Robust error bounds for ultrasonic models in the inviscid limit

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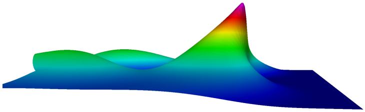
joint work with Benjamin Dörich (KIT)



2024 SCS Springmeeting

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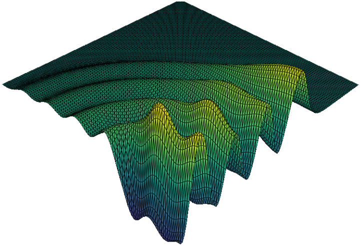
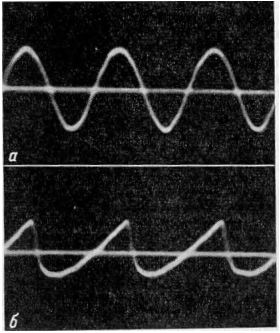
Based on

- Robust fully discrete error bounds for the Kuznetsov equation in the inviscid limit

  with Benjamin Dörich (KIT), arXiv:2401.06492, 2024.

Ultrasonics



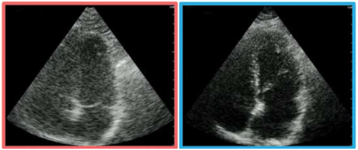


[Naugol'nykh, Romanenko 1958]

High amplitude-to-wavelength ratio $\rightsquigarrow$ Nonlinear behavior

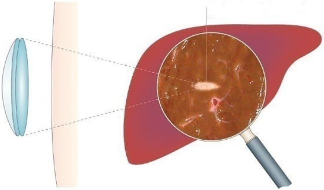
Ultrasound applications

- Diagnostic ultrasound



[Sapozhnikov et al. 2019]

- Therapeutic ultrasound



[Kennedy 2005]

Modeling

The Kuznetsov wave equation

$$u_{tt} - c^2\Delta u - \beta\Delta u_t = \left(\kappa u_t^2 + \ell\nabla u \cdot \nabla u\right)_t$$

[Kuznetsov 1973]

u ... acoustic velocity potential, $c > 0$ ... sound speed

κ, $\ell \in \mathbb{R}$ ... nonlinearity coefficients

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[Kuznetsov 1973]

u ... acoustic velocity potential, $c > 0$ ... sound speed

κ, $\ell \in \mathbb{R}$ ... nonlinearity coefficients

$\beta \geq 0$ ... sound diffusivity

Properties

The Kuznetsov wave equation

$$(1 - 2\kappa u_t)u_{tt} - c^2\Delta u - \beta\Delta u_t = 2\ell\nabla u \cdot \nabla u_t$$

- Quasilinear wave evolution
- Needed to avoid degeneracy:

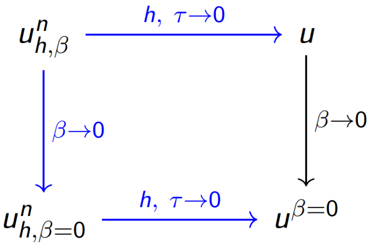
$$0 < \underline{\mathfrak{a}} \leq 1 - 2\kappa u_t \leq \overline{\mathfrak{a}} \quad \text{in } \Omega \times (0, T)$$

- Strongly damped if $\beta > 0$

Central question: Stability as $\beta \searrow 0$

Stability in the inviscid limit

- Preserving asymptotics



- Continuous setting: [Kaltenbacher & N., SIMA 2022]

Conforming FE approximation

- $\Omega \subset \mathbb{R}^d$ is a polygonal/polyhedral convex domain, $d \in \{1, 2, 3\}$
- $\Omega = \bigcup_{K \in \mathcal{T}_h} K$, K triangles/tetrahedrons
- $\{\mathcal{T}_h\}_{h \in (0,\bar{h})}$ regular family of quasi-uniform partitions

$$V_h = \{u_h \in H_0^1(\Omega) : \ u_h|_K \in P_k(K), \forall K \in \mathcal{T}_h\}, \ k \geq 2$$

- Useful: Inverse estimates and discrete Sobolev embeddings

$$\|\Delta_h \varphi_h\|_{L^2(\Omega)} \leq Ch^{-1} \|\nabla \varphi_h\|_{L^2(\Omega)}, \ \|\varphi_h\|_{L^\infty(\Omega)} \leq C \|\Delta_h \varphi_h\|_{L^2(\Omega)}$$

