Measuring risk in engineering systems and financial networks

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1 Overview

The aim of these notes is to present a short overview of the methods that can be used to compute risk in financial systems, in particular focusing on methods that can be used as adaptations of the numerical simulations used in modeling engineering systems. When modeling an engineering system one considers the different components making up the system, determines the interactions between these components and uses this to simulate quantities of interest with regard to the total system. The quantities of interest when working with reliability analysis are e.g. system failure, what leads to the failure of the system, effects of maintenance. We start with describing a general Monte Carlo (MC) algorithm which can be used in order to compute the state of the system as a whole through time in Section 2.

The MC methods used in engineering and in particular the modeling of the engineering system bear similarities with modeling interbank networks in financial systems as described in Section 3. The interbank network consists of a large number of nodes. These nodes interact with each other according to a predefined interaction term. This interaction represents in the simplest case the lending and borrowing that goes on in the system between the nodes. Each node then has a continuous state function representing the amount of financial assets held in its possession. Due to the lending and borrowing interaction the amount of assets changes through time. When the number of assets drops below zero we can consider a node to have defaulted. A common quantity of interest is then to compute the total expected loss in the system obtained by running a large number of Monte Carlo simulations. Furthermore, by changing the parameters in the network one can get an idea of the networks sensitivity to a default cascade (when a default of one bank causes the banks it interacts with to default propagating the default through the system).

In Section 4 notes we present a Monte Carlo method for modeling the loss in a financial system. In particular, it has been observed empirically that the assets of a bank are – additionally to the interaction through lending – also subjected to self- and cross-exciting shocks. In other words, a negative shock to the asset value of a bank might lead to an increase in the probability of an additional shock to the bank itself, or to other banks with which it is connected. In this way the default propagation through the system is fueled additionally by this self- and cross-exciting component, which can increase the total loss in the system. We

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model this component by means of a Hawkes process. This then allows one to investigate the sensitivity of the system to the excitement factor.

Finally, in Section 5 we present a way of computing the risk in the system analytically. While the Monte Carlo algorithms are a relatively easy and flexible modeling tool, when considering systems with a large amount of nodes the simulation can take a large amount of time (especially when additionally modeling the Hawkes process excitement factor). Alternatively to the numerical simulation we can compute the so-called mean-field behavior of the system by considering the behavior of the network as the number of nodes, or banks, grows to infinity. This allows us to compute a weak convergence limit and to obtain analytical estimates for various risk measures of interest.

2 Engineering systems reliability analysis

When working with engineering systems one of the main concerns is to be able to protect the system from failure behavior of its components. In particular, it is of importance to understand what sequences of events could turn into damage to the system or failure, the probability of these sequences and the consequences. The model of the engineering system is based on the interaction between the various components, the interaction of the system with the environment and data about the properties of the components themselves. The movement of the components that make up the system between their possible states thus determine how the system as a whole behaves. Having determined this model one can then investigate possible scenarios of failure, repairs, maintenance and other aspects. Monte Carlo simulation is a powerful tool for analyzing the complex systems by simulating possible scenarios. For an overview of Monte Carlo methods in system reliability and risk analysis we refer to [24].

2.1 The Monte Carlo method

The general algorithm A Monte Carlo simulation consists of sampling different histories of the system defined through its mission time T and computing the number of events of interest, i.e. failed instances of the system or its components or total time of failure. We define a number N_S to be the amount of simulations to be performed. The mission time is then split into a time intervals of time step $\Delta t := t_{i+1} - t_i$ for $i = \{1, ..., N\}$. We define a system state vector $v = (v_1, ..., v_M)$ where M is the total number of components of the system and each v_i represents the state of the ith component of the system, in the basic case represented by a boolean $[0, 1]$ state, with 0 being a working state and 1 representing the failed state. Then we determine a system state function $\Phi(v)$ which maps the individual states to the quantity of interest with regard to the total system. In order to model the behavior of the system we need to specify the stochastic behavior of the constituent components. A common way of modeling this component behavior is by using a probability density function $f(t)$ that governs the failure and repair of the component through e.g. a Weibull distribution function. In other words we assume that throughout the mission time of the system the component makes transitions between failed and working states in which the time between transitions is governed by the predefined density function or alternatively a transition rate λ defined through

$$\lambda = \frac{f(t)}{1 - F(t)},$$

where $F(t)$ represents the cumulative density function. This transition rate then governs the rate of transition of component i between its possible states.

We assume we have a way of simulating random variables from the given distribution (for more on this see Section 3 in [24]). We focus on the so-called direct method of Monte Carlo estimation. This method simulates the evolution of the system by simulating the evolution of its single components. We begin with all components in their nominal states at time $t_0 = 0$. For each component a sequence of transitions times is sampled using the samples from the failure and repair densities. In this way we can obtain a history of the components' transitions and can compute the system evolution determined by $\Phi(v)$ between two subsequent transitions. After running the simulation for N_S simulations we can then obtain estimates of the system unreliability or unavailability through the average of $\Phi(v)$ over the N_S simulations. See also [19] for details.

Aging When a component has been in use for a long time it can start to show signs of aging, i.e. a deteriating performance characterized by an increasing transition rate between the working and failed state of the component. The Weibull density function given by

$$f(t) = \frac{k}{\beta}\left(\frac{t - t_0}{\beta}\right)^{k-1} e^{-\left(\frac{t-t_0}{\beta}\right)^k}.$$

The parameter k allows for changes in transition rate, in particular $k > 1$ allows for modeling component aging. In other words, the Weibull distribution allows for modelling a time-dependent failure rate which allows to take into account a changing default intensity of the components (see also [6]).

Maintenance The reliability estimates of a system can be performed by taking into account various maintenance and repair strategies, in this way obtaining knowledge of the optimal strategies so as to reduce the plant downtime and costs of maintenance and repair. The optimization of the maintenance strategy can be performed with genetic algorithms, see e.g. [20]. Alternatively, one can perform a Monte Carlo simulation with a particular maintenance and repair scheme and compute both the costs as well as the estimation of the system availability, in order to obtain an idea of optimal maintenance strategies. Again we refer to [6] for more on this.

