CDS calibration under an extended JDCEV model

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Abstract

We propose a new methodology for the calibration of a hybrid credit-equity model to CDS spreads and survival probabilities. We consider an extended Jump to Default Constant Elasticity of Variance model incorporating stochastic and possibly negative interest rates. Our approach is based on a perturbation technique that provides an explicit asymptotic expansion of the CDS spreads. The robustness and efficiency of the method is confirmed by several calibration tests on real market data.

Keywords: credit default swap, hybrid credit-equity model, JDCEV model, asymptotic expansion

1 Introduction

The purpose of this paper is provide a robust and efficient method to calibrate a hybrid credit-equity model to the CDS market. Credit Default Swaps (CDS) are the most influential and traded credit derivatives. They played an important role in the recent financial scandals: in the sub-prime crisis in 2007-2008 or the trading losses by the "London Whale" at JP Morgan Chase in 2012. On the other hand, large global banks have been successfully exploiting the CDS market in their trading activities: for example, JP Morgan has several trillions of dollars of CDS notional outstanding. In parallel, the academic research on CDS, liabilities and derivatives in general has quickly expanded in the recent years. Among the most important contributions, the Jump to Default Constant Elasticity of Variance (JDCEV) model by Carr and Linetsky [1], [7] is one of the first attempts to unify credit and equity models into the framework of deterministic and positive interest rates. The authors of [1] claim that credit models should not ignore information on stocks and there exists a connection among stock prices, volatilities and default intensities. Indeed, earlier research on credit models (e.g. [2], [3], [4]) was more focused on how to palliate the absence of bankruptcy possibility in classical option pricing theory and take into account that in real world firms have a positive probability of default in finite time.

Nowadays the restrictive assumption of *positive and deterministic interest rates* of the JDCEV model is not realistic and contradicts market observations. The purpose of this study is, on the one hand, to incorporate stochastic and possibly negative interest rates into the JDCEV model; on the other hand,

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we aim at providing a fast and efficient technique to compute CDS spreads and default probabilities for calibration purposes. In doing this we employ a recent methodology introduced in [6, 8], which consists in an asymptotic expansion of the solution to the pricing partial differential equation. Our method allows to calibrate the extended JDCEV model to real market data in real time. To assess the robustness of the approximation method and the capability of the model of reproducing price dynamics, we provide several tests on UBS and BNP Paribas CDS spreads.

This paper is organized as follows. In Section 2 we set the notations and review the jump to default diffusion model. In Section 3 we introduce an extended JDCEV model with stochastic interest rates and provide explicit approximation formulas for the CDS spreads and the risk-neutral survival probabilities. Section 4 contains the numerical tests: we consider both the cases of correlated or uncorrelated spreads and interest rates; we calibrate the model to market data of CDS spreads and compute the risk-neutral survival probabilities: a comparison with standard Monte Carlo methods is provided as well. Appendix 5 contains auxiliary results and technical proofs.

2 CDS spread and default probability

We consider a probability space $(\Omega, \mathcal{G}, \mathbb{Q})$ carrying a standard Brownian motion W and an exponential random variable $e \sim \text{Exp}(1)$ independent of W. We assume, for simplicity, a frictionless market, no arbitrage and take an equivalent martingale measure $\mathbb{Q}$ as given. All stochastic processes defined below live on this probability space and all expectations are taken with respect to $\mathbb{Q}$.

We assume that the pre-default stock log-price dynamics is given by

$$\begin{cases} \mathrm{d}X_t = \left(r_t - \frac{1}{2}\sigma^2\left(t, X_t\right) + \lambda\left(t, X_t\right)\right)\mathrm{d}t + \sigma\left(t, X_t\right)\mathrm{d}W_t^1, \\ \mathrm{d}r_t = \kappa\left(\theta - r_t\right)\mathrm{d}t + \delta \mathrm{d}W_t^2, \\ \mathrm{d}W_t^1 \mathrm{d}W_t^2 = \rho \mathrm{d}t, \end{cases}$$

where the interest rate r_t follows the Vasicek dynamics with parameters $\kappa, \theta, \delta > 0$. The time- and state-dependent stock volatility $\sigma = \sigma(t, X)$ and default intensity $\lambda = \lambda(t, X)$ are assumed to be differentiable with respect to X and uniformly bounded. In general the price can become worthless in two scenarios: either the process e^X hits zero via diffusion or a jump-to-default occurs from a positive value. The default time ζ can be modeled as $\zeta = \zeta_0 \wedge \tilde{\zeta}$, where $\zeta_0 = \inf\{t > 0 \mid e^{X_t} = 0\}$ is the first hitting time of zero for the stock price and $\tilde{\zeta} = \inf\{t \geq 0 \mid \Lambda_t \geq e\}$ is the jump-to-default time with intensity λ and hazard rate $\Lambda_t = \int_0^t \lambda(s, X_s)\mathrm{d}s$. In what follows, we denote by $\mathbb{F} = \{\mathcal{F}_t,\ t \geq 0\}$ the filtration generated by the pre-default stock price and by $\mathbb{D} = \{\mathcal{D}_t,\ t \geq 0\}$ the filtration generated by the process $D_t = \mathbb{1}_{\{\zeta \leq t\}}$. Eventually, $\mathbb{G} = \{\mathcal{G}_t,\ t \geq 0\},\ \mathcal{G}_t = \mathcal{F}_t \vee \mathcal{D}_t$ is the enlarged filtration.

