stochastic, correlated to the defaultable stock price and can possibly take negative values. In order to calibrate the model, we have derived an approximate analytic expression of the CDS spread (and the survival probability) based on an asymptotic approximation of solutions to parabolic partial differential equations. The numerical test on different corporates have shown that our method gives very good fit to real market data for short maturities, i.e. up to six years maturity. Moreover we have seen this approximation formula give an efficient and fast calibration to the market CDS spreads, in both correlated and uncorrelated models.

In Chapter 4, we have considered the valuation of a non callable and callable defaultable coupon bond where the underlying stochastic factors are the interest rate and the defaultable stock price follow the extended JDCEV model introduced in Chapter 3. In the case of non callable bond, the pricing problem is posed as a sequence of IBVPs. More precisely, two PDE problems with different initial conditions with maturity each coupon payment date need to be solved. Once the numerical solution of these problems is carried out, the value of the bond is computed by means of an expression which also involves the computation of an integral term.

To obtain a numerical solution of the PDE problems, we have proposed appropriate numerical methods based on Lagrange-Galerkin formulations. More precisely, we combine a Crank-Nicolson semi-Lagrangian scheme for time discretization with biquadratic Lagrange finite elements for space discretization. Moreover, the integral term which is involved in the computation of the bond value is approximated by means of the classical composite trapezoidal rule. Finally, we show some numerical results in order to illustrate the behavior of the proposed methods. we obtained very good approximations as shown in Table 4.7 which compare the method to Monte Carlo simulation.

In Chapter 5, we have presented a hybrid sovereign risk model in which the intensity of default of the sovereign is based on the jump to default extended CEV model with a deterministic interest rate. We have assumed that the solvency of a sovereign follows the Jump to Default Constant Elasticity Variance model introduced in [9]. The model captures the interrelationship between creditworthiness of a sovereign, its intensity to default and the correlation with the exchange rate between the bond's currency and the currency in which the CDS spread are quoted. We have analyzed the differences between the default intensity under the domestic and foreign measure and we have computed the default-survival probabilities in the bond's currency measure. We have also given an approximation formula to sovereign CDS spread obtained by using the same technique as in the Chapter 3. The numerical test on Republics of Italy, France, Portugal and Spain have shown that the model gives a very good fit to real market data, both at the outbreak of the government crisis and at the present date.

With regard to future research, a relevant question to address is if these model and CDS approximation give good results for a dynamic CDS quote. Therefore, further empirical research is needed to extend the approximation technique introduced in this dissertation.

Furthermore, it will be highly relevant to apply the numerical method in Chapter 4 on callable defaultable bonds. i.e a bond with an embedded call option. Indeed, instead of parabolic partial differential equations, we would face to obstacle problems. By definition, for a callable bond the issuer preserves the right to call back the bond and pay a fixed price for it. When interest rates decline (bond prices rise) after the initial issue, the firm can refund the bonds at the fixed price instead of the market value. Then, adding the call option to a bond should make it less attractive to buyers, since it reduces the potential upside on the bond. As interest rates go down, and the bond price increases, the bonds are more likely to be called back.

Let us assume that the defaultable coupon bond introduced in Chapter 4 now contains a call option with $R = \{t_c, \cdots, T_M\}$ the set of all possible call dates and $[0, t_c]$ the protection period. At the time of call $t \in R$, the issuer re-purchases the bond from the bondholder for a call price $K(t)$. We denote by Ψ the payoff to the bondholder at maturity T. Let $\mathcal{T}$ be the set of all $\mathbb{F}$-stopping time. We define

$$\mathcal{T}_{t,T} = \{\tau \in \mathcal{T} \colon \tau \in [t, T] \;\; a.s.\}$$

Let $\tau \in \mathcal{T}_{t \vee t_c, T}$ be the time modelling the time of call by the issuer. From [26] one can see that the t-value of the bond assuming that no default occurs prior to t is given by

$$V(\tau; t, S, r) = \mathbb{E}\left[\exp\left(-\int_t^T (r_u + \lambda(u, S_u))\,\mathrm{d}u\right)\Psi(S_T)\,\mathbb{1}_{\{T \leq \tau\}}|\mathcal{F}_t\right]$$

$$
\begin{aligned}
&+ FV \cdot \mathbb{E}\left[\sum_{i=1}^{M} \exp\left(-\int_t^{t_i} (r_u + \lambda(u, S_u))\,\mathrm{d}u\right) c_i \mathbb{1}_{\{t_i<\tau\}} | \mathcal{F}_t\right] \\
&+ \mathbb{E}\left[\exp\left(-\int_t^{\tau} (r_u + \lambda(u, S_u))\,\mathrm{d}u\right) K(\tau)\,\mathbb{1}_{\{\tau\in R\setminus\{\ T\}\}} | \mathcal{F}_t\right] \\
&+ FV \cdot \mathbb{E}\left[\int_t^{\tau} \exp\left(-\int_t^{s} (r_u + \lambda(u, S_u))\,\mathrm{d}u\right) \eta\lambda(s, S_s)\,\mathrm{d}s | \mathcal{F}_t\right]
\end{aligned}
$$

The first term is the present value of the terminal payoff in case of no default and no call prior to maturity. The second term is the present value of the coupons stream, where $c_i \cdot FV$ is received if no default and call occur prior to t_i. The third term is the present value of the payoff if the issuer calls the bond before default and maturity date. The fourth term is the present value of the recovery payment in case of default before call and maturity. We set $C(t) := \sum_{i=1}^{M-1} c_i \cdot FV \cdot \delta(t - t_i)$, where $\delta(x)$ is the Dirac delta function, and

$$
h(t, S_t) = \begin{cases} K(t) \quad \text{if} \ \ t \in R \setminus T, \\ \Psi(S_T) \quad \text{if} \ \ t = T, \\ 0 \text{ otherwise.} \end{cases} \tag{6.0.1}
$$

Then we have

$$
\begin{aligned}
V(\tau; t, S, r) = &\ \mathbb{E}\left[\exp\left(-\int_t^{\tau} (r_u + \lambda(u, S_u))\,\mathrm{d}u\right) h(\tau, S_\tau)\right] \\
&+ \mathbb{E}\left[\int_t^{\tau} \exp\left(-\int_t^{s} (r_u + \lambda(u, S_u))\,\mathrm{d}u\right) C(s)\,\mathrm{d}s\right] \\
&+ Fv \cdot \mathbb{E}\left[\int_t^{\tau} \exp\left(-\int_t^{s} (r_u + \lambda(u, S_u))\,\mathrm{d}u\right) \eta\lambda(s, S_s)\,\mathrm{d}s\right].
\end{aligned}
$$

By setting $f(t, S_t) = -\left(C(t) + \eta \cdot FV \cdot \lambda(t, S_t)\right)$, (6.0.2) become

$$
V(\tau; t, S, r) = \mathbb{E}\left[h(\tau, S_\tau) \exp\left(-\int_t^{\tau} (r_u + \lambda(u, S_u))\,\mathrm{d}u\right) - \int_t^{\tau} f(s, S_s) \exp\left(-\int_t^{s} (r_u + \lambda(u, S_u))\,\mathrm{d}u\right) \mathrm{d}s\right].
$$

