Portfolio risk and the quantum majorization of correlation matrices

Andrea Fontanari*
Delft University of Technology and CWI Amsterdam

Iddo Eliazar
Independent Researcher

Pasquale Cirillo
Delft University of Technology

Cornelis W. Oosterlee
CWI Amsterdam and Delft University of Technology

Abstract

We propose quantum majorization as a way of comparing and ranking correlation matrices, with the aim of assessing portfolio risk in a unified framework. Quantum majorization is a partial order in the space of correlation matrices, which are evaluated through their spectra.

We discuss the connections between quantum majorization and an important class of risk functionals, and we define two new risk measures able to capture interesting characteristics of portfolio risk.

1 Introduction

Consider a portfolio $\mathcal{P}$ containing n assets. A common way to deal with the dependence structure of $\mathcal{P}$–and thus with portfolio risk–is to consider its $n \times n$ correlation matrix $\mathbf{C}$ [6].

We refer to the correlation matrix in the most general sense, as a real, symmetric, positive semidefinite, Hermitian matrix, whose entries are correlation coefficients according to some definition, all lying in the interval $[-1, 1]$, with all ones on the main diagonal.

The approach here proposed is indeed applicable to all correlation matrices, and not only to the classical Pearson's correlation matrix: one can take into consideration matrices based on Spearman's ρ, Kendall τ, Gini correlation, etc. [13, 25, 38, 43]. This allows to deal with more general forms of dependence, beyond the standard linear one of Pearson's correlation [23].

The main idea of this work is to find a way of comparing correlation matrices, and of extracting the portfolio risk information they contain, so that they can be ranked. The comparison can be over time, if one studies the evolution of the correlation matrix for a given portfolio, but it can also be cross-sectional, comparing different portfolios at the same time, for example to look for the one minimizing portfolio risk. The only requirement is that the size of the matrices is the same, i.e. only portfolios having the same number of assets are considered.

Our proposal is to use an ordering, developed in the field of quantum mechanics [1, 33, 32], called quantum majorization, to study the dynamics of the entropy of a quantum system, applying it to correlation matrices. To the best of our knowledge, this is the first time such an ordering is used in finance.

Quantum majorization is a partial order to rank matrices looking at their eigenvalues. The use of eigensystems to study multivariate dependence is not at all new [23], but we show how the

*Corresponding Author: Andrea Fontanari, Applied Probability Group, EEMCS Faculty, Delft University of Technology, Building 28, Van Mourik Broekmanweg 6, 2628 XE Delft, The Netherlands. Phone:+31.152.782.589. Mail: A.Fontanari@tudelft.nl

spectrum of a correlation matrix can be used to capture relevant features of portfolio risk in a brand new way.

We introduce the $\mathcal{M}_\lambda$ class of risk functionals, which are isotonic to quantum majorization, and whose aim is to capture the (monotonic) dependence embedded into portfolio correlation matrices. An important property of such a class, stated in Proposition 2, is that under quantum majorization, i.e. when it is possible to rank correlation matrices according to the order introduced in Definition 3, all risk functionals in $\mathcal{M}_\lambda$ are comonotonic. The implication is that, if we are able to identify majorization, then the choice of the risk functional becomes secondary, as they will all behave in the same way, indicating an increase (or a decrease) of portfolio risk. It is when the ordering does not hold–as we shall see–that risk functionals may give inconsistent information.

With respect to single risk measures, quantum majorization thus provides a stronger characterization of risk and dependence among correlation matrices. We are therefore able to provide a unifying approach to the analysis of portfolio risk and correlation: we introduce several tools, we discuss their properties, and we show how to use them in practice. In doing so, we will avoid all unnecessary sophistication, giving space to financial interpretability and usability.

The paper is organized as follows: Section 2 introduces the concept of quantum majorization for correlation matrices; Section 3 deals with the $\mathcal{M}_\lambda$ class of isotonic risk measures, analyzing its properties, and discussing its link with quantum majorization; Section 4 introduces the empirical majorization matrix approach as a fully data-driven methodology to deal with portfolio risk; Section 5 is devoted to an application to empirical data related to the Industrial Dow Jones, in which we show how to use the tools introduced in the previous sections; finally Section 6 builds a connection between our approach and network analysis, which can open the path for future research.

2 The quantum majorization of correlation matrices

The aim is to deal with portfolio risk, as represented by correlation matrices. We thus look for an order relation allowing us to compare and rank them. To guarantee financial applicability and interpretability, such an order should respect the following conditions:

- C1 *Minimal Element*: According to the order, the least risky element among all correlation matrices should be the identity matrix;
- C2 *Maximal Element(s)*: The riskiest elements among all correlation matrices should be the set of the matrices of rank 1;
- C3 *Monotonocity*: The order should not increase in the rank of the correlation matrices;
- C4 *Convexity*: The correlation matrix obtained as convex combination of two correlation matrices should not be riskier than the convex combination of the two original ones.

Conditions C1 and C2 fix the two extremes of the ordered set, i.e. the minimal and the maximal elements. C1 identifies the case of no correlation (the identity matrix) as the least risky one; while C2 finds the riskiest situation in a portfolio whose assets are all monotonic transformations of one of them (comonotonicity and countermonotonicity), therefore all correlation matrices of rank 1. From a portfolio risk prospective, we can think about the propagation of market shocks: if assets in our portfolio are completely uncorrelated, shocks on one of them do not propagate to any of the others; while in a comonotonic (countermonotonic) portfolio, shocks on one single asset affect all the others, possibly increasing the overall risk.

Notice that while the minimal element is necessarily unique (the identity), the maximal one corresponds to a set of correlation matrices, given that in a comonotonic portfolio all assets can represent the leading term.

Condition C3 implies that, given two portfolios $\mathcal{P}_1$ and $\mathcal{P}_2$, such that $\mathcal{P}_1$ has more monotonically dependent assets than $\mathcal{P}_2$ (hence its correlation matrix a smaller rank), then the risk associated to $\mathcal{P}_1$ should never be lower than that of $\mathcal{P}_2$.

Finally, condition C4 reflects the usual financial postulate that diversification decreases risk [7]. While the condition is stated for correlation matrices, the object of our analysis, in a sense we are requiring that, by taking convex combinations of portfolios–not necessarily uncorrelated, the overall risk should decrease.

Studying the available literature, a powerful partial order, compatible with our conditions C1-C4, and thus useful to account for portfolio risk, was introduced in the field of quantum mechanics by Alberti and Uhlmann [1], and further analyzed by Ando [2]. This order, here called quantum majorization, was developed to study entropy increasing dynamics on density matrices, and it relies on the spectra (the vectors containing the eigenvalues) of the matrices under scrutiny, which are required to be Hermitian and with equal trace.

Being real symmetric, a correlation matrix is a special type of Hermitian matrix, and its trace is equal to the number of assets the matrix represents: n. In other words, the sum of the eigenvalues of a correlation matrix–equal to its trace–is nothing but the number of elements it deals with. Therefore, if we consider two different $n \times n$ correlation matrices, we know for sure that their spectra will sum to the same value n. This apparently simple property justifies the use of quantum majorization on correlation matrices, and not on variance-covariance ones.

It is important to stress that, in introducing their order, Alberti and Uhlmann [1] did not consider the C1-C4 conditions stated above, which are just relevant to us in terms of portfolio risk[1].

The conditions above can be easily restated in terms of the spectra of correlation matrices, a trick that will prove essential to deal with quantum majorization as per Theorem 1 below.

As stressed later, this "eigenrepresentation" of the axiomatic conditions also bridges towards the spectral study of random matrices [5, 35], and classical multivariate analysis à la Wilks [23], further supporting the approach we are proposing.

For instance, for a portfolio containing n uncorrelated assets, C1 requires the spectrum of the correlation matrix to be the n-dimensional vector of ones $\boldsymbol{\lambda} = [1, ..., 1]$. An $n \times n$ identity matrix (the minimal element according to C1) has indeed a single eigenvalue equal to 1, with an algebraic multiplicity of n. On the opposite side, C2 requires the spectrum of the correlation matrix of a comonotonic/countermonotonic portfolio to be equal to $\boldsymbol{\lambda} = [n, 0, ..., 0]$, which is the case of an $n \times n$ matrix of rank 1.

Before formally introducing quantum majorization for correlation matrices, we first need some basic results from majorization theory [3], an important field of linear algebra and order theory.

Introduced by the works of Polya, Hardly, Littlewood, Dalton, Muirhead and Schur [22, 34], majorization is a way to define a partial order in the space of vectors in $\mathbb{R}^n$, ranking them in terms of their intrinsic variability, i.e. how scattered they are with respect to their common vector of averages. Given a vector $\mathbf{x} = [x_1, x_2, ..., x_n] \in \mathbb{R}^n$, such that $\sum_{i=1}^n x_i = d$, the vector of averages is defined as

$$\bar{\mathbf{x}} = \left[\frac{\sum_{i=1}^n x_i}{n}, ..., \frac{\sum_{i=1}^n x_i}{n} \right] = \left[\frac{d}{n}, ..., \frac{d}{n} \right],$$

that is the n-dimensional vector whose entries correspond to the average of $\mathbf{x}$.

Definition 1 (Majorization of vectors)**.** Take two vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. We say that $\mathbf{x}$ majorizes $\mathbf{y}$, in symbols $\mathbf{x} \succ \mathbf{y}$, if

$$\sum_{i=1}^n x_i = \sum_{i=1}^n y_i$$

and

$$\sum_{i=1}^k x_{[i]} \geq \sum_{i=1}^k y_{[i]}, \quad \text{for all } k = \{1, ..., n-1\}, \tag{1}$$

where $x_{[1]}, ..., x_{[n]}$ are the coordinates of the vector $\mathbf{x}$ sorted in descending order, so that $x_{[1]} \geq x_{[2]} \geq ... \geq x_{[n]}$. If the condition $\sum_{i=1}^n x_i = \sum_{i=1}^n y_i$ is not satisfied, we speak of weak majorization, $\mathbf{x} \succ_w \mathbf{y}$.

Being a partial order [3],it may not be possible to verify the majorization between $\mathbf{x}$ and $\mathbf{y}$, as per Definition 1. In these cases, we write $\mathbf{x} \not\succ \mathbf{y}$. The fact that it is a partial order, however, guarantees that majorization respects the transitivity property: if $\mathbf{x} \succ \mathbf{y}$ and $\mathbf{y} \succ \mathbf{z}$, then $\mathbf{x} \succ \mathbf{z}$.