A CDS is an agreement between two parties, called the protection buyer and the protection seller, typically designed to transfer to the protection seller the financial loss that the protection buyer would suffer if a particular default event happened to a third party, called the reference entity. The protection seller delivers a protection payment to the protection buyer at the time of the default event. In exchange the protection buyer makes periodic premium payments at time intervals α at the credit default swap rate up to the default event or the expiry maturity, whichever comes first. The protection payment is the specified percentage $(1-\eta)$ of the CDS notional amount $\mathcal{N}$, called *loss-given-default*. The valuation problem is to determine the arbitrage-free CDS rate R that makes the present value of the CDS contract equal to zero. This rate equates the present value of the protection payoff to the present value of all the premium payments.

Proposition 2.1. *Let T be the expiry date of the CDS contract, M be the total number of premium payments and t_i be the i-th periodic premium payment date, so that $t_{i+1}-t_i=\frac{T}{M}$. Then we have*

$$R=\frac{(1-\eta)\left(1-E\left[e^{-\int_0^T(r_u+\lambda(u,X_u))\mathrm{d}u}\right]-\int_0^T E\left[e^{-\int_0^s(r_u+\lambda(u,X_u))\mathrm{d}u}r_s\right]\mathrm{d}s\right)}{\frac{T}{M}\sum_{i=1}^{M}E\left[e^{-\int_0^{t_i}(r_u+\lambda(u,X_u))\mathrm{d}u}\right]}. \tag{2.1}$$

Proof. By Corollary 5.5 in Appendix 5 and assuming a unit notional, the protection and premium legs at time t are given by:

$$\begin{aligned}\mathrm{PV}\,(\text{Protection leg})&=E\left[e^{-\int_t^{\zeta}r_u\mathrm{d}u}\,(1-\eta)\,\mathbb{1}_{\{\zeta\le T\}}|\mathcal{G}_t\right]\\&=\mathbb{1}_{\{\zeta>t\}}\,(1-\eta)\int_t^T E\left[e^{-\int_t^s(r_u+\lambda(u,X_u))\mathrm{d}u}\lambda(s,X_s)|\mathcal{F}_t\right]\mathrm{d}s,\\\mathrm{PV}\,(\text{Premium leg})&=\sum_{i=1}^{M}E\left[e^{-\int_t^{t_i}r_u\mathrm{d}u}\,\tfrac{T}{M}R_t\mathbb{1}_{\{\zeta>t_i\}}|\mathcal{G}_t\right]\\&=\tfrac{T}{M}R_t\mathbb{1}_{\{\zeta>t\}}\sum_{i=1}^{M}E\left[e^{-\int_t^{t_i}(r_u+\lambda(u,X_u))\mathrm{d}u}|\mathcal{F}_t\right].\end{aligned}$$

The CDS spread at time $t<T$ is given by equating the protection and premium legs and thus we get

$$R_t=\frac{\mathbb{1}_{\{\zeta>t\}}\,(1-\eta)\int_t^T E\left[e^{-\int_t^s(r_u+\lambda(u,X_u))\mathrm{d}u}\lambda(s,X_s)|\mathcal{F}_t\right]\mathrm{d}s}{\frac{T}{M}\mathbb{1}_{\{\zeta>t\}}\sum_{i=1}^{M}E\left[e^{-\int_t^{t_i}(r_u+\lambda(u,X_u))\mathrm{d}u}|\mathcal{F}_t\right]},$$

and in particular

$$R\equiv R_0=\frac{(1-\eta)\,E\left[\int_0^T e^{-\int_0^s(r_u+\lambda(u,X_u))\mathrm{d}u}\lambda(s,X_s)\mathrm{d}s\right]}{\frac{T}{M}\sum_{i=1}^{M}E\left[e^{-\int_0^{t_i}(r_u+\lambda(u,X_u))\mathrm{d}u}\right]}.$$

Next we use the identities

$$e^{-\int_0^s(r_u+\lambda(u,X_u))\mathrm{d}u}\lambda(s,X_s)=-\frac{\partial}{\partial_s}\left(e^{-\int_0^s(r_u+\lambda(u,X_u))\mathrm{d}u}\right)-r_se^{-\int_0^s(r_u+\lambda(u,X_u))\mathrm{d}u},$$

and

$$\int_0^T e^{-\int_0^s(r_u+\lambda(u,X_u))\mathrm{d}u}\lambda(s,X_s)\mathrm{d}s=1-\left(e^{-\int_0^T(r_u+\lambda(u,X_u))\mathrm{d}u}+\int_0^T e^{-\int_0^s(r_u+\lambda(u,X_u))\mathrm{d}u}r_s\mathrm{d}s\right).$$

The thesis easily follows. □

Remark 2.2. The default intensity $\lambda(t, X_t)$ can be considered as the instantaneous probability that the stock will default between t and $t + \mathrm{d}t$, conditioned on the fact that no default has happened before:

$$\lambda(t, X_t) = \mathbb{Q}\left(t \leq \zeta < t + \mathrm{d}t \mid \zeta \geq t\right).$$

The survival probability up to time t is defined as

$$Q\left(t\right) := E\left[e^{-\int_0^t \lambda(u, X_u)\mathrm{d}u}\right]. \tag{2.2}$$