The issuer chooses the call time to maximize the bond value V. Hence the t-value $U(t,S,r;T)$ of the callable bond at time t is given by

$$\begin{aligned} U(t,S,r;T) &= \sup_{\tau\in\mathcal{T}_{t\vee c,T}} V(\tau;t,S,r) \qquad (6.0.2)\\ &= \sup_{\tau\in\mathcal{T}_{t\vee c,T}} \mathbb{E}\left[h(\tau,S_\tau)\exp\left(-\int_t^\tau (r_u+\lambda(u,S_u))\,\mathrm{d}u\right)-\int_t^\tau f(s,S_s)\exp\left(-\int_t^s (r_u+\lambda(u,S_u))\,\mathrm{d}u\right)\mathrm{d}s|\mathcal{F}_t\right]. \end{aligned}$$

It can determined in terms of a complementarity problem as follows:

$$\begin{cases} \max\{(\partial_t+\mathcal{A})\,u-f,h-u\}=0 & \text{in } (0,T)\times(0,\infty)\times(-\infty,\infty)\,,\\ u(T,\cdot,\cdot)=h(T,S)\,, & (0,\infty)\times(-\infty,\infty)\,, \end{cases}$$

with a change a variable $r_t=e^{-\kappa t}y$. Moreover, the operator $\mathcal{L}$ is defined as follows

$$\begin{aligned} \mathcal{A}u &= \frac{1}{2}\sigma^2(t,S)\,S^2\partial_{SS}u+\rho\delta\sigma(t,S)\exp(\kappa t)S\partial_{Sy}u+\frac{1}{2}\delta^2\exp(2\kappa t)\partial_{yy}u\\ &+(\exp(-\kappa t)y+\lambda(t,S))\,S\partial_S u+\kappa\theta\exp(\kappa t)\partial_y u-(\exp(-\kappa t)y+\lambda(t,S))\,u. \end{aligned}$$

Remark 6.0.1. In the case of American callable defaultable bond, i.e $R=[0,T]$, the function h in (6.0.1) is defined as

$$h(t,S_t)=\begin{cases} K(t) & \text{if } t\in R\setminus T\\ \Psi(S_T) & \text{if } t=T. \end{cases}$$

Chapter 7

Appendix

7.1 Appendix A: Asymptotic approximation for solution to Cauchy problem

We present an asymptotic approximation method introduce in [37] for solution to Cauchy problem. We consider the following Cauchy problem

$$\begin{cases} (\partial_t + \mathcal{A})\, u\,(t,x) = 0, \quad t \in [0,T),\ x \in \mathbb{R}^d, \\ u\,(T,x) = \varphi\,(x)\,, \quad x \in \mathbb{R}^d, \end{cases} \tag{7.1.1}$$

where $\mathcal{A}$ is the second order elliptic differential operator with variable coefficients

$$\mathcal{A} = \sum_{i,j=1}^{d} a_{ij}\,(t,x)\,\partial_{x_i x_j} + \sum_{i=1}^{d} a_i\,(t,x)\,\partial_{x_i} + a\,(t,x)\,, \quad t \in \mathbb{R},\ x \in \mathbb{R}^d. \tag{7.1.2}$$

For any $nin\mathbb{N}_0$, we denote by $C_b^{n,1}\left(\mathbb{R}^d\right)$ the class of bounded functions with (globally) Lipschitz continuous derivatives of order less than equal to n, and by $||f||_{C_b^{n,1}}$ the sum of the L^∞-norms of the derivatives of f up to order n and the Lipschitz constants of derivatives of order n of f. We also denote by $C_b^{-1,1} = L^\infty$ the class of bounded and measurable functions and set $||\cdot||_{C_b^{-1,1}} = ||\cdot||_{L^\infty}$.

We assume that $\bar{T} > 0$ and $N \in \mathbb{N}_0$ are fixed and the coefficients of the operators $\mathcal{A}$ in (7.1.1) satisfy the following assumption.

Assumption 7.1.1. There exists a positive constant M such that the following hold:

1. Uniformly ellipticity:

$$M^{-1}|\xi|^2 \leq \sum_{i,j=1}^{d} a_{i,j}(t,x)\,\xi_i\xi_j \leq M|\xi|^2, \quad t\int[0,\bar{T}).\ x,\xi \in \mathbb{R}^d.$$

2. Regularity and boundedness: the coefficients $a_{ij},\ a_i,\ a \in C\left([0,\bar{T}] \times \mathbb{R}^d\right)$ and forany $t \in [0,\bar{T}]$ we have $a_{ij}(t,\cdot), a_i(t,\cdot), a(t,\cdot) \in C_b^{N,1}\left(\mathbb{R}^d\right)$ with their $||\cdot||_{C_b^{N,1}}$-norms bounded by M

Under Assumption 7.1.1, for any $T \in]0,\bar{T}]$ and $\varphi in C_b^{-1,1}$, the backward parabolic Cauchy problem (7.1.1) admits a classical solution u that must be computed numerically for practical purposes since the closed-form is unknown.

We rewrite the differential operator (7.1.2) in the more compact form

$$\mathcal{A} = \sum_{|\alpha|} a_\alpha(t,x)\,D_x^\alpha, \quad t\int\mathbb{R},\ x \in \mathbb{R}^d, \tag{7.1.3}$$

where by standard notation

$$\alpha = (\alpha_1 \ldots, \alpha_d), \quad |\alpha| = \sum_{i=1}^{d} \alpha_i, \quad D_x^\alpha = D^\alpha = \partial_{x_1}^{\alpha_1}\cdots\partial_{x_d}^{\alpha_d}.$$

One of the main steps of the method is the expansion schemes of the operator $\mathcal{A}$ based on the expansions schemes of the coefficients $(a_\alpha)_{\alpha\leq 2}$. In what follows we only consider the Taylor polynomial expansion but one could choose the enhanced Taylor polynomial expansion, the time-dependent Taylor polynomial expansion or the Hermite polynomial expansion.

Definition 7.1.2. We say that $(a_{\alpha,n})_{0\leq n\leq N}$ is an N th order polynomial expansion if, for any $t \in [0,\bar{T}]$, the functions $a_{\alpha,n}(t,\cdot)$ are polynomials with $a_{\alpha,0}(t,\cdot) = a_{\alpha,0}(t)$.

