Inverse methods for freeform optical design

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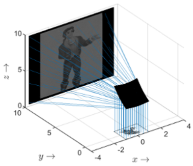
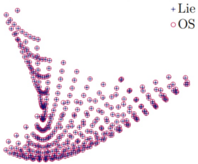
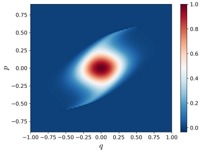
Dutch-Flemish Scientific Computing Society
Spring Meeting
May 24, 2024, Rijksuniversiteit Groningen

Computational illumination optics at TU/e

- computational illumination optics group in Math. Department
- research tracks
  - Nonimaging freeform optics
  - Imaging optics
  - Improved direct methods
- collaboration with local high-tech industry
- `https://www.win.tue.nl/~martijna/Optics`







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2024 Woudschoten conference

Computational Optics is a theme

Speakers

- Martijn Anthonissen (TU Eindhoven)
- Boris Thibert (U. Grenoble Alpes)

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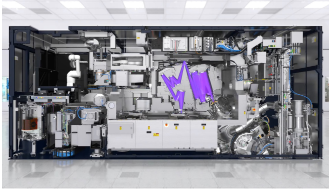
Illumination optics

- optics for illumination purposes
- geometrical optics
- two branches: nonimaging vs. imaging optics
- our goal: develop simulation tools for optical design



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Geometrical optics

- describes light propagation in terms of rays, ray optics
- Fermat's principle: the optical path length (OPL)

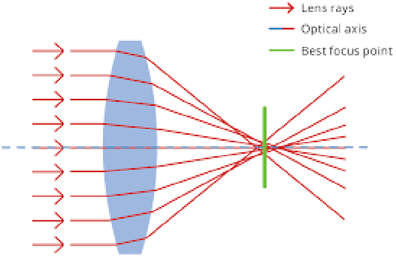
$$\mathrm{OPL} = \int_{\mathcal{C}} n \, \mathrm{d}s$$

  of a ray connecting two points attains a stationary value
- laws of reflection/refraction
- Euler-Lagrange equations: ray equation
- alternative description: Hamiltonian system

- transfer of light between source and target
- *not* concerned with image formation
- devices: reflectors, lenses, TIR-collimator, light guides etc.
- mathematics: optimal transport
- application: lighting → Signify
- related problem: antenna design

Imaging optics

- goal is to form a precise image of an object
- study and reduce aberrations (imperfections in image)
- devices: reflectors, lenses
- mathematics: Lie transformations
- application: EUV lithography systems → ASML

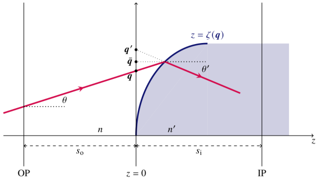


- typical mathematical model

$$\boldsymbol{q}' = -[H,\cdot]\boldsymbol{q}, \quad \boldsymbol{p}' = -[H,\cdot]\boldsymbol{p}$$
$$\boldsymbol{q}(z) = \exp(-z[H,\cdot])\boldsymbol{q}(0), \quad \boldsymbol{p}(z) = \exp(-z[H,\cdot])\boldsymbol{p}(0)$$
$$\bar{\boldsymbol{q}} = \boldsymbol{q} + \zeta(\bar{\boldsymbol{q}})\frac{\boldsymbol{p}}{\sqrt{n^2-|\boldsymbol{p}|^2}}, \quad \bar{\boldsymbol{p}} = \boldsymbol{p} + \nabla\zeta(\bar{\boldsymbol{q}})\sqrt{n^2-|\boldsymbol{p}|^2}$$

etc.