Strictly related to majorization, and fundamental for our work, is the class of Schur-convex or isotonic functions. These functions have indeed the property of preserving majorization when applied.

[1]A successful alternative to rank correlation matrices with conditions similar to C1-C4 has been developed by Giovagnoli and Romanazzi [20], using a special type of G-majorization [21]. However we believe that their approach, despite being theoretically fascinating, it is difficult to use in practice, especially for risk management purposes. Moreover, one can prove that the order we consider here is richer, as it orders at least as many elements as Giovagnoli and Romanazzi's one.

Definition 2 (Schur-convex function)**.** Let ϕ be a real valued function defined on $\mathbb{R}^n$, ϕ is Schur-convex if, whenever $\mathbf{x} \succ \mathbf{y}$, then $\phi(\mathbf{x}) \geq \phi(\mathbf{y})$. When the inequality is strict, we speak of strictly Schur-convex function.

If $\mathbf{x} \succ \mathbf{y}$ and $\phi(\mathbf{x}) \leq \phi(\mathbf{y})$ we call ϕ Schur-concave, such that $-\phi$ is Schur-convex.

Schur-convex functions can thus be seen as summary measures of the variability of a vector, when variability is defined in terms of majorization. Interestingly, several quantities commonly used in statistics and science to represent variability are Schur-convex: the variance, the coefficient of variation, the entropy, the arithmetic and the geometric means, the mean absolute deviation and inequality indices like the Gini and the Pietra [22]. These common measures of variability are therefore nothing more than functions naturally related to the concept of majorization, as we shall also see in the multivariate framework.

Let $\mathcal{C}$ be the class of correlation matrices of a given size, say $n \times n$. The following definition introduces what we will call quantum majorization.

Definition 3 (Quantum Majorization)**.** Consider two correlation matrices $\mathbf{C}_1, \mathbf{C}_2 \in \mathcal{C}$, and denote their spectra by $\boldsymbol{\lambda}(\mathbf{C}_1)$ and $\boldsymbol{\lambda}(\mathbf{C}_2)$ respectively. We say that $\mathbf{C}_1$ quantum majorizes $\mathbf{C}_2$, i.e. $\mathbf{C}_1 \overset{\lambda}{\succ} \mathbf{C}_2$, if the spectrum of $\mathbf{C}_1$ majorizes the spectrum of $\mathbf{C}_2$, as per Definition 1, that is $\boldsymbol{\lambda}(\mathbf{C}_1) \succ \boldsymbol{\lambda}(\mathbf{C}_2)$.

It is important to remark that Definition 3 induces a partial order in the spectra of the correlation matrices, but only an equivalence relation on the space of correlation matrices themselves. In fact, if $\mathbf{C}_1 \overset{\lambda}{\succ} \mathbf{C}_2$ and $\mathbf{C}_2 \overset{\lambda}{\succ} \mathbf{C}_1$, we can conclude that $\boldsymbol{\lambda}(\mathbf{C}_1) = \boldsymbol{\lambda}(\mathbf{C}_2)$, that is to say that $\mathbf{C}_1$ and $\mathbf{C}_2$ are similar. This, in terms of risk, implies that $\mathbf{C}_1$ and $\mathbf{C}_2$ have the same riskiness. As an example, just consider all the rank 1 matrices involved in condition C2.

Theorem 1 below shows that the majorization[2] of Definition 3 satisfies conditions C1-C4, and it is therefore suitable to build comparisons among portfolio correlation matrices, providing a bridge between quantum mechanics and finance.

Theorem 1. The partial order of Definition 3 is consistent with the requirements of minimal and maximal element, monotonicity and convexity of conditions C1-C4.

Proof. The first two requirements are easily checked. Their proof follows from the definition of quantum majorization as majorization of the eigenvalues of the correlation matrix $\mathbf{C}$, and by noticing that all correlation matrices of size $n \times n$ have trace equal to $n = \sum_{i=1}^{n} \lambda_i(\mathbf{C})$.

Since every correlation matrix is positive semi-definite, we have that its eigenvalues are non-negative, so that $\boldsymbol{\lambda}(\mathbf{C}) \in \mathbb{R}^n_+$. Therefore, using a result for vector majorization (Proposition C1 page 192 in [34]), we have that

$$\boldsymbol{\lambda}_{\max}(\mathbf{C}) \overset{\lambda}{\succ} \boldsymbol{\lambda}(\mathbf{C}) \overset{\lambda}{\succ} \boldsymbol{\lambda}_{\min}(\mathbf{C}),$$

where $\boldsymbol{\lambda}_{\max}(\mathbf{C}) = [n, 0, ..., 0]$ and $\boldsymbol{\lambda}_{\min}(\mathbf{C}) = [1, 1, ..., 1]$.

To verify C1, notice that the only diagonalizable matrix with spectrum equal to $\boldsymbol{\lambda}_{\min}(\mathbf{C})$ is the identity matrix. For C2, observe that the only matrices with spectrum equal to $\boldsymbol{\lambda}_{\max}(\mathbf{C})$ are those of rank 1.

To prove C3 recall that, given two correlation matrices $\mathbf{C}_1$ and $\mathbf{C}_2$ of equal size, if $Rank(\mathbf{C}_1) \geq Rank(\mathbf{C}_2)$ then $\mathbf{C}_1$ must have a smaller number of zeros in its spectrum. Therefore, by applying Definition 1, we have that $\sum_{i=1}^{Rank(\mathbf{C}_2)} \lambda_{[i]}(\mathbf{C}_2) > \sum_{i=1}^{Rank(\mathbf{C}_2)} \lambda_{[i]}(\mathbf{C}_1)$, meaning that $\boldsymbol{\lambda}(\mathbf{C}_1)$ can never majorize $\boldsymbol{\lambda}(\mathbf{C}_2)$, thus verifying C3.

To check C4, we want to verify that, given two correlation matrices $\mathbf{C}_1$ and $\mathbf{C}_2$ of equal size, and a number $\alpha \in [0, 1]$, we have

$$\sum_{i=1}^{k} \lambda_{[i]}(\mathbf{C}) \leq \alpha \sum_{i=1}^{k} \lambda_{[i]}(\mathbf{C}_1) + (1 - \alpha) \sum_{i=1}^{k} \lambda_{[i]}(\mathbf{C}_2),$$

for every $k = 1, ..., n$, where $\mathbf{C} = \alpha \mathbf{C}_1 + (1 - \alpha)\mathbf{C}_2$.

[2] From now on, to avoid excessive repetitions, we just speak of majorization for both vectors and matrices. Naturally, when dealing with matrices we will refer to Definition 3, while for vectors we will imply Definition 1.

By the Fan representation theorem [15], we have that $\max_{\mathbf{U}\mathbf{U}^T=\mathbf{I}_k} Tr(\mathbf{U}\mathbf{C}\mathbf{U}^T) = \sum_{i=1}^k \lambda_{[i]}(\mathbf{C})$, with $\mathbf{U}$ a $k \times n$ unitary matrix and $\mathbf{I}_k$ the $k \times k$ identity. By the linearity of the trace operator and the subadditivity of the max, we then have

$$\max_{\mathbf{U}\mathbf{U}^T=\mathbf{I}_k} Tr(\mathbf{U}\mathbf{C}\mathbf{U}^T) \leq \alpha \max_{\mathbf{U}\mathbf{U}^T=\mathbf{I}_k} Tr(\mathbf{U}\mathbf{C}_1\mathbf{U}^T) + (1-\alpha) \max_{\mathbf{U}\mathbf{U}^T=\mathbf{I}_k} Tr(\mathbf{U}\mathbf{C}_2\mathbf{U}^T).$$

By applying the Fan representation theorem on both sides of the inequality above the desired result is obtained. □

The following proposition underlines an important connection between quantum majorization and the correlation matrices on which it is verified, providing a strong argument in favor of the use of majorization in studying portfolio risk.

Proposition 1. Take two correlation matrices $\mathbf{C}_1, \mathbf{C}_2 \in \mathcal{C}$. If $\mathbf{C}_1 \overset{\lambda}{\succ} \mathbf{C}_2$, then

$$\mathbf{C}_2 = \sum_{i=1}^{K} \alpha_i \mathbf{U}_i \mathbf{C}_1 \mathbf{U}_i^T \tag{2}$$

where $(\alpha_i)_{i=1}^K$ is a vector of probabilities summing to 1 and $(\mathbf{U}_i)_{i=1}^K$ is a collection of unitary matrices.

A general proof of the proposition can be found in [2, 33], not only for correlation matrices. What is relevant to us is that Equation (2) builds a connection between majorization and the related correlations. In fact, if $\mathbf{C}_1$ majorizes $\mathbf{C}_2$, then the latter can be expressed in terms of the former, expliciting the underlying dependence. In the context of portfolio risk, this tells us that quantum majorization unveils a deep and powerful link among correlation matrices, usually not visible by only looking at the traditional risk measures.

3 The $\mathcal{M}_\lambda$ class of monotonic portfolio risk measures

Now that we have identified the quantum majorization of Definition 3 as a proper (risk) order in the space of correlation matrices, we are ready to quantify the amount of portfolio risk embedded in a given correlation matrix. To tackle this problem, we introduce a special class of matrix risk functionals, the $\mathcal{M}_\lambda$ class.

Definition 4 ($\mathcal{M}_\lambda$ class)**.** Given the set of $n \times n$ correlation matrices $\mathcal{C}$, the class $\mathcal{M}_\lambda$ contains all the $\overset{\lambda}{\succ}$-monotone (isotonic) functionals that are real-valued matrix functions $\phi : \mathcal{C} \to \mathbb{R}$ such that, for $\mathbf{C}_1, \mathbf{C}_2 \in \mathcal{C}$, if $\mathbf{C}_1 \overset{\lambda}{\succ} \mathbf{C}_2$, then $\phi(\mathbf{C}_1) \geq \phi(\mathbf{C}_2)$.

While a complete characterization of the functionals in the $\mathcal{M}_\lambda$ class is not available at the moment, later in Lemma 1 we provide a sufficient condition for a matrix function to be in $\mathcal{M}_\lambda$.

The following proposition collects important properties of the $\mathcal{M}_\lambda$ class.