Let $(a_{\alpha,n})_{0\le n\le N}$ be an N th order polynomial expansion of a_α. That is

$$a_\alpha(t,x)=\sum_{n=0}^{N}a_{\alpha,n}(t,x)$$

Let us consider a polynomial expansion $(a_{\alpha,n})_{0\le n\le N}$, that is $a_\alpha(t,x)=\sum_{n=0}^{N}a_{\alpha,n}(t,x)$, and let us assume that the operator $\mathcal{A}$ in (7.1.3) can be formally written as

$$\mathcal{A}=\sum_{n=0}^{\infty}\mathcal{A}_n,\qquad \mathcal{A}_n:=\sum_{|\alpha|\le 2}a_{\alpha,n}(t,x)\,D_n^{\alpha}. \tag{7.1.4}$$

We now follow the classical approach and expand the solution u of (7.1.1) as follows:

$$u=\sum_{n=0}^{\infty}u_n \tag{7.1.5}$$

Inserting (7.1.4) and (7.1.5) into (7.1.1), we find that $(u_n)_{n\ge 0}$ satisfy the following sequence of nested Cauchy problems:

$$\begin{cases}(\partial_t+\mathcal{A}_0)\,u_0(t,x)=0, & t\in[0,T),\ x\in\mathbb{R}^d,\\ u_0(T,x)=\varphi(x), & x\in\mathbb{R}^d,\end{cases} \tag{7.1.6}$$

and

$$\begin{cases}(\partial_t+\mathcal{A}_0)\,u_n(t,x)=-\sum_{h=1}^{n}\mathcal{A}_h u_{n-h}(t,x), & t\in[0,T),\ x\in\mathbb{R}^d,\\ u_n(T,x)=0, & x\in\mathbb{R}^d.\end{cases} \tag{7.1.7}$$

Since, by assumption, the function $a_{\alpha,0}$ depend only on t, the operator $\mathcal{A}_0$ is elliptic with time-dependent coefficients. It will be useful to write the operator $\mathcal{A}_0$ in the following form:

$$\mathcal{A}_0=\frac{1}{2}\sum_{i,j=1}^{d}C_{ij}\partial_{x_ix_j}+\langle m(t),\nabla_x\rangle+\gamma(t),\qquad \langle m(t),\nabla_x\rangle=\sum_{i=1}^{d}m_i(t)\,\partial_{x_i}.$$

where the $d\times d$-matrix C is positive definite, uniformly for $t\in[0,T]$, and m and γ are d-dimensional vector and a scalar function, respectively.

It is clear the leading term u_0 in the expansion (7.1.6) is explicitly given by

$$u_0(t,x) = e^{\int_t^T \gamma(s)\mathrm{d}s} \int_{\mathbb{R}^d} \Gamma_0(t,x;T,y)\,\varphi(y)\,\mathrm{d}y, \quad t < T,\ x \in \mathbb{R}^d, \tag{7.1.8}$$

where Γ_0 is the d-dimensional Gaussian density

$$\Gamma_0(t,x;T,y) = \frac{1}{\sqrt{2\pi^d \det C(t,T)}} \exp\left(-\frac{1}{2}\langle C^{-1}(t,T)(y-x-m(t,T)),(y-x-m(t,T))\rangle\right),$$

with covariance matrix $C(t,T)$ and mean vector $z + m(t,T)$ given by

$$C(t,T) = \int_t^T C(s)\mathrm{d}s, \qquad m(t,T) = \int_t^T m(s)\mathrm{d}s. \tag{7.1.9}$$

It turns out that, for any $n \geq 0$, u_n can be computed explicitly, as the following result shows.

Theorem 7.1.3. *For any $n \geq 1$, the nth term u_n in (7.1.7) is given by*

$$u_n(t,x) = \mathcal{L}_n^x(t,T)\,u_0(t,x), \qquad t < T,\ x \in \mathbb{R}^d. \tag{7.1.10}$$

In (7.1.10), $\mathcal{L}_n^x(t,T)$ denotes the differential operator acting on the x-variable and defined as

$$\mathcal{L}_n^x(t,T) = \sum_{h=1}^{n} \int_t^T \mathrm{d}s_1 \int_{s_1}^T \mathrm{d}s_2 \ldots \int_{s_{h-1}}^T \mathrm{d}s_h \sum_{i \in I_{n,h}} \mathcal{G}_{i_1}^x(t,s_1)\ldots\mathcal{G}_{i_h}^x(t,s_h), \tag{7.1.11}$$

where

$$I_{n,h} = \{i = (i_1,\ldots,i_h) \in \mathbb{N}^h \mid i_1 + i_2 + \ldots + i_h = n\} \tag{7.1.12}$$

and the operators $\mathcal{G}_n^x(t,s)$ are defined as

$$\mathcal{G}_n^x(t,s) = \sum_{|\alpha| \leq 2} a_{\alpha,n}(s, \mathcal{M}^x(t,s))\,D_x^\alpha, \tag{7.1.13}$$

with

$$\mathcal{M}^x(t,s) = x + m(t,s) + C(t,s)\nabla_x. \tag{7.1.14}$$

The next proposition and corollary are key for the proof of Theorem 7.1.3.

Proposition 7.1.4. *For any $t < s < T,\ x, y \in \mathbb{R}^d$ and $n \geq 1$, we have*

$$\int_{\mathbb{R}^d} \Gamma_0\left(t,x;s,\xi\right) \mathcal{A}_n^{\xi}\left(s\right) f\left(\xi\right) \mathrm{d}\xi = \mathcal{G}_n^{x}\left(t,s\right) \int_{\mathbb{R}^d} \Gamma_0\left(t,x;s,\xi\right) f\left(\xi\right) \mathrm{d}\xi,$$
$$\int_{\mathbb{R}^d} f\left(\xi\right) \mathcal{A}_n^{\xi}\left(s\right) \Gamma_0\left(t,\xi;T,y\right) \mathrm{d}\xi = \bar{\mathcal{G}}_n^{y}\left(s,T\right) \int_{\mathbb{R}^d} f\left(\xi\right) \Gamma_0\left(s,\xi;T,y\right) \mathrm{d}\xi,$$

for any $f \in C_0^2\left(\mathbb{R}^d\right)$ with

$$\bar{\mathcal{G}}_n^{y}\left(t,s\right) = \sum_{|\alpha| \leq 2} \left(-1\right)^{|\alpha|} D_y^{\alpha} a_{\alpha,n}\left(s, \bar{\mathcal{M}}^{y}\left(t,s\right)\right),$$
$$\bar{\mathcal{M}}^{y}\left(t,s\right) = y - m\left(t,s\right) + C\left(t,s\right) \nabla_y.$$

Furthermore, the following relation holds:

$$\mathcal{G}_n^{x}\left(t,s\right) \Gamma_0\left(t,x;T,y\right) = \bar{\mathcal{G}}_n^{y}\left(s,T\right) \Gamma_0\left(t,x;T,y\right).$$

For the proof cf [29]. □

Corollary 7.1.5. *Let u_0 be as in (7.1.8) with $\gamma = 0$. For any $t < s < T,\ x, y \in \mathbb{R}^d,\ n \geq 1$, we have*

$$\int_{\mathbb{R}^d} \Gamma_0\left(t,x;s,\xi\right) \mathcal{G}_{i_1}^{\xi}\left(s,s_1\right) \cdots \mathcal{G}_{i_n}^{\xi}\left(s,s_n\right) u_0\left(s,\xi\right) = \mathcal{G}_{i_1}^{x}\left(s,s_1\right) \cdots \mathcal{G}_{i_n}^{x}\left(s,s_n\right) u_0\left(t,x\right), \tag{7.1.15}$$

for any $i \in \mathbb{N}^n$ and $s < s_1 < \cdots < s_n < T$.