- promising alternative to classical aberration theory



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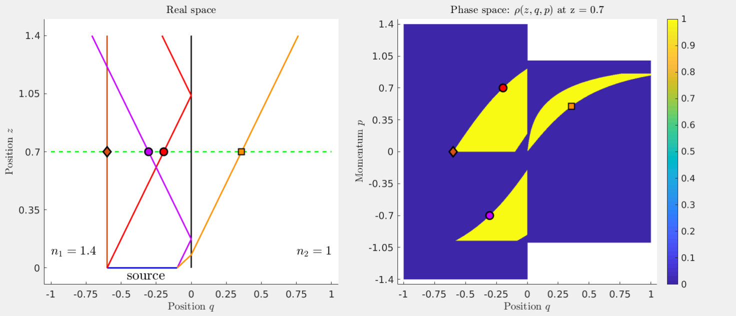
Novel simulation techniques

- current standard: Monte Carlo ray tracing
- alternative description based on phase space (space and direction coordinates)
- light ray = 'point' moving in phase space, light beam = 'flow' in phase space
- governing equation: Liouville's eq.

$$\frac{\partial \rho}{\partial z} + \frac{\partial H}{\partial \boldsymbol{p}} \cdot \frac{\partial \rho}{\partial \boldsymbol{q}} - \frac{\partial H}{\partial \boldsymbol{q}} \cdot \frac{\partial \rho}{\partial \boldsymbol{p}} = 0$$

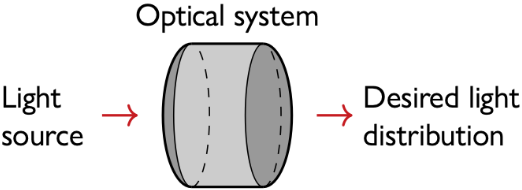
- numerical methods from CFD available
- outperforms standard ray tracing (in 2D)

Novel simulation techniques



Freeform optical design

- given: source and target distributions
- ideal source: parallel, point (zero-étendue)
- goal: design optical surface(s) (reflector/lens)
- freeform surfaces, no symmetries



Three model systems

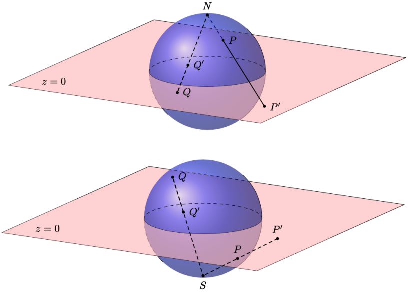
- parallel-to-far-field reflector
- parallel-to-parallel lens
- point-to-near-field reflector

Nomenclature

- ray direction vectors: $\hat{\boldsymbol{s}} \in \mathrm{S}^2$ (emitted), $\hat{\boldsymbol{t}} \in \mathrm{S}^2$ (transmitted)
- ray coordinates at source/target: $\boldsymbol{q} \in \mathrm{R}^2$ (position), $\boldsymbol{p} \in \mathrm{R}^2$ (momentum)
- parametrization optical system: $\boldsymbol{x} \in \mathcal{S}$ (source), $\boldsymbol{y} \in \mathcal{T}$ (target)

Stereographic projections

- source
  - planar $\boldsymbol{x} = \boldsymbol{q}_{\mathrm{s}}$
  - point $\boldsymbol{x} = \boldsymbol{S}[\hat{\boldsymbol{s}}]$
- target
  - screen $\boldsymbol{y} = \boldsymbol{q}_{\mathrm{t}}$
  - far field $\boldsymbol{y} = \boldsymbol{S}[\hat{\boldsymbol{t}}]$
- optical map $\boldsymbol{y} = \boldsymbol{m}(\boldsymbol{x})$



- parametrization: $\boldsymbol{x} = \boldsymbol{q}_{\mathrm{s}}$, $\boldsymbol{y} = \boldsymbol{S}[\hat{\boldsymbol{t}}]$

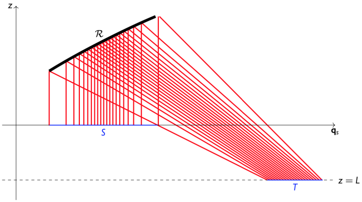
$$\mathcal{R} : z = u(\boldsymbol{x}), \quad \boldsymbol{x} \in \mathcal{S}$$