Proposition 2 (Properties of the $\mathcal{M}_\lambda$ class)**.** The class $\mathcal{M}_\lambda$ exhibits the following properties:

1. *Comonotonicity with respect to majorization*: When applied to correlation matrices that are ordered according to the quantum majorization, all functionals in $\mathcal{M}_\lambda$ are comonotonic, i.e. they move in the same direction in terms of risk. When majorization does not hold, this is no longer guaranteed;

2. *Closure under increasing functions*: The class $\mathcal{M}_\lambda$ is closed under increasing functions. If $\{\phi_1, ..., \phi_k\} \in \mathcal{M}_\lambda$ and $h : \mathbb{R}^k \to \mathbb{R}$ is an increasing function in its arguments, then $h(\phi_1, ..., \phi_k)$ belongs to $\mathcal{M}_\lambda$;

3. *Bounds*: Every function in $\mathcal{M}_\lambda$ is bounded from below when evaluated in the identity matrix, and from above when evaluated in any of the rank 1 correlation matrices.

Proof. The proof of the first property easily derives from the definition of the $\mathcal{M}_\lambda$ class itself.

In order to prove the second statement notice that if $\mathbf{C}_1 \overset{\lambda}{\succ} \mathbf{C}_2$, then $[\phi_1(\mathbf{C}_1), ..., \phi_k(\mathbf{C}_1)] \geq [\phi_1(\mathbf{C}_2), ..., \phi_k(\mathbf{C}_2)]$, where $\geq$ denotes the standard product order. The result follows by h being increasing in all its arguments.

The last statement simply follows from conditions C1 and C2 and Theorem 1. □

The first property tells us that, given ϕ and ψ in $\mathcal{M}_\lambda$, if $\mathbf{C}_1 \overset{\lambda}{\succ} \mathbf{C}_2$, then $\phi(\mathbf{C}_1) \geq \phi(\mathbf{C}_2)$ and $\psi(\mathbf{C}_1) \geq \psi(\mathbf{C}_2)$. This means that if two correlation matrices can be ordered according to majorization, all functionals in $\mathcal{M}_\lambda$ will show the same behavior, assigning a higher portfolio risk to the majorizing correlation matrix ($\mathbf{C}_1$), and a lower risk to the majorized one ($\mathbf{C}_2$).

The practical implications of such a property are evident: if it is possible to observe majorization on the market, all the risk measures belonging to $\mathcal{M}_\lambda$ will move in the same direction, making every discussion about which measure is better less relevant. One can verify that, when majorization is lost, functionals in $\mathcal{M}_\lambda$ can show different behavior, moving in different directions: one measure may see increasing risks, while another may indicate a decrease. Therefore, the consistent and inconsistent behavior of risk functionals is a way of identifying majorization (and vice versa). Empirical investigations, as the one we offer in Section 5, suggest that financial data commonly show majorization, especially during periods of crisis, when the phenomenon appears to be quite strong.

The second property simply extends the same behavior that Schur-convex functions exhibit for real vectors [34], and it thus allows to build new risk measures starting from existing ones.

Finally, the last property can be used to introduce a trivial standardization for the functions in $\mathcal{M}_\lambda$, that is

$$\bar{\phi}(\mathbf{X}) = \frac{\phi(\mathbf{X}) - \phi(\mathbf{I})}{\phi(\mathbf{J}) - \phi(\mathbf{I})} \tag{3}$$

where the $\mathbf{J}$ matrix is the special rank 1 matrix with $J_{i,j} = 1$ for every i, j.

In the next Subsections, we discuss some examples of matrix functions belonging to $\mathcal{M}_\lambda$. Some of these measures represent new tools for the analysis of financial data, which–we hope–other researchers will further test and develop.

3.1 The quantum Lorenz curve and the inequality functionals

An important set of functionals belonging to the $\mathcal{M}_\lambda$ class is given by the matrix representation of some famous measures of economic inequality [11, 16].

Mimicking the approach used in the study of economic size distributions and inequality [26], the starting point is the definition of a multivariate version of the well-known Lorenz curve [30].

The (univariate) Lorenz function was introduced to study the distribution of wealth among individuals, but it later developed into a powerful tool of statistical analysis [43].

Given a vector of ordered positive quantities $x_{[1]} \geq x_{[2]} \geq ... \geq x_{[n]}$, say incomes [30], the classical Lorenz curve is defined as

$$L_k(\mathbf{x}) = \frac{\sum_{i=1}^{k} x_{[i]}}{\sum_{i=1}^{n} x_i} \quad \text{for } k = \{1, ..., n\}. \tag{4}$$

This function has several useful properties, for which we refer to [12, 43]. For our purposes, it is sufficient to notice that if $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n_+$ and $L_k(\mathbf{x}) \geq L_k(\mathbf{y})$ for every $k \in \{1, ..., n\}$, then $\mathbf{x} \succ \mathbf{y}$ and vice versa. For this reason the majorization relation between two vectors $\mathbf{x}$ and $\mathbf{y}$ is sometimes interpreted as $\mathbf{x}$ being more unequal than $\mathbf{y}$.

Several multivariate extensions of the Lorenz curve have been proposed in the literature to deal with multivariate inequality, and for a review we suggest [26, 43]. Here we introduce a brand new one, suitable for portfolio risk management and compatible with the quantum majorization of Definition 3. For this reason, we call it quantum Lorenz curve.

Definition 5 (Quantum Lorenz curve)**.** Let $\mathbf{C}$ be an $n \times n$ correlation matrix, its quantum Lorenz curve is the matrix function $\mathbf{L} : \mathcal{C} \to \mathbb{R}^n$, such that

$$L_k(\mathbf{C}) := \frac{\max_{\mathbf{U}\mathbf{U}^T = \mathbf{I_k}} Tr(\mathbf{U}\mathbf{C}\mathbf{U}^T)}{Tr(\mathbf{C})} = \frac{\sum_{i=1}^{k} \lambda_{[i]}(\mathbf{C})}{\sum_{i=1}^{n} \lambda_i(\mathbf{C})} \quad \text{for } k = \{1, ..., n\}, \tag{5}$$

where $\mathbf{U}$ is a $k \times n$ unitary invariant matrix, Tr is the trace operator, and $\lambda_{[1]} \geq \lambda_{[2]} \geq ... \geq \lambda_{[n]}$ are the ordered eigenvalues of C.

The definition can be easily extended[3] to any $n \times m$ matrix $\mathbf{C}$ that admits a singular value decomposition $\mathbf{C} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}$ by setting the numerator in (5) equal to $\max_{\mathbf{U}\mathbf{U}^T = \mathbf{V}\mathbf{V}^T = \mathbf{I_k}} Tr(\mathbf{U}\mathbf{A}\mathbf{V})$.

[3]This would be particularly useful in the study of economic inequality, together with many other properties of the quantum Lorenz, which nevertheless go beyond the scope of this paper.

The quantum Lorenz has a nice interpretation in terms of portfolio risk: for a given k, it represents the percentage of the total portfolio variance explained by the first k eigenvalues [42]. Alternatively, in the context of dimensionality reduction, $1 - L_k(\mathbf{C})$ provides an estimate of the percentage of information that is lost by taking a rank k approximation of the matrix $\mathbf{C}$ [23].

Proposition 3 (Properties of the quantum Lorenz)**.** A quantum Lorenz curve $\mathbf{L}(\mathbf{C})$ has the following properties:

1. *Unitary invariance*: $\mathbf{L}(\mathbf{U}\mathbf{C}\mathbf{U}^T) = \mathbf{L}(\mathbf{C})$ with $\mathbf{U}$ being a unitary matrix;
2. *Convexity and sub-additivity*: $L_k(\alpha\mathbf{C}_1+(1-\alpha)\mathbf{C}_2) \leq \alpha L_k(\mathbf{C}_1)+(1-\alpha)L_k(\mathbf{C}_2)$ and $L_k(\mathbf{C}_1+\mathbf{C}_2) \leq L_k(\mathbf{C}_1) + L_k(\mathbf{C}_2)$ for every $k = \{1, ..., n\}$;
3. *Perfect equality*: $L_k(\mathbf{C}) = \frac{k}{n}$ for every $k = \{1, ..., n\}$ if and only if $\mathbf{C}$ is the identity matrix.
4. *Perfect inequality*: $L_k(\mathbf{C}) = 1$ for every $k = \{1, ..., n\}$ if and only if $\mathbf{C}$ is a rank 1 matrix.

Proof. Property 1 simply follows from basic properties of the trace operator: linearity and invariance to cyclic permutations.

Property 2 follows from the sub-additivity of the max and the fact that, for $n \times n$ correlation matrices $\mathbf{C}_1$ and $\mathbf{C}_2$, $Tr(\mathbf{C}_1) = Tr(\mathbf{C}_2)$.

To prove the "if part" of Property 3, just recall that, for $\mathbf{U}$ a $k \times n$ unitary matrix, we have that $Tr(\mathbf{UIU}) = Tr(\mathbf{I_k}) = k$. The "only if" part once again follows from the Fan representation theorem [15], and it mimics the same reasoning used in the proof of Theorem 1.

Property 4 can be proven in a similar way, after noticing that the set of rank 1 matrices is writable as $\{\mathbf{C} \in \mathcal{C} : \mathbf{C} = \boldsymbol{\rho}\boldsymbol{\rho}^T,\ \boldsymbol{\rho} \in \{-1, 1\}^n\}$. Therefore $Tr(\boldsymbol{\rho}\boldsymbol{\rho}^T) = n$, since $\rho_i\rho_i = 1$ for every $\boldsymbol{\rho} \in \{-1, 1\}^n$, and $Tr(\mathbf{U}(\boldsymbol{\rho}\boldsymbol{\rho}^T)\mathbf{U^T}) = n$ for every $k \times n$ matrix $\mathbf{U}$. □

The following proposition identifies a close connection between quantum majorization and the quantum Lorenz, further justifying the choice of the name.

Proposition 4 (Quantum Lorenz and quantum majorization)**.** Given two correlation matrices $\mathbf{C}_1, \mathbf{C}_2$, we have that $\mathbf{C}_1 \overset{\lambda}{\succ} \mathbf{C}_2$ if and only if $\mathbf{L}(\mathbf{C}_1) \geq \mathbf{L}(\mathbf{C}_2)$, with $\geq$ being the standard product order for vectors.