3 CDS spread approximation under extended JDCEV model

In the JDCEV model the stock volatility is of the form

$$\sigma(t, X) = a(t)e^{(\beta-1)X}$$

where $\beta < 1$ and $a(t) > 0$ are the so-called elasticity parameter and scale function. The default intensity is expressed as a function of the stock volatility and the stock log-price, as follows

$$\lambda(t, X) = b(t) + c\,\sigma(t, X)^2 = b(t) + c\,a(t)^2 e^{2(\beta-1)X} \tag{3.1}$$

where $b(t) \geq 0$ and $c \geq 0$ govern the sensitivity of the default intensity with respect to the volatility. The risk-neutral dynamics of the defaultable stock price $S_t = \{S_t,\ t \geq 0\}$ are then given by

$$\begin{cases} S_t = S_0 e^{X_t} \mathbb{1}_{\{\zeta \geq t\}}, \qquad S_0 > 0, \\ \mathrm{d}X_t = \left(r_t - \frac{1}{2}\sigma^2\left(t, X_t\right) + \lambda\left(t, X_t\right)\right)\mathrm{d}t + \sigma\left(t, X_t\right)\mathrm{d}W_t^1, \\ \mathrm{d}r_t = \kappa\left(\theta - r_t\right)\mathrm{d}t + \delta \mathrm{d}W_t^2, \\ \zeta = \inf\{t \geq 0 \mid \int_0^t \lambda\left(t, X_t\right) \geq e\}, \\ \mathrm{d}W_t^1 \mathrm{d}W_t^2 = \rho \mathrm{d}t. \end{cases} \tag{3.2}$$

Let us consider a European claim on the defaultable asset, paying $h(X_T)$ at maturity T if no default happens and without recovery in case of default. In case of *constant interest rates*, one deduces the value of the European claim from the following result proved in [1].

Theorem 3.1. *Let r be a non-negative constant and h be a continuous and bounded function. Then, for any $0 \leq t \leq T$, we have*

$$E\left[\exp\left(-c\int_t^T a(u)^2 e^{2(\beta-1)X_u}\mathrm{d}u\right) h\left(X_T\right)|X_t = X_0\right] = E\left[\left(\frac{Z_\tau}{x}\right)^{-\frac{1}{|\beta-1|}} h\left(e^{\int_t^T \alpha(s)\mathrm{d}s}\left(|\beta-1|Z_\tau\right)^{\frac{1}{|\beta-1|}}\right)\right], \tag{3.3}$$

where $\{Z_t,\ t \geq 0\}$ is a Bessel process starting from x, of index $\nu = \frac{c+1/2}{|\beta-1|}$, and τ is the deterministic time change defined as

$$\tau\left(t\right) = \int_0^t a^2(u) e^{2|\beta-1|\int_0^s \alpha_s \mathrm{d}s}\mathrm{d}u, \qquad \alpha(t) = r + b(t). \tag{3.4}$$

By Theorem 3.1 and standard results from enlargement filtration theory (cf. [4]), the value of the European claim at time $t < T$ is given by

$$
\begin{aligned}
E\left[e^{-\int_t^T r_u \mathrm{d}u} h\left(X_T\right)|\mathcal{G}_t\right] &= \mathbb{1}_{\{\zeta > t\}} E\left[e^{-\int_t^T (r_u+\lambda(u,X_u))\mathrm{d}u} h\left(X_T\right)|\mathcal{F}_t\right] \\
&= \mathbb{1}_{\{\zeta > t\}} e^{-\int_t^T (r_u+b_u)\mathrm{d}u} E\left[e^{-c\int_t^T a_u^2 e^{2(\beta-1)X_u}\mathrm{d}u} h\left(X_T\right)|\mathcal{F}_t\right] \\
&= \mathbb{1}_{\{\zeta > t\}} e^{-\int_t^T (r_u+b_u)\mathrm{d}u} E\left[\left(\frac{Z_\tau}{x}\right)^{-\frac{1}{|\beta-1|}} h\left(e^{\int_t^T \alpha_s \mathrm{d}s}\left(|\beta-1|Z_\tau\right)^{\frac{1}{|\beta-1|}}\right)\right].
\end{aligned}
\tag{3.5}
$$

The validity of the second and third equalities above is based on the assumption of deterministic interest rates. In the general case of stochastic rates, the time-change function (3.4) is not deterministic anymore and the expectation (3.3) cannot be computed analytically. For this reason, to deal with the general case, we adopt a completely different approach and introduce a perturbation technique which provides an explicit asymptotic expansion of the building block (3.5). Specifically, we base our analysis on the recent results in [6, 8] on the approximation of solution to parabolic partial differential equations and we derive approximations of the CDS spread (2.1) and the risk-neutral survival probability (2.2).

To present our main results, we consider the following general backward Cauchy problem

$$
\begin{cases}
\left(\partial_t + \mathcal{A}\right) u\left(t,z\right) = 0, & t < T,\ z \in \mathbb{R}^d, \\
u\left(T,z\right) = h\left(z\right), & z \in \mathbb{R}^d,
\end{cases}
\tag{3.6}
$$

where $\mathcal{A} = \mathcal{A}(t,z)$ is a (locally) parabolic differential operator of the form

$$
\mathcal{A}\left(t,z\right) = \sum_{|\alpha|\le 2} a_\alpha\left(t,z\right) D_z^\alpha, \qquad t \in \mathbb{R}^+,\ z \in \mathbb{R}^d,
$$

where

$$
\alpha = \left(\alpha_1,\dots,\alpha_d\right),\ |\alpha| = \sum_{i=1}^d \alpha_1 + \ldots + \alpha_d,\ D_z^\alpha = \partial_{z_1}^{\alpha_1}\ldots\partial_{z_d}^{\alpha_d}.
$$

In our specific setting, we will consider $\mathcal{A}$ to be the infinitesimal generator of the stochastic processes (X,r) in (3.2), whose precise expression in given in formula (3.14).

