CDS calibration under an extended JDCEV model

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Abstract
We propose a new methodology for the calibration of a hybrid credit-equity model to credit default swap (CDS) spreads and survival probabilities. We consider an extended Jump to Default Constant Elasticity of Variance model incorporating stochastic and possibly negative interest rates. Our approach is based on a perturbation technique that provides an explicit asymptotic expansion of the credit default swap spreads. The robustness and efficiency of the method is confirmed by several calibration tests on real market data.

KEYWORDS
credit default swap; hybrid credit-equity model; Constant Elasticity of Variance model; asymptotic expansion

1. Introduction

The purpose of this paper is to provide a robust and efficient method to calibrate a hybrid credit-equity model to the CDS market. Credit Default Swaps (CDS) are the most influential and traded credit derivatives. They played an important role in the recent financial scandals: in the sub-prime crisis in 2007-2008 or the trading losses by the “London Whale” at JP Morgan Chase in 2012. On the other hand, large global banks have been successfully exploiting the CDS market in their trading activities: for example, JP Morgan has several trillions of dollars of CDS notional outstanding. In parallel, the academic research on CDS, liabilities and derivatives in general has quickly expanded in the recent years. Among the most important contributions, the Jump to Default Constant Elasticity of Variance (JDCEV) model by Carr and Linetsky [2, 9, 10] is one of the first attempts to unify credit and equity models into the framework of deterministic and positive interest rates. The authors of [2] claim that credit models should

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not ignore information on stocks and there exists a connection among stock prices, volatilities and default intensities. Indeed, earlier research on credit models (e.g. [3, 4, 6]was more focused on how to palliate the absence of bankruptcy possibility in classical option pricing theory and take into account that in real world firms have a positive probability of default in finite time.

Nowadays the restrictive assumption of *positive and deterministic interest rates* of the JDCEV model is not realistic and contradicts market observations. The purpose of this study is, first, to incorporate stochastic and possibly negative interest rates into the JDCEV model and then, we propose a fast and efficient technique to compute CDS spreads and default probabilities for calibration purposes. In doing this we employ a recent methodology introduced in [8, 11], which consists of asymptotic expansion of the solution to the pricing partial differential equation. Our method allows to calibrate the extended JDCEV model to real market data in real time. To assess the robustness of the approximation method and the capability of the model of reproducing price dynamics, we provide several tests on UBS AG and BNP Paribas CDS spreads.

This paper is organized as follows. In Section 2 we set the notations and review the jump to default diffusion model. In Section 3 we introduce an extended JDCEV model with stochastic interest rates and provide explicit approximation formulas for the CDS spreads and the risk-neutral survival probabilities. Section 4 contains the numerical tests: we consider both the cases of correlated or uncorrelated spreads and interest rates; we calibrate the model to market data of CDS spreads and compute the risk-neutral survival probabilities: a comparison with standard Monte Carlo methods is provided as well. Appendix 5 contains auxiliary results and technical proofs.

2. CDS spread and default probability

We consider a probability space $(\Omega, \mathcal{G}, \mathbb{Q})$ carrying a standard Brownian motion W and an exponential random variable $\varepsilon \sim \text{Exp}(1)$ independent of W. We assume, for simplicity, a frictionless market, no arbitrage and take an equivalent martingale measure $\mathbb{Q}$ as given. All stochastic processes defined below live on this probability space and all expectations are taken with respect to $\mathbb{Q}$.

Let $\widetilde{S}$ be the pre-default stock price. We assume that the dynamics of $X = \log \widetilde{S}$ is given by

$$\begin{cases} \mathrm{d}X_t = \left(r_t - \frac{1}{2}\sigma^2(t, X_t) + \lambda(t, X_t)\right)\mathrm{d}t + \sigma(t, X_t)\,\mathrm{d}W_t^1, \\ \mathrm{d}r_t = \kappa(\theta - r_t)\,\mathrm{d}t + \delta \mathrm{d}W_t^2, \\ \mathrm{d}W_t^1 \mathrm{d}W_t^2 = \rho \mathrm{d}t, \end{cases} \tag{2.1}$$

where the interest rate r_t follows the Vasicek dynamics with parameters $\kappa, \theta, \delta > 0$. The time- and state-dependent stock volatility $\sigma = \sigma(t, X)$ and default intensity $\lambda = \lambda(t, X)$ are assumed to be differentiable with respect to X and uniformly bounded. In general the price can become worthless in two scenarios:

either the process e^X hits zero via diffusion or a jump-to-default occurs from a positive value. The default time ζ can be modeled as $\zeta = \zeta_0 \wedge \tilde{\zeta}$, where $\zeta_0 = \inf\{t > 0 \mid \widetilde{S}_t = 0\}$ is the first hitting time of zero for the stock price and $\tilde{\zeta} = \inf\{t \geq 0 \mid \Lambda_t \geq \varepsilon\}$ is the jump-to-default time with intensity λ and hazard rate $\Lambda_t = \int_0^t \lambda(s, X_s)\mathrm{d}s$. In what follows, we denote by $\mathbb{F} = \{\mathcal{F}_t,\ t \geq 0\}$ the filtration generated by the pre-default stock price and by $\mathbb{D} = \{\mathcal{D}_t,\ t \geq 0\}$ the filtration generated by the process $D_t = \mathbb{1}_{\{\zeta \leq t\}}$. Eventually, $\mathbb{G} = \{\mathcal{G}_t,\ t \geq 0\},\ \mathcal{G}_t = \mathcal{F}_t \vee \mathcal{D}_t$ is the enlarged filtration.

A CDS is an agreement between two parties, called the protection buyer and the protection seller, typically designed to transfer to the protection seller the financial loss that the protection buyer would suffer if a particular default event happened to a third party, called the reference entity. The protection seller delivers a protection payment to the protection buyer at the time of the default event. In exchange the protection buyer makes periodic premium payments at time intervals α at the credit default swap rate up to the default event or the expiry maturity, whichever comes first. The protection payment is the specified percentage $(1-\eta)$ of the CDS notional amount $\mathcal{N}$, called *loss-given-default*. The valuation problem is to determine the arbitrage-free CDS rate R that makes the present value of the CDS contract equal to zero. This rate equates the present value of the protection payoff to the present value of all the premium payments.

Proposition 2.1. *Let T be the expiry date of the CDS contract, M be the total number of premium payments and t_i be the i-th periodic premium payment date, so that $t_{i+1} - t_i = \frac{T}{M}$. Then we have*

$$R = \frac{(1-\eta)\left(1 - E\left[e^{-\int_0^T (r_u + \lambda(u, X_u))\mathrm{d}u}\right] - \int_0^T E\left[e^{-\int_0^s (r_u + \lambda(u, X_u))\mathrm{d}u} r_s\right] \mathrm{d}s\right)}{\frac{T}{M} \sum_{i=1}^{M} E\left[e^{-\int_0^{t_i} (r_u + \lambda(u, X_u))\mathrm{d}u}\right]}. \tag{2.2}$$