Proof. The proof is trivial if one notices that the spectral representation of the quantum Lorenz curve implies Definition 3. □

On the basis of the quantum Lorenz, we can finally introduce a class of inequality (risk) functionals.

Definition 6 (Inequality functionals)**.** Given a correlation matrix $\mathbf{C}$ and a real valued function $\psi : \mathbb{R}^n \to \mathbb{R}$ increasing in all its arguments, we define an inequality functional as

$$\mathcal{D}_\psi(\mathbf{C}) = \psi(\mathbf{L}(\mathbf{C})), \tag{6}$$

where $\mathbf{L}(\mathbf{C})$ is the quantum Lorenz curve associated to $\mathbf{C}$.

A consequence of Propositions 4 and 2 is that the set of inequality functionals belongs to the $\mathcal{M}_\lambda$ class. The following lemma gives us a way of constructing these functionals.

Lemma 1 (Construction of $\mathcal{M}_\lambda$ inequality functionals)**.** Any real-valued matrix functional built via a Schur-convex function ϕ applied to the spectrum of a correlation matrix $\mathbf{C}$, i.e. $\phi(\boldsymbol{\lambda}(\mathbf{C}))$, is an inequality functional in $\mathcal{M}_\lambda$.

Proof. Consider $\Delta_k := L_k(\mathbf{C}) - L_{k-1}(\mathbf{C})$. The definition of the quantum Lorenz tells that $\Delta_k = \lambda_{[k]}/n$.

Given the properties of Schur-convex functions [34], the application of an increasing function to $L(\mathbf{C})$ is equivalent to the application of a Schur-convex function to Δ_k. Now, set $D_\phi(\mathbf{C}) = \phi(\boldsymbol{\lambda}(\mathbf{C}))$, where ϕ is a Schur-convex function mapping $\boldsymbol{\lambda}(\mathbf{C})$ onto the reals. The use of Equation (5) concludes the proof. □

Lemma 1 generalizes a result by Alberti and Uhlmann [1], who proved the isotonicity to quantum majorization only for symmetric convex functions of the spectra of Hermitian matrices, clearly a subset of the larger Schur-convex class [34].

Let us now consider some examples of inequality functionals obtained via Definition 6. These functionals generalize some of the most famous socio-economic inequality indices [4, 12].

Example 3.1 (Matrix distance indices)**.** Let $\psi = ||.||_p$ be the L^p vector distance between the quantum Lorenz curve $\mathbf{L}(\mathbf{C})$ of the correlation matrix $\mathbf{C}$ and its lower bound $\mathbf{L}(\mathbf{I})$. Then

$$\mathcal{D}_{||.||_p}(C) = ||\mathbf{L}(\mathbf{C}) - \mathbf{L}(\mathbf{I})||_p \tag{7}$$

is the matrix extension of the univariate distance indices of [12].

For $p = 1$, we obtain the (correlation) matrix version of the Gini index [19], i.e.

$$\mathcal{D}_{||.||_1}(\mathbf{C}) = \frac{1}{n}\sum_{k=1}^{n}\sum_{i=1}^{k}|\lambda_{[i]} - k|. \tag{8}$$

For $p = \infty$, we get the matrix version of the Pietra index, or [36]

$$\mathcal{D}_{||.||_\infty}(\mathbf{C}) = \max_k |L_k(\mathbf{C}) - L_k(\mathbf{I})|. \tag{9}$$

Example 3.2 (Ratio indices)**.** If $\psi = \pi_k$, where π_k is the projection operator onto the k-th coordinate, we have

$$\mathcal{D}_{\pi_k}(\mathbf{C}) = L_k(\mathbf{C}). \tag{10}$$

This represents the matrix extension of the top-$\alpha\%$ inequality index used in the social sciences to measure the proportion of the total wealth owned by the top $\alpha\%$ richest individuals [11] in the economy. A particular case of ratio index, called absorption ratio, has already been used as portfolio risk measure for correlation matrices in [28, 27]. It is obtained by setting $k = \lfloor 0.05 \times n \rfloor$.

3.2 Entropy-based functionals

Following the seminal work of Alberti and Uhlmann [1], a large class of real-valued matrix functions for Hermitian matrices, called entropy-like functionals, was introduced in the physical literature to study the behaviour of Gibbs densities [33, 32].

Since correlation matrices are trivially Hermitian, these functionals can be also applied in our portfolio risk framework, and they naturally fall in the $\mathcal{M}_\lambda$ class.

Definition 7 (Entropy-like functionals)**.** Let $\mathbf{H}$ be an Hermitian matrix and $f : \mathbb{R} \to \mathbb{R}$ a convex function, the class of entropy-like matrix functional S_f is defined as

$$S_f(\mathbf{H}) = Tr(f(\mathbf{H})), \tag{11}$$

where $f(\mathbf{H})$ is the usual notation for matrix functions, i.e.

$$f(\mathbf{H}) := \mathbf{U}\begin{bmatrix} f(\lambda_1(\mathbf{H})) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & f(\lambda_n(\mathbf{H})) \end{bmatrix}\mathbf{U}^T,$$

with $\mathbf{U}$ being a unitary matrix that diagonalizes $\mathbf{H}$.

It should be clear that, for a Schur-convex function ϕ, one has $\phi(\boldsymbol{\lambda}(\mathbf{H})) = \sum_{i=1}^{n} f(\lambda_i(\mathbf{H}))$. As a consequence, Lemma 1 ensures that S_f is monotone with respect to quantum majorization.

An useful property of entropy-like functionals is their convexity, that is

$$S_f(\alpha\mathbf{C_1} + (1-\alpha)\mathbf{C_2}) \leq \alpha S_f(\mathbf{C_1}) + (1-\alpha)S_f(\mathbf{C_2}),$$

where $\alpha \in [0,1]$ and $\mathbf{C_1}$, $\mathbf{C_2}$ are $n \times n$ correlation matrices. This further justifies their possible use in evaluating portfolio risk.

Example 3.3 (Von Neumann entropy)**.** By taking $f(\lambda_i) = -\lambda_i \log(\lambda_i)$ in (11), we obtain the Von Neumann entropy [40] of a correlation matrix,

$$S_{VN}(\mathbf{H}) := \sum_{i=1}^{n} \lambda_i(\mathbf{H}) \log \lambda_i(\mathbf{H}). \tag{12}$$

The function f is trivially concave, therefore it is sufficient to define S_{-f} to preserve the $\overset{\lambda}{\succ}$ order.

Example 3.4 (Renyi α-entropies and effective rank)**.** This class of entropies à la Renyi [1, 26] is obtained as a generalization of S_f, by taking the function $g(S_f) := \frac{1}{1-\alpha} \log S_f$ and by setting $f(x) = x^\alpha$, $\alpha \geq 0$. One has

$$S^\alpha(\mathbf{H}) := \frac{1}{1-\alpha} \log Tr(\mathbf{H}^\alpha) = \frac{1}{1-\alpha} \log \sum_{i=1}^{n} \lambda_i(\mathbf{H})^\alpha, \tag{13}$$

where $\mathbf{H}^\alpha$ is the matrix power function.

This class is once again concave, hence one defines $-S^\alpha$ to ensure convexity. To verify the $\overset{\lambda}{\succ}$ monotonicity, just observe that g is an increasing function and apply Proposition 2, Property 2, with $k = 1$.

By defining $g(S_f) := e^{-S_f}$ with $f(\lambda_i) = \lambda_i \log(\lambda_i)$, one obtains another entropy-based index commonly called effective rank [37], that is

$$ER(\mathbf{H}) := e^{-Tr(f(\mathbf{H}))} = e^{-\sum_{i=1}^{n} \lambda_i(\mathbf{H}) \log \lambda_i(\mathbf{H})}$$

This measure was developed as an attempt to propose a real-valued proxy for the rank of a correlation matrix. Clearly $-ER$ belongs to the $\mathcal{M}_\lambda$ class.

3.3 Other quantum majorization preserving functionals

Many other functionals related to quantum majorization can be identified and proposed. Here we only cite two additional possibilities: determinant-based measures and matrix norms.

Determinant-based measures originate from the works of Wilks [42], and use the determinant of the covariance matrix as a summary tool for the dispersion of an n-dimensional random vector. More recently, the determinant of the correlation matrix has also been taken into consideration [10]. It is not difficult to show that all these measures belong to the $\mathcal{M}_\lambda$ class.

Matrix norms were introduced by Von Neumann [40, 41] as the set of all norms $||.||$ such that, given a matrix $\mathbf{C}$, one has $||\mathbf{C}|| = ||\mathbf{UCU}^T||$, with $\mathbf{U}$ being a unitary matrix. A notable example is the Frobenius norm $||\mathbf{C}||_F = \sqrt{\sum_{i=1}^{n} \lambda_i^2(\mathbf{C})}$ [18, 41], often used in portfolio optimization and management [14].

Mirsky [31] showed that there is a one-to-one correspondence between matrix norms and a special subclass of Schur-convex functions–symmetric gauge functions–on the spectrum of the matrix. Thanks to this connection, Lemma 1 guarantees that matrix norms belong to the $\mathcal{M}_\lambda$ class.

4 The quantum majorization matrix

Let $\mathbf{C}_t$ be the $n \times n$ correlation matrix of an portfolio $\mathcal{P}$ with n assets, which is observed over time[4], for $t = 1, ..., T$.

Our aim is to introduce a concise way of representing the majorization among all the $\mathbf{C}_t$ matrices as time passes. In fact, if during a certain period the correlation matrix $\mathbf{C}_t$ majorizes the previous one, $\mathbf{C}_{t-1}$, we can read this as an overall increase of portfolio risk (Section 5 shows that this actually happens). And, in general, if $\mathbf{C}_t \overset{\lambda}{\succ} \mathbf{C}_{t-s}$, for some s, then the portfolio at time t is riskier than what it was at time $t - s$. Further, if $\mathbf{C}_t \overset{\lambda}{\succ} \mathbf{C}_{t-1} \overset{\lambda}{\succ} ... \overset{\lambda}{\succ} \mathbf{C}_{t-s}$, with no interruption in the majorization series, risk has increased for s periods, and so on. Naturally, on the opposite side, if $\mathbf{C}_t \overset{\lambda}{\prec} \mathbf{C}_{t-1}$ risk is decreasing. It is important to notice that, however, if $\mathbf{C}_t \overset{\lambda}{\not\succ} \mathbf{C}_{t-1}$, it is not possible to state that risk has either increased or decreased with confidence.