Next, we consider the formal expansions $\mathcal{A} = \sum_n \mathcal{A}_n$ and $u = \sum_n u_n$, where the u_n's, for $n \ge 0$, are defined recursively by

$$
\begin{cases}
(\partial_t + \mathcal{A}_0)u_0(t,z) = 0, & t < T,\ z \in \mathbb{R}^d, \\
u_0(T,z) = h(z), & z \in \mathbb{R}^d,
\end{cases}
$$

and

$$
\begin{cases}
(\partial_t + \mathcal{A}_0)u_n(t,z) = -\sum_{k=1}^n \mathcal{A}_k u_{n-k}(t,z), & t < T,\ z \in \mathbb{R}^d, \\
u_n(T,z) = 0, & z \in \mathbb{R}^d,
\end{cases}
$$

where

$$\mathcal{A}_n = \sum_{|\alpha|\leq 2} a_{\alpha,n}(t,z) D_z^{\alpha}. \tag{3.7}$$

In (3.7), $(a_{\alpha,n})_{0\leq n\leq N}$ is the N-th order Taylor expansion of a_α, in the spatial variables, around a fixed point $\bar{z}$. Notice that the functions $a_{\alpha,0}$ depend only on t: hence $\mathcal{A}_0$ is a heat operator with time-dependent coefficients and can be written in the form

$$\mathcal{A}_0 = \frac{1}{2}\sum_{i,j=1}^{d} C_{ij}(t)\,\partial_{z_i z_j} + \sum_{i=1}^{d} m_i(t)\,\partial_{z_i} + \gamma(t),$$

for some $C = (C_{ij})_{i,j\leq d} \in \mathbb{R}^{d\times d}$, $m = (m_i)_{i\leq d} \in \mathbb{R}^d$ and $\gamma \in \mathbb{R}$. It turns out that, for any $n \geq 0$, u_n can be computed explicitly, as the following result shows.

Theorem 3.2. *We have*

$$u_n(t,z;T) = \mathcal{L}_n^z(t,T)\,u_0(t,z;T), \qquad t<T,\ z\in\mathbb{R}^d,\ n\geq 1, \tag{3.8}$$

where

$$u_0(t,z;T) = e^{\int_t^T \gamma(s,z)\mathrm{d}s}\int_{\mathbb{R}^d}\Gamma_0(t,z;T,\xi)\,h(\xi)\,\mathrm{d}\xi, \quad t<T,\ z\in\mathbb{R}^d,$$

and Γ_0 is the d-dimensional Gaussian density

$$\Gamma_0(t,z;T,\xi) = \frac{1}{\sqrt{2\pi^d \det C(t,T)}}\exp\left(-\frac{1}{2}\langle C^{-1}(t,T)(\xi - z - m(t,T)),(\xi - z - m(t,T))\rangle\right),$$

with covariance matrix $C(t,T)$ and mean vector $z + m(t,T)$ given by

$$C(t,T) = \int_t^T C(s)ds, \qquad m(t,T) = \int_t^T m(s)ds. \tag{3.9}$$

In (3.8), $\mathcal{L}_n^z(t,T)$ denotes the differential operator acting on the z-variable and defined as

$$\mathcal{L}_n^z(t,T) = \sum_{h=1}^{n}\int_t^T \mathrm{d}s_1\int_{s_1}^T \mathrm{d}s_2 \ldots \int_{s_{h-1}}^T \mathrm{d}s_h \sum_{i\in I_{n,h}} \mathcal{G}_{i_1}^z(t,s_1)\ldots\mathcal{G}_{i_h}^z(t,s_h), \tag{3.10}$$

where

$$I_{n,h} = \{i = (i_1,\ldots,i_h)\in\mathbb{N}^h \mid i_1 + i_2 + \ldots + i_h = n\}$$

and the operators $\mathcal{G}_n^z(t,s)$ are defined as

$$\mathcal{G}_n^z(t,s) = \sum_{|\alpha|\leq 2} a_{\alpha,n}(s, M^z(t,s))\,D_z^{\alpha},$$

with

$$M^z(t,s) = z + m(t,s) + C(t,s)\,D_z.$$

Proof. See [6]. □

Under rather general assumptions on $\mathcal{A}$, the following estimate for the approximation error holds:

$$|u(t,z) - u_N(t,z;T)| \le C(T-t)^{\frac{N+2}{2}} \tag{3.11}$$

where $u_N(t,z;T)$ is the N-th order approximation in (3.8) and C is a positive constant dependent on n. Formula (3.11) ensures the short-time asymptotic convergence of the approximation u_n to the exact solution u of problem (3.6). This theoretical result can be proved by adapting the arguments of [8], Theor. 3.1, and will be confirmed by the numerical tests in Section 4.

Going back to CDS spread approximation, we see from (2.1) that we have to evaluate expectations of the form

$$u(0, X_0, r_0; T) = E\left[e^{-\int_0^T (r_u + \lambda(u, X_u))\mathrm{d}u} h(r_T)\right], \tag{3.12}$$

with $h(r) = 1$ or $h(r) = r$. By the change of variable $r_t = e^{-\kappa t} y_t$ and from the Feynman-Kac formula (cf., for instance, [9]) it follows that u in (3.12) is solution to the Cauchy problem

$$\begin{cases} (\partial_t + \mathcal{A})\, u(t,x,y) = 0, & t < T,\ x, y \in \mathbb{R} \\ u(T,x,y) = h(y), & x, y \in \mathbb{R}, \end{cases} \tag{3.13}$$

where

$$\begin{aligned} \mathcal{A} = &\frac{1}{2}\sigma^2(t,x)\,\partial_{xx} + \rho\delta\sigma(t,x)e^{\kappa t}\partial_{xy} + \frac{1}{2}\delta^2 e^{2\kappa t}\partial_{yy} \\ &+ \left(e^{-\kappa t}y + \lambda(t,x) - \frac{1}{2}\sigma^2(t,x)\right)\partial_x + \kappa\theta e^{\kappa t}\partial_y - \left(e^{-\kappa t}y + \lambda(t,x)\right). \end{aligned} \tag{3.14}$$