Conforming FE approximation

$$((1-2\kappa\partial_t u_h)\partial_t^2 u_h, \varphi_h)_{L^2} - (c^2\Delta_h u_h, \varphi_h)_{L^2} - (\beta\Delta_h\partial_t u_h, \varphi_h)_{L^2} - 2\ell(\nabla u_h \cdot \nabla\partial_t u_h, \varphi_h)_{L^2} = 0$$

for all $\varphi_h \in V_h$, with $(u_h, \partial_t u_h)|_{t=0} = (\mathrm{R}_h u_0, \mathrm{R}_h v_0)$

- Needed to avoid degeneracy:

$$0 < \underline{\mathfrak{a}} \leq 1 - 2\kappa\partial_t u_h \leq \overline{\mathfrak{a}} \quad \text{in } \Omega \times (0, T)$$

Approach in the analysis

Existence is first proven on an h-dependent interval $[0, t_h^*]$ and then extended to $[0, T]$.

[Hochbruck, Maier, IMAJNA 2022], [Dörich, Found. Comput. Math. 2023]

Main steps

1. Local existence on $[0, t_h^*]$ via the Picard–Lindelöf theorem
2. Uniform estimate for $e_h = u_h - \mathrm{R}_h u$

Approach in the analysis

$$\boxed{(1-2\kappa u_t)u_{tt} - c^2\Delta u - \beta\Delta u_t = 2\ell\nabla u \cdot \nabla u_t}$$

- Problematic term:

$$\ldots = 2\ell(\nabla u_h \cdot \nabla u_{h,tt}, u_{h,tt})_{L^2}$$

- Idea: Set up the analysis to exploit

$$2\ell(\nabla u \cdot \nabla e_{htt}, e_{htt})_{L^2} = -\ell(\Delta u, e_{htt}^2)_{L^2}$$

with $e_h = u_h - \mathrm{R}_h u$

Approach in the analysis

$$(1-2\kappa u_t)u_{tt} - c^2\Delta u - \beta\Delta u_t = 2\ell\nabla u \cdot \nabla u_t$$

- Testing strategy:

$$(P_{e_h}) \cdot (-\Delta_h e_h) + (P_{e_h})_t \cdot e_{h,tt}$$

- Smooth exact solution u needed and $k \geq 2$

Robust FE error bounds

- Set $\partial_t^2 u_h(0)$ as the solution to

$$
\begin{aligned}
&((1 - 2\kappa \partial_t u_h(0))\partial_t^2 u_h(0), \varphi_h)_{L^2} - (c^2 \Delta_h u_h(0), \varphi_h)_{L^2} \\
&\quad - (\beta \Delta_h \partial_t u_h(0), \varphi_h)_{L^2} - 2\ell(\nabla u_h(0) \cdot \nabla \partial_t u_h(0), \varphi_h)_{L^2} = 0
\end{aligned}
$$

Then there exists $h_0 > 0$ and $C > 0$, independent of h and β, such that for all $h \leq h_0$

$$
\left\| \partial_t^2 u(t) - \partial_t^2 u_h(t) \right\|_{L^2(\Omega)}^2 + \|\nabla \partial_t u(t) - \nabla \partial_t u_h(t)\|_{L^2(\Omega)}^2 \leq C h^{2k}
$$

for all $t \in [0, T]$.

Limiting behavior

The difference $\bar{u}_h = u_h^{\beta=0} - u_h$ solves

$$\begin{aligned}((1 - 2\kappa\partial_t u_h^{\beta=0})\partial_t^2 \bar{u}_h - 2\kappa\partial_t \bar{u}_h \partial_t^2 u_h - c^2 \Delta_h \bar{u}_h - 2\ell\nabla \bar{u}_h \cdot \nabla\partial_t u_h \\ - 2\ell\nabla u_h^{\beta=0} \cdot \nabla\partial_t \bar{u}_h, \varphi_h)_{L^2} = -\beta(\Delta_h \partial_t u_h, \varphi_h)_{L^2}\end{aligned}$$

Limiting behavior

The difference $\bar{u}_h = u_h^{\beta=0} - u_h$ solves

$$((1 - 2\kappa\partial_t u_h^{\beta=0})\partial_t^2 \bar{u}_h - 2\kappa\partial_t \bar{u}_h \partial_t^2 u_h - c^2 \Delta_h \bar{u}_h - 2\ell\nabla \bar{u}_h \cdot \nabla\partial_t u_h \\ - 2\ell\nabla u_h^{\beta=0} \cdot \nabla\partial_t \bar{u}_h, \varphi_h)_{L^2} = -\beta(\Delta_h \partial_t u_h, \varphi_h)_{L^2}$$