Importance measures Another component of interest in the reliability assessments is the importance measures which allow one to estimate sensitivity measures of the simulation output to various parameters of interest in the model. The Monte Carlo method allows one to compute first-order sensitivities at little additional cost by perturbing the parameters and computing the effect on the simulation output function of interest. This in turn gives an idea of the relevance of the component structure to the reliability of the system. For more details we refer to [19].

3 Financial systems

Interbank markets allow financial institutions to borrow and lend from and to each other in this way earning a profit or increasing liquidity. Financial systems thus consist of individual banks connected to each other most commonly through financial liabilities owed by one firm to another. In this way the value of the firm is dependent on the payments they receive from their claims on other firms. This value in turn depends on the financial health of again other firms in the network. In this way the lending agreements cause the banks to be interconnected with each other in an intricate way. A default of one firm might then cause a default cascade through the network causing defaults at other banks it is connected to, the connections being represented by these loan agreements. Systemic risk, defined as the risk of collapse of an entire system triggered by a default of one or a few firms, is therefore a very important measure to understand in these kind of networks.

Since the global financial crisis the concern of policy makers with regard to being able to identify, monitor and address systemic risk present in interbank networks has increased. One way of modeling the interconnectedness of financial systems can be using a network representation in which the nodes represent the institutions, while the connections represents counterparty exposures. There are multiple papers that have studied networks like this, see e.g. [13] where the authors consider a one-period network in which the liabilities are cleared consistently with the laws of bankruptcy and a fictitious default algorithm is developed to illustrate how contagion propagates through the network, [22] in which the authors extend this framework by additionally considering liquidation costs arising at default, [1] and [4] where the authors study the role of the architecture of the network in the propagation of the contagion when the institutions suffer a negative shock in asset value. It is common to consider a static network and study the robustness of the system by perturbing single nodes, see e.g. [4]. In this Section we will discuss several common ways of computing the risk in a financial system, in particular focussing on ones that bear similartities with the risk computation algorithm in engineering.

3.1 The Algorithms

In this section we present several common ways of computing risk in interconnected systems. We do not go into detail into the algorithms until later in Section 4 in which we present our own multi-period simulation model for assessing the risk in the system.

Network simulation and risk computation The fictituous default algorithm proposed in [13] computes a so-called clearing vector representing the payments made by each node in the system so that these satisfy the bankruptcy laws. A bank is set to be defaulted once it is no longer able to repay its obligations. The algorithm goes as follows: determine each node's payout, assuming all other nodes satisfy their obligations. If under the assumption that all nodes pay fully, all obligations are satisfied then we terminate the algorithm and the payout vector is given by the obligations that each bank has. If some nodes default even if all others pay in full we try to solve the system again in which the payment vector if given by the payments made by the defaulted institutions (not in full) and the solvent ones. If this causes no further defaults we terminate the algorithm. If further defaults occur we repeat this algorithm until no further defaults occur.

This algorithm is therefore an iteration allowing to determine the payment vector. However, clearly this method is static in time, in the sense that the payment vector determined is for a fixed point in time, and no changes in liabilities, shocks or interactions occur.

In [16] the authors assess the systemic risk in a network using an interbank network simulation in combination with the clearing algorithm from [13]. Using historic data on the of individual banks' activities one can compute the probability that a bank makes an interbank placement to another bank in the system. The network is generated in a random way where the connections between two nodes in the network is determined through the previously defined probability. Additionally a number is generated which determines what percentage of liabilities of one bank comes from another bank it is connected to. In this way we can construct a large number of these networks, ones in which the connections and the corresponding loan agreement values are fixed. Analysing many interbank structures instead of one specific one (as was done previously in e.g. [13]) can account for the very dynamic and unstable nature of interbank structures and allows to compute the systemic risk statistics in a more generalized way. A shock is then applied to one or a set of banks that is subsequently transmitted throughout this system.

Multi-period clearing Static models with fixed connections and liabilities provide insight about immediate consequences caused by certain shocks and defaults but they do not capture the propagation and aftershocks of default events. Similar to the Monte Carlo algorithms used in engineering we would be interested in determining a multi-period algorithm in which the banks' financial health changes through time in a predefined manner. This would allow us to assess the financial health of a system throughout a given time period. In [7] the authors consider a multi-period framework in which they stochastically evolve the interbank liabilities. For the clearing of the model in each time step they use the fictituous default algorithm from [13]. The evolving of the network through time allows to model the dynamic component of systemic risk. We remark that the connections in the network, i.e. the network structure, is kept fixed and the authors consider in particular two different types of networks in which they evolve the interbank liabilities through time to assess the systemic risk in the two types of networks throughout a given time period.

Another paper in which the network dynamics are modelled through time is the one by [23] where the authors use a so-called agent based modeling in which the agents (i.e. the banks) interact following optimal selfish strategies. The model is designed to mimic the behavior and dynamics of the interbank lending market during financial crises and the banks' balance sheet is used to determine the behavior of the banks.

Systemic risk mitigation Similar to the maintenance in engineering systems, one might be interested in systemic risk mitigation in financial networks. Consider the usual interconnected network of banks, where the banks have outstanding loan agreements with other banks in the system and suppose we model the network through time with the multi-period model discussed above. Consider also the presence of some monitoring entity, one that can give an influx of cash to any node in the system in order to avoid a default cascade. The Monte Carlo algorithms used for maintenance studies in engineering systems can be used to also assess the effects of cash influx. In particular, the monitoring entity would want to optimize the systemic risk mitigation by providing liquidity assistance to the node(s) which targets the systemically important banks. Depending on the network structure, these can be e.g. the largest banks in the system or the ones with the most connections to other banks. In [7] the authors consider a network in which the liabilities of the banks evolve stochastically through time and analyse two liquidity assistance strategies.