Theorem 3.3. *Under the assumptions of Proposition 2.1 and under the general dynamics (3.2), the N-th order approximation of the CDS spread in (2.1) is given by*

$$R_N = \frac{(1-L)\left(1 - \sum_{n=0}^{N} \mathcal{L}_n^{(x,y)}(0,T)\, u_0^1(0,x,y;T) - \int_0^T e^{-\kappa s} \sum_{n=0}^{N} \mathcal{L}_n^{(x,y)}(0,s)\, u_0^2(0,x,y;s)\,\mathrm{d}s\right)}{\frac{T}{M}\sum_{i=1}^{M}\sum_{n=0}^{N} \mathcal{L}_n^{(x,y)}(0,t_i)\, u_0^1(0,x,y;t_i)}, \tag{3.15}$$

where

$$u_0^1(t,x,y,s) = e^{-\int_t^s \left(e^{-\kappa u}y + \lambda(u,x)\right)\mathrm{d}u},$$
$$u_0^2(t,x,y;s) = e^{-\int_t^s \left(e^{-\kappa u}y + \lambda(u,x)\right)\mathrm{d}u}\left(y + m_2(t,s)\right),$$

$m_2(t,s)$ is the second component of the vector $m(t,s)$ in (3.9) and the differential operators $\mathcal{L}_n^{(x,y)}$ can be computed explicitly as in Theorem (3.2).

Proof. In formula (2.1) there appear terms of the form

$$E\left[e^{-\int_0^t (r_u + \lambda(u,X_u))\mathrm{d}u}\right]$$

in the numerator and denominator, that are solutions to problem (3.13) with $h(y) = 1$. On the other hand, in (2.1) there also appear terms of the form

$$E\left[e^{-\int_0^t (r_u+\lambda(u,X_u))\mathrm{d}u} r_t\right]$$

which are solutions to the same problem with $h(y) = e^{-\kappa t} y$. Theorem 3.2 and (3.11) yield the approximations

$$E\left[e^{-\int_0^t (r_u+\lambda(u,X_u))\mathrm{d}u}\right] = \sum_{n=0}^{N} \mathcal{L}_n^{(x,y)}(0,t)\, u_0^1(0,x,y;t) + \mathrm{O}\left(t^{\frac{N+2}{2}}\right),$$

$$E\left[e^{-\int_0^t (r_u+\lambda(u,X_u))\mathrm{d}u} r_t\right] = e^{-\kappa t}\sum_{n=0}^{N} \mathcal{L}_n^{(x,y)}(0,t)\, u_0^2(0,x,y;t) + \mathrm{O}\left(t^{\frac{N+2}{2}}\right),$$

as $t \to 0^+$, where

$$\begin{aligned} u_0^1(0,x,y,t) &= e^{-\int_0^t \left(e^{-\kappa u}y+\lambda(u,x)\right)\mathrm{d}u} \int_{\mathbb{R}^2} \Gamma_0(0,x,y;t,\xi_1,\xi_2)\,\mathrm{d}\xi_1\mathrm{d}\xi_2, \\ &= e^{-\int_0^t \left(e^{-\kappa u}y+\lambda(u,x)\right)\mathrm{d}u}, \end{aligned}$$

and

$$\begin{aligned} u_0^2(0,x,y;t) &= u_0^1(0,x,y,t) \int_{\mathbb{R}^2} \Gamma_0(0,x,y;t,\xi_1,\xi_2)\, h(\xi_2)\,\mathrm{d}\xi_1\mathrm{d}\xi_2, \\ &= u_0^1(0,x,y,t) \int_{\mathbb{R}^2} \Gamma_0(0,x,y;t,\xi_1,\xi_2)\, \xi_2\mathrm{d}\xi_1\mathrm{d}\xi_2 \\ &= u_0^1(0,x,y,t)\,(y + m_2(0,t)). \end{aligned}$$

□

Remark 3.4. We have an analogous approximation result for the survival probability in (2.2). Since it can be expressed as the solution to the problem

$$\begin{cases} (\partial_t + \mathcal{A})\, v(t,x,y) = 0, & t < T,\ x,y \in \mathbb{R} \\ v(T,x,y) = 1, & x,y \in \mathbb{R}, \end{cases}$$

then, by Theorem 3.2, we have

$$Q(t) = \sum_{n\geq 0} \mathcal{L}_n^x(0,t)\, v_0(0,x,y;T), \tag{3.16}$$

where $v_0(t,x,y;T) = \exp\left(-\int_t^T \lambda(u,x)\mathrm{d}u\right)$ and the operators $\mathcal{L}_n^x(0,t)$ are defined as in (3.10). As an example, we give here the explicit first order approximation of the survival probability in the case of constant parameters $a(t) = a$ and $b(t) = b$:

$$\begin{aligned} v_0(0,x,y;T) &= e^{-\left(b+ca^2e^{2(\beta-1)x}\right)T}, \\ v_1(t,x,y;T) &= v_0(t,x,y;T) - ce^{2(\beta-1)x-T\left(b+ca^2e^{(\beta-1)x}\right)}(\beta-1)\,a^2\cdot \\ &\quad\cdot \frac{-4y+4\theta+e^{-T\kappa}\left(4y-4\theta+e^{T\kappa}T\kappa\left(4y-4\theta+e^{T\kappa}\left(2(b+\theta)+(-1+2c)\,e^{2(\beta-1)x}a^2\right)\right)\right)}{2\kappa^2}. \end{aligned}$$

4 CDS calibration and numerical tests

In this section we apply the method developed in Section 3 to calibrate the model to market CDS spreads. We use quotations for different companies (specifically, UBS and BNP Paribas) in order to check the robustness of our methodology. The calibration is based on a two-step procedure: we first calibrate separately the interest rate model to daily yields curves for zero-coupon bonds (ZCB), generated using Libor swap curve. Subsequently, we consider CDS contracts with different maturity dates. We use the approximation formulas (3.15) and (3.16) for the CDS spreads and survival probabilities, respectively. We use second-order approximations: we have found these to be sufficiently accurate by numerical experiments and theoretical error estimates. The formulas for the second-order approximation are simple, making the method easy to implement.