Main steps

1. Testing the difference equation with $\phi_h = \partial_t \bar{u}_h$
2. Relying on the obtained uniform bounds and

$$\beta \int_0^t (\nabla\partial_t u_h, \nabla\partial_t \bar{u}_h)_{L^2}\, \mathrm{d}s \\ = \beta(\nabla\partial_t u_h(t), \nabla\bar{u}_h(t))_{L^2} - \int_0^t (\nabla\partial_t^2 u_h, \nabla\bar{u}_h)_{L^2}\, \mathrm{d}s$$

Limiting behavior

Theorem. [Dörich & N., 2024] Under the previous assumptions, for $h \in (0, h_0]$, the family $\{u_h\}_{\beta \in (0, \bar{\beta}]}$ converges in the energy norm to $u_h^{\beta=0}$ at a linear rate as $\beta \to 0$:

$$\|\partial_t u_h - \partial_t u_h^{\beta=0}\|_{L^\infty(L^2(\Omega))} + \|\nabla(u_h - u_h^{\beta=0})\|_{L^\infty(L^2(\Omega))} \leq C\beta,$$

where the constant $C > 0$ is independent of β and h.

A fully discrete problem

- Semi-implicit time discretization

$$((1-2\kappa\partial_\tau u_h^n)\partial_\tau^2 u_h^{n+1},\varphi_h)_{L^2} - c^2(\Delta_h u_h^{n+1},\varphi_h)_{L^2} - \beta(\Delta_h\partial_\tau u_h^{n+1},\varphi_h)_{L^2} - 2(\ell\nabla u_h^n\cdot\nabla\partial_\tau u_h^{n+1},\varphi_h)_{L^2} = 0$$

for all $\varphi_h \in V_h$, $1 \leq n \leq N$, where

$$\partial_\tau a^n = \frac{1}{\tau}(a^n - a^{n-1}),\ n \geq 1, \qquad \partial_\tau^{k+1} a^n = \partial_\tau \partial_\tau^k a^n,\ k \geq 0$$

A fully discrete problem

$$((1-2\kappa\partial_\tau u_h^n)\partial_\tau^2 u_h^{n+1}, \varphi_h)_{L^2} - c^2(\Delta_h u_h^{n+1}, \varphi_h)_{L^2} - \beta(\Delta_h \partial_\tau u_h^{n+1}, \varphi_h)_{L^2} - 2(\ell \nabla u_h^n \cdot \nabla \partial_\tau u_h^{n+1}, \varphi_h)_{L^2} = 0$$

- Analogous strategy in the analysis
- The estimates are derived for the fully discrete error

$$e_h^n = \mathrm{R}_h \widehat{u}^n - u_h^n, \quad \widehat{u}^n = u(t_n)$$

- Under the CFL condition

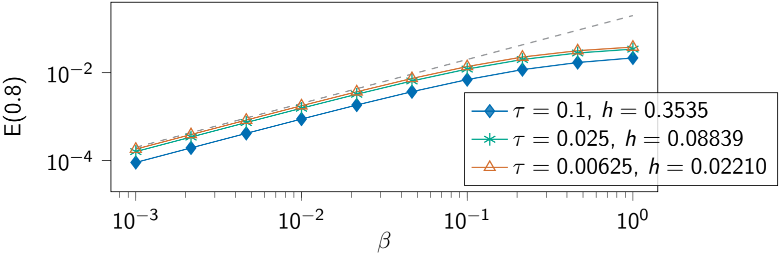
$$\boxed{\tau \leq Ch^{1+d/6+2\varepsilon}}$$

Preserving asymptotics on a fully discrete level

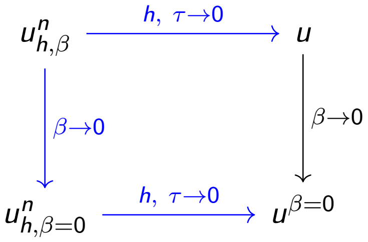
Theorem [Dörich & N., 2024] Under previous assumptions, for $h \in (0, \overline{h}]$ and $n \in \{2, \ldots, N+1\}$ fixed, the following bound holds:

$$E(t_n) := \|\partial_\tau u^n_{h,\beta} - \partial_\tau u^n_{h,\beta=0}\|_{L^2(\Omega)} + \|\nabla(u^n_{h,\beta} - u^n_{h,\beta=0})\|_{L^2(\Omega)} \leq C\beta,$$

for all $n = 1, \ldots, N+1$, where the constant $C > 0$ is independent of β, h, and τ.



1. Robust error bounds in the inviscid limit
2. Convergence rates in the vanishing β limit



Thank you!