Proof. By Corollary 5.11 in Appendix 5 and assuming a unit notional, the protection and premium legs at time t are given by:

$$\mathrm{PV}\,(\text{Protection leg}) = E\left[e^{-\int_t^\zeta r_u \mathrm{d}u}\,(1-\eta)\,\mathbb{1}_{\{\zeta \leq T\}} | \mathcal{G}_t\right] \tag{2.3}$$

$$= \mathbb{1}_{\{\zeta > t\}}\,(1-\eta) \int_t^T E\left[e^{-\int_t^s (r_u + \lambda(u, X_u))\mathrm{d}u} \lambda(s, X_s) | \mathcal{F}_t\right] \mathrm{d}s, \tag{2.4}$$

$$\mathrm{PV}\,(\text{Premium leg}) = \sum_{i=1}^{M} E\left[e^{-\int_t^{t_i} r_u \mathrm{d}u}\, \tfrac{T}{M} R_t \mathbb{1}_{\{\zeta > t_i\}} | \mathcal{G}_t\right] \tag{2.5}$$

$$= \tfrac{T}{M} R_t \mathbb{1}_{\{\zeta > t\}} \sum_{i=1}^{M} E\left[e^{-\int_t^{t_i} (r_u + \lambda(u, X_u))\mathrm{d}u} | \mathcal{F}_t\right]. \tag{2.6}$$

The CDS spread at time $t < T$ is given by equating the protection and premium legs and thus we get

$$R_t = \frac{\mathbb{1}_{\{\zeta > t\}} \left(1-\eta\right) \int_t^T E\left[e^{-\int_t^s (r_u + \lambda(u,X_u))\mathrm{d}u} \lambda(s,X_s) | \mathcal{F}_t\right] \mathrm{d}s}{\frac{T}{M} \mathbb{1}_{\{\zeta > t\}} \sum_{i=1}^{M} E\left[e^{-\int_t^{t_i} (r_u + \lambda(u,X_u))\mathrm{d}u} | \mathcal{F}_t\right]}, \tag{2.7}$$

and in particular

$$R \equiv R_0 = \frac{\left(1-\eta\right) E\left[\int_0^T e^{-\int_0^s (r_u + \lambda(u,X_u))\mathrm{d}u} \lambda(s,X_s)\mathrm{d}s\right]}{\frac{T}{M} \sum_{i=1}^{M} E\left[e^{-\int_0^{t_i} (r_u + \lambda(u,X_u))\mathrm{d}u}\right]}. \tag{2.8}$$

Next we use the identities

$$e^{-\int_0^s (r_u + \lambda(u,X_u))\mathrm{d}u} \lambda(s,X_s) = -\frac{\partial}{\partial_s}\left(e^{-\int_0^s (r_u + \lambda(u,X_u))\mathrm{d}u}\right) - r_s e^{-\int_0^s (r_u + \lambda(u,X_u))\mathrm{d}u}, \tag{2.9}$$

and

$$\int_0^T e^{-\int_0^s (r_u + \lambda(u,X_u))\mathrm{d}u} \lambda(s,X_s)\mathrm{d}s = 1 - \left(e^{-\int_0^T (r_u + \lambda(u,X_u))\mathrm{d}u} + \int_0^T e^{-\int_0^s (r_u + \lambda(u,X_u))\mathrm{d}u} r_s \mathrm{d}s\right). \tag{2.10}$$

The statement easily follows. □

Remark 2.2. The default intensity $\lambda(t, X_t)$ can be considered as the instantaneous probability that the stock will default between t and $t + \mathrm{d}t$, conditioned on the fact that no default has happened before:

$$\lambda(t, X_t)\mathrm{d}t = \mathbb{Q}\left(t \le \zeta < t + \mathrm{d}t \mid \zeta \ge t\right). \tag{2.11}$$

The survival probability up to time t is defined as

$$Q\left(t\right) := E\left[e^{-\int_0^t \lambda(u,X_u)\mathrm{d}u}\right]. \tag{2.12}$$

3. CDS spread approximation under extended JDCEV model

In the JDCEV model the stock volatility is of the form

$$\sigma(t, X) = a(t)e^{(\beta-1)X} \tag{3.1}$$

where $\beta < 1$ and $a(t) > 0$ are the so-called elasticity parameter and scale function. The default intensity is expressed as a function of the stock volatility and the stock log-price, as follows

$$\lambda(t, X) = b(t) + c\,\sigma(t, X)^2 = b(t) + c\,a(t)^2 e^{2(\beta-1)X} \tag{3.2}$$

where $b(t) \geq 0$ and $c \geq 0$ govern the sensitivity of the default intensity with respect to the volatility. The risk-neutral dynamics of the defaultable stock price $S_t = \{S_t,\ t \geq 0\}$ are then given by

$$\begin{cases} S_t = S_0 e^{X_t} \mathbb{1}_{\{\zeta \geq t\}}, \qquad S_0 > 0, \\ \mathrm{d}X_t = \left(r_t - \frac{1}{2}\sigma^2(t, X_t) + \lambda(t, X_t)\right)\mathrm{d}t + \sigma(t, X_t)\,\mathrm{d}W_t^1, \\ \mathrm{d}r_t = \kappa(\theta - r_t)\,\mathrm{d}t + \delta \mathrm{d}W_t^2, \\ \zeta = \inf\{t \geq 0 \mid \int_0^t \lambda(t, X_t) \geq e\}, \\ \mathrm{d}W_t^1 \mathrm{d}W_t^2 = \rho \mathrm{d}t. \end{cases} \tag{3.3}$$

Let us consider a European claim on the defaultable asset, paying $h(X_T)$ at maturity T if no default happens and without recovery in case of default. In case of *constant interest rates*, one deduces the value of the European claim from the following result proved in [2].

Theorem 3.1. *Let r be a non-negative constant and h be a continuous and bounded function. Then, for any $0 \leq t \leq T$, we have*

$$E\left[\exp\left(-c\int_t^T a(u)^2 e^{2(\beta-1)X_u}\mathrm{d}u\right) h(X_T)\,|X_t = X_0\right] = E\left[\left(\frac{Z_{\tau(t)}}{x}\right)^{-\frac{1}{|\beta-1|}} h\left(e^{\int_t^T \alpha(s)\mathrm{d}s}\left(|\beta-1|Z_{\tau(t)}\right)^{\frac{1}{|\beta-1|}}\right)\right], \tag{3.4}$$

where $\{Z_t,\ t \geq 0\}$ is a Bessel process starting from x, of index $\nu = \frac{c+1/2}{|\beta-1|}$, and τ is the deterministic time change defined as

$$\tau(t) = \int_0^t a^2(u) e^{2|\beta-1|\int_0^u \alpha_s \mathrm{d}s}\mathrm{d}u, \qquad \alpha(t) = r + b(t). \tag{3.5}$$

By Theorem 3.1 and standard results from enlargement filtration theory (cf. [6]), the value of the European claim at time $t < T$ is given by

$$E\left[e^{-\int_t^T r_u \mathrm{d}u} h(X_T)\,|\mathcal{G}_t\right] = \mathbb{1}_{\{\zeta > t\}} E\left[e^{-\int_t^T (r_u + \lambda(u, X_u))\mathrm{d}u} h(X_T)\,|\mathcal{F}_t\right] \tag{3.6}$$

$$= \mathbb{1}_{\{\zeta > t\}} e^{-\int_t^T (r_u + b_u)\mathrm{d}u} E\left[e^{-c\int_t^T a_u^2 e^{2(\beta-1)X_u}\mathrm{d}u} h(X_T)\,|\mathcal{F}_t\right] \tag{3.7}$$

$$= \mathbb{1}_{\{\zeta > t\}} e^{-\int_t^T (r_u + b_u)\mathrm{d}u} E\left[\left(\frac{Z_{\tau(t)}}{x}\right)^{-\frac{1}{|\beta-1|}} h\left(e^{\int_t^T \alpha_s \mathrm{d}s}\left(|\beta-1|Z_{\tau(t)}\right)^{\frac{1}{|\beta-1|}}\right)\right]. \tag{3.8}$$

The validity of the second and third equalities above is based on the assumption of deterministic interest rates. In the general case of stochastic rates, the time-change function (3.5) is not deterministic anymore and the expectation (3.4) cannot be computed analytically. For this reason, to deal with the general case, we adopt a completely different approach and introduce a perturbation technique which provides an explicit asymptotic expansion of the building block (3.6). Specifically, we base our analysis on the recent results in [8, 11] on the approximation of solution to parabolic partial differential equations and we derive approximations of the CDS spread (2.2) and the risk-neutral survival probability (2.12).