Following the results of Section 3, one could be inclined to choose a measure $\phi \in \mathcal{M}_\lambda$ to evaluate risk on every correlation matrix $\mathbf{C}_t$, for $t = 1, ..., T$, thus creating a sequence $(\phi(\mathbf{C}_t))_{t=1}^T$

[4] In view of Section 5 we will compare correlation matrices over time, but–as said–the quantum majorization approach can naturally be applied cross-sectionally, to compare and rank portfolios in terms of risk.

to assess portfolio risk over time. However, given Propositions 1 and 2, it seems more natural to deal with majorization directly, given that: 1) it represents a strong connection among correlations; and 2) when it manifests itself, all functionals in $\mathcal{M}_\lambda$ are comonotonic and give the same type of information about changes in portfolio risk.

The tool we propose is the *quantum majorization matrix*, that is a matrix of indicators, where each element points out the majorization relation between two different correlation matrices.

Definition 8 (Quantum majorization matrix)**.** Consider a collection of correlation matrices $\{\mathbf{C}_t\}_{t=1}^T$ for a portfolio $\mathcal{P}$. Let $\mathbf{A}$ be a $T \times T$ matrix such that

$$A_{i,j} = \begin{cases} 1 & \textit{if } \mathbf{C_i} \stackrel{\lambda}{\succ} \mathbf{C_j} \\ 0 & \textit{otherwise} \end{cases}, \tag{14}$$

for $i, j = 0, 1, ..., T$.

The matrix $\mathbf{A}$ is called quantum majorization matrix.

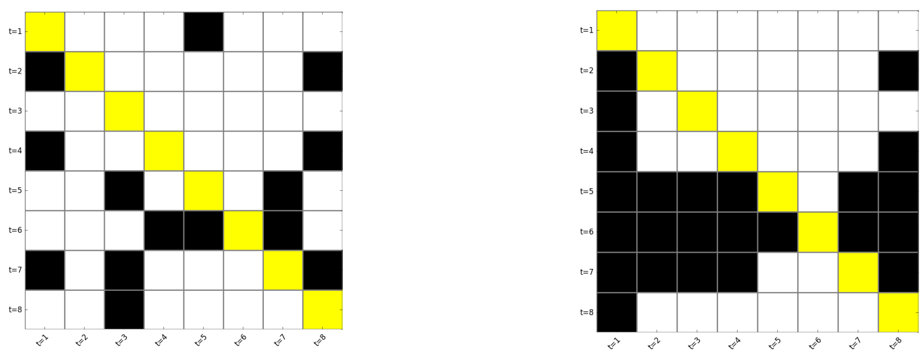
By definition the quantum majorization matrix keeps track of all the $\stackrel{\lambda}{\succ}$ ordering relations in the collection $\{\mathbf{C}_t\}_{t=1}^T$. Recalling Proposition 1, $\mathbf{A}$ thus helps in noticing the presence of a non-trivial dependence structure in the portfolio.

Fix a row i in $\mathbf{A}$. The majorized (or dominated) set $\mathcal{D}_i := \{\mathbf{C}_k : A_{i,k} = 1, \forall k \in \{1, ..., T\}\}$ corresponds to the set of the all correlation matrices that are majorized by $\mathbf{C}_i$, being less risky than $\mathbf{C}_i$. The majorizing (or dominating) set $\mathcal{ND}_i := \{\mathbf{C}_k : A_{\{k,i\}} = 1, \forall k \in \{1, ..., T\}\}$, conversely, contains all the correlation matrices that majorize $\mathbf{C}_i$, and are thus riskier. As a consequence, $\mathcal{S}_i := \mathcal{D}_i \cap \mathcal{ND}_i$ is the set of all the correlation matrices that are similar to $\mathbf{C}_i$, while $\mathcal{N}_i := \mathcal{D}_i^c \cap \mathcal{ND}_i^c$ is the set of all the correlation matrices that have no ordering relation with $\mathbf{C}_i$.

From now on, without any loss of generality[5], we impose not to have similar correlation matrices. Therefore, the set $\mathcal{S}_i$ only contains $\mathbf{C}_i$, and the total number of majorizations in $\mathbf{A}$ lies in the interval $\left[0, \frac{T(T+1)}{2}\right]$. Also observe that

$$\sum_{k=1}^{T} (A_{i,k} + A_{k,i}) - 1 \leq T. \tag{15}$$

Figure 1 gives two graphical representations of possible 8×8 quantum majorization matrices. To represent the different $A_{i,j}$'s in the matrices we choose black for 1, and white for 0. Yellow is used to indicate the trivial diagonal of each correlation matrix majorizing and being majorized by itself.



(a) Case 1 (b) Case 2

Figure 1: Two examples of 8×8 quantum majorization matrices. Black squares represent ones, white squares zeros, and the yellow diagonal shows the trivial situation of a correlation matrix majorizing itself.

Consider Subfigure 1a: cell (t_2, t_1) is black, indicating that $A_{2,1} = 1$. This means that the correlation matrix $\mathbf{C}_2$ majorizes $\mathbf{C}_1$, i.e. $\mathbf{C}_2 \stackrel{\lambda}{\succ} \mathbf{C}_1$: the embedded portfolio risk has thus increased. As expected cell (t_1, t_2) is white ($A_{1,2} = 0$).

[5]In applications it is not plausible to observe two correlation matrices with the same exact spectrum.

Let us now look at (t_5, t_1): it is white, while the symmetric cell (t_1, t_5) is black. In this case, we have $\mathbf{C}_1 \overset{\lambda}{\succ} \mathbf{C}_5$, i.e. $\mathbf{C}_5$ is majorized.

Take now (t_3, t_1): it is white, but also (t_1, t_3) is. Therefore, between $\mathbf{C}_1$ and $\mathbf{C}_3$ there is no majorization, $\mathbf{C}_1 \overset{\lambda}{\not\succ} \mathbf{C}_3$, and no unique comment on risk can be made.

All in all, if one of the two symmetric cells is black, majorization takes place[6], if both cells are white, no majorization can be observed.

Consider now Subfigure 1b, and let us focus our attention on the correlation matrix $\mathbf{C}_6$, represented by row t_6. It is clear that this correlation matrix majorizes all the others, thus representing the up-to-date maximum in terms of risk (as time passes, the ranking could change). Matrix $\mathbf{C}_5$ represents the second riskiest portfolio (up-to-date), which majorizes all other portfolios apart from the one represented by $\mathbf{C}_6$, by which it is majorized.

Looking at patterns in a quantum majorization matrix we can therefore identify periods of higher and lower risk, and we can compare them. As we shall see in Section 5, the presence of large black areas in the plot of a quantum majorization matrix will indicate times of financial distress for our portfolio. When a portfolio is granular and representative of a market, majorization can be used to look for and analyze increases in systemic risk and the emergence of financial crises.

4.1 Two simple risk measures on the quantum majorization matrix

In applications, when T is large, it is common to obtain datasets with a very big number of correlation matrices. This makes it cumbersome to just explore the associated quantum majorization matrix graphically, as we did a few lines above[7]. A natural solution is to introduce some summary measures based on the majoritazion matrix $\mathbf{A}$.

A simple tool, which proves very useful in applications (see Section 5), is the function $\theta : \mathbf{C}_i \rightarrow \mathbb{R}$, defined as

$$\theta(\mathbf{C}_i) = \frac{1}{2} + \frac{\#\mathcal{D}_i}{2T} - \frac{\#\mathcal{ND}_i}{2T}, \tag{16}$$

where # indicates the cardinality (i.e. the number of elements) of the set.

The counting measure in Equation (16) expresses the risk embedded in each correlation matrix as a function of its majorized and majorizing sets. Interestingly, $\theta(\mathbf{C}_i)$ is also order preserving with respect to quantum majorization, so that if $\mathbf{C_i} \overset{\lambda}{\succ} \mathbf{C_j}$ then $\theta(\mathbf{C_i}) \geq \theta(\mathbf{C_j})$.

It is easy to observe that $\theta(\mathbf{C}_i) \in [0, 1]$. The case $\theta(\mathbf{C}_i) = 0$ indicates that the correlation $\mathbf{C}_i$ is majorized by all the other matrices in the collection, while $\theta(\mathbf{C}_i) = 1$ tells us that $\mathbf{C}_i$ is the up-to-date maximum in terms of risk.

Given its definition, the value $\frac{1}{2}$ represents for $\theta(\mathbf{C}_i)$ a threshold. If $\theta(\mathbf{C}_i) \leq \frac{1}{2}$, the correlation matrix $\mathbf{C}_i$ can be classified as non risky with respect to the great part of the remaining matrices, while the opposite happens for $\theta(\mathbf{C}_i) > \frac{1}{2}$.

From the quantum majorization matrix $\mathbf{A}$ we can also obtain a quantity that bridges towards Proposition 2, Property 1.

Recall that $\mathcal{N}_i$ is the set, with cardinality $\#\mathcal{N}_i$, of all the correlation matrices that have no ordering relation with $\mathbf{C}_i$. If we sum over all possible $i = 1, ..., T$, we obtain the total number of non-ordered (non-majorized) pairs in $\{\mathbf{C}_t\}_{t=1}^{T}$. This number clearly lies between 0 and $T(T-1)$. When it is 0, all correlation matrices are ordered, while when it is $T(T-1)$ no majorization structure is present in the collection.

It seems therefore natural to introduce the quantity

$$U = \frac{\sum_{i=1}^{T} \#\mathcal{N}_i}{T(T-1)} \in [0, 1]. \tag{17}$$

U is an index telling us how much majorization takes place in a collection of correlation matrices. The lower the index the more often majorization occurs, therefore the more often all functionals in the $\mathcal{M}_\lambda$ class are comonotonic. The higher U the more we can expect the measures in $\mathcal{M}_\lambda$ to behave in a not necessarily coherent way.

[6]Recall that we have excluded similarity among correlation matrices, otherwise we could have two corresponding black cells.

[7]Yet, as discussed later in Section 5, we always suggest to plot the quantum majorization matrix, to have a quick heuristic idea of portfolio risk.