For the proof cf [29]. □

Proof of Theorem 7.1.3. Proceeding by induction on n, we first prove the case $n = 1$. By definition, u_1 is unique solution of the non-homogeneous Cauchy problem (7.1.7) with $n = 1$. Thus, by Duhamel's principle, we have

$$u_1\left(t,x\right) = \int_t^T \int_{\mathbb{R}^d} \Gamma_0\left(t,x;s,\xi\right) \mathcal{A}_1^{\xi}\left(s\right) u_0\left(s,\xi\right) \mathrm{d}\xi \ \mathrm{d}s$$

$$
\begin{aligned}
&= \int_t^T \mathcal{G}_1^x(t,s) \int_{\mathbb{R}^d} \Gamma_0(t,x;s,\xi)\, u_0(s,\xi)\, \mathrm{d}\xi\ \mathrm{d}s \qquad \text{(by (7.1.16) with } n=1\text{)} \\
&= \int_t^T \mathcal{G}_1^x(t,s) \int_{\mathbb{R}^d} \Gamma_0(t,x;s,\xi) \int_{\mathbb{R}^d} \Gamma_0(t,\xi;T,y)\, \mathrm{d}y\ \mathrm{d}\xi\ \mathrm{d}s \qquad \text{(by (7.1.8))} \\
&= \int_t^T \mathcal{G}_1^x(t,s) \int_{\mathbb{R}^d} \varphi(y) \int_{\mathbb{R}^d} \Gamma_0(t,x;s,\xi)\, \Gamma_0(t,\xi;T,y)\, \mathrm{d}\xi\ \mathrm{d}y\ \mathrm{d}s \qquad \text{Fubini's theorem} \\
&= \int_t^T \mathcal{G}_1^x(t,s)\, \mathrm{d}s\ u_0(t,x) \qquad \text{(Chapman-Kolmogorov and (7.1.8))} \\
&= \mathcal{L}_1^x(t,T)\, u_0(t,x) \qquad \text{(by (7.1.11)} - \text{(7.1.12))}\,.
\end{aligned}
$$

For the general case, let us assume that (7.1.10) holds for $n \geq 1$ and prove it holds for $n+1$. By definition, u_{n+1} is the unique solution of the non-homogeneous Cauchy problem (7.1.7). Thus, by Duhamel's principle, we have

$$
\begin{aligned}
u_{n+1}(t,x) &= \int_t^T \int_{\mathbb{R}^d} \Gamma_0(t,x;s,\xi) \sum_{h=1}^{n+1} \mathcal{A}_h^{\xi}(s)\, u_{n+1-h}(s,\xi)\, \mathrm{d}\xi\ \mathrm{d}s \\
&= \sum_{h=1}^{n+1} \int_t^T \mathcal{G}_h^x(t,s) \int_{\mathbb{R}^d} \Gamma_0(t,x;s,\xi)\, u_{n+1-h}(s,\xi)\, \mathrm{d}\xi\ \mathrm{d}s \qquad \text{(by (7.1.16) with } n=h\text{)} \\
&= \sum_{h=1}^{n+1} \int_t^T \mathcal{G}_h^x(t,s) \int_{\mathbb{R}^d} \Gamma_0(t,x;s,\xi)\, \mathcal{L}_{n+1-h}^{\xi}(s,T)\, u_0(s,\xi)\, \mathrm{d}\xi\ \mathrm{d}s \qquad \text{(by induction hypothesis)}
\end{aligned}
\tag{7.1.16}
$$

Now, by definition (7.1.11)-(7.1.12), we have

$$
\begin{aligned}
&\int_{\mathbb{R}^d} \Gamma_0(t,x;s,\xi)\, \mathcal{L}_{n+1-h}^{\xi}(s,T)\, u_0(s,\xi)\, \mathrm{d}\xi\ \mathrm{d}s \\
&= \sum_{j=1}^{n+1-h} \int_{\mathbb{R}^d} \Gamma_0(t,x;s,\xi) \int_s^T \mathrm{d}s_1 \cdots \int_{s_j-1}^T \mathrm{d}s_j \sum_{i\in I_{n+1-h,j}} \mathcal{G}_{i_1}^{\xi}(s,s_1) \cdots \mathcal{G}_{i_j}^{\xi}(s,s_j)\, u_0(s,\xi)\, \mathrm{d}\xi \\
&= \sum_{j=1}^{n+1-h} \int_s^T \mathrm{d}s_1 \cdots \int_{s_j-1}^T \mathrm{d}s_j \sum_{i\in I_{n+1-h,j}} \int_{\mathbb{R}^d} \Gamma_0(t,x;s,\xi)\, \mathcal{G}_{i_1}^{\xi}(s,s_1) \cdots \mathcal{G}_{i_j}^{\xi}(s,s_j)\, u_0(s,\xi)\, \mathrm{d}\xi \qquad \text{(Fubini's theorem)} \\
&= \sum_{j=1}^{n+1-h} \int_s^T \mathrm{d}s_1 \cdots \int_{s_j-1}^T \mathrm{d}s_j \sum_{i\in I_{n+1-h,j}} \mathcal{G}_{i_1}^{\xi}(s,s_1) \cdots \mathcal{G}_{i_j}^{\xi}(s,s_j)\, u_0(s,\xi)\, u_0(t,x) \qquad \text{(by (7.1.15))}\,.
\end{aligned}
\tag{7.1.17}
$$

Next, by inserting (7.1.16) into (7.1.17), we obtain

$$
u_{n+1}(t,x) = \widetilde{\mathcal{L}}_n^x(t,T)\, u_0(t,x)
$$

where

$$\begin{aligned}\widetilde{\mathcal{L}}_n^x(t,T) &= \int_t^T \mathcal{G}_{n+1}^x(t,s_0)\,\mathrm{d}s_0 \\ &\quad + \sum_{h=1}^{n}\sum_{j=1}^{n+1-h}\int_t^T \mathrm{d}s_0 \int_{s_0}^T \mathrm{d}s_1 \cdots \int_{s_{j-1}}^T \mathrm{d}s_j \sum_{i\in I_{n+1-h,j}} \mathcal{G}_h^x(t,s_0)\,\mathcal{G}_{i_1}^x(t,s_i)\cdots\mathcal{G}_{i_j}^x(t,s_j)\,.\end{aligned}$$

In order to conclude the proof, it is enough to check that $\widetilde{\mathcal{L}}_n^x(t,T) = \mathcal{L}_{n+1}^x(t,T)$. By exchanging the indexes in the sums, we obtain