4 Interbank networks with self-exciting shocks

The correlation between assets held by institutions on their balance sheets and interconnectedness through lending agreements can be significant sources of default contagion. Many previous research focuses on contagion through interbank lending agreements, however contagion can occur through multiple other channels, e.g. linked balance sheets that may result in fire sales or financial acceleration.

A known source of systemic risk in financial networks is the propagation of default due to interbank exposures such as loans, where the failure of a borrowing node to repay its loans, may consequently cause a loss in liquidity of the lenders as well, in this way propagating the default through the network. Besides interbank exposures, another common cause of default propagation are fire sales. If one institution decides to liquidate a large part of its assets, depressing the price, this causes a loss at the institutions holding the same assets, creating a *cross-exciting* spiral across the institutions. Therefore, institutions that do not have mutual counterparty exposure can still suffer financial distress if they have holdings of common assets on their balance sheets. As illustrated by [15], the effects from these so-called fire sales can be even greater than the contagion effects due to counterparty exposures. The effects of these fire sales and the amplified effects of an initial shock due to further liquidations by counterparties are investigated in e.g. [8] and [9].

A *self-exciting* effect present in financial networks is known as financial acceleration and refers to the fact that current variations in the asset side of the balance sheet depend on past variations in the assets themselves. In other words, a shock affecting the banks portfolio can cause creditors to claim their funds back or tighten the credit conditions, in this way causing an additional shock for the bank. In [4] the authors incorporated this effect by including a jump in the SDE for robustness where the jump at time t depends also on the realization of the robustness at $t-1$. The penalty associated to this jump depends on the change in the robustness. This means that in case a bank is hit by a negative shock its partners react only if this shock is perceived as abnormal given the current market conditions.

As has been mentioned in [10], while interbank lending itself may not be a significant cause of default propagation, it is important to account for both the correlated effects of default contagion through lending agreements as well as exposure to common market events. Here, we choose to model the correlated effects of the fire sales, financial acceleration and the interbank lending structure on both the default propagation as well as overall loss in the network through a Hawkes counting process. The shocks affecting the portfolio of the institution arrive conditional on the infinite history of previous shocks to both the institutions own assets as well as those of the other nodes in the network provided that they share common assets.

In this section we will investigate the correlated effects of self-exciting shocks affecting the asset value of the bank – where the excitement is due to both a financial acceleration, so that current variations in the asset value of a bank can be self-excited by the movements of the asset value in the past, as well as the effect of other banks' asset movements due to common asset holdings – and the default propagation through these shocks on quantities such as the total loss in the network. We derive a multi-period network of interbank loans and and present several ways of modeling the evolvement of the connections when the asset values of the nodes are subjected to the Hawkes-distributed shock. Using a Monte Carlo simulation we derive several results on the total loss in the network and show how this depends on both the dynamics of the self-exciting shock as well as on the specification of the interaction term in the network. This multi-period framework with a dynamic connection evolvement could help one to understand what strategies and network structures are more robust to the effects of the Hawkes shock.

4.1 Hawkes processes

Specific types of events that are observed in time do not always arrive in evenly spaced intervals, but can show signs of clustering, e.g. the arrival of trades in an order book, or the contagious default of financial institutions. Therefore, assuming that these events happen independently is not a valid assumption. A Hawkes process (HP), also known as a self-exciting process, has an intensity function whose current value, unlike in the Poisson process, is influenced by past events. In particular, if an arrival causes the conditional intensity to increase, the process is said to be self-exciting, causing a temporal clustering of arrivals. Hawkes processes can be used for modelling credit default events in a portfolio of securities, as has been done in e.g. [14] or for modelling asset prices using a mutually exciting jump component to model the contagion of financial shocks over different markets ([2]). An overview of other applications of Hawkes processes in finance, in particular in modelling the market microstructure, can be found in e.g. [3].

Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a complete filtered probability space where the filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ satisfies the usual condition. Hawkes processes ([17]) are a class of multi-variate counting processes $(N_t^1, ..., N_t^M)_{t\geq 0}$ characterized by a stochastic intensity vector $(\lambda_t^1, ..., \lambda_t^M)_{t\geq 0}$ which describes the $\mathcal{F}_t$-conditional mean jump rate per unit of time, where $\mathcal{F}_t$ is the filtration generated by $(N^i)_{1\leq i\leq M}$ up to time t. Consider the set of nodes $I_M := \{1, \ldots, M\}$. Define the kernel $g(t) = (g^{i,j}(t), (i,j) \in I_M \times I_M)$ with $g^{i,j}(t) : \mathbb{R}_+ \to \mathbb{R}$ and the constant intensity $\mu = (\mu^i, i \in I_M)$ with $\mu^i \in \mathbb{R}_+$.

Definition 4.1 (Hawkes process)**.** A linear Hawkes process with parameters (g, μ) is a family of $\mathcal{F}_t$-adapted counting processes $(N_t^i)_{i\in I_M, t\geq 0}$ such that:

1. almost surely for all $i \neq j$, $(N_t^i)_{t\geq 0}$ and $(N_t^j)_{t\geq 0}$ never jump simultaneously,
2. for every $i \in I_M$, the compensator Λ_t^i of N_t^i has the form $\Lambda_t^i = \int_0^t \lambda_s^i ds$, where the intensity process $(\lambda_t^i)_{t\geq 0}$ is given by

$$\lambda_t^i = \mu^i + \sum_{j=1}^{M} \int_{[0,t[} g^{i,j}(t-s) dN_s^j. \tag{4.1}$$