If we come back to the two examples in Figure 1, we can compute $U = 0.5$ in the first case, and $U = 0.14$ in the second. This is consistent with what we have discussed before, given that the second case visually shows a higher level of majorization.

5 An example on actual data

Our dataset–available for download–consists of 4563 daily log-returns for 28 components of the Industrial Dow Jones (INDJ), between January 03 2000 and February 21 2018. Using 100-day moving windows, with a 90-day overlap, we have constructed a series of 447 correlation matrices, which we will compare and study using majorization. It is worth underlying that changing the size of the moving windows and of the overlaps does not affect our conclusions, until it is possible to play with a sufficient number of correlation matrices (≥ 50). The combination here shown is the one providing the best results from a graphical point of view. For each correlation matrix in the collection, we have computed the associated eigenvalues via a standard singular value decomposition.

For each 100-day moving window, Figure 2 shows the average log-returns of the INDJ index; superimposed we also provide the time series of the corresponding matrix Gini index ($\mathcal{D}_{||.||_1}$), computed on the different correlation matrices. The chosen measure seems definitely capable of identifying some major events in the observation period, something not always possible by just looking at the log-returns.

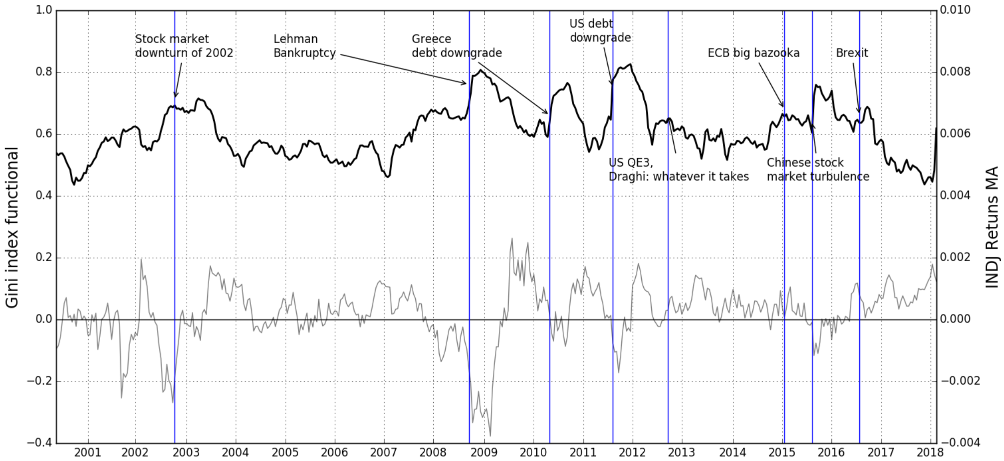


Figure 2: The average INDJ index (gray) in the period between January 03 2000 and February 21 2018, over rolling windows of 100 days, with a 90-day overlap. For the same windows we provide the matrix Gini index (black) computed on the corresponding correlation matrices. Major events in the observation period are indicated.

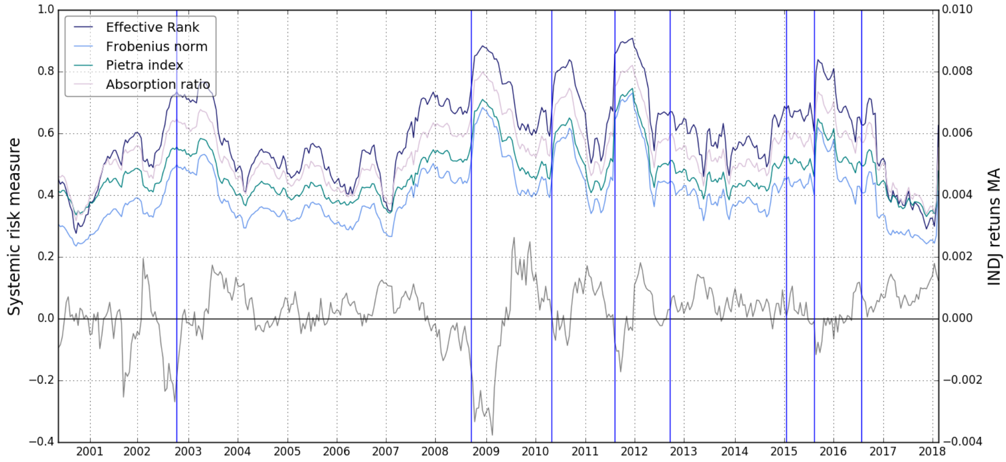


Figure 3: Time series of four $\overset{\lambda}{\succ}$ monotonic measures applied to the dataset.

The choice of the matrix Gini in Figure 2 is arbitrary, that is why in Figure 3 we also provide other possible measures of risk in $\mathcal{M}_\lambda$: the Frobenius norm, the effective rank, the absorption ratio and the Pietra index. Evidently, all these measures tend to behave similarly, suggesting the presence of several majorizations in the portfolio evolution, consistently with Proposition 2, Property 1.

A measure of the amount of majorization in the portfolio is provided by the U measure of Equation (17): the closer U to zero, the more often majorization occurs. The average U in our data is 0.15, with a standard deviation of 0.03; this indicates that quantum majorization is a rather common phenomenon in the INDJ portfolio. Therefore we can expect the different risk measures to behave similarly quite often, as per Figure 3.

Figure 4 shows the behavior of U over 100-day rolling windows. Interestingly, during the riskiest moments of the last twenty years, like the 2007-2009 crisis or the second wave of 2011-2012, the U index decreases down to 0.1, indicating an increase in majorization–and thus risk–on the market. Conversely, during the periods of good market conditions, like 2006-2007 majorization decreases, and U grows up to 0.3.

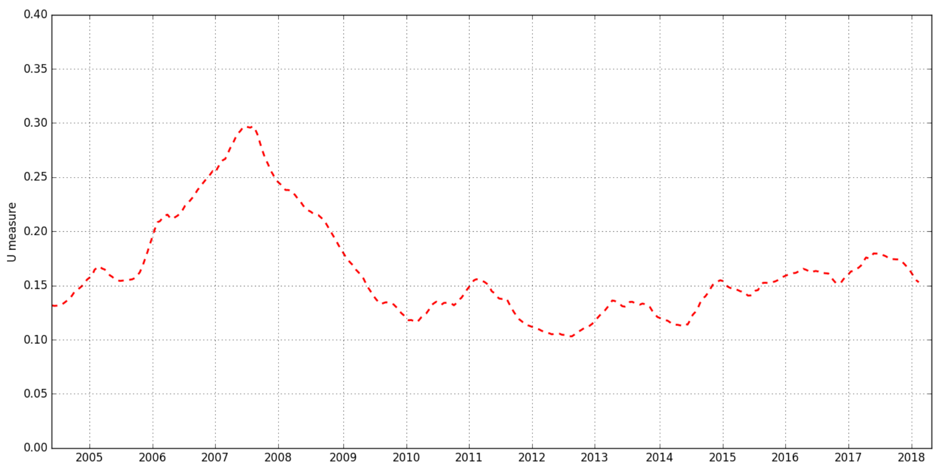


Figure 4: Time evolution of the U measure over a 100-day rolling window.

Let us now have a look at Figure 5, which presents the quantum majorization matrix $\mathbf{A}$ of the INDJ sample. An interesting fact, consistent with the behavior of U, can be observed: the correlation matrices of the 2007-2009 crisis period are the up-to-date maximal elements for the dataset. In fact, as clear from the red band in the picture, they majorize most of the other correlation matrices, with just a few exceptions of non-majorization ($\not\succ$). Conversely, the correlation matrices of the years 2005, 2006 or 2014 are majorized by almost all the other correlations, representing the least risky set.

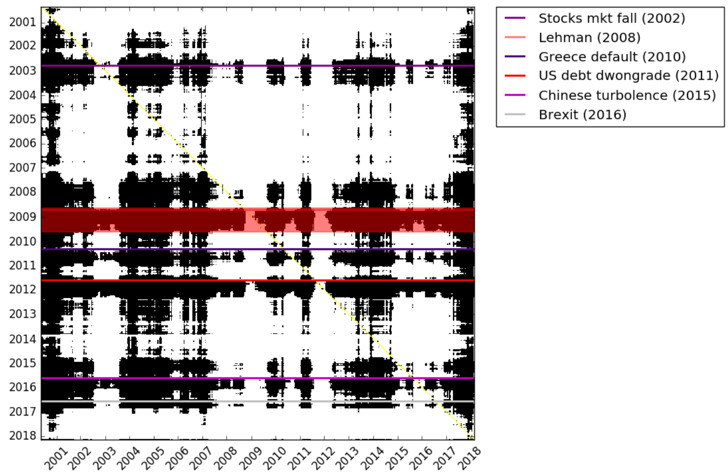


Figure 5: Quantum majorization matrix of the INDJ dataset. Black squares represent quantum majorizations between correlation matrices, the red lines indicate major market events during the observation period, while the red band underlines the riskiest period (in term of majorizations) in the dataset.

In Figure 6, we provide the behavior of another risk measure we have discussed in Subsection 4.1, the quantity $\theta(\mathbf{C}_i)$, together with the average log-returns of the INDJ portfolio. For every correlation matrix $\{\mathbf{C}_t\}_{t=1}^{447}$ the corresponding $\theta(\mathbf{C}_t)$ is computed. Looking at the graph, we can see that all periods of crisis (see again Figure 2) are characterized by riskier correlation matrices, i.e. $\theta(\mathbf{C}_t) > 0.5$, while less turbolent periods do show $\theta(\mathbf{C}_t) < 0.5$. Once again the maximal levels of majorization are observed in 2007-2009 and 2011-2012, where θ almost reaches 1.