$$\begin{aligned}\widetilde{\mathcal{L}}_n^x(t,T) &= \int_t^T \mathcal{G}_{n+1}^x(t,s_0)\,\mathrm{d}s_0 \\ &\quad + \sum_{j=1}^{n}\sum_{h=1}^{n+1-j}\int_t^T \mathrm{d}s_0 \int_{s_0}^T \mathrm{d}s_1 \cdots \int_{s_{j-1}}^T \mathrm{d}s_j \sum_{i\in I_{n+1-h,j}} \mathcal{G}_h^x(t,s_0)\,\mathcal{G}_{i_1}^x(t,s_i)\cdots\mathcal{G}_{i_j}^x(t,s_j) \\ &\quad (\text{ settint } l=j+1) \\ &= \int_t^T \mathcal{G}_{n+1}^x(t,s_0)\,\mathrm{d}s_0 \\ &\quad + \sum_{l=2}^{n+1}\sum_{h=1}^{n+2-l}\int_t^T \mathrm{d}s_0 \int_{s_0}^T \mathrm{d}s_1 \cdots \int_{s_{l-2}}^T \mathrm{d}s_{l-1} \sum_{i\in I_{n+1-h,l-1}} \mathcal{G}_h^x(t,s_0)\,\mathcal{G}_{i_1}^x(t,s_i)\cdots\mathcal{G}_{i_{l-1}}^x(t,s_{l-1}) \\ &\quad (\text{replacing the integration variables: } (\mathrm{d}s_0,\mathrm{d}s_1,\cdots,\mathrm{d}s_{l-1}) \mapsto (\mathrm{d}r_1,\mathrm{d}r_2,\cdots,\mathrm{d}r_l)) \\ &= \int_t^T \mathcal{G}_{n+1}^x(t,s_0)\,\mathrm{d}s_0 \\ &\quad + \sum_{l=2}^{n+1}\sum_{h=1}^{n+2-l}\int_t^T \mathrm{d}r_1 \int_{r_1}^T \mathrm{d}r_2 \cdots \int_{r_{l-1}}^T \mathrm{d}r_l \sum_{i\in I_{n+1-h,l-1}} \mathcal{G}_h^x(t,r_1)\,\mathcal{G}_{i_1}^x(t,r_2)\cdots\mathcal{G}_{i_{l-1}}^x(t,r_l) \\ &= \int_t^T \mathcal{G}_{n+1}^x(t,s_0)\,\mathrm{d}s_0 \\ &\quad + \sum_{l=2}^{n+1}\int_t^T \mathrm{d}r_1 \int_{r_1}^T \mathrm{d}r_2 \cdots \int_{r_{l-1}}^T \mathrm{d}r_l \sum_{h=1}^{n+2-l}\sum_{i\in I_{n+1-h,l-1}} \mathcal{G}_h^x(t,r_1)\,\mathcal{G}_{i_1}^x(t,r_2)\cdots\mathcal{G}_{i_{l-1}}^x(t,r_l) \\ &\quad (\text{by definition(7.1.12)}) \\ &= \sum_{l=1}^{n+1}\int_t^T \mathrm{d}r_1 \int_{r_1}^T \mathrm{d}r_2 \cdots \int_{r_{l-1}}^T \mathrm{d}r_l \sum_{Z\in I_{n+1,l}} \mathcal{G}_{z_1}^x(t,r_1)\,\mathcal{G}_{z_2}^x(t,r_2)\cdots\mathcal{G}_{z_l}^x(t,r_l) \\ &\quad (\text{by definition } (7.1.11)) \\ &= \mathcal{L}_{n+1}^x(t,T)\,,\end{aligned}$$

which concludes the proof. $\square$

The second main result consists of local-in-time error bounds for the Nth order Taylor expansion. Let $\bar{x}$ be the expansion point of the Taylor series. We set, for $n \leq N$ and $\bar{x} \in \mathbb{R}^d$,

$$\mathcal{A}_n^{(\bar{x})} = \sum_{|x| \leq 2} a_{\alpha,n}^{(\bar{x})}(t,x) D_x^{\alpha}, \qquad a_{\alpha,n}^{(\bar{x})}(t,x) = \sum_{|\beta|=n} \frac{D^{\beta} a_{\alpha}(t,\bar{x})}{\beta!} (x-\bar{x})^{\beta}$$

The approximation term $u_n = u_n^{(\bar{x})}$ in the expansion (7.1.5) solve

$$\begin{cases} \left(\partial_t + \mathcal{A}_0^{(\bar{x})}\right) u_0^{(\bar{x})}(t,x) = 0, \quad t \in [0,T), \ x \in \mathbb{R}^d, \\ u_0^{(\bar{x})}(T,x) = \varphi(x), \quad x \in \mathbb{R}^d, \end{cases}$$

and for $1 \leq n \leq N$

$$\begin{cases} \left(\partial_t + \mathcal{A}_0^{(\bar{x})}\right) u_n^{(\bar{x})}(t,x) = -\sum_{h=1}^{n} \mathcal{A}_h^{(\bar{x})} u_{n-h}^{(\bar{x})}(t,x), \quad t \in [0,T), \ x \in \mathbb{R}^d, \\ u_n^{(\bar{x})}(T,x) = 0, \quad x \in \mathbb{R}^d. \end{cases}$$

Next, we define the approximate solution at order N for the Taylor expansion centered at $\bar{x}$ as

$$\bar{u}_N^{(\bar{x})}(t,x) := \sum_{n=0}^{N} \bar{u}_n^{(\bar{x})}(t,x). \tag{7.1.18}$$

For the particular choice $\bar{x} = x$, we simply set

$$\bar{u}_N(t,x) := \bar{u}_N^{(\bar{x})}(t,x).$$

We call $\bar{u}_N$ the Nth order Taylor approximation of u. Analogously for the fundamental solution Γ of $(\partial_t + \mathcal{A})$, we set

$$\widetilde{\Gamma}_N(t,x;T,y) = \widetilde{\Gamma}_N^{(\bar{x})}(t,x;T,y).$$

Theorem 7.1.6. *Let Assumption 7.1.1 holds and let $0 < T < \bar{T}$. Assume also the initial datum*

$\varphi \in C_b^{k-1,1}\left(\mathbb{R}^d\right)$ *for some* $0 \le k \le 2$. *Then we have*

$$|u\left(t,x\right) - \bar{u}_N\left(t,x\right)| \le C\left(T-t\right)^{\frac{N+k+1}{2}}, \qquad 0 \le t < T,\ x \in \mathbb{R}^d, \tag{7.1.19}$$

where the constant C *only depends on* $M,\ N,\ \bar{T}$, *and* $||\varphi||_{C_b^{k-1,1}}$. *Moreover, for any* $\varepsilon > 0$ *we have*

$$|\Gamma\left(t,x;T,y\right) - \widetilde{\Gamma}_N\left(t,x;T,y\right)| \ge C\left(T-t\right)^{\frac{N+1}{2}}\Gamma^{M+\varepsilon}\left(t,x;T,y\right), \quad 0 \le t < T,\ x,y \in \mathbb{R}^d, \tag{7.1.20}$$

where $\Gamma^{M+\varepsilon}\left(t,x;T,y\right)$ *is the fundamental solution of the* d*-dimensional heat operator*

$$H^{M+\varepsilon} = \left(M+\varepsilon\right)\sum_{i=1}^{d}\partial_{x_i}^2 + \partial_t, \tag{7.1.21}$$

and C *is a positive constant that depends on* $M,\ N,\ \bar{T},\ \varepsilon$.

The following lemma are key for the proof of Theorem 7.1.6. For the proof, the reader can refer to [29].