To present our main results, we consider the following general backward Cauchy problem

$$\begin{cases} (\partial_t + \mathcal{A})\, u\,(t,z) = 0, & t < T,\ z \in \mathbb{R}^d, \\ u\,(T,z) = h\,(z)\,, & z \in \mathbb{R}^d, \end{cases} \tag{3.9}$$

where $\mathcal{A} = \mathcal{A}(t,z)$ is a (locally) parabolic differential operator of the form

$$\mathcal{A}\,(t,z) = \sum_{|\alpha|\leq 2} a_\alpha\,(t,z)\, D_z^\alpha, \qquad t \in \mathbb{R}^+,\ z \in \mathbb{R}^d, \tag{3.10}$$

where

$$\alpha = (\alpha_1, \ldots, \alpha_d)\,,\ |\alpha| = \sum_{i=1}^{d} \alpha_1 + \ldots + \alpha_d,\ D_z^\alpha = \partial_{z_1}^{\alpha_1} \ldots \partial_{z_d}^{\alpha_d}. \tag{3.11}$$

In our specific setting, we will consider $\mathcal{A}$ to be the infinitesimal generator of the stochastic processes (X, r) in (3.3), whose precise expression is given in formula (3.27).

Next, we consider the formal expansions $\mathcal{A} = \sum\limits_n \mathcal{A}_n$ and $u = \sum\limits_n u_n$, where the u_n's, for $n \geq 0$, are defined recursively by

$$\begin{cases} (\partial_t + \mathcal{A}_0) u_0(t,z) = 0, & t < T,\ z \in \mathbb{R}^d, \\ u_0(T,z) = h(z), & z \in \mathbb{R}^d, \end{cases} \tag{3.12}$$

and

$$\begin{cases} (\partial_t + \mathcal{A}_0) u_n(t,z) = -\sum\limits_{k=1}^{n} \mathcal{A}_k u_{n-k}(t,z), & t < T,\ z \in \mathbb{R}^d, \\ u_n(T,z) = 0, & z \in \mathbb{R}^d, \end{cases} \tag{3.13}$$

where

$$\mathcal{A}_n = \sum_{|\alpha|\leq 2} a_{\alpha,n}\,(t,z)\, D_z^\alpha. \tag{3.14}$$

In (3.14), $(a_{\alpha,n})_{0\leq n\leq N}$ is the N-th order Taylor expansion of a_α, in the spatial variables, around a fixed point $\bar{z}$. Notice that the functions $a_{\alpha,0}$ depend only on t: hence $\mathcal{A}_0$ is a heat operator with time-dependent coefficients and can be written in the form

$$\mathcal{A}_0 = \frac{1}{2}\sum_{i,j=1}^{d} C_{ij}\left(t\right)\partial_{z_i z_j} + \sum_{i=1}^{d} m_i\left(t\right)\partial_{z_i} + \gamma\left(t\right), \tag{3.15}$$

for some $C = (C_{ij})_{i,j\le d} \in \mathbb{R}^{d\times d}$, $m = (m_i)_{i\le d} \in \mathbb{R}^d$ and $\gamma \in \mathbb{R}$. By Duhamel's principle, the solution u_0 to the pde (3.12) is

$$u_0\left(t, z; T\right) = e^{\int_t^T \gamma(s,z)\mathrm{d}s}\int_{\mathbb{R}^d}\Gamma_0\left(t, z; T, \xi\right)h\left(\xi\right)\mathrm{d}\xi, \quad t < T,\ z \in \mathbb{R}^d, \tag{3.16}$$

where Γ_0 is the d-dimensional Gaussian density

$$\Gamma_0\left(t, z; T, \xi\right) = \frac{1}{\sqrt{2\pi^d \det C\left(t, T\right)}}\exp\left(-\frac{1}{2}\langle C^{-1}\left(t, T\right)\left(\xi - z - m\left(t, T\right)\right), \left(\xi - z - m\left(t, T\right)\right)\rangle\right), \tag{3.17}$$

with covariance matrix $C(t, T)$ and mean vector $z + m(t, T)$ given by

$$C(t, T) = \int_t^T C(s)ds, \qquad m(t, T) = \int_t^T m(s)ds. \tag{3.18}$$

It turns out that, for any $n \geq 0$, u_n can be computed explicitly, as the following result shows.

Theorem 3.2. *For any $n \geq 1$, the solution u_n to the Cauchy problem (3.13) is given by*

$$u_n\left(t, z; T\right) = \mathcal{L}_n^z\left(t, T\right)u_0\left(t, z; T\right), \qquad t < T,\ z \in \mathbb{R}^d. \tag{3.19}$$

In (3.19), $\mathcal{L}_n^z(t, T)$ denotes the differential operator acting on the z-variable and defined as

$$\mathcal{L}_n^z\left(t, T\right) = \sum_{h=1}^{n}\int_t^T \mathrm{d}s_1\int_{s_1}^T \mathrm{d}s_2 \ldots \int_{s_{h-1}}^T \mathrm{d}s_h \sum_{i\in I_{n,h}} \mathcal{G}_{i_1}^z\left(t, s_1\right)\ldots\mathcal{G}_{i_h}^z\left(t, s_h\right), \tag{3.20}$$

where

$$I_{n,h} = \{i = (i_1, \ldots, i_h) \in \mathbb{N}^h \mid i_1 + i_2 + \ldots + i_h = n\} \tag{3.21}$$

and the operators $\mathcal{G}_n^z(t, s)$ are defined as

$$\mathcal{G}_n^z\left(t, s\right) = \sum_{|\alpha|\le 2} a_{\alpha,n}\left(s, M^z\left(t, s\right)\right)D_z^\alpha, \tag{3.22}$$

with

$$M^z\left(t, s\right) = z + m\left(t, s\right) + C\left(t, s\right)D_z. \tag{3.23}$$

Proof. See [8]. □

Under rather general assumptions on $\mathcal{A}$, the following estimate for the approximation error holds:

$$|u(t,z) - u_N(t,z;T)| \leq C_N (T-t)^{\frac{N+2}{2}} \tag{3.24}$$

where $u_N(t,z;T)$ is the N-th order approximation in (3.19) and C_N is a positive constant dependent on N but not on $T-t$. Formula (3.24) ensures the short-time asymptotic convergence of the approximation u_n to the exact solution u of Cauchy problem (3.9). This theoretical result can be proved by adapting the arguments of [11], Theor. 3.1, and will be confirmed by the numerical tests in Section 4.