In other words, $g^{i,j}$ denotes the influence of an event of type j on the arrival of i: each previous event dN_s^j raises the jump intensity $(\lambda_t^i)_{i\in I_M}$ of its neighbors through the function $g^{i,j}$. The compensated jump process $N_t - \int_0^t \lambda_s ds$ is a $\mathcal{F}_t$-local martingale. For g a positive and a decreasing function of time t, the influence of a jump decreases and tends to 0 as time evolves. By introducing the pair $\{t_k, n_k\}_{k=1}^{K_t}$, where t_k denotes the time of event k, $n_k \in I_M$ is the event type and $K_t = \sum_{i=1}^{M} N_t^i$ is total number of event arrivals up to time t, we can rewrite the intensity as

$$\lambda_t^i = \mu^i + \sum_{k=1}^{K_t} g^{i,n_k}(t - t_k), \qquad i \in I_M.$$

A common choice for $g^{i,j}(t)$ is an exponential decay function defined as

$$g^{i,j}(t) = \alpha^{i,j} e^{-\beta^i t}, \tag{4.2}$$

so that λ_t^i jumps by $\alpha^{i,j}$ when a shock in j occurs, and then decays back towards the mean level μ^i at speed β^i. Note that this function satisfies the local integrability property, i.e. $g^{i,j} \in L^1_{\text{loc}}(\mathbb{R}_+)$. If $g^{i,j}$ is exponential then the couple (N_t, λ_t) is a Markov process. The simulation of a Hawkes process can be done using what is known as Ogata's modified thinning algorithm, see for more details [21] and [11] and Section 6.1.

4.2 Banks' financial structure

Let us consider an interbank network representation of the mean-field model developed in the previous section. Fix a finite time horizon T divided into discrete intervals $[t_i, t_{i+1})$, $i \in \{1, ..., N\}$ and times $t_m \in \{t_1, ..., t_N\}$ corresponding to the interbank loan payments. The network is then given by (B, E), where B denotes the M nodes in the system b_i, $i = 1, ..., M$, and E the directed connections between the nodes given by $x^{i,j}$, representing the loan given by j to i at time $t-1$ which needs to be repaid at time t for $i \neq j$ and $x_t^{i,i} = 0$. We will discuss the formation of these connections in a later section. We denote by o_t^i the operating cash inflow of bank i at time t, e.g. the income from the assets minus the liabilities. Let l_t denote the liabilities vector at time t with elements $l_t^i = \sum_{j=1}^M x_t^{i,j}$ denoting the total obligations from bank i to the other nodes in the system. Define

$$\pi_t^{i,j} = \begin{cases} \frac{x_t^{i,j}}{l_t^i} & \text{if } x_t^i > 0 \\ 0 & \text{else,} \end{cases}$$

the relative size of liabilities owed by i to j at time t.

4.2.1 Liquidity shocks

As mentioned before, besides default due to a connected interbanking network, fire sales and financial acceleration are also common causes of default propagation. These result in self- and cross-exciting shocks, respectively. We model shocks through time due to an unspecified market event affecting i's operating cash value, e.g. if one bank decides to liquidate a large part of its assets causing a cross-exciting effect at institutions holding similar assets or a self-exciting effect when a past shock causes creditors to claim back their funds resulting in another liquidity shock. The shock occurs at time t with the intensity λ_t^i given by a Hawkes process in (4.1). Then $g^{i,j}(t)$ for $i \neq j$ reflects the general commonness of the asset holdings on the balance sheets of i and j, so that a shock received by bank b_j can also influence the balance sheet of b_i, while the term $g^{i,i}$ models the financial acceleration, so that current variations in the asset value can be self-excited by what happened in the past due to the counterparties of i penalizing i in case of sudden robustness decline. Using the Hawkes process thus allows us to simulate the self-exciting characteristics of the financial market in combination with a default propagation.

4.2.2 Clearing payments

At time t suppose that bank i is hit by a shock $c^i Z_t$ affecting the non-lending related assets. As a consequence, depending on the size of the shock, i might become unable to repay its interbank liabilities. The network is thus stochastic and the randomness comes from the random shocks affecting the balance sheet distributed as Hawkes processes. After each shock we use a clearing payment system similar to the one-period fictituous default algorhtm in [13]. A clearing payment vector represents the payments made by each node in the system so that these satisfy the standard conditions imposed by bankruptcy laws: limited liability of equity, priority of liability over equity, and proportional repayments of liabilities in default. The clearing vector (i.e. the payment vector) for node i at time t is given by p_t^i defined as

$$p_t^i = \begin{cases} (1+r_t^i)l_t^i & \text{if } \tau^i > t \\ (1-\alpha)(l_t^i + c_t^i) & \text{if } \tau^i = t \\ 0 & \text{if} \tau^i < t \end{cases}$$

where τ^i is defined as the default time of node i, α denotes the cost of default and r_t^i is the interest rate paid on the loan. Each non-defaulted node therefore repays the full liabilities plus an interest rate specific to that node given by r_t^i. The defaulted nodes repay their liabilities proportionally. Define the cash available to node i at time t after entering into the loan agreements signed at time t with maturity dates $t_m > t$ as

$$v_t^i = c_{t-1}^i - \sum_{j=1}^{M} x_t^{j,i} + \sum_{j=1}^{M} x_t^{i,j}.$$

Then, on a repayment date, having computed the payment vector, the cash available to node i at time t after clearing is given by

$$c_t^i := (1+r)v_{t-1}^i + o_t^i + \sum_{j=1}^{M} \pi_t^{j,i} p_t^j - (1+r_t^i)l_t^i,$$

i.e. the cash surplus from the previous time step after entering the loans on which a risk-free rate r is earned, the operating cash inflow at time t, the repayments on the loans given out by node i and the repaid liabilities owed by i to the nodes it borrowed from. Note that in this case if a node i did not default, it pays a rate r_t^i on its borrowing activities, and earns r_t^j on the loans given out to non-defaulted nodes $j = 1, ..., M$ for $i \neq j$. When entering into the loans, we expect o_t^i to be sufficient enough to be able to repay the liabilities. However, this operating cash is exposed to a Hawkes-distributed shock

$$o_t^i := \sigma^i dW_t^i + c^i Z_t dN_t^i,$$

which might result in the node i no longer being able to repay its liabilities in full. In particular, a bank i is considered to be in default if $c_t^i \leq 0$. The cash assets c_t^i thus contain any losses and gains from the interbank loans entered into at $t-1$ and the shocks from financial acceleration and/or fire sales at t.