Lemma 7.1.7. *For any* $\varepsilon > 0$ *and* β, ν *in* $\mathbb{N}_0^d$ *with* $|\nu| \le N+2$, *we have*

$$|\left(x-y\right)^{\beta} D_x^{\nu}\Gamma\left(t,x;T,y\right)| \le C \cdot \left(T-t\right)^{\frac{|\beta|-|\nu|}{2}}\Gamma^{M+\varepsilon}\left(t,x;T,y\right), \quad 0 \le t < T \le \bar{T},\ x,y \in \mathbb{R}^d,$$

where $\Gamma^{M+\varepsilon}$ *is the fundamental solution of the heat operator* (7.1.21) *and* C *is a positive constant, only depends on* $M, N, \bar{T}, \varepsilon$ *and* $|\beta|$.

Lemma 7.1.8. *Under the hypothesis of Theorem 7.1.6, for any* $\bar{x} \in \mathbb{R}^d$ *and* $N \in \mathbb{N}_0$, *we have*

$$\begin{aligned} u\left(t,x\right) - &\bar{u}_N^{(\bar{x})}\left(t,x\right) \\ &= \int_t^T \int_{\mathbb{R}^d} \Gamma\left(t,x;s,\xi\right)\sum_{n=0}^{N}\left(\mathcal{A} - \bar{\mathcal{A}}_n^{(\bar{x})}\right)u_{N-n}^{(\bar{x})}\left(s,\xi\right)\mathrm{d}\xi\mathrm{d}s, \quad t < T,\ x \in \mathbb{R}^d, \end{aligned}$$

where the function u *is the solution of* (7.1.1), *the function* $\bar{u} : N^{(\bar{x})}$ *is the* N*th order approximation in* (7.1.18), *and*

$$\bar{\mathcal{A}}_n^{(\bar{x})} = \sum_{i=0}^{n}\mathcal{A}_i^{(\bar{x})}.$$

Lemma 7.1.9. *Under the assumptions of Theorem 7.1.6, for any multi-index $\beta \in \mathbb{N}_0^d$, we have*

$$|D_x^\beta u_0^{(\bar{x})}(t,x)| \leq C \cdot (T-t)^{\frac{\min\{k-|\beta|,0\}}{2}}, \quad 0 \leq t < T \leq \bar{T},\ x, \bar{x} \in \mathbb{R}^d. \tag{7.1.22}$$

Moreover, if $N \geq 1$, then for any $n \in \mathbb{N},\ n \leq N$, we have

$$|D_x^\beta u_n^{(\bar{x})}(t,x)| \leq C \cdot (T-t)^{\frac{n+k-|\beta|}{2}} \left(1 + |x-\bar{x}|^n (T-t)^{-\frac{n}{2}}\right), \tag{7.1.23}$$
$$0 \leq t < T \leq \bar{T},\ x, \bar{x} \in \mathbb{R}^d.$$

The constants in (7.1.22) and (7.1.23) depend only on $M, N, \bar{T}, |\beta|$ and $||\varphi||_{C_b^{k-1,1}}$.

Proof of Theorem 7.1.6. In this proof, $\{C_i\}_{i\geq 1}$ denote some positive constants dependent only on $M,\ N,\ \bar{T}$, and $||\varphi||_{C_b^{k-1,1}}$. By Lemma 7.1.8, we have

$$u - \bar{u}_N = \sum_{n=0}^{N} I_n, \quad I_n(t,x) = \int_t^T \int_{\mathbb{R}^d} \Gamma(t,x;s,\xi) \left(\mathcal{A} - \sum_{i=0}^{n} \mathcal{A}_i^x\right) u_{N-n}^x(s,\xi)\, \mathrm{d}\xi \mathrm{d}s.$$

Moreover, $I_n = I_{n,1} + I_{n,2}$ with (cf 6.1)

$$I_{n,1}(t,x) = \sum_{|\alpha|\leq 1} \int_t^T \int_{\mathbb{R}^d} \left(a_\alpha(s,\xi) - \mathbf{T}_{x,n}^{a_\alpha(s,\cdot)}(\xi)\right) \Gamma(t,x;s,\xi)\, D_\xi^\alpha u_{N-n}^x(s,\xi)\, \mathrm{d}\xi \mathrm{d}s,$$
$$I_{n,2}(t,x) = \sum_{|\alpha|=2} \int_t^T \int_{\mathbb{R}^d} \left(a_\alpha(s,\xi) - \mathbf{T}_{x,n}^{a_\alpha(s,\cdot)}(\xi)\right) \Gamma(t,x;s,\xi)\, D_\xi^\alpha u_{N-n}^x(s,\xi)\, \mathrm{d}\xi \mathrm{d}s.$$

Now by Lemma 7.1.9, we have

$$\begin{aligned}
|I_{n,1}(t,x)| &\leq C_1 \sum_{|\alpha|\leq 1} \int_t^T \int_{\mathbb{R}^d} |\xi - x|^{n+1} \Gamma(t,x;s,\xi) (T-s)^{\frac{N-n-|\alpha|+k}{2}} \left(1 + (T-s) - \frac{N-n}{2} |x-\xi|^{N-n}\right) \mathrm{d}\xi \mathrm{d}s \\
&\leq C_2 \sum_{|\alpha|\leq 1} \int_t^T \left((T-s)^{\frac{N-n+|\alpha|+k}{2}} (s-t)^{\frac{n+1}{2}} + (T-s)^{\frac{-|\alpha|+k}{2}} (s-t)^{\frac{N+1}{2}}\right) \int_{\mathbb{R}^d} \Gamma^{M+\varepsilon}(t,x;s,\xi)\, \mathrm{d}\xi \mathrm{d}s \\
&\leq C_3 \cdot (T-t)^{\frac{N+k+2}{2}},
\end{aligned}$$

where we have used Lemma 7.1.7 and the identity

$$\int_t^T (T-s)\ n\,(s-t)^k\,\mathrm{d}s = \frac{\Gamma_E\,(k+1)\,\Gamma_E\,(n+1)}{\Gamma_E\,(k+n+2)}\,(T-t)^{k+n+1}$$

with Γ_E denoting the Euler Gamma function. To estimate $I_{n,2}$, we first integrate by parts and obtain

$$I_{n,2}\,(t,x) = -\sum_{|\alpha_1|=1}\sum_{|\alpha_2|=1}\int_t^T\int_{\mathbb{R}^d} D_\xi^{\alpha_1}$$
$$\left(\left(a_{\alpha_1+\alpha_2}\,(t,\xi) - \mathbf{T}_{x,n}^{a_{\alpha_1+\alpha_2}(t,\cdot)}\,(\xi)\right)\Gamma\,(t,x;s,\xi)\right) D_\xi^{\alpha_2} u_{N-n}^x\,(s,\xi)\,\mathrm{d}\xi\mathrm{d}s.$$

Using the same arguments as above, one can show that

$$|I_{n,2}\,(t,x)\,| \le C_4\cdot(T-t)^{\frac{N+k+1}{2}}\,.$$

Finally, estimating (7.1.20) is obtained by a straightforward modification of the proof of (7.1.19) for $k=0$, by mean of the application of Lemma 6.1 and the Chapman-Kolmogorov equation. We omit the details for simplicity. □

7.2 Appendix B: Further calibration tests I

As said, in this section, we perform more calibration tests on more different real market CDS spreads from large corporates (See Tables 7.1, 7.2, 7.3, 7.4 and 7.5) to confirm the efficiency and robustness of our method.

Table 7.1: Calibration to Caixa Bank SA CDS spreads (Correlated case).