Going back to CDS spread approximation, we see from (2.2) that we have to evaluate expectations of the form

$$u(0, X_0, r_0; T) = E\left[e^{-\int_0^T (r_u + \lambda(u, X_u))\mathrm{d}u} h(r_T)\right], \tag{3.25}$$

with $h(r) = 1$ or $h(r) = r$. By the change of variable $r_t = e^{-\kappa t} y_t$ and from the Feynman-Kac formula (cf., for instance, [12]) it follows that u in (3.25) is solution to the Cauchy problem

$$\begin{cases} (\partial_t + \mathcal{A})\, u(t,x,y) = 0, & t < T,\ x, y \in \mathbb{R} \\ u(T,x,y) = h(y), & x, y \in \mathbb{R}, \end{cases} \tag{3.26}$$

where

$$\begin{aligned} \mathcal{A} = {} & \frac{1}{2}\sigma^2(t,x)\,\partial_{xx} + \rho\delta\sigma(t,x)e^{\kappa t}\partial_{xy} + \frac{1}{2}\delta^2 e^{2\kappa t}\partial_{yy} \\ & + \left(e^{-\kappa t}y + \lambda(t,x) - \frac{1}{2}\sigma^2(t,x)\right)\partial_x + \kappa\theta e^{\kappa t}\partial_y - \left(e^{-\kappa t}y + \lambda(t,x)\right). \end{aligned} \tag{3.27}$$

Theorem 3.3. *Under the assumptions of Proposition 2.1 and under the general dynamics (3.3), the N-th order approximation of the CDS spread in (2.2) is given by*

$$R_N = \frac{(1-L)\left(1 - \sum_{n=0}^{N} \mathcal{L}_n^{(x,y)}(0,T)\, u_0^1(0,x,y;T) - \int_0^T e^{-\kappa s} \sum_{n=0}^{N} \mathcal{L}_n^{(x,y)}(0,s)\, u_0^2(0,x,y;s)\,\mathrm{d}s\right)}{\frac{T}{M}\sum_{i=1}^{M}\sum_{n=0}^{N} \mathcal{L}_n^{(x,y)}(0,t_i)\, u_0^1(0,x,y;t_i)}, \tag{3.28}$$

where

$$u_0^1(t,x,y,s) = e^{-\int_t^s \left(e^{-\kappa u} y + \lambda(u,x)\right)\mathrm{d}u}, \tag{3.29}$$

$$u_0^2(t,x,y;s) = e^{-\int_t^s \left(e^{-\kappa u} y + \lambda(u,x)\right)\mathrm{d}u}\,(y + m_2(t,s)), \tag{3.30}$$

$m_2(t,s)$ is the second component of the vector $m(t,s)$ in (3.18) and the differential operators $\mathcal{L}_n^{(x,y)}$ can be computed explicitly as in Theorem (3.2).

Proof. In formula (2.2) there appear terms of the form

$$E\left[e^{-\int_0^t (r_u+\lambda(u,X_u))\mathrm{d}u}\right]$$

in the numerator and denominator, that are solutions to problem (3.26) with $h(y)=1$. On the other hand, in (2.2) there also appear terms of the form

$$E\left[e^{-\int_0^t (r_u+\lambda(u,X_u))\mathrm{d}u} r_t\right]$$

which are solutions to the same problem with $h(y)=e^{-\kappa t}y$. Theorem 3.2 and (3.24) yield the approximations

$$E\left[e^{-\int_0^t (r_u+\lambda(u,X_u))\mathrm{d}u}\right] = \sum_{n=0}^{N} \mathcal{L}_n^{(x,y)}(0,t)\, u_0^1(0,x,y;t) + \mathrm{O}\left(t^{\frac{N+2}{2}}\right), \tag{3.31}$$

$$E\left[e^{-\int_0^t (r_u+\lambda(u,X_u))\mathrm{d}u} r_t\right] = e^{-\kappa t}\sum_{n=0}^{N} \mathcal{L}_n^{(x,y)}(0,t)\, u_0^2(0,x,y;t) + \mathrm{O}\left(t^{\frac{N+2}{2}}\right), \tag{3.32}$$

as $t\to 0^+$, where

$$u_0^1(0,x,y,t) = e^{-\int_0^t \left(e^{-\kappa u}y+\lambda(u,x)\right)\mathrm{d}u}\int_{\mathbb{R}^2}\Gamma_0(0,x,y;t,\xi_1,\xi_2)\,\mathrm{d}\xi_1\mathrm{d}\xi_2, \tag{3.33}$$

$$= e^{-\int_0^t \left(e^{-\kappa u}y+\lambda(u,x)\right)\mathrm{d}u}, \tag{3.34}$$

and

$$u_0^2(0,x,y;t) = u_0^1(0,x,y,t)\int_{\mathbb{R}^2}\Gamma_0(0,x,y;t,\xi_1,\xi_2)\,h(\xi_2)\,\mathrm{d}\xi_1\mathrm{d}\xi_2, \tag{3.35}$$

$$= u_0^1(0,x,y,t)\int_{\mathbb{R}^2}\Gamma_0(0,x,y;t,\xi_1,\xi_2)\,\xi_2\mathrm{d}\xi_1\mathrm{d}\xi_2 \tag{3.36}$$

$$= u_0^1(0,x,y,t)\left(y+m_2(0,t)\right). \tag{3.37}$$

□

Remark 3.4. We have an analogous approximation result for the survival probability in (2.12). Since it can be expressed as the solution to the problem

$$\begin{cases}(\partial_t+\mathcal{A})\,v(t,x,y)=0, & t<T,\ x,y\in\mathbb{R}\\ v(T,x,y)=1, & x,y\in\mathbb{R},\end{cases} \tag{3.38}$$

then, by Theorem 3.2, we have

$$Q\left(t\right)=\sum_{n\geq 0}\mathcal{L}_{n}^{(x,y)}\left(0,t\right)v_{0}\left(0,x,y;T\right), \tag{3.39}$$

where $v_0(t,x,y;T)=\exp\left(-\int_t^T\lambda(u,x)\mathrm{d}u\right)$ and the operators $\mathcal{L}_n^{(x,y)}\left(0,t\right)$ are defined as in (3.20). As an example, we give here the explicit first order approximation of the survival probability in the case of constant parameters $a(t)=a$ and $b(t)=b$:

$$\begin{aligned} v_{0}\left(0,x,y;T\right) &= e^{-\left(b+ca^{2}e^{2(\beta-1)x}\right)T}, \\ v_{1}\left(t,x,y;T\right) &= -ce^{2(\beta-1)x-T\left(b+ca^{2}e^{(\beta-1)x}\right)}\left(\beta-1\right)a^{2}\cdot \\ &\quad\cdot\frac{-4y+4\theta+e^{-T\kappa}\left(4y-4\theta+e^{T\kappa}T\kappa\left(4y-4\theta+e^{T\kappa}\left(2\left(b+\theta\right)+\left(-1+2c\right)e^{2(\beta-1)x}a^{2}\right)\right)\right)}{2\kappa^{2}}. \end{aligned}$$

These expressions are obtained by using Mathematica symbolic programming. Indeed, by replacing $\mathcal{G}_n^{(x,y)}(0,s)$ and $M^{(x,y)}(0,s)$ by their expressions (3.22) and (3.23) in (3.20), we can write $v_n(0,x,y;T)$ as follow:

$$v_{n}\left(0,x,y;T\right)=\sum_{h=1}^{n}\sum_{i=0}^{3n}\sum_{j=0}^{3n-i}\sum_{k=0}^{n}\sum_{l=0}^{n}\left(x-\bar{x}\right)^{k}\left(y-\bar{y}\right)^{l}\partial_{x}^{i}\partial_{y}^{j}v_{0}\left(0,x,y;T\right)F_{i,j,k,l}^{(n,h)}\left(0,T\right), \tag{3.40}$$

with

$$F_{i,j,k,l}^{(n,h)}\left(0,T\right)=\int_{0}^{T}\int_{s_{1}}^{T}\cdots\int_{s_{h-1}}^{T}f_{i,j,k,l}^{(n,h)}\left(0,s_{1},\cdots,s_{h}\right)\mathrm{d}s_{1}\cdots\mathrm{d}s_{h},$$

and $(\bar{x},\bar{y})$ are chosen and can be time-dependent. The coefficients $f_{i,j,k,l}^{(n,h)}$ have already been computed by symbolic programming with Mathematica and only depend on the coefficients in (3.27), $\bar{x}$ and $\bar{y}$. The final expressions are very long but very simple and easy to compute for any $n>0$. It follows, from integrations of $f_{i,j,k,l}^{(n,h)}$ and partial derivatives of (3.16), the expressions of v_n.