We distinguish between the uncorrelated and correlated cases: in the first case, i.e. when $\rho = 0$, the survival probability, which is not quoted from the market, can be inferred from the CDS spreads through a bootstrapping formula and therefore it is possible to calibrate directly to the survival probabilities. In the general case when $\rho \neq 0$, we calibrate to the market spreads using formula (3.15).

To add more flexibility to the model, we assume that the coefficients $a(t)$ and $b(t)$ in (3.1) are linearly dependent on time: more precisely, we assume that

$$a(t) = a_1 t + a_2, \qquad b(t) = b_1 t + b_2,$$

for some constants $a_1,\ a_2,\ b_1$, and b_2.

As defined in (3.2), the stochastic interest rate is described by a Vasicek model

$$\mathrm{d}r_t = \kappa\left(\theta - r_t\right)\mathrm{d}t + \delta \mathrm{d}W_t^2.$$

Apart from its simplicity, one of the advantages of this model is that interest rates can take negative values. For the calibration, we use the standard formula for the price $P_t\left(T\right)$ of a T-bond, which we recall here for convenience:

$$P_t\left(T\right) = A_t\left(T\right) e^{-B_t(T) r_t},$$

where

$$A_t\left(T\right) = e^{\left(\theta - \frac{\delta^2}{2\kappa^2}\right)\left(B_t(T) - T + t\right) - \frac{\delta^2}{4\kappa} B_t(T)}, \qquad B_t\left(T\right) = \frac{1 - e^{-\kappa(T-t)}}{\kappa}.$$

The results of the interest rate calibration are given in Table 1.

4.1 CDS calibration

We consider UBS and BNP Paribas senior CDS contracts with expiry dates up to 6 years and paid quarterly with a recovery rate of 40% at the event of default. We first consider the uncorrelated case ($\rho = 0$): in Tables 2 and 4 we present the calibration of the model to the market UBS and BNP Paribas CDS spreads. Columns 2 and 3 contain the market and the model CDS spreads respectively; in the last

Maturity (years)	market ZCB	model ZCB	error
1	1.00229	1.00675	0.389355 %
2	1.00372	1.00906	0.532726 %
3	1.00333	1.0075	0.415164 %
4	1.00099	1.00257	0.157386 %
5	0.995825	0.994877	-0.0952038%
6	0.987805	0.984878	-0.296311%
7	0.976833	0.972974	-0.395081%
8	0.963223	0.959587	-0.377492%
9	0.947687	0.945069	-0.276223%
10	0.932845	0.929643	-0.343245 %

Table 1: Calibration to ZCB: $\kappa = 0.045$, $\theta = 0.103$, $\delta = 0.021$ and $r_0 = -0.009$

Maturity	market spread (bps)	model spread (bps)	error
0.5	21.88	21.7414	-0.633298 %
1	25.72	26.0506	1.2855 %
2	35.105	34.81	-0.840299%
3	43.97	43.73	-0.545896 %
4	52.3	52.7784	0.914736 %
5	61.91	61.9249	0.0240213 %
6	71.285	71.1412	-0.201668 %

Table 2: Calibration to UBS CDS spreads (uncorrelated case): $a_1 = 0.034,\ a_2 = 0.05,\ \beta = 0.73,\ b_1 = 0.003,\ b_2 = 0.003,\ c = 0.044$

Maturity	market probability	model probability	Monte Carlo
0.5	0.99818	0.998189	$[0.998193, 0.998193]$
1	0.99572	0.995667	$[0.995681, 0.995681]$
2	0.98837	0.988462	$[0.988521, 0.988521]$
3	0.97823	0.978346	$[0.978483, 0.978485]$
4	0.96564	0.965305	$[0.965557, 0.965562]$
5	0.94944	0.94935	$[0.949772, 0.949784]$
6	0.93056	0.930517	$[0.931157, 0.931178]$

Table 3: Risk-neutral UBS survival probabilities (uncorrelated case)

column the corresponding relative errors are given. In Tables 3 and 5 the results are compared with a standard Monte Carlo approximation.

Next we consider the general case when ρ is not necessarily null. Tables 6 and 8 show the results of

Maturity	market spread (bps)	model spread (bps)	error
0.5	29.885	29.676	-0.699438 %
1	34.615	34.6479	0.0950748 %
2	45.115	45.3808	0.589209%
3	56.11	57.1068	1.77644 %
4	72.59	69.7438	-3.92092 %
5	82.27	83.2102	1.14283 %
6	96.705	97.4355	0.755409 %

Table 4: Calibration to BNP Paribas CDS spreads (uncorrelated case): $a_1 = 0.04,\ a_2 = 0.0,\ \beta = 0.5,\ b_1 = 0.003,\ b_2 = 0.004,\ c = 0.16$

Maturity	market probability	model probability	Monte Carlo
0.5	0.99425	0.997529	$[0.997533, 0.997533]$
1	0.99425	0.994242	$[0.994258, 0.994258]$
2	0.98508	0.984982	$[0.985055, 0.985056]$
3	0.97230	0.971786	$[0.971978, 0.971982]$
4	0.95254	0.954274	$[0.954678, 0.954694]$
5	0.93328	0.93212	$[0.932902, 0.932944]$
6	0.90887	0.905038	$[0.906404, 0.9065]$

Table 5: Risk-neutral BNP Paribas survival probabilities (uncorrelated case)

the calibration to the market UBS and BNP Paribas CDS spreads, respectively. We use the calibrated parameters to compute the survival probabilities via the approximation formula (3.16) and compare them with the ones calculated by Monte Carlo simulation, see Tables 7 and 9.