The clearing vector p_t^i is determined through the *multi-period fictitious default algorithm*, a simple multi-period extension of the algorithm defined in [13]. After each shock we try to clear the system. We check whether the banks in the system still possess enough liquidity, i.e. if $c_t^i(p_t^i) \geq 0$. Starting with $\bar{p}_t^0 = (1+r_t)l_t$, let $\bar{p}_t^{n+1} = (1+r_t)l_t \wedge (1-\alpha)(l_t + c_t(\bar{p}_t^n))$, the minimum between the liabilities and the assets under the assumption that the repayments are done as determined in the previous round $\bar{p}_t^n$. We continue wit $n = 1, ...$ until no further defaults occur and the payment vector is given as $p_t = \bar{p}_t^n$. We remark that each shock thus results in lower asset values for the bank receiving the shock and in case of a default also for the claimants, in turn making them more susceptible to default. Note that defining the defaults in this way we consider sequential defaults, not just first-order defaults (as is done in [7]). We define the systemic risk as the expected total loss in the system relative to the total size of liabilities as

$$S(T) = \mathbb{E}\left[\frac{\sum_{t=0}^{T}\sum_{i\in\mathcal{D}_t}(l^i(t) - p^i(t))}{\sum_{t=0}^{T}\sum_{i=1}^{M} l^i(t)}\right],$$

where we assume $\mathcal{D}_t$ to be the set of defaulted nodes at time t.

4.3 Dynamic connection evolvement

Throughout the simulation of the model we can assume that the interactions, i.e. lending agreements, are static and pre-determined upon initialisation of the simulation. Alternatively, there exist methods of letting these interactions vary throughout the simulation period. In this section we present two different models for modeling the interaction term between the nodes.

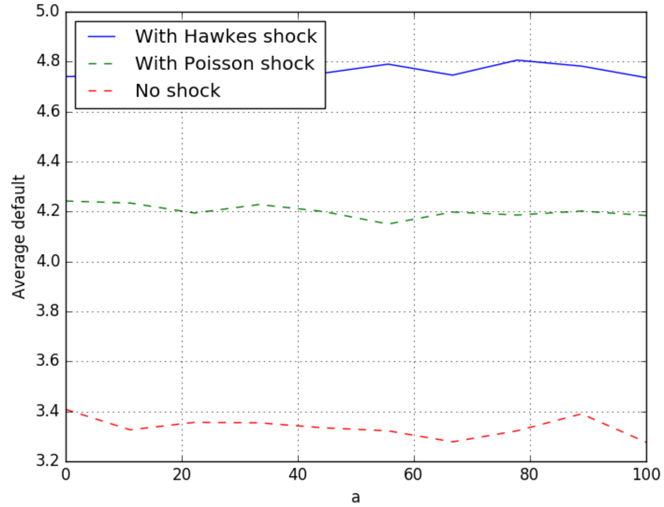
4.3.1 Mean-field interaction

The first and simplest model we consider is similar to that of a mean-field interaction term. We assume

$$x_t^{i,j} = \frac{a}{M}\max(c_{t-1}^j - c_{t-1}^i, 0),$$

so that if the available cash of j is higher than that of i, in other words j has a surplus in its accounts, it gives a loan to i the size of which is controlled trough the normalized constant a/M and vice versa. This corresponds to the actual lending and borrowing motivations of banks in a system. In case a bank has too much liquidity, it can lend this to other banks earning an interest rate over the loan. Similarly in the case of a liquidity shortage, the bank needs to borrow from other banks against the interest rate in order to fulfill the reserve requirements. In case of the loans being overnight this mean-field interaction is fairly representative of the way banks lend and borrow in a system, since for an overnight loan, one can disregard the probability of default of the counterparties in the loan since an overnight default occurs rarely. The constant controlling the size of the loans a_i also controls the total systemic risk in the network. In Figure 4.1 we see the average number of defaults systemic risk for an independent Poisson jump and a Hawkes jump for $a \in [0, 100]$. As we can see, the systemic risk in case of a Hawkes process is higher than in the independent case, as expected, and more lending at first increases the systemic risk, but after a certain point reaches a stable level. In Figure 4.2 we simulate the cumulative total loss in the network through time for two different values of the lending coefficient a. One can clearly see in the top figure for $a = 1$ the self-exciting property of the Hawkes processes being reflected in the total loss resulting in an additional factor that can propagate the default through the network resulting in more clustered increases in the loss. The Poisson shock loss is smaller and contains less of these clusters. As a becomes larger ($a = 10$) the Hawkes loss remains larger than the Poisson loss but the clustering of the defaults becomes less visible.

Figure 4.1: The average number of defaults and the systemic risk in the network for both an independent Poisson shock as well as the Hawkes shock for $a \in [0, 100]$. The parameters in the Monte Carlo simulated based on a discretized Euler-Maruyama scheme are $M = 10$, $T = 1$, 500 simulations, 100 time steps, $\sigma = 0.05$, $\alpha = 0.2$, $r, r^i = 0$ and the jump size is $\sim N(0, 1)$ with $\mu^i = 10/M$ and G exponential as in (4.2) with $\beta^{i,j} = 1/M$, $\alpha^{i,j} = 1/M$.