4. CDS calibration and numerical tests

In this section we apply the method developed in Section 3 to calibrate the model to market CDS spreads. We use quotations for different companies (specifically, UBS AG and BNP Paribas) in order to check the robustness of our methodology. The calibration is based on a two-step procedure: we first calibrate separately the interest rate model to daily yields curves for zero-coupon bonds (ZCB), generated using Libor swap curve. Subsequently, we consider CDS contracts with different maturity dates. We use the approximation formulas (3.28) and (3.39) for the CDS spreads and survival probabilities, respectively. We use second-order approximations: we have found these to be sufficiently accurate by numerical experiments and theoretical error estimates. The formulas for the second-order approximation are simple, making the method easy to implement.

We distinguish between the uncorrelated and correlated cases: in the first case, i.e. when $\rho = 0$, the survival probability, which is not quoted from the market, can be inferred from the CDS spreads through a bootstrapping formula and therefore it is possible to calibrate directly to the survival probabilities. In the general case when $\rho \neq 0$, we calibrate to the market spreads using formula (3.28).

To add more flexibility to the model, we assume that the coefficients $a(t)$ and $b(t)$ in (3.2) are linearly dependent on time: more precisely, we assume that

$$a(t) = a_1 t + a_2, \qquad b(t) = b_1 t + b_2, \tag{4.1}$$

for some constants $a_1,\ a_2,\ b_1$, and b_2.

As defined in (3.3), the stochastic interest rate is described by a Vasicek model

$$\mathrm{d}r_t = \kappa \left(\theta - r_t\right) \mathrm{d}t + \delta \mathrm{d}W_t^2. \tag{4.2}$$

Apart from its simplicity, one of the advantages of this model is that interest rates can take negative values. For the calibration, we use the standard formula for the price $P_t\left(T\right)$ of a T-bond, which we recall here for convenience:

$$P_t\left(T\right) = A_t\left(T\right) e^{-B_t(T) r_t}, \tag{4.3}$$

where

$$A_t\left(T\right) = e^{\left(\theta - \frac{\delta^2}{2\kappa^2}\right)\left(B_t(T) - T + t\right) - \frac{\delta^2}{4\kappa} B_t(T)^2}, \qquad B_t\left(T\right) = \frac{1 - e^{-\kappa(T-t)}}{\kappa}. \tag{4.4}$$

The results of the interest rate calibration are given in Table 1.

Table 1. Calibration to ZCB.

Times to maturity (years)	market ZCB	model ZCB	error
1.	1.00229	1.00641	-0.410925 %
2	1.00371	1.00844	-0.470717 %
3	1.00333	1.00667	-0.333553 %
4	1.00099	1.00165	-0.0660915 %
5	0.995836	0.993987	0.185689 %
6	0.987867	0.984129	0.378397 %
7	0.977005	0.972476	0.463518 %
8	0.963596	0.959441	0.431229 %
9	0.948371	0.945363	0.317189 %
10	0.933829	0.930451	0.361748 %

$\kappa = 0.06,\ \theta = 0.09,\ \delta = 0.024,\ r_0 = -0.009$

4.1. CDS calibration

The problem of calibrating the model (3.3) is formulated as an optimization problem. We want to minimize the error between the model CDS spread and the market CDS spreads. Our approach is to use the square difference between market and model CDS spreads. This leads to the nonlinear least square method

$$\inf_{\Theta} F(\Theta), \qquad F(\Theta) = \sum_{i=1}^{N} \omega_i |R_i - \widehat{R}_i|^2, \tag{4.5}$$

where N is the number of spreads used, ω_i is a weight, $\widehat{R}_i$ is the market CDS spreads of the considered reference entity observed at time $t = 0$ and $\Theta = (a_1, a_2, b_1, b_2, \beta, c, \rho)$, with

$$a_2 \geq 0, \quad a_1 \geq -\frac{a_2}{T}, \quad b_2 \geq 0, \quad b_1 \geq -\frac{b_2}{T} \quad c > 0, \quad \beta < 1 \text{ and } -1 < \rho < 1. \tag{4.6}$$

In order to calibrate our model to data from real market, we received data from Bloomberg for two large credit derivatives dealers: UBS AG and BNP Paribas.

4.1.1. Calibration results

In this section, we present the results of calibrating of the model to set of data covering the period from January, 1st, 2017 to January, 1st , 2023. In both uncorrelated and correlated cases, we can see, in tables 2, 4, 3 and 10, that the model gives very good fit to the market data, particularly to the most liquid market CDS spreads (2Y, 3Y and 5Y maturities). However we can still observe high relative errors for the BNP Paribas CDS spread with maturity 4 years due to the market incompleteness or the non-liquidity of its 4Y maturity CDS observed at January, 1st, 2017. The interesting fact is that the model gives very

Table 2. Calibration to UBS AG CDS spreads (uncorrelated case).

Time to Maturities	Market CDS spread (bps)	Model CDS spread (bps)	Rel. Errors
1.	25.72	25.914	0.754262 %
1.5	30.2627	30.3312	0.226099 %
2.	35.105	34.7814	-0.921671 %
2.5	39.6922	39.2606	-1.08729 %
3.	43.97	43.7645	-0.467301 %
3.5	48.0616	48.289	0.473158 %
4.	52.3	52.8299	1.01328 %
4.5	56.9646	57.3833	0.73514 %
5.	61.91	61.9452	0.0568152 %
5.5	66.8408	66.5115	-0.492607 %
6.	71.285	71.0784	-0.289867 %

$a_1 = 0.005,\ a_2 = 0.001,\ \beta = 0.91,\ b_1 = 0.003,\ b_2 = 0.003,\ c = 1.4$

Table 3. Calibration to UBS AG CDS spreads (Correlated case).

Time to Maturities	Market CDS spread (bps)	Model CDS spread (bps)	Rel. Errors
1.	25.72	25.9622	0.941777 %
1.5	30.2627	30.3606	0.323276 %
2.	35.105	34.787	-0.905712 %
2.5	39.6922	39.2422	-1.13365 %
3.	43.97	43.7262	-0.554424 %
3.5	48.0616	48.2386	0.368363 %
4.	52.3	52.7785	0.914999 %
4.5	56.9646	57.3445	0.666967 %
5.	61.91	61.9345	0.0396039 %
5.5	66.8408	66.5461	-0.44087 %
6.	71.285	71.1762	-0.152671 %

$a_1 = -0.035,\ a_2 = 0.23,\ \beta = 0.66,\ b_1 = 0.003,\ b_2 = 0.0005,\ c = 0.045,\ \rho = 0.9$

good fit to liquid market CDS and this is confirmed, in the appendix 5.2, by more calibration tests on CDS spreads of other different companies.