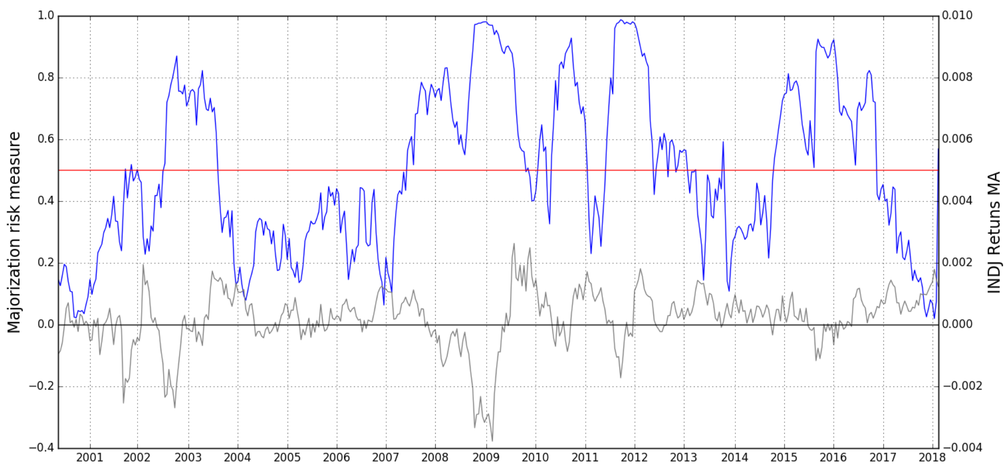


Figure 6: Time evolution of the risk measure $\theta(\mathbf{C}_i)$ for $\{\mathbf{C}_t\}_{t=1}^{T}$, $T = 447$, with its 0.5 threshold level. The INDJ average log-returns are also provided for readers' convenience.

On the basis of the quantum majorization matrix of Figure 5, we can also give the intuition of a simple alarm system for portfolio risk. The idea is to cluster the correlation matrices $\{\mathbf{C}_t\}_{t=1}^{447}$ according to the information embedded in the matrix $\mathbf{A}$, so to group together the correlations with similar majorization patters. For the collected matrices we can then compute the corresponding θ index, and verify whether particular values manifest themselves.

As common in cluster analysis [23], to group observations we need some notion of distance among the majorization patterns in $\mathbf{A}$, as induced by each $\mathbf{C_t}$. A standard way to deal with this type of problems is spectral embedding [17], which associates each $\mathbf{C_t}$–that for us is a data point in the space of correlation matrices–with a position in a multidimensional space, whose dimension and coordinates are defined by the left and right eigenvectors of $\mathbf{A}$.

Figure 7 shows the position of each correlation matrix in a 2-dimensional space, where the x-coordinates are given by the first entries of the left eigenvectors of $\mathbf{A}$, and the y-coordinates by the first entries of the right eigenvectors[8].

The identification of clusters can then be performed using several different methods; here we rely on the standard k-means algorithm [23], with the goal of identifying 2 and 3 possible clusters (a larger number of clusters becomes difficult to interpret in this framework). The obtained groups are visible in the subplots of Figure 7, and their separation is definitely good. In the 2-cluster case of Figure 7, the red dots on the right identify the risky correlation matrices, those which tend to majorize the others, while the white dots represent the majorized correlations, associated to the less risky periods. In the 3-cluster situation, the yellow group appears, taking elements from the two original clusters. This group represents a situation of intermediate risk, possibly corresponding to a transition between more and less risky periods.

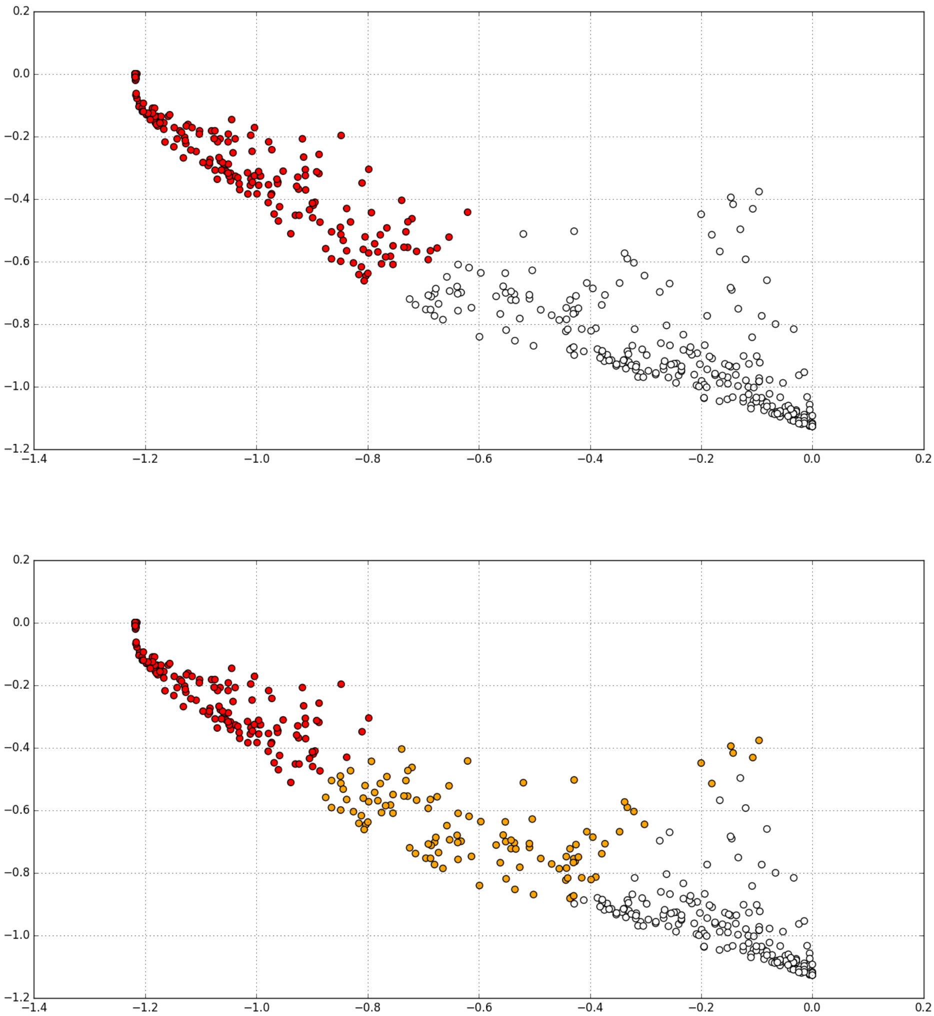


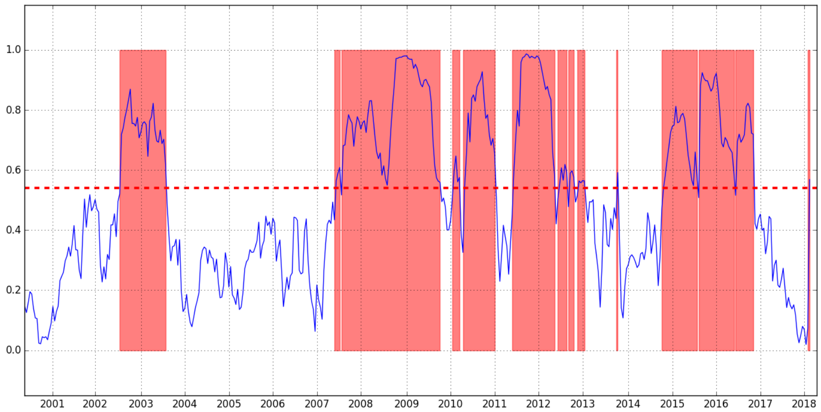
Figure 7: Projection of the INDJ dataset on a 2D Euclidean plane, on the basis of the eigendecomposition of $\mathbf{A}$. The separation into 2 (top) and 3 (bottom) clusters via the k-means algorithm is evident.

[8]Plots in higher dimensions do not add much information, that is why we restrict our attention to the 2d case.

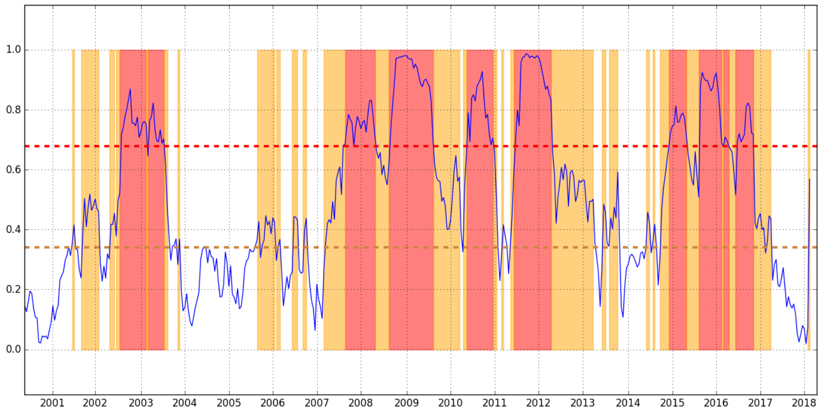
In Figure 8 the clusters of Figure 7 are superimposed to the time series of the $\theta(\mathbf{C}_t)$ risk index also given in Figure 6.

In Subfigure 8a, the red areas clearly indicate the majorizing correlation matrices (the red dots in Figure 7). The empirical threshold for $\theta(\mathbf{C}_t)$ that distinguishes majorizing from majorized matrices is 0.54. This value is close to the theoretical one (0.5) we expect from Equation (16). The periods of high portfolio risk are quite evident, and consistent with the recent economic history.

In the 3-cluster case of Subfigure 8b, the yellow areas show those correlation matrices that correspond to transitions between the majorizing/riskier (red) and the majorized/less risky (white) situations. In the INDJ dataset, a correlation matrix belongs to the yellow cluster if its θ falls in the interval $[0.34, 0.68]$.



(a) 2-cluster case, discriminating threshold at 0.54.



(b) 3-cluster case, discriminating thresholds at 0.34 and 0.68.

Figure 8

Subfigure 8b can thus represent a first example of alarm system based on quantum majorization. If we monitor the θ index over time, the closer (from below) the value to the 0.68 threshold, the higher the chance of entering into a risky period of strong quantum majorization. The lower θ, the smaller our portfolio risk.

As in all alarm systems [29], false alarms represent a problem, which requires serious treatment. For example, in Subfigure 8b, around year 2013, we see that θ oscillates a lot in the yellow area. This could possibly generate a number of false alarms regarding a possible increase in quantum majorization, which we should treat.