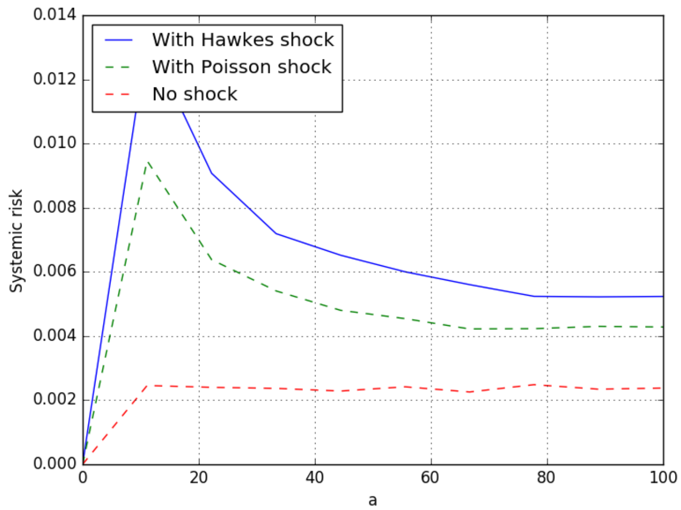
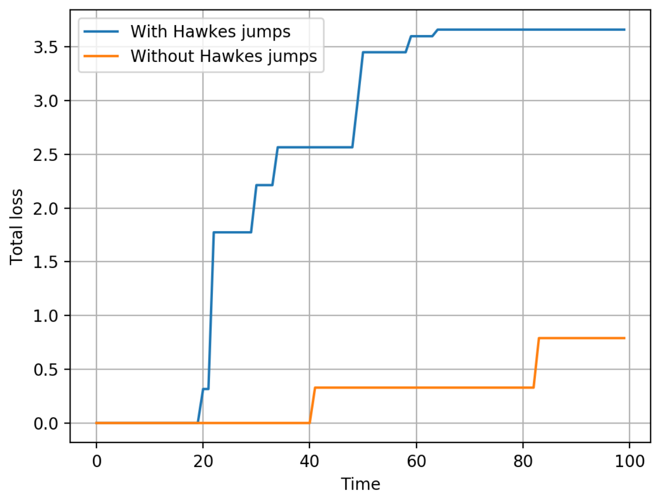
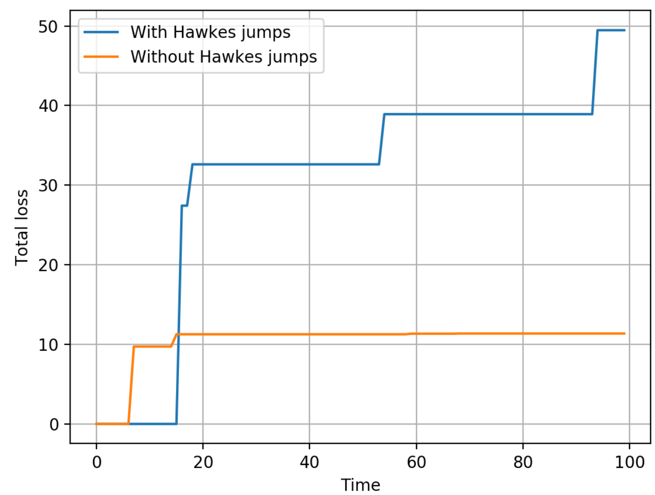


Figure 4.2: The total loss in the network through time for both an independent Poisson shock as well as the Hawkes shock. The parameters in the Monte Carlo simulated based on a discretized Euler-Maruyama scheme are $M = 10$, $T = 1$, 1000 simulations, 100 time steps, $a = 1$ (T) and $a = 10$ (B), $\sigma = 0.2$, $\alpha = 0.2$, $r, r^i = 0$ and the jump size is $\sim N(0, 1)$ with $\mu^i = 10/M$ and G exponential as in (4.2) with $\beta^{i,j} = 1/M$, $\alpha^{i,j} = 5/M$.





4.3.2 Default-based interaction

The downside of the mean-field interaction is that each node i enters into a loan agreement at time t with every bank j for which $c_t^i > c_t^j$, regardless of the ability of j to repay this loan at the due date. As mentioned, while for overnight loans this default probability is very small, for long-term lending this is something that each bank should be taking into account. Therefore, we follow here [18] and present a more dynamic connection evolvement in which each bank i considers the expected profit $\Pi^{j,i}$ from giving a loan to j to be given by

$$\mathbb{E}(\Pi_t^{j,i}) = (1-p_t^j)r(t)c_t^{j,i} + p_t^j \alpha a_t^j - \delta a_t^j,$$

where p^j is the probability of j defaulting, r is the interest rate received on the loan, δ is the opportunity cost of entering into the lending agreement and $c_{j,i}$ is the maximum that i is willing to lend to j and is given by $(1-h_t^j)a_t^j$ where a^j are j's assets and $h^j = h^{\max}/\sqrt{d_{t-1}^j + 1}$ is the haircut which is inversely proportional to the number of incoming connections of the agent in the previous time step. In other words the maximum one bank is willing to lend to another is its lending capacity given by its asset values minus a haircut. This number of connections can be interpreted as the credit rating, since the higher the number of banks willing to lend to j the safer he must be. Then each node i cuts its outgoing link with k and forms a link with j with probability

$$P_t^i = \frac{1}{1+e^{-\gamma(\varphi_t^j - \varphi_t^k}},$$

where

$$\varphi_t^j = \frac{\tilde{p}_t^j}{\tilde{p}_t^{\max}}, \quad \tilde{p}_t^j = \frac{r(1-h_t^j)-\delta}{r(1-h_t^j)-\alpha},$$

where γ generates the different network structures. In particular, when increasing γ the network becomes more centralized with a few banks being particularly attractive borrowers. Therefore, changing γ gives an idea of the network structure most robust against default propagation. This model thus allows to model a changing network structure over time. A similar methodology of connection making is considered in [12]. The authors use a preferential attachment mechanism and show that by varying the parameters in the network creation several different interbank networks are built.

5 An analytical method for risk computation

Assume that, for $i \in I_M$ the log-monetary reserves of the i-th bank satisfies the following stochastic differential equation (SDE)

$$dX_t^i = \frac{a^i}{M}\sum_{k=1}^{M}(X_t^k - X_t^i)dt + \sigma^i dW_t^i + c^i dN_t^i,$$