Table 4. Calibration to BNP Paribas CDS spreads (uncorrelated case).

Time to Maturities	Market CDS spread (bps)	Model CDS spread (bps)	Rel. Errors
1.	34.615	34.0535	-1.62217 %
1.5	39.8758	39.6736	-0.50717 %
2.	45.115	45.4635	0.77257 %
2.5	49.9935	51.4145	2.84236 %
3.	56.11	57.517	2.50765 %
3.5	64.5726	63.7612	-1.25665 %
4.	72.59	70.1364	-3.38005 %
4.5	77.6516	76.6317	-1.31333 %
5.	82.27	83.2357	1.17384 %
5.5	89.1455	89.9366	0.887389 %
6.	96.705	96.7222	0.0177715 %

$a_1 = 0.018,\ a_2 = 0.085,\ \beta = 0.88,\ b_1 = 0.002,\ b_2 = 0.0,\ c = 0.53$

Table 5. Calibration to BNP CDS spreads (Correlated case).

Time to Maturities	Market CDS spread (bps)	Model CDS spread (bps)	Rel. Errors
1.	34.615	34.0073	-1.7557 %
1.5	39.8758	39.6668	-0.524173 %
2.	45.115	45.4813	0.811846 %
2.5	49.9935	51.4436	2.90047 %
3.	56.11	57.5465	2.56009 %
3.5	64.5726	63.7827	-1.22334 %
4.	72.59	70.1449	-3.36833 %
4.5	77.6516	76.6259	-1.32086 %
5.	82.27	83.2184	1.15283 %
5.5	89.1455	89.9156	0.863835 %
6.	96.705	96.7105	0.00566037 %

$a_1 = 0.023,\ a_2 = 0.08,\ \beta = 0.6,\ b_1 = 0.002,\ b_2 = 0.002,\ c = 0.3,\ \rho = 0.96$

For the calibration, we used a global optimizer, *NMinimize*, from Mathematica's optimization toolbox on a PC with 1$\times$ Intel i7-6599U 2.50 GHz CPU and 8GB RAM. We present in table 6 the computational times of the calibration of the model to our two corporates in both uncorrelated and correlated cases. One can conclude that the approximation formula (3.28) gives an efficient and fast calibration.

Table 6. Computational Times

	Uncorrelated (second)	Correlated (second)
UBS AG	116.856	168.12
BNP Paribas	124.216	161.408

We check the results obtained from the calibration by computing the risk-neutral survival probabilities with the approximation formula (3.39). In tables 7 and 9, by comparing the real market survival probability (column 2), our method (column 3) and the Monte Carlo (MC) simulation (column 4), we observe that the method provides results as good as the MC. The latter is performed with 100000 iterations and a confident interval of 95%. We do the same test for the correlated case for both corporates and present the results in tables 8 and 10.

Table 7. Risk-neutral UBS AG survival probabilities (uncorrelated case).

Times to maturity	Market probabilities	Model probabilities	Monte Carlo
1.	0.995724	0.99569	[0.995704, 0.995704]
2.	0.988367	0.988473	[0.988531, 0.988531]
3.	0.978233	0.978335	[0.978468, 0.978468]
4.	0.965644	0.965293	[0.965531, 0.965531]
5.	0.949437	0.949391	[0.949765, 0.949766]
6.	0.93056	0.930706	[0.931249, 0.93125]

Table 8. Risk-neutral UBS AG survival probabilities (correlated case).

Times to maturity	model probability	Monte Carlo	
1.	0.995724	0.995682	$[0.995695, 0.995697]$
2.	0.988367	0.988471	$[0.988523, 0.988528]$
3.	0.978233	0.978353	$[0.978479, 0.978486]$
4.	0.965644	0.965324	$[0.965553, 0.965563]$
5.	0.949437	0.949399	$[0.949759, 0.94977]$
6.	0.93056	0.930619	$[0.931142, 0.931152]$

Table 9. Risk-neutral BNP Paribas survival probabilities (uncorrelated case).

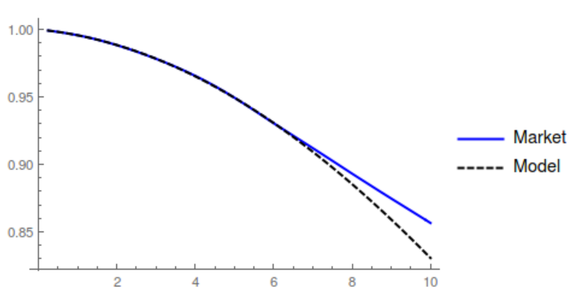
Times to maturity	Market probabilities	Model probabilities	Monte Carlo
1.	0.994253	0.994339	$[0.994358, 0.994359]$
2.	0.985077	0.984948	$[0.98503, 0.985034]$
3.	0.972302	0.97157	$[0.971779, 0.971788]$
4.	0.952539	0.954016	$[0.954445, 0.954462]$
5.	0.933285	0.932175	$[0.932925, 0.932955]$
6.	0.908874	0.906025	$[0.907231, 0.90728]$

Table 10. Risk-neutral BNP Paribas survival probabilities (correlated case).

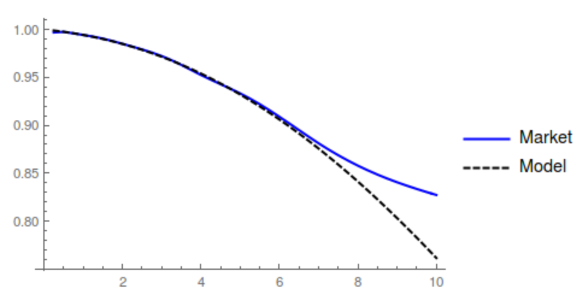
Times to maturity	market probability	model probability	Monte Carlo
1.	0.994253	0.994347	$[0.994363, 0.994368]$
2.	0.985077	0.98494	$[0.985012, 0.985034]$
3.	0.972302	0.971544	$[0.971742, 0.9718]$
4.	0.952539	0.953977	$[0.95432, 0.954448]$
5.	0.933285	0.932115	$[0.932728, 0.932973]$
6.	0.908874	0.9059	$[0.906818, 0.907245]$

However, as mentioned above in (3.24), the convergence of the method is in the asymptotic sense; that is it is asymptotically exact as the maturity goes to zero. To show the dependence of the errors on the maturity, we plot the market and model survival probabilities in function of maturity in Figure 1. We observe that, after 6Y, the errors between the market and model survival probabilities start increasing, as expressed by the error bounds (3.24).