- optical map $\boldsymbol{q}_{\mathrm{s}} \mapsto \boldsymbol{p}_{\mathrm{t}} = \begin{pmatrix} t_1 \\ t_2 \end{pmatrix}$
- Hamilton's mixed characteristic

$$W(\boldsymbol{q}_{\mathrm{s}}, \boldsymbol{p}_{\mathrm{t}}) = V(\boldsymbol{q}_{\mathrm{s}}, \boldsymbol{q}_{\mathrm{t}}) - \boldsymbol{q}_{\mathrm{t}} \cdot \boldsymbol{p}_{\mathrm{t}}$$
$$V(\boldsymbol{q}_{\mathrm{s}}, \boldsymbol{q}_{\mathrm{t}}) = u(\boldsymbol{x}) + d$$
$$d = \sqrt{|\boldsymbol{q}_{\mathrm{t}} - \boldsymbol{x}|^2 + (L - u(\boldsymbol{x}))^2}$$

- condition at source

$$\frac{\partial W}{\partial \boldsymbol{q}_{\mathrm{s}}} = \boldsymbol{0} \quad \Rightarrow \quad W = W(\boldsymbol{y})$$



- W independent of L
- geometrical equation $(u_1 = u, u_2 = f(W))$

$$u_1(\boldsymbol{x}) + u_2(\boldsymbol{y}) = \boldsymbol{x} \cdot \boldsymbol{y}$$

- separation of variables in LHS, quadratic cost function $\boldsymbol{x} \cdot \boldsymbol{y}$
- multiple solution pairs $(u_1(\boldsymbol{x}), u_2(\boldsymbol{y}))$
- possible solution: Legendre transform

$$u_1(\boldsymbol{x}) = \max_{\boldsymbol{y} \in \mathcal{T}} (\boldsymbol{x} \cdot \boldsymbol{y} - u_2(\boldsymbol{y})), \quad u_2(\boldsymbol{y}) = \max_{\boldsymbol{x} \in \mathcal{S}} (\boldsymbol{x} \cdot \boldsymbol{y} - u_1(\boldsymbol{x}))$$

- necessary condition: stationary point

$$\boldsymbol{y} - \nabla u_1(\boldsymbol{x}) = \boldsymbol{0}$$

Parallel-to-far-field reflector

- conservation of luminous flux ($\mathcal{A} \subset \mathcal{S}$), far-field approximation

$$\int_{\mathcal{A}} M(\boldsymbol{x})\,\mathrm{d}A(\boldsymbol{x}) = \int_{\hat{\boldsymbol{t}}(\mathcal{A})} I(\hat{\boldsymbol{t}})\,\mathrm{d}S(\hat{\boldsymbol{t}})$$

- transform to stereographic coordinates, substitute $\boldsymbol{y} = \boldsymbol{m}(\boldsymbol{x})$
- differential form, assume $\det(\mathrm{D}\boldsymbol{m}) > 0$

$$\det(\mathrm{D}\boldsymbol{m}) = \underbrace{\tfrac{1}{4}\left(|\boldsymbol{m}(\boldsymbol{x})|^2 + 1\right)^2}_{\boldsymbol{y}\mapsto\hat{\boldsymbol{t}}} \frac{M(\boldsymbol{x})}{I(\boldsymbol{m}(\boldsymbol{x}))} =: F(\boldsymbol{x}, \boldsymbol{m}(\boldsymbol{x}))$$

- transport boundary condition $\boldsymbol{m}(\partial\mathcal{S}) = \partial\mathcal{T}$