with $X_0^i \in \mathbb{R}_+$ the initial reserves for each bank and where $a^i \geq 0$, $\sigma^i \geq 0$ and $c^i := \hat{c}^i/M < 0$ are constants for each $i \in I_M$. The process $W(t) = \{W_t^i\}_{i=1}^M$ is a M-dimensional uncorrelated Brownian motion, and $N_t = \{N_t^i\}_{i=1}^M$ is the vector of Hawkes processes with self-exciting intensity λ_t^i as defined in 4.1. With the drift term defined in this way, if bank k has more (less) log-monetary reserves than bank i, i.e. $X_t^k > X_t^i$ ($X_t^k < X_t^i$), bank k is assumed to lend (borrow) a proportion of the surplus to (deficit from) bank i, with proportionality factor a^i/M. A jump in the Hawkes process i affects the corresponding X_t^i through the proportionality factor c^i and increases the intensity λ_t^j for $j \in I_M$ if $g^{i,j}(t) \neq 0$. In this way the jump activity varies over time resulting in a clustering of the arrival of the jumps and the shocks propagate through the network in a contagious manner through the contagion function $g^{i,j}(t)$. We thus interpret the jump term $c^i dN_t^i$ as a self- and cross-exciting *negative* effect that occurs due to financial acceleration and fire sales, resulting in a decrease in a banks monetary reserve. In [5] the authors considered a similar mean-field model for the monetary reserves but assumed the jumps to occur at independent Poisson distributed random times. However, not accounting for the clustering effect of the jumps might cause a significant underestimation of the systemic risk present in the network. We define a default level $D \leq 0$ and say that bank i is in a default state at time T if its log-monetary reserve reached the level D at time T. We remark that in our model even if bank i has defaulted, i.e. its monetary reserve reaches a negative level, it continues to participate in the interbank activities borrowing from the counterparties until it e.g. reaches a positive reserve level again. In other words, the level of monetary reserves takes in values in $\mathbb{R}$. We will work in the following setting:

Assumption 5.2 (Parameters)**.** We collect the parameters associated with the dynamics of the i-th monetary reserve process $i \in I_M$ as

$$p^i := (a^i, \sigma^i, c^i) \in (\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_-).$$

We assume that $p^i \to p^* := (a, \sigma, c)$ as $i \to \infty$.

Defining the reserve average as

$$\bar{X}_t = \frac{1}{M} \sum_{i=1}^{M} X_t^i,$$

we can rewrite the SDE as a mean-field interaction SDE

$$dX_t^i = a^i(\bar{X}_t - X_t^i)dt + \sigma^i dW_t^i + c^i dN_t^i. \tag{5.3}$$

From (5.3) we see that the processes (X_t^i) are mean-reverting to their ensemble average $(\bar{X}_t)$ at rate a^i. Furthermore define

$$\nu_t^M(f) = \frac{1}{M} \sum_{i=1}^{M} f(p^i, X_t^i), \qquad t \geq 0.$$

Then we have $\bar{X}_t = \nu_t^M(I)$ where $I(x) = x$. Define then

$$\nu_t(A) := \mathbb{P}(X_t(\mathbf{p}) \in A), \tag{5.4}$$

where $A \in \mathcal{B}(\mathbb{R})$ and the underlying limiting state process $X(\mathbf{p}) = (X_t(\mathbf{p}))_{t\geq 0}$ is a diffusion with time-varying coefficients given by

$$X_t(\mathbf{p}) = x + \int_0^t \left(a\left(Q_1(s) - X_s(\mathbf{p})\right) + c\bar{\lambda}_s\right) ds + \sigma \int_0^t dW_s, \qquad t \geq 0,$$

with $\bar{\lambda}_t$ is defined as

$$\bar{\lambda}_t := \mu + \int_0^t g(t-s) d\mathbb{E}[\bar{N}_s],$$

and

$$Q_1(t) = x + c\int_0^t \bar{\lambda}_s ds.$$

Then one can prove the following Theorem.

Theorem 5.3 (Limiting process)**.** *The empirical measure-valued process ν^M admits the weak convergence $\nu^M \to \nu$, as $M \to \infty$, where ν is defined as in* (5.4)*. Furthermore, $\nu^M(I) \to Q_1$.*

In other words, the system evolution when the number of nodes tends to infinity is governed by the measue ν_t (the empirical measure of the system at its limit) and the SDE $X_t(\mathbf{p})$ determining the empirical measure.

Risk indicators We propose to compute systemic risk in the mean-field model based on the fraction of banks that have transitioned from a normal to a defaulted state. We define the risk indicator as the expected value of the fraction of banks that throughout time $t \in [0,T]$ have dropped below the default level D,

$$\mathrm{SR}^M := \mathbb{E}\left[\frac{1}{M}\sum_{i=1}^{M} \mathbb{1}_{\left\{\min\limits_{0\le t\le T} X_t^i \le D\right\}}\right].$$

We have the following result that is able to give a sense of the average defaults in a large system

Lemma 5.4. *For $t \in [0,T]$ we have*

$$\lim_{M\to\infty} \mathrm{SR}^M = \mathbb{E}\left[\mathbb{1}_{\left\{\min\limits_{0\le t\le T} X_t(\mathbf{p}) \le D\right\}}\right].$$

This is the price of a barrier option on the underlying limiting process $X_t(\mathbf{p})$.