Figure 1. Dependence of the error on the maturity



(a) UBS AG Survival Probability



(b) BNP Paribas Survival Probability

4.1.2. Influence of the correlation

To see the influence of the correlation in our model, we adopt the test done in [1]. Indeed the authors consider four different payoffs that appear in credit derivatives and compare their present values in very positive and negative correlation cases, i.e. $\rho = 1$ and $\rho = -1$.

$$A = D(0, 5Y)\, L(4Y, 5Y)\, \mathbb{1}_{\{\zeta < 5Y\}}, \qquad B = D(0, 5Y)\, \mathbb{1}_{\{\zeta < t\}}, \tag{4.7}$$

$$C = D(0, \min(\zeta, 5Y)), \quad H = D(0, 5Y)\, L(4Y, 5Y)\, \mathbb{1}_{\{\zeta \in [4Y, 5Y]\}}, \tag{4.8}$$

where ζ is the time of default and $L(S, T)$ is the market LIBOR rate $T > S$. We consider the UBS AG corporate. First we calibrate the model (3.3) to the UBS AG market CDS spreads in both very positive and negative correlation cases. We obtain the following parameters:

$$\rho = 1: \ a_1 = 0.008,\ a_2 = 0.008,\ \beta = 0.5,\ b_1 = 0.003,\ b_2 = 0.003,\ c = 0.68, \tag{4.9}$$

and

$$\rho = -1: \ a_1 = 0.006,\ a_2 = 0.04,\ \beta = 0.624,\ b_1 = 0.002,\ b_2 = 0.0004,\ c = 1.325. \tag{4.10}$$

Table 11 shows, on one hand, that the correlation has no impact in the payoff of the form B and C. Since the CDS spread and the risk-neutral survival probability expressions are written as function in terms of B and C, the correlation has no influence in the computations of the CDS spreads and the risk-neutral survival probabilities. On the other hand, higher effect can be seen in the values of derivatives including LIBOR rates (A and H). This explains why in both cases (non-correlation and correlation), our model gives a very good fit to the market data. It follows that when we want to use the model for pricing derivatives of types A, H or pricing in general, it is better and much more accurate to consider the model with correlation.

Table 11. Impact of the correlation

	$\rho = -1$	$\rho = 1$	Rel. Errors	Abs. Errors
A	22.61 bps	90.986 bps	+148.50%	+0.00135
B	505.482 bps	505.058 bps	+0.083%	+0.00004
C	9947.149 bps	9948.279 bps	-0.011%	-0.00011
$\mathbb{D}$	16.244 bps	-0.361 bps	-548.883%	0.00019

In Section 5.2, we confirm the robustness of our methodology by presenting the results of the calibration of the model to real market CDS spreads of large credit derivative dealers like CaixaBank SA (Table 12), Citigroup Inc. (Table 13), Commerzbank AG (Table 14), Deutsche Bank AG (Table 16) and Mediobanca

S.p.A (Table 15).

5. Appendix

5.1. Mathematical Background

We collect some results on hazard rate and conditional expectation with respect to enlarged filtrations. We present the key formula which relates the conditional expectation with respect to a "big" filtration to the conditional expectation with respect to a "small" filtration. For more about filtration enlargement, we refer for instance to [7].

Let ζ be a non-negative random variable on a probability space $(\Omega, \mathcal{G}, \mathbb{Q})$, such that $\mathbb{Q}(\zeta = 0) = 0$ and $\mathbb{Q}(\zeta > t) > 0$ for any $t \geq 0$. We introduce a right-continuous process D defined as $D_t = \mathbb{1}_{\{\zeta \leq t\}}$, and we denote by $\mathbb{D}$ the filtration generated by D; that is $\mathcal{D}_t = \sigma(D_u \mid u \leq t)$. Let $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ be a given filtration on $(\Omega, \mathcal{G}, \mathbb{Q})$ such that $\mathbb{G} := \mathbb{D} \vee \mathbb{F}$; that is we set $\mathcal{G}_t := \mathcal{D}_t \vee \mathcal{F}_t$ for every $t \in \mathbb{R}_+$. Since $\mathcal{D}_t \subseteq \mathcal{G}_t$ for any t, the random variable ζ is a stopping time with respect to $\mathbb{G}$. The financial interpretation is that the filtration $\mathbb{F}$ models the flow of observations available to the investors prior to the default time ζ. For any $t \in \mathbb{R}_+$, we write $F_t = \mathbb{Q}(\zeta \leq t | \mathcal{F}_t)$, so that $1 - F_t = \mathbb{Q}(\zeta > t | \mathcal{F}_t)$: notice that F is a bounded and non-negative $\mathbb{F}$-submartingale. We may thus deal with its right-continuous modification.

Definition 5.1. The $\mathbb{F}$-*hazard process* of ζ, denoted by Γ, is defined through the formula $1 - F_t = e^{-\Gamma_t}$ for every $t \in \mathbb{R}_+$.

Lemma 5.2. *We have $\mathcal{G}_t \subset \mathcal{G}_t^*$, where*

$$\mathcal{G}_t^* := \{A \in \mathcal{G} \mid \exists B \in \mathcal{F}_t \quad A \cap \{\zeta > t\} = B \cap \{\zeta > t\}\}. \tag{5.1}$$

Proof. Observe that $\mathcal{G}_t = \mathcal{D}_t \vee \mathcal{F}_t = \sigma(\mathcal{D}_t, \mathcal{F}_t) = \sigma(\{\zeta \leq u\}, u \leq t, \mathcal{F}_t)$. Also, it is easily seen that the the class $\mathcal{G}_t^*$ is a sub$-\sigma$-field of $\mathcal{G}$. Therefore, it is enough to check that if either $A = \{\zeta \leq u\}$ for $u \leq t$ or $A \in \mathcal{F}_t$, then there exists an event $B \in \mathcal{F}_t$ such that $A \cap \{\zeta > t\} = B \cap \{\zeta > t\}$. Indeed, in the former case we may take $B = \emptyset$, in the latter $B = A$. □

Lemma 5.3. *For any $\mathcal{G}$-measurable random variable Y we have, for any $t \in \mathbb{R}_+$*

$$E\left[\mathbb{1}_{\{\zeta > t\}} Y | \mathcal{G}_t\right] = \mathbb{1}_{\{\zeta > t\}} \frac{E\left[Y | \mathcal{F}_t\right]}{\mathbb{Q}(\zeta > t | \mathcal{F}_t)} = \mathbb{1}_{\{\zeta > t\}} e^{\Gamma_t} E\left[\mathbb{1}_{\{\zeta > t\}} Y | \mathcal{F}_t\right]. \tag{5.2}$$

Proof. Let us fix $t \in \mathbb{R}_+$. In view of the Lemma 5.2. any $\mathcal{G}_t$-measurable random variable coincides on the set $\{\zeta > t\}$ with some $\mathcal{F}_t$-measurable random variable. Therefore

$$E\left[\mathbb{1}_{\{\zeta > t\}} Y | \mathcal{G}_t\right] = \mathbb{1}_{\{\zeta > t\}} E\left[Y | \mathcal{G}_t\right] = \mathbb{1}_{\{\zeta > t\}} X, \tag{5.3}$$

where X is an $\mathcal{F}_t$-measurable random variable. Taking the conditional expectation with respect to $\mathcal{F}_t$, we obtain

$$E\left[\mathbb{1}_{\{\zeta>t\}}Y|\mathcal{F}_t\right] = \mathbb{Q}(\zeta > t|\mathcal{F}_t)X. \tag{5.4}$$

□

Proposition 5.4. *Let Z be a bounded $\mathbb{F}$-predictable process. Then for any $t < s \leq \infty$*

$$E\left[\mathbb{1}_{\{t<\zeta\leq s\}}Z_\zeta|\mathcal{G}_t\right] = \mathbb{1}_{\{\zeta>t\}}e^{\Gamma_t}E\left[\int_{]t,s]} Z_u dF_u|\mathcal{F}_t\right]. \tag{5.5}$$