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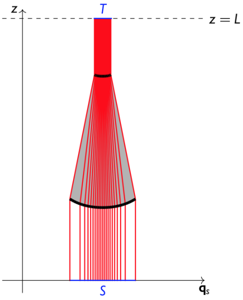
- double freeform lens
- parametrization: $\boldsymbol{x} = \boldsymbol{q}_{\mathrm{s}}$, $\boldsymbol{y} = \boldsymbol{q}_{\mathrm{t}}$

$$\mathcal{L}_1 : z = u_1(\boldsymbol{x}), \quad \boldsymbol{x} \in \mathcal{S}$$
$$\mathcal{L}_2 : z = L - u_2(\boldsymbol{y}), \quad \boldsymbol{y} \in \mathcal{T}$$



- optical map $\boldsymbol{q}_{\mathrm{s}} \mapsto \boldsymbol{q}_{\mathrm{t}}$
- Hamilton's point characteristic (OPL)

$$V(\boldsymbol{q}_{\mathrm{s}}, \boldsymbol{q}_{\mathrm{t}}) = u_1(\boldsymbol{q}_{\mathrm{s}}) + nd + u_2(\boldsymbol{q}_{\mathrm{t}})$$
$$d = \sqrt{|\boldsymbol{q}_{\mathrm{t}} - \boldsymbol{q}_{\mathrm{s}}|^2 + (L - u_2(\boldsymbol{q}_{\mathrm{t}}) - u_1(\boldsymbol{q}_{\mathrm{s}}))^2}$$

- conditions at source and target

$$\frac{\partial V}{\partial \boldsymbol{q}_{\mathrm{s}}} = \boldsymbol{0}, \quad \frac{\partial V}{\partial \boldsymbol{q}_{\mathrm{t}}} = \boldsymbol{0}$$

Parallel-to-parallel lens

$\boldsymbol{x} = \boldsymbol{q}_\mathrm{s}, \boldsymbol{y} = \boldsymbol{q}_\mathrm{t}$

- OPL is constant, $V(\boldsymbol{q}_\mathrm{s}, \boldsymbol{q}_\mathrm{t}) = \mathrm{Const}$
- geometrical equation

$$u_1(\boldsymbol{x}) + u_2(\boldsymbol{y}) = c(\boldsymbol{x}, \boldsymbol{y})$$

- separation of variables $\boldsymbol{x}$ and $\boldsymbol{y}$ in LHS
- cost function $(\beta = V - L)$

$$c(\boldsymbol{x}, \boldsymbol{y}) = -\frac{V - n^2 L}{n^2 - 1} - \frac{n}{n^2 - 1}\sqrt{\beta^2 - (n^2 - 1)|\boldsymbol{y} - \boldsymbol{x}|^2}$$

- multiple solution pairs $(u_1(\boldsymbol{x}), u_2(\boldsymbol{y}))$

Parallel-to-parallel lens

- possible solution: c-transform

$$u_1(\boldsymbol{x}) = \max_{\boldsymbol{y}\in\mathcal{T}}\big(c(\boldsymbol{x},\boldsymbol{y}) - u_2(\boldsymbol{y})\big), \quad u_2(\boldsymbol{y}) = \max_{\boldsymbol{x}\in\mathcal{S}}\big(c(\boldsymbol{x},\boldsymbol{y}) - u_1(\boldsymbol{x})\big)$$

- necessary condition: stationary point

$$\nabla_{\boldsymbol{x}} c(\boldsymbol{x},\boldsymbol{y}) - \nabla u_1(\boldsymbol{x}) = \boldsymbol{0} \quad (*)$$

- substitute $\boldsymbol{y} = \boldsymbol{m}(\boldsymbol{x})$ in $(*)$, take gradient with respect to $\boldsymbol{x}$

$$\boldsymbol{C}(\boldsymbol{x},\boldsymbol{m}(\boldsymbol{x}))\mathrm{D}\boldsymbol{m}(\boldsymbol{x}) = \boldsymbol{P}(\boldsymbol{x})$$

$$\boldsymbol{C}(\boldsymbol{x},\boldsymbol{y}) = \left(\frac{\partial^2 c}{\partial x_i \partial y_j}\right), \quad \boldsymbol{P}(\boldsymbol{x}) = \mathrm{D}^2 u_1(\boldsymbol{x}) - \mathrm{D}_{\boldsymbol{x}\boldsymbol{x}} c(\boldsymbol{x},\boldsymbol{m}(\boldsymbol{x}))$$