Time to Maturities	Market CDS spread (bps)	Model CDS spread (bps)	Rel. Errors
1.	76.655	76.8783	0.291366 %
1.5	85.1622	83.5574	-1.88441 %
2.	90.115	90.4517	0.373677 %
2.5	96.1837	97.5516	1.42216 %
3.	103.465	104.847	1.33529 %
3.5	111.251	112.325	0.965738 %
4.	120.19	119.976	-0.177871 %
4.5	130.524	127.787	-2.09744 %
5.	139.515	135.743	-2.70359 %
5.5	144.723	143.832	-0.615569 %
6.	147.885	152.039	2.80874 %

$a_1 = 0.029,\ a_2 = 0.1,\ \beta = 0.9,\ b_1 = 0.002,\ b_2 = 0.007,\ c = 0.28,\ \rho = 0.9$

Table 7.2: Calibration to Citigroup Inc CDS spreads (Correlated case).

Time to Maturities	Market CDS spread (bps)	Model CDS spread (bps)	Rel. Errors
1.	23.86	24.0532	0.80977 %
1.5	28.1143	28.2493	0.480115 %
2.	33.45	32.6986	-2.24632 %
2.5	37.9995	37.393	-1.5961 %
3.	42.135	42.3335	0.471192 %
3.5	46.6976	47.5358	1.79501 %
4.	52.165	53.0361	1.66997 %
4.5	58.7451	58.8977	0.259815 %
5.	65.93	65.2171	-1.08133 %
5.5	73.0353	72.1307	-1.23864 %
6.	79.385	79.8203	0.548391 %

$a_1 = 0.01,\ a_2 = 0.04,\ \beta = -2.67,\ b_1 = 0.0006,\ b_2 = 0.0,\ c = 1.9,\ \rho = 0.9$

Table 7.3: Calibration to Commerzbank AG CDS spreads (Correlated case).

Time to Maturities	Market CDS spread (bps)	Model CDS spread (bps)	Rel. Errors
1.	44.69	44.7819	0.205608 %
1.5	53.9328	54.0445	0.207137 %
2.	63.175	63.0747	-0.158831 %
2.5	71.8376	71.8844	0.0651909 %
3.	80.285	80.4911	0.256673 %
3.5	88.977	88.9161	-0.0684516 %
4.	97.81	97.1835	-0.640533 %
4.5	106.475	105.319	-1.08584 %
5.	114.405	113.349	-0.923244 %
5.5	121.117	121.298	0.149436 %
6.	126.73	129.192	1.94273 %

$a_1 = -0.002,\ a_2 = 0.08,\ \beta = -2.7,\ b_1 = 0.005,\ b_2 = 0.0,\ c = 0.6,\ \rho = 0.6$

Table 7.4: Calibration to Mediobanca SpA spreads (Correlated case).

Time to Maturities	Market CDS spread (bps)	Model CDS spread (bps)	Rel. Errors
1.	87.545	87.4557	-0.101987 %
1.5	96.831	96.8623	0.0323408 %
2.	106.715	106.688	-0.0252551 %
2.5	116.657	116.678	0.0176859 %
3.	126.405	126.621	0.170711 %
3.5	135.88	136.343	0.340403 %
4.	145.42	145.703	0.19438 %
4.5	155.159	154.586	-0.369086 %
5.	164.01	162.903	-0.674879 %
5.5	170.956	170.583	-0.218563 %
6.	176.485	177.572	0.615901 %

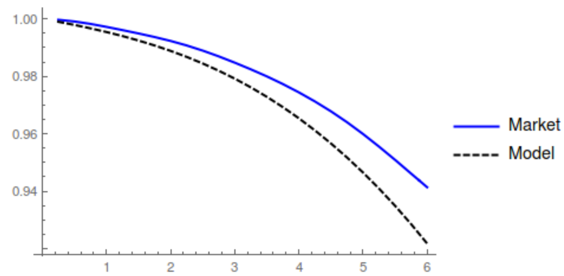
$a_1 = -0.04,\ a_2 = 0.93,\ \beta = -1.7,\ b_1 = -0.002,\ b_2 = 0.01,\ c = 0.0004,\ \rho = 0.82$

Table 7.5: Calibration to Deutsche Bank AG spreads (Correlated case).

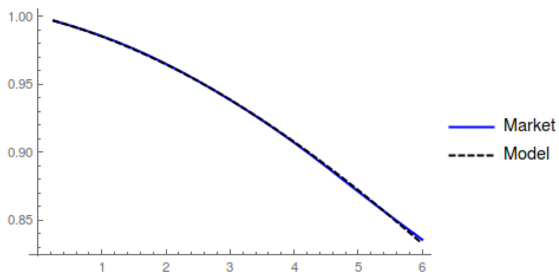
Time to Maturities	Market CDS spread (bps)	Model CDS spread (bps)	Rel. Errors
1.	67.02	66.42	-0.895209 %
1.5	77.7864	79.1848	1.79767 %
2.	92.015	92.5289	0.558545 %
2.5	107.233	105.796	-1.33972 %
3.	120.505	118.488	-1.67348 %
3.5	129.985	130.235	0.192996 %
4.	138.645	140.774	1.53567 %
4.5	149.244	149.928	0.458061 %
5.	158.86	157.592	-0.798277 %
5.5	164.376	163.718	-0.400654 %
6.	167.59	168.305	0.426501 %

$a_1 = -0.1,\ a_2 = 1.55,\ \beta = -0.08,\ b_1 = -0.001,\ b_2 = 0.006,\ c = 0.0003,\ \rho = 0.83$

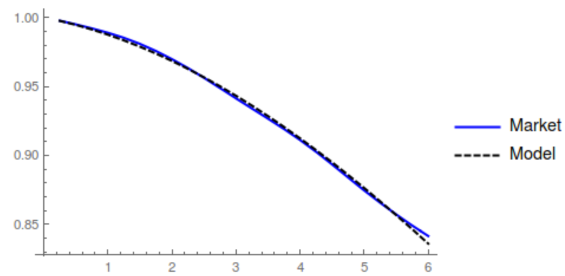
In tables— , in compute the survival probabilities for each CDS and we compare them to those obtained by bootstrapping and used in the real market. One can see that our model give a very good fit to the real market data and with the calibrated parameters one obtains survival probabilities close to the real market ones.



(a) Citigroup Inc Survival Probability



(b) Mediobanca SpA Survival Probability



(c) Deutsche Bank Survival Probability

Figure 7.1: Dependence of the error on the maturity

7.3 Appendix C: Further calibration tests II

Table 7.6: Calibration to France USD CDS quoted as COB November, 15th, 2011.