Maturity	market spread (bps)	model spread (bps)	error
0.5	21.88	21.6822	1.37983 %
1	25.72	26.0781	1.37983 %
2	35.105	34.8897	-0.720097%
3	43.97	43.7306	-0.580432 %
4	52.3	52.6546	0.769752 %
5	61.91	61.7995	-0.0627898 %
6	71.285	71.4283	0.0279619%

Table 6: Calibration to UBS CDS spreads: $a_1 = 0.08,\ a_2 = 0.0,\ \beta = -1.3,\ b_1 = 0.003,\ b_2 = 0.003,\ c = 0.001$ and $\rho = -0.018$

Maturity	approx.	Monte Carlo
0.5	0.998194	$[0.998198, 0.998198]$
1	0.995663	$[0.995677, 0.995677]$
2	0.988437	$[0.98828, 0.988505]$
3	0.97835	$[0.974545, 0.9758]$
4	0.965397	$[0.958105, 0.95988]$
5	0.949471	$[0.942493, 0.944291]$
6	0.930256	$[0.923795, 0.925681]$

Table 7: Risk-neutral UBS survival probabilities (correlated case): comparison between the value obtained via the approximation formula (3.16) and standard Monte Carlo simulation

Maturity	market spread (bps)	model spread (bps)	error
0.5	29.885	29.7742	-0.370808 %
1	34.615	34.6073	-0.0222804 %
2	45.115	45.2691	0.341665 %
3	56.11	57.1111	1.78411 %
4	72.59	69.8972	-3.70955 %
5	82.27	83.3487	1.31112 %
6	96.705	97.176	0.487021 %

Table 8: Calibration to BNP Paribas CDS spreads: $a_1 = 0.02,\ a_2 = 0.02,\ \beta = -0.017,\ b_1 = 0.002,\ b_2 = 0.004,\ c = 0.86$ and $\rho = -0.9$

Maturity	approx.	Monte Carlo
0.5	0.997521	$[0.997525, 0.997525]$
1	0.994247	$[0.994264, 0.994265]$
2	0.984991	$[0.98509, 0.985093]$
3	0.971631	$[0.971968, 0.971979]$
4	0.953636	$[0.954551, 0.954581]$
5	0.930552,	$[0.932707, 0.932774]$
6	0.901985	$[0.906558, 0.90669]$

Table 9: Risk-neutral BNP Paribas survival probabilities (correlated case)

5 Appendix

We collect some results on hazard rate and conditional expectation with respect to enlarged filtrations. We present the key formula which relates the conditional expectation with respect to a "big" filtration to the conditional expectation with respect to a "small" filtration. For more about filtration enlargement, we refer for instance to [5].

Let ζ be a non-negative random variable on a probability space $(\Omega, \mathcal{G}, \mathbb{Q})$, such that $\mathbb{Q}(\zeta = 0) = 0$ and $\mathbb{Q}(\zeta > t) > 0$ for any $t \geq 0$. We introduce a right-continuous process D defined as $D_t = \mathbb{1}_{\{\zeta \leq t\}}$, and we denote by $\mathbb{D}$ the filtration generated by D; that is $\mathcal{D}_t = \sigma(D_u \mid u \leq t)$. Let $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ be a given filtration on $(\Omega, \mathcal{G}, \mathbb{Q})$ such that $\mathbb{G} := \mathbb{D} \vee \mathbb{F}$; that is we set $\mathcal{G}_t := \mathcal{D}_t \vee \mathcal{F}_t$ for every $t \in \mathbb{R}_+$. Since $\mathcal{D}_t \subseteq \mathcal{G}_t$ for any t, the random variable ζ is a stopping time with respect to $\mathbb{G}$. The financial interpretation is that the filtration $\mathbb{F}$ models the flow of observations available to the investors prior to the default time ζ. For any $t \in \mathbb{R}_+$, we write $F_t = \mathbb{Q}(\zeta \leq t | \mathcal{F}_t)$, so that $1 - F_t = \mathbb{Q}(\zeta > t | \mathcal{F}_t)$: notice that F is a bounded and non-negative $\mathbb{F}$-submartingale. We may thus deal with its right-continuous modification.

Definition 5.1. The $\mathbb{F}$-*hazard process* of ζ, denoted by Γ, is defined through the formula $1 - F_t = e^{-\Gamma_t}$ for every $t \in \mathbb{R}_+$.