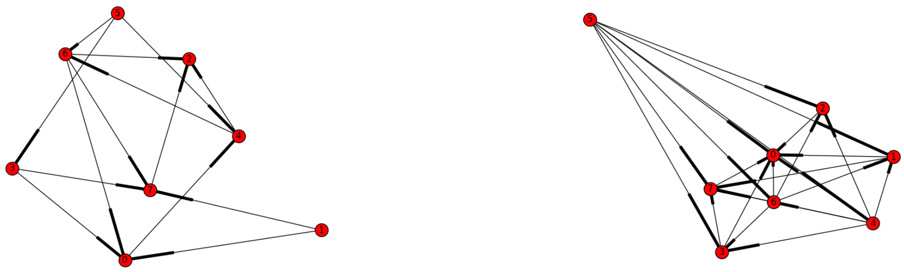
The development of a full alarm system based on quantum majorization goes beyond the scope of the present paper, but it will surely be object of future research. A fascinating possibility could be to build a urn-based alarm system, similar to the one introduced in [8].

6 A new insight

In studying quantum majorization we have noticed an important connection with network analysis, something that, in our knowledge, has never been considered before. The fascinating consequence of this liaison is that it could facilitate the graphical exploration of portfolio risk.

Just notice that a quantum majorization matrix $\mathbf{A}$ can be seen as the adjacency matrix of a directed graph [24]. If the non-similarity assumption holds, then $\bar{\mathbf{A}} = \mathbf{A} - \mathbf{I}$ is the adjacency matrix of a directed acyclical graph.

We call the graph associated to $\mathbf{A}$ majorization graph, and the space of all $n \times n$ correlation matrices $\mathcal{C}$ is now the vertex space containing all the possible vertices the graph can have. The collection $\{\mathbf{C}_t\}_{t=1}^{T}$ represents the set of vertices that actually compose the graph in reality, and the set of directed edges among the vertices is given by the corresponding quantum majorization matrix. If $\mathbf{C_i} \overset{\lambda}{\succ} \mathbf{C_j}$, there is a directed edge from $\mathbf{C_i}$ to $\mathbf{C_j}$, while for $\mathbf{C_i} \prec^{\lambda} \mathbf{C_j}$ the directed edge goes from $\mathbf{C_j}$ to $\mathbf{C_i}$. In case of no majorization, $\mathbf{C_i} \not\succ \mathbf{C_j}$, no edge is present. This makes the majorization graph a non-trivial lattice[9] [24].



(a) Case 1

(b) Case 2

Figure 9: Examples of graphs associated to the quantum majorization matrices of Figure 1. Thick lines represent the direction of the edges between the nodes, and can ve read as arrows.

Figure 9 shows an example of graphs associated to the two quantum majorization matrices presented in Figure 1.

A majorization graph can be thought as a special case of directed inhomogeneous graph, in which the connection probability is different for each vertex, and it depends on the size of the permutraedon spanned by the spectrum of the correlation matrix associated to that vertex [9].

Thanks to the graph representation of majorization, one can immediately observe some interesting connections with what we have developed in this paper. For example, the risk measure $\theta(\mathbf{C_i})$ corresponds to the difference between the out-degree and the in-degree centralities of the vertex associated to $\mathbf{C_i}$, while measure U is nothing but the density of the graph.

A majorization graph could also be used to build and calibrate a stochastic model to predict the dynamics of a correlation matrix, for instance by exploiting the approach of [24].

Another active field of research in networks theory is the detection of communities [39]. In a majorization graph, communities can be understood as nodes corresponding to correlation matrices with a similar behaviour. Their individuation could therefore lead to results compatible with those discussed in Section 5, also extending them towards the development of a taxonomy of communities for crisis detection.

In conclusion, we believe that the link between quantum majorization and random graphs represents a fruitful line of future research, which we are willing to explore.

Acknowledgments The first, third and forth authors gladly acknowledge the generous support of the EU H2020 Marie Sklodowska-Curie Grant, The Netherlands Agreement No 643045 WakE-UpCall. Most of the contents in the article have been developed during the research visit of the first author at Tel Aviv University supported by the EU H2020 Marie Sklodowska-Curie Grant.

[9] Note that, unlike most direct acyclical graphs in the literature[24], the flow to and from the vertices in the graph does not depend on their time index, but rather on their ordering.

References

[1] Peter M Alberti and Armin Uhlmann. *Stochasticity and partial order.* Deutscher Verlag der Wissenschaften, 1982.

[2] Tsuyoshi Ando. Majorizations and inequalities in matrix theory. *Linear algebra and its Applications*, 199:17–67, 1994.

[3] Barry C Arnold and José María Sarabia. *Majorization and the Lorenz order with applications in applied mathematics and economics.* Springer, 2018.

[4] Anthony B Atkinson. On the measurement of inequality. *Journal of economic theory*, 2(3):244–263, 1970.

[5] Zhidong Bai and Jack W Silverstein. *Spectral Analysis of Large Dimensional Random Matrices.* Springer, 2009.

[6] Emilio Barucci. *Financial markets theory : equilibrium, efficiency and information.* Springer, 2003.

[7] Hans Bühlmann. *Mathematical Methods in Risk Theory.* Grundlehren der mathematischen Wissenschaften. Springer Berlin Heidelberg, 2005.

[8] Pasquale Cirillo, Jürg Hüsler, and Pietro Muliere. Alarm systems and catastrophes from a diverse point of view. *Methodology and Computing in Applied Probability*, 15(4):821–839, 2013.

[9] Geir Dahl. Majorization permutahedra and (0, 1)-matrices. *Linear Algebra and its Applications*, 432(12):3265–3271, 2010.

[10] Iddo Eliazar. How random is a random vector? *Annals of Physics*, 363:164–184, 2015.

[11] Iddo Eliazar. A tour of inequality. *Annals of Physics*, 389:306–332, 2017.

[12] Iddo Eliazar and Igor M Sokolov. Measuring statistical evenness: A panoramic overview. *Physica A: Statistical Mechanics and its Applications*, 391(4):1323–1353, 2012.

[13] Paul Embrechts, Claudia Klüppelberg, and Thomas Mikosch. *Modelling extremal events: for insurance and finance*, volume 33. Springer Science & Business Media, 2003.

[14] Frank J Fabozzi, Petter N Kolm, Dessislava A Pachamanova, and Sergio M Focardi. *Robust Portfolio Optimization and Management.* Wiley, 2007.

[15] Ky Fan. On a theorem of weyl concerning eigenvalues of linear transformations: Ii. *Proceedings of the National Academy of Sciences*, 36(1):31–35, 1950.

[16] Andrea Fontanari, Pasquale Cirillo, and Cornelis W Oosterlee. From concentration profiles to concentration maps. new tools for the study of loss distributions. *Insurance: Mathematics and Economics*, 78:13–29, 2018.

[17] Santo Fortunato and Darko Hric. Community detection in networks: A user guide. *Physics Reports*, 659:1–44, 2016.

[18] James E Gentle. *Numerical Linear Algebra for Applications in Statistics.* Statistics and Computing. Springer, 1998.

[19] Corrado Gini. Variabilità e mutabilità. *Reprinted in Memorie di metodologica statistica (Ed. Pizetti E, Salvemini, T). Rome: Libreria Eredi Virgilio Veschi*, 1912.

[20] Alessandra Giovagnoli and Mario Romanazzi. A group majorization ordeting for correlation matrices. *Linear Algebra and its Applications*, 127:139–155, 1990.

[21] Alessandra Giovagnoli and Henry P Wynn. G-majorization with applications to matrix orderings. *Linear algebra and its applications*, 67:111–135, 1985.

[22] Godfrey Harold Hardy, John Edensor Littlewood, and George Pólya. *Inequalities.* Cambridge university press, 1952.

[23] Richard A Johnson and Dean M Wichern. *Applied Multivariate Statistical Analysis 6th edition*. Pearson, 2007.

[24] Brian Karrer and Mark EJ Newman. Random graph models for directed acyclic networks. *Physical Review E*, 80(4):046110, 2009.

[25] Maurice G Kendall. A new measure of rank correlation. *Biometrika*, 30(1/2):81–93, 1938.

[26] Christian Kleiber and Samuel Kotz. *Statistical Size Distributions in Economics and Actuarial Sciences*. Wiley, 2003.

[27] Mark Kritzman. Risk disparity. *The Journal of Portfolio Management*, 40(1):40–48, 2013.

[28] Mark Kritzman, Yuanzhen Li, Sébastien Page, and Roberto Rigobon. Principal components as a measure of systemic risk. *The Journal of Portfolio Management*, 37(4):112–126, 2011.

[29] Georg Lindgren. Model processes in nonlinear prediction with application to detection and alarm. *The Annals of Probability*, 8(4):775–792, 1980.

[30] Max O Lorenz. Methods of measuring the concentration of wealth. *Publications of the American statistical association*, 9(70):209–219, 1905.

[31] Leon Mirsky. Symmetric gauge functions and unitarily invariant norms. *The quarterly journal of mathematics*, 11(1):50–59, 1960.

[32] Michael A Nielsen. Probability distributions consistent with a mixed state. *Physical Review A*, 62(5):052308, 2000.

[33] Michael A Nielsen and Isaac L Chuang. *Quantum computation and quantum information*. Cambridge University Press, Cambridge, 2000.

[34] Ingram Olkin and Albert W Marshall. *Inequalities: Theory of Majorization and Its Applications*, volume 143. Academic Press, 2016.

[35] Leonid A Pastur and Mariya Shcherbina. *Eigenvalue Distribution of Large Random Matrices*. American Mathematical Society, 2011.

[36] Gaetano Pietra. *Delle relazioni tra gli indici di variabilità*. Atti Del Reale Istituto Veneto Di Scienze, Lettere Ed Arti, Tomo LXXIV Parte II., 1915.

[37] Olivier Roy and Martin Vetterli. The effective rank: A measure of effective dimensionality. In *Signal Processing Conference, 2007 15th European*, pages 606–610. IEEE, 2007.

[38] Edna Schechtman and Shlomo Yitzhaki. On the proper bounds of the gini correlation. *Economics letters*, 63(2):133–138, 1999.

[39] Remco Van Der Hofstad. *Random graphs and complex networks*, volume 1. Cambridge university press, 2016.

[40] Leon van Hove. Von neumann's contributions to quantum theory. *Bull. Amer. Math. Soc.*, 64:95–99, 05 1958.

[41] John Von Neumann. Some matrix inequalities and metrization of metric space. *Tomsk Univ. Rev*, 1:286–296, 1937.

[42] Samuel S Wilks. Multidimensional statistical scatter. *Contributions to probability and statistics*, 2:486, 1960.

[43] Shlomo Yitzhaki and Edna Schechtman. *The Gini Methodology*. Springer, 2013.