Time to Maturities (Year)	Market CDS spread (bps)	Model CDS spread (bps)	Rel. Errors
1.25	153.882	153.479	-0.261987 %
1.5	158.258	159.02	0.481714 %
1.75	164.035	164.579	0.331552 %
2.	170.14	170.156	0.00938846 %
2.25	175.829	175.751	-0.0445177 %
2.5	181.045	181.363	0.175564 %
2.75	186.464	186.993	0.283517 %
3.	192.24	192.64	0.208105 %
3.25	198.586	198.304	-0.141846 %
3.5	205.021	203.986	-0.504832 %
3.75	211.605	209.684	-0.907797 %
4.	217.74	215.397	-1.07585 %
4.25	223.104	221.127	-0.886165 %
4.5	227.526	226.872	-0.287417 %
4.75	231.179	232.631	0.627802 %
5.	233.936	238.404	1.90985 %

$a_1 = -0.03,\ a_2 = 0.009,\ \beta = 0.6,\ b_1 = -0.004,\ b_2 = 0.01,\ c = 0.14,\ \eta = 0.08\ \rho = -0.6,\ \gamma = 0.84,\ 52.428\ seconds$

Table 7.7: Calibration to France USD CDS quoted as COB May, 30th 2017.

Time to Maturities (Year)	Market CDS spread (bps)	Model CDS spread (bps)	Rel. Errors
1.25	5.01852	4.98324	-0.702846 %
1.5	6.04391	6.12876	1.40401 %
1.75	7.28556	7.32835	0.587229 %
2.	8.64	8.58375	-0.651094 %
2.25	9.99688	9.89661	-1.00297 %
2.5	11.3472	11.2685	-0.693951 %
2.75	12.7118	12.7007	-0.0867337 %
3.	14.14	14.1946	0.386329 %
3.25	15.6773	15.7512	0.471642 %
3.5	17.3094	17.3713	0.357688 %
3.75	18.9923	19.0557	0.333576 %
4.	20.775	20.8046	0.142628 %
4.25	22.651	22.6184	-0.144168 %
4.5	24.5785	24.4968	-0.332471 %
4.75	26.4888	26.4395	-0.185999 %
5.	28.42	28.4459	0.0910832 %

$a_1 = 0.03,\ a_2 = 0.3,\ \beta = 0.79,\ b_1 = 0.0003,\ b_2 = 0.0,\ c = 0.5,\ \eta = 0.4,\ \rho = 0.07,\ \gamma = 0.7,\ 86.056\ seconds$

Table 7.8: Calibration to Portugal USD CDS quoted as COB November, 15th, 2011.

Time to Maturities (Year)	Market CDS spread (bps)	Model CDS spread (bps)	Rel. Errors
1.25	1528.93	1543.47	0.951086 %
1.5	1566.41	1562.36	-0.258228 %
1.75	1582.78	1570.83	-0.754834 %
2.	1579.19	1567.94	-0.712081 %
2.25	1558.57	1553.04	-0.354768 %
2.5	1525.58	1525.99	0.0268333 %
2.75	1481.35	1487.44	0.411513 %
3.	1430.24	1439.13	0.621161 %
3.25	1375.66	1383.93	0.601337 %
3.5	1322.47	1325.79	0.251204 %
3.75	1270.06	1269.29	-0.060548 %
4.	1223.28	1218.89	-0.359159 %
4.25	1184.4	1177.92	-0.547237 %
4.5	1153.2	1147.57	-0.487785 %
4.75	1126.69	1126.17	-0.0466888 %
5.	1104.02	1109.19	0.467984 %

$a_1 = 0.012,\ a_2 = 0.1,\ \beta = -0.64,\ b_1 = 0.05,\ b_2 = 0.1,\ c = 10.7,\ \eta = 0.013\ \rho = -0.66,\ \gamma = 0.99,\ 96.196\ seconds$

Table 7.9: Calibration to Portugal USD CDS quoted as COB May, 30th 2017.

Time to Maturities (Year)	Market CDS spread (bps)	Model CDS spread (bps)	Rel. Errors
1.25	55.7676	56.6391	1.56276 %
1.5	67.7586	68.2853	0.777373 %
1.75	80.3718	79.9073	-0.57795 %
2.	92.975	91.4484	-1.64194 %
2.25	104.727	102.853	-1.78851 %
2.5	115.529	114.07	-1.2627 %
2.75	125.543	125.049	-0.393047 %
3.	135.15	135.747	0.441821 %
3.25	144.688	146.124	0.992659 %
3.5	154.187	156.147	1.27133 %
3.75	163.548	165.787	1.36936 %
4.	173.19	175.025	1.05927 %
4.25	183.136	183.842	0.385523 %
4.5	192.933	192.231	-0.364032 %
4.75	201.949	200.186	-0.873256 %
5.	210.07	207.707	-1.12481 %

$a_1 = 0.015,\ a_2 = 0.5,\ \beta = 0.7,\ b_1 = 0.0,\ b_2 = 0.0,\ c = 0.53,\ \eta = 0.78,\ \rho = -0.65,\ \gamma = 0.7,\ 79.192\ \textit{seconds}$

Table 7.10: Calibration to Spain USD CDS quoted as COB November, 15th, 2011.

Time to Maturities (Year)	Market CDS spread (bps)	Model CDS spread (bps)	Rel. Errors
1.25	443.089	447.5	0.995382 %
1.5	455.942	452.898	-0.667565 %
1.75	461.191	457.574	-0.784297 %
2.	462.81	461.585	-0.264677 %
2.25	464.557	464.99	0.0931244 %
2.5	466.775	467.842	0.228544 %
2.75	469.216	470.197	0.209013 %
3.	471.4	472.106	0.149761 %
3.25	473.03	473.626	0.125969 %
3.5	474.222	474.819	0.125775 %
3.75	475.314	475.759	0.0935208 %
4.	476.525	476.544	0.0040035 %
4.25	478.011	477.313	-0.146031 %
4.5	479.465	478.267	-0.24985 %
4.75	480.572	479.705	-0.180317 %
5.	480.911	482.072	0.24148 %

$a_1 = -0.06,\ a_2 = 0.44,\ \beta = 0.52,\ b_1 = -0.002,\ b_2 = 0.044,\ c = 0.4,\ \eta = 0.03\ \rho = -0.6,\ \gamma = 0.5,\ 58.548\ \textit{seconds}$

Table 7.11: Calibration to Spain USD CDS quoted as COB May, 30th 2017.

Time to Maturities (Year)	Market CDS spread (bps)	Model CDS spread (bps)	Rel. Errors
1.25	33.5073	33.4062	-0.301704 %
1.5	36.6139	36.5987	-0.0413123 %
1.75	39.678	39.7374	0.149748 %
2.	42.74	42.8222	0.192309 %
2.25	45.7535	45.8532	0.218024 %
2.5	48.7076	48.8305	0.252401 %
2.75	51.6109	51.7541	0.277466 %
3.	54.535	54.624	0.163207 %
3.25	57.5301	57.4403	-0.156104 %
3.5	60.4933	60.2031	-0.479832 %
3.75	63.2717	62.9123	-0.56793 %
4.	65.87	65.5682	-0.458181 %
4.25	68.2415	68.1708	-0.103697 %
4.5	70.5227	70.7201	0.279874 %
4.75	72.8767	73.2164	0.466098 %
5.	75.59	75.6597	0.0922325 %

$a_1 = -0.014,\ a_2 = 0.073,\ \beta = 0.62,\ b_1 = 0.002,\ b_2 = 0.001,\ c = 0.28,\ \eta = 0.6,\ \rho = -0.38,\ \gamma = 0.96,\ 63.304\ seconds$

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