Proof. We start by assuming that Z is a piecewise constant $\mathbb{F}$-predictable process, so that (we are interested only in values of Z for $u \in]t, s]$)

$$Z_u = \sum_{i=0}^{n} Z_{t_i}\mathbb{1}_{]t_i,t_{i+1}]}(u), \tag{5.6}$$

where $t = t_0 < \ldots < t_{n+1} = s$ and the random variable Z_{t_i} is $\mathcal{F}_{t_i}$-measurable. In the view of (5.2), for any i we have

$$E\left[\mathbb{1}_{\{t_i<\zeta\leq t_{i+1}\}}Z_\zeta|\mathcal{G}_t\right] = \mathbb{1}_{\{\zeta>t\}}e^{\Gamma_t}E\left[\mathbb{1}_{\{t_i<\zeta\leq t_{i+1}\}}Z_{t_i}|\mathcal{F}_t\right] \tag{5.7}$$

$$= \mathbb{1}_{\{\zeta>t\}}e^{\Gamma_t}E\left[Z_{t_i}(F_{t_{i+1}} - F_{t_i})|\mathcal{F}_t\right]. \tag{5.8}$$

In the second step we approximate an arbitrary bounded $\mathbb{F}$-predictable process by a sequence of piecewise constant $\mathbb{F}$-predictable process. □

Corollary 5.5. *Let Y be a $\mathcal{G}$-measurable random variable. Then, for any $t \leq s$, we have*

$$E\left[\mathbb{1}_{\{\zeta>s\}}Y|\mathcal{G}_t\right] = \mathbb{1}_{\{\zeta>t\}}E\left[\mathbb{1}_{\{\zeta>s\}}e^{\Gamma_t}Y|\mathcal{F}_t\right]. \tag{5.9}$$

Furthermore, for any F_s-measurable random variable Y we have

$$E\left[\mathbb{1}_{\{\zeta>s\}}Y|\mathcal{G}_t\right] = \mathbb{1}_{\{\zeta>t\}}E\left[e^{\Gamma_t-\Gamma_s}Y|\mathcal{F}_t\right]. \tag{5.10}$$

If F (and thus Γ) is a continuous increasing process then for any $\mathbb{F}$-predictable bounded process Z we have

$$E\left[\mathbb{1}_{\{t<\zeta\leq s\}}Z_\zeta|\mathcal{G}_t\right] = \mathbb{1}_{\{\zeta>t\}}E\left[\int_t^s Z_u e^{\Gamma_t-\Gamma_u}\mathrm{d}\Gamma_u|\mathcal{F}_t\right]. \tag{5.11}$$

Proof. In view of (5.2), to show that (5.9) holds, it is enough to observe that $\mathbb{1}_{\{\zeta>s\}} = \mathbb{1}_{\{\zeta>t\}}\mathbb{1}_{\{\zeta>s\}}$. Equality (5.10) is a straightforward consequence of (5.9). Formula (5.11) follows from (5.5) since, when F is increasing, $\mathrm{d}F_u = e^{-\Gamma_u}\mathrm{d}\Gamma_u$. □

5.2. Further calibration tests

In this section, we collect some calibration tests of the model with correlation to real market data of some large companies.

Table 12. Calibration to Caixa Bank SA CDS spreads (Correlated case).

Time to Maturities	Market CDS spread (bps)	Model CDS spread (bps)	Rel. Errors
1.	76.655	76.8783	0.291366 %
1.5	85.1622	83.5574	-1.88441 %
2.	90.115	90.4517	0.373677 %
2.5	96.1837	97.5516	1.42216 %
3.	103.465	104.847	1.33529 %
3.5	111.251	112.325	0.965738 %
4.	120.19	119.976	-0.177871 %
4.5	130.524	127.787	-2.09744 %
5.	139.515	135.743	-2.70359 %
5.5	144.723	143.832	-0.615569 %
6.	147.885	152.039	2.80874 %

$a_1 = 0.029,\ a_2 = 0.1,\ \beta = 0.9,\ b_1 = 0.002,\ b_2 = 0.007,\ c = 0.28,\ \rho = 0.9$

Table 13. Calibration to Citigroup Inc CDS spreads (Correlated case).

Time to Maturities	Market CDS spread (bps)	Model CDS spread (bps)	Rel. Errors
1.	23.86	24.0532	0.80977 %
1.5	28.1143	28.2493	0.480115 %
2.	33.45	32.6986	-2.24632 %
2.5	37.9995	37.393	-1.5961 %
3.	42.135	42.3335	0.471192 %
3.5	46.6976	47.5358	1.79501 %
4.	52.165	53.0361	1.66997 %
4.5	58.7451	58.8977	0.259815 %
5.	65.93	65.2171	-1.08133 %
5.5	73.0353	72.1307	-1.23864 %
6.	79.385	79.8203	0.548391 %

$a_1 = 0.01,\ a_2 = 0.04,\ \beta = -2.67,\ b_1 = 0.0006,\ b_2 = 0.0,\ c = 1.9,\ \rho = 0.9$

Table 14. Calibration to Commerzbank AG CDS spreads (Correlated case).

Time to Maturities	Market CDS spread (bps)	Model CDS spread (bps)	Rel. Errors
1.	44.69	44.7819	0.205608 %
1.5	53.9328	54.0445	0.207137 %
2.	63.175	63.0747	-0.158831 %
2.5	71.8376	71.8844	0.0651909 %
3.	80.285	80.4911	0.256673 %
3.5	88.977	88.9161	-0.0684516 %
4.	97.81	97.1835	-0.640533 %
4.5	106.475	105.319	-1.08584 %
5.	114.405	113.349	-0.923244 %
5.5	121.117	121.298	0.149436 %
6.	126.73	129.192	1.94273 %

$a_1 = -0.002,\ a_2 = 0.08,\ \beta = -2.7,\ b_1 = 0.005,\ b_2 = 0.0,\ c = 0.6,\ \rho = 0.6$

Table 15. Calibration to Mediobanca SpA spreads (Correlated case).

Time to Maturities	Market CDS spread (bps)	Model CDS spread (bps)	Rel. Errors
1.	87.545	87.4557	-0.101987 %
1.5	96.831	96.8623	0.0323408 %
2.	106.715	106.688	-0.0252551 %
2.5	116.657	116.678	0.0176859 %
3.	126.405	126.621	0.170711 %
3.5	135.88	136.343	0.340403 %
4.	145.42	145.703	0.19438 %
4.5	155.159	154.586	-0.369086 %
5.	164.01	162.903	-0.674879 %
5.5	170.956	170.583	-0.218563 %
6.	176.485	177.572	0.615901 %

$a_1 = -0.04,\ a_2 = 0.93,\ \beta = -1.7,\ b_1 = -0.002,\ b_2 = 0.01,\ c = 0.0004,\ \rho = 0.82$

Table 16. Calibration to Deutsche Bank AG spreads (Correlated case).

Time to Maturities	Market CDS spread (bps)	Model CDS spread (bps)	Rel. Errors
1.	67.02	66.42	-0.895209 %
1.5	77.7864	79.1848	1.79767 %
2.	92.015	92.5289	0.558545 %
2.5	107.233	105.796	-1.33972 %
3.	120.505	118.488	-1.67348 %
3.5	129.985	130.235	0.192996 %
4.	138.645	140.774	1.53567 %
4.5	149.244	149.928	0.458061 %
5.	158.86	157.592	-0.798277 %
5.5	164.376	163.718	-0.400654 %
6.	167.59	168.305	0.426501 %

$a_1 = -0.1,\ a_2 = 1.55,\ \beta = -0.08,\ b_1 = -0.001,\ b_2 = 0.006,\ c = 0.0003,\ \rho = 0.83$

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