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- conservation of luminous flux
- integral form, $\mathcal{A} \subset \mathcal{S}$

$$\int_{\mathcal{A}} M(\boldsymbol{x})\,\mathrm{d}A(\boldsymbol{x}) = \int_{\boldsymbol{m}(\mathcal{A})} E(\boldsymbol{y})\,\mathrm{d}A(\boldsymbol{y})$$
$$= \int_{\mathcal{A}} E(\boldsymbol{m}(\boldsymbol{x}))\left|\det(\mathrm{D}\boldsymbol{m}(\boldsymbol{x}))\right|\mathrm{d}A(\boldsymbol{x})$$

- differential form, assume $\det(\mathrm{D}\boldsymbol{m}) > 0$

$$\det(\mathrm{D}\boldsymbol{m}) = \frac{M(\boldsymbol{x})}{E(\boldsymbol{m}(\boldsymbol{x}))} =: F(\boldsymbol{x}, \boldsymbol{m}(\boldsymbol{x}))$$

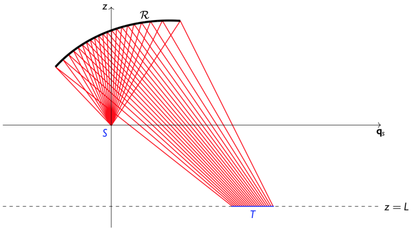
- transport boundary condition $\boldsymbol{m}(\partial\mathcal{S}) = \partial\mathcal{T}$

- parametrization: $\boldsymbol{x} = \boldsymbol{S}[\hat{\boldsymbol{s}}]$, $\boldsymbol{y} = \boldsymbol{q}_\mathrm{t}$

$$\mathcal{R} : \boldsymbol{r} = u(\boldsymbol{x})\hat{\boldsymbol{e}}_r, \quad \boldsymbol{x} \in \mathcal{S}$$

- optical map $\boldsymbol{p}_\mathrm{s} = \begin{pmatrix} s_1 \\ s_2 \end{pmatrix} \mapsto \boldsymbol{q}_\mathrm{t}$
- second mixed Hamilton's characteristic function

$$W^*(\boldsymbol{p}_\mathrm{s}, \boldsymbol{q}_\mathrm{t}) = V(\boldsymbol{q}_\mathrm{s}, \boldsymbol{q}_\mathrm{t}) + \boldsymbol{q}_\mathrm{s} \cdot \boldsymbol{p}_\mathrm{s}$$
$$V(\boldsymbol{q}_\mathrm{s}, \boldsymbol{q}_\mathrm{t}) = u(\hat{\boldsymbol{s}}) + d, \quad d = \sqrt{|\boldsymbol{q}_\mathrm{t} - u(\hat{\boldsymbol{s}})\boldsymbol{p}_\mathrm{s}|^2 + (L - u(\hat{\boldsymbol{s}})s_3)^2}$$



POINT-TO-NEAR FIELD REFLECTOR

$\boldsymbol{x} = \boldsymbol{S}[\hat{s}], \boldsymbol{y} = \boldsymbol{q}_{\mathrm{t}}$

- condition at source

$$\frac{\partial W^*}{\partial \boldsymbol{p}_{\mathrm{s}}} = \boldsymbol{0} \quad \Longrightarrow \quad W^* = W^*(\boldsymbol{y})$$

- geometrical equation $(u_1 = u, u_2 = W^*)$

$$u_2(\boldsymbol{y}) = H(\boldsymbol{x}, \boldsymbol{y}, u_1(\boldsymbol{x}))$$
$$H(\boldsymbol{x}, \boldsymbol{y}, z) = z + \sqrt{|\boldsymbol{y} - z\boldsymbol{p}_{\mathrm{s}}(\boldsymbol{x})|^2 + (L - z s_3(\boldsymbol{x}))^2}$$