Furthermore, similar to [5] we can define the average distance to default as

$$\mathrm{ADD}^M(t) := \mathbb{E}\left[\frac{1}{M}\sum_{i=1}^{M} X_t^i\right],$$

for which we have the following result

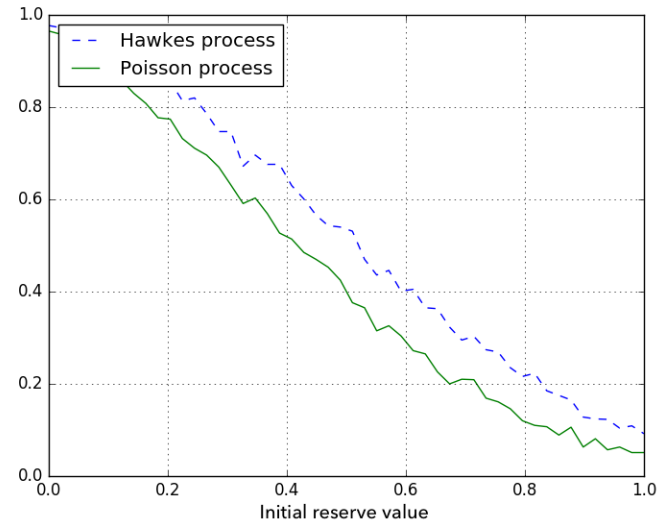
Lemma 5.5. *We have*

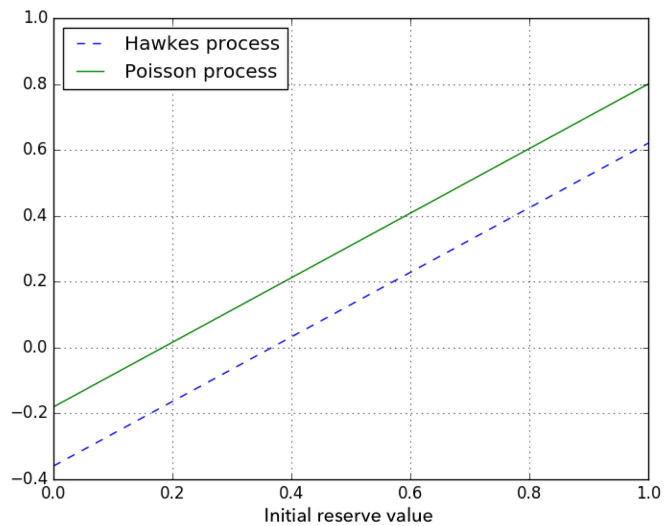
$$\lim_{M\to\infty} \mathrm{ADD}^M(t) = \nu_t(I).$$

In other words, the expected value of the average of the monetary reserve processes is given by $\nu_t(I) = \mathbb{E}[X_t(\mathbf{p})]$.

One can compute the risk measures defined above using a Monte Carlo simulation in which we simulate the stochastic evolution of each node separately using the SDE in (5.3) and compute the risk measures for the system. Alternatively, when the system of components is large enough (i.e. a large number of banks in the network) we can use the results obtained above, in which case we have analytical expressions for the risk measures. In Figure 5 we present the LLN estimates for the systemic risk and average distance to default for a Poisson process (i.e. independent jumps) and a Hawkes process (i.e. jumps excite each other). The results clearly show that the Hawkes process, as expected, results in a higher risk in the system.

Figure 5.1: LLN estimates for the systemic risk (T) and LLN estimates for the average distance to default (B) for a network of $M = 300$ bankts at time $T = 1$ with $\mu = 0.2$, $\alpha = 1.2$, $\beta = 1.2$, $a = 0.5$, $\sigma = 0.5$, $c = -1$ and $D = 0$ for a independent Poisson process and the Hawkes process for $x_0 \in [0, 1]$





6 Appendix

6.1 Hawkes process simulation

In a Hawkes process the intensity $\lambda(t)$ is assumed to be a monotonically decreasing function of time t after each event. The general idea behind these so-called thinning algorithms is to consider a point process generated with a constant intensity function which dominates the desired rate function, $\lambda(t) \leq \hat{\lambda}$ and then accept an event at time t_k with probability $\lambda(t_k)/\hat{\lambda}$ so that the remaining points form a point process with the desired intensity. The efficacy of the algorithm depends on how closely the dominating intensity approximates the desired rate. In particular, the algorithm for simulating a multi-variate Hawkes process is based on the following proposition.

Proposition 6.6. *(Ogata, 1981) Consider the M-variate point process $N(t)$ with $\mathcal{F}_t$-conditional intensity $\lambda^i(t)$, $i \in I_M$. Suppose we have a $\mathcal{F}_t$-predictable process $\hat{\lambda}(t)$ such that $\sum_{i=1}^{M} \lambda^i(t) \leq \hat{\lambda}(t)$ for $0 \leq t \leq T$ and set $\lambda^0(t) = \hat{\lambda}(t) - \sum_{i=1}^{M} \lambda^i(t)$ Let $\hat{t}_1 \leq ... \leq \hat{t}_{\hat{N}(t)}$ be the points of the process $\hat{N}(t)$ with piecewise constant intensity $\hat{\lambda}(t)$ changing according to $\mathcal{F}_t$. For each point $\hat{t}_j$, assign to it a mark $i \in I_M$ with probability $\lambda^i(\hat{t}_j)/\hat{\lambda}(\hat{t}_j)$. Then the points with marks $i \in I_M$ correspond to the points generated by a M-variate Hawkes process with intensities $\lambda^i(t)$.*

Proof. For a proof we refer to [21]. □

Based on this proposition, the algorithm for simulating an M-variate Hawkes process on the interval $[0, T]$ where we assume an empty initial state can be formulated as follows:

1. Define $t_0 = 0$.
2. For $k \geq 1$:
   (a) Define $\tau_{0,k} = t_{k-1}$
   (b) For $r \geq 1$ define $\hat{\lambda} = \sum_{i=1}^{M} \lambda^i(\tau_{r-1,k}|k-1)$, sample $U \sim U(0,1)$ and set $E = -\ln(U)/\hat{\lambda}$. Define the arrival time of the next candidate point by $\tau_{r,k} = \tau_{r-1,k} + E$, so that the interarrival time is exponentially distributed with intensity $\hat{\lambda}$. Sample $D \sim U(0,1)$. If $D\hat{\lambda} \leq \sum_{i=1}^{M} \lambda^i(\tau_{r,k})$ stop the iteration so that $\tau_{r,k}$ is accepted with probability $\sum_{i=1}^{M} \lambda^i(\tau_{r,k})/\hat{\lambda}$.
   (c) Find an $n \in I_M$ such that $\sum_{i=1}^{n-1} \lambda^i(\tau_{r,k}) < D\hat{\lambda} \leq \sum_{i=1}^{n} \lambda^i(\tau_{r,k})$ so that $\tau_{r,k}$ is assigned to dimension n with probability $\lambda^n(\tau_{r,k})/\hat{\lambda}$.
3. Define $t_k = \tau_k$ and $n_k = k$.
4. If $t_k > T$ return $\{(t_0, n_0), ..., (t_{k-1}, n_{k-1})\}$, else continue with (2).

We denote with $\lambda^i(t|k)$ the intensity at time t given the first k events.

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