Lemma 5.2. *We have $\mathcal{G}_t \subset \mathcal{G}_t^*$, where*

$$\mathcal{G}_t^* := \{A \in \mathcal{G} \mid \exists B \in \mathcal{F}_t \quad A \cap \{\zeta > t\} = B \cap \{\zeta > t\}\}.$$

Proof. Observe that $\mathcal{G}_t = \mathcal{D}_t \vee \mathcal{F}_t = \sigma(\mathcal{D}_t, \mathcal{F}_t) = \sigma(\{\zeta \leq u\}, u \leq t, \mathcal{F}_t)$. Also, it is easily seen that the the class $\mathcal{G}_t^*$ is a sub$-\sigma$-field of $\mathcal{G}$. Therefore, it is enough to check that if either $A = \{\zeta \leq u\}$ for $u \leq t$ or $A \in \mathcal{F}_t$, then there exists an event $B \in \mathcal{F}_t$ such that $A \cap \{\zeta > t\} = B \cap \{\zeta > t\}$. Indeed, in the former case we may take $B = \emptyset$, in the latter $B = A$. □

Lemma 5.3. *For any $\mathcal{G}$-measurable random variable Y we have, for any $t \in \mathbb{R}_+$*

$$E\left[\mathbb{1}_{\{\zeta > t\}} Y | \mathcal{G}_t\right] = \mathbb{1}_{\{\zeta > t\}} \frac{E\left[Y | \mathcal{F}_t\right]}{\mathbb{Q}(\zeta > t | \mathcal{F}_t)} = \mathbb{1}_{\{\zeta > t\}} e^{\Gamma_t} E\left[\mathbb{1}_{\{\zeta > t\}} Y | \mathcal{F}_t\right]. \tag{5.1}$$

Proof. Let us fix $t \in \mathbb{R}_+$. In view of the Lemma 5.2. any $\mathcal{G}_t$-measurable random variable coincides on the set $\{\zeta > t\}$ with some $\mathcal{F}_t$-measurable random variable. Therefore

$$E\left[\mathbb{1}_{\{\zeta > t\}} Y | \mathcal{G}_t\right] = \mathbb{1}_{\{\zeta > t\}} E\left[Y | \mathcal{G}_t\right] = \mathbb{1}_{\{\zeta > t\}} X,$$

where X is an $\mathcal{F}_t$-measurable random variable. Taking the conditional expectation with respect to $\mathcal{F}_t$, we obtain

$$E\left[\mathbb{1}_{\{\zeta > t\}} Y | \mathcal{F}_t\right] = \mathbb{Q}(\zeta > t | \mathcal{F}_t) X.$$

□

Proposition 5.4. *Let Z be a bounded $\mathbb{F}$-predictable process. Then for any $t < s \leq \infty$*

$$E\left[\mathbb{1}_{\{t < \zeta \leq s\}} Z_\zeta | \mathcal{G}_t\right] = \mathbb{1}_{\{\zeta > t\}} e^{\Gamma_t} E\left[\int_{]t,s]} Z_u dF_u | \mathcal{F}_t\right]. \tag{5.2}$$

Proof. We start by assuming that Z is a piecewise constant $\mathbb{F}$-predictable process, so that (we are interested only in values of Z for $u \in]t, s]$)

$$Z_u = \sum_{i=0}^{n} Z_{t_i} \mathbb{1}_{]t_i, t_{i+1}]}(u),$$

where $t = t_0 < \ldots < t_{n+1} = s$ and the random variable Z_{t_i} is $\mathcal{F}_{t_i}$-measurable. In the view of (5.1), for any i we have

$$\begin{aligned} E\left[\mathbb{1}_{\{t_i<\zeta\le t_{i+1}\}} Z_\zeta | \mathcal{G}_t\right] &= \mathbb{1}_{\{\zeta>t\}} e^{\Gamma_t} E\left[\mathbb{1}_{\{t_i<\zeta\le t_{i+1}\}} Z_{t_i} | \mathcal{F}_t\right] \\ &= \mathbb{1}_{\{\zeta>t\}} e^{\Gamma_t} E\left[Z_{t_i}(F_{t_{i+1}} - F_{t_i}) | \mathcal{F}_t\right]. \end{aligned}$$

In the second step we approximate an arbitrary bounded $\mathbb{F}$-predictable process by a sequence of piecewise constant $\mathbb{F}$-predictable process. □

Corollary 5.5. *Let Y be a $\mathcal{G}$-measurable random variable. Then, for any $t \le s$, we have*

$$E\left[\mathbb{1}_{\{\zeta>s\}} Y | \mathcal{G}_t\right] = \mathbb{1}_{\{\zeta>t\}} E\left[\mathbb{1}_{\{\zeta>s\}} e^{\Gamma_t} Y | \mathcal{F}_t\right]. \tag{5.3}$$

Furthermore, for any F_s-measurable random variable Y we have

$$E\left[\mathbb{1}_{\{\zeta>s\}} Y | \mathcal{G}_t\right] = \mathbb{1}_{\{\zeta>t\}} E\left[e^{\Gamma_t - \Gamma_s} Y | \mathcal{F}_t\right]. \tag{5.4}$$

If F (and thus Γ) is a continuous increasing process then for any $\mathbb{F}$-predictable bounded process Z we have

$$E\left[\mathbb{1}_{\{t<\zeta\le s\}} Z_\zeta | \mathcal{G}_t\right] = \mathbb{1}_{\{\zeta>t\}} E\left[\int_t^s Z_u e^{\Gamma_t - \Gamma_u} \mathrm{d}\Gamma_u | \mathcal{F}_t\right]. \tag{5.5}$$

Proof. In view of (5.1), to show that (5.3) holds, it is enough to observe that $\mathbb{1}_{\{\zeta>s\}} = \mathbb{1}_{\{\zeta>t\}}\mathbb{1}_{\{\zeta>s\}}$. Equality (5.4) is a straightforward consequence of (5.3). Formula (5.5) follows from (5.2) since, when F is increasing, $\mathrm{d}F_u = e^{-\Gamma_u}\mathrm{d}\Gamma_u$. □

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