- no separation of variables $\boldsymbol{x}$ and $\boldsymbol{y}$ in expression for H
- for fixed $\boldsymbol{x}$ and $\boldsymbol{y}$, H invertible: $H^{-1}(\boldsymbol{x}, \boldsymbol{y}, \cdot) = G(\boldsymbol{x}, \boldsymbol{y}, \cdot)$

Point-to-near field reflector

- generating function $G = G(\boldsymbol{x}, \boldsymbol{y}, z)$
- location of reflector

$$u_1(\boldsymbol{x}) = G(\boldsymbol{x}, \boldsymbol{y}, u_2(\boldsymbol{y}))$$

$$G(\boldsymbol{x}, \boldsymbol{y}, z) = \frac{1}{2}\frac{z^2 - |\tilde{\boldsymbol{y}}|^2}{z - \hat{\boldsymbol{s}}(\boldsymbol{x}) \cdot \tilde{\boldsymbol{y}}}, \quad \tilde{\boldsymbol{y}} = \begin{pmatrix} \boldsymbol{y} \\ L \end{pmatrix}$$

- multiple solution pairs $(u_1(\boldsymbol{x}), u_2(\boldsymbol{y}))$

Point-to-near field reflector

- possible solution: G-convex, H-concave pair

$$u_1(\boldsymbol{x}) = \max_{\boldsymbol{y}\in\mathcal{T}} G(\boldsymbol{x},\boldsymbol{y},u_2(\boldsymbol{y})), \quad u_2(\boldsymbol{y}) = \min_{\boldsymbol{x}\in\mathcal{S}} H(\boldsymbol{x},\boldsymbol{y},u_1(\boldsymbol{x}))$$

- necessary condition, $\widetilde{H}(\boldsymbol{x},\boldsymbol{y}) := H(\boldsymbol{x},\boldsymbol{y},u_1(\boldsymbol{x}))$

$$\nabla_{\boldsymbol{x}}\widetilde{H}(\boldsymbol{x},\boldsymbol{y}) = \boldsymbol{0} \quad (*)$$

- substitute $\boldsymbol{y} = \boldsymbol{m}(\boldsymbol{x})$ in $(*)$, take gradient with respect to $\boldsymbol{x}$

$$\boldsymbol{C}(\boldsymbol{x},\boldsymbol{m}(\boldsymbol{x}),u_1(\boldsymbol{x}))\mathrm{D}\boldsymbol{m}(\boldsymbol{x}) = \boldsymbol{P}(\boldsymbol{x},u_1(\boldsymbol{x}))$$

$$\boldsymbol{C}(\boldsymbol{x},\boldsymbol{y}) = \left(\frac{\partial^2\widetilde{H}}{\partial x_i \partial y_j}\right), \quad \boldsymbol{P}(\boldsymbol{x},u_1(\boldsymbol{x})) = -\mathrm{D}_{\boldsymbol{x}\boldsymbol{x}}\widetilde{H}(\boldsymbol{x},\boldsymbol{m}(\boldsymbol{x}))$$

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- conservation of luminous flux
- integral form, $\hat{\boldsymbol{s}} \in A \subset \mathrm{S}^2, \boldsymbol{x} \in \mathcal{A} \subset \mathcal{S}$

$$\int_A I(\hat{\boldsymbol{s}})\,\mathrm{d}S(\hat{\boldsymbol{s}}) = \int_{\boldsymbol{m}(\mathcal{A})} E(\boldsymbol{y})\,\mathrm{d}A(\boldsymbol{y})$$

- transform to stereographic coordinates, substitute $\boldsymbol{y} = \boldsymbol{m}(\boldsymbol{x})$
- differential form, assume $\det(\mathrm{D}\boldsymbol{m}) > 0$

$$\det(\mathrm{D}\boldsymbol{m}) = \underbrace{\frac{4}{(1+|\boldsymbol{x}|^2)^2}}_{\boldsymbol{x}\mapsto\hat{\boldsymbol{s}}} \frac{I(\boldsymbol{x})}{E(\boldsymbol{m}(\boldsymbol{x}))} =: F(\boldsymbol{x}, \boldsymbol{m}(\boldsymbol{x}))$$

- transport boundary condition $\boldsymbol{m}(\partial\mathcal{S}) = \partial\mathcal{T}$

Summary of mathematical models

- optical systems
  - parallel-to-far-field reflector (Pa2FFR)
  - parallel-to-parallel lens (Pa2PaL)
  - point-to-near-field reflector (Po2NFR)
- geometrical equation

Pa2FFR	$u_1(\boldsymbol{x}) + u_2(\boldsymbol{y}) = \boldsymbol{x} \cdot \boldsymbol{y}$
Pa2PaL	$u_1(\boldsymbol{x}) + u_2(\boldsymbol{y}) = c(\boldsymbol{x}, \boldsymbol{y})$
Po2NFR	$u_2(\boldsymbol{y}) = H(\boldsymbol{x}, \boldsymbol{y}, u_1(\boldsymbol{x})) \quad \Leftrightarrow$
	$u_1(\boldsymbol{x}) = G(\boldsymbol{x}, \boldsymbol{y}, u_2(\boldsymbol{y}))$

Summary of mathematical models

- matrix equation optical map: $\boldsymbol{C}\mathrm{D}\boldsymbol{m} = \boldsymbol{P}$

	stationary point	$\boldsymbol{C}$	$\boldsymbol{P}$
Pa2FFR	$\boldsymbol{m} - \nabla u_1 = \mathbf{0}$	$\boldsymbol{I}$	$\mathrm{D}^2 u_1$
Pa2PaL	$\nabla_{\boldsymbol{x}} c(\cdot, \boldsymbol{m}) - \nabla u_1 = \mathbf{0}$	$\mathrm{D}_{\boldsymbol{xy}} c$	$\mathrm{D}^2 u_1 - \mathrm{D}_{\boldsymbol{xx}} c$
Po2NFR	$\nabla_{\boldsymbol{x}} \widetilde{H}(\cdot, \boldsymbol{m}) = \mathbf{0}$	$\mathrm{D}_{\boldsymbol{xy}} \widetilde{H}$	$-\mathrm{D}_{\boldsymbol{xx}} \widetilde{H}$

- constraint: conservation of luminous flux

$$\det(\mathrm{D}\boldsymbol{m}) = F(\boldsymbol{x}, \boldsymbol{m}(\boldsymbol{x}))$$

Numerical solution methods

- least-squares algorithm for quadratic cost function
- two-stage algorithm

compute optical map $\boldsymbol{m} : \mathcal{S} \rightarrow \mathcal{T}$ from

$$\det(\mathrm{D}\boldsymbol{m}) = F(\cdot, \boldsymbol{m}), \quad \boldsymbol{m}(\partial\mathcal{S}) = \partial\mathcal{T}$$

compute optical surface from $\quad \boldsymbol{m} - \nabla u_1 = \mathbf{0}$

- requirement: $\mathrm{D}\boldsymbol{m} = \boldsymbol{P}$ with $\boldsymbol{P}^{\mathrm{T}} = \boldsymbol{P}$ and $\det(\boldsymbol{P}) = F$

Numerical solution methods

- functionals $(0 < \alpha < 1)$

$$J_{\mathrm{I}}[\boldsymbol{m}, \boldsymbol{P}] = \frac{1}{2} \int_{\mathcal{S}} \|\mathrm{D}\boldsymbol{m} - \boldsymbol{P}\|_{\mathrm{F}}^2 \, \mathrm{d}\boldsymbol{x}$$

$$J_{\mathrm{B}}[\boldsymbol{m}, \boldsymbol{b}] = \frac{1}{2} \oint_{\partial\mathcal{S}} |\boldsymbol{m} - \boldsymbol{b}|_2^2 \, \mathrm{d}s$$

$$J[\boldsymbol{m}, \boldsymbol{P}, \boldsymbol{b}] = \alpha J_{\mathrm{I}}[\boldsymbol{m}, \boldsymbol{P}] + (1 - \alpha) J_{\mathrm{B}}[\boldsymbol{m}, \boldsymbol{b}]$$

$$I[u_1] = \frac{1}{2} \int_{\mathcal{S}} |\boldsymbol{m} - \nabla u_1|_2^2 \, \mathrm{d}\boldsymbol{x}$$

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- iteration scheme to compute $\boldsymbol{m}$

$$\begin{aligned}
\boldsymbol{P}^{k+1} &= \mathrm{argmin}_{\boldsymbol{P}\in\mathcal{P}(\boldsymbol{m}^k)} J_{\mathrm{I}}[\boldsymbol{m}^k, \boldsymbol{P}] \\
\boldsymbol{b}^{k+1} &= \mathrm{argmin}_{\boldsymbol{b}\in\mathcal{B}} J_{\mathrm{B}}[\boldsymbol{m}^k, \boldsymbol{b}] \\
\boldsymbol{m}^{k+1} &= \mathrm{argmin}_{\boldsymbol{m}\in\mathcal{M}} J[\boldsymbol{m}, \boldsymbol{P}^{k+1}, \boldsymbol{b}^{k+1}]
\end{aligned}$$

- computation optical surface: $u_1 = \mathrm{argmin}_{v\in\mathcal{U}} I[v]$
- function spaces

$$\begin{aligned}
\mathcal{P}(\boldsymbol{m}) &= \{\boldsymbol{P} \in C^1(\mathcal{S})^{2\times 2} \,|\, \boldsymbol{P}^{\mathrm{T}} = \boldsymbol{P}, \det(\boldsymbol{P}) = F\} \\
\mathcal{B} &= \{\boldsymbol{b} \in C(\partial\mathcal{S})^2 \,|\, \boldsymbol{b}(\boldsymbol{x}) \in \partial\mathcal{T}\} \\
\mathcal{M} &= C^2(\mathcal{S})^2 \\
\mathcal{U} &= C^2(\mathcal{S})
\end{aligned}$$

Numerical solution methods

- computation $\boldsymbol{P}$
  - point-wise constrained minimization
  - constraints on $\det(\boldsymbol{P})$ and $\operatorname{tr}(\boldsymbol{P})$
  - exact solution possible
- computation $\boldsymbol{b}$
  - point-wise projection of $\boldsymbol{m}$ on $\partial\mathcal{T}$
- computation $\boldsymbol{m}$: set $\delta J[\boldsymbol{m}, \boldsymbol{P}, \boldsymbol{b}](\boldsymbol{\eta}) = 0$
- resulting BVP $(l = 1, 2)$

$$\nabla^2 m_l = \nabla \cdot \boldsymbol{p}_l, \quad \boldsymbol{x} \in \mathcal{S}$$

$$(1-\alpha)m_l + \alpha \nabla m_l \cdot \hat{\boldsymbol{n}} = (1-\alpha)b_l + \alpha \boldsymbol{p}_l \cdot \hat{\boldsymbol{n}}, \quad \boldsymbol{x} \in \partial\mathcal{S}$$

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NUMERICAL SOLUTION METHODS

- computation u_1: set $\delta I[u_1](v) = 0$
- resulting Neumann BVP

$$\nabla^2 u_1 = \nabla \cdot \boldsymbol{m}, \quad \boldsymbol{x} \in \mathcal{S}$$
$$\nabla u_1 \cdot \hat{\boldsymbol{n}} = \boldsymbol{m} \cdot \hat{\boldsymbol{n}}, \quad \boldsymbol{x} \in \partial\mathcal{S}$$

- least-squares algorithm, modifications for non-quadratic cost function
- two-stage algorithm

compute optical map $\boldsymbol{m} : \mathcal{S} \to \mathcal{T}$ from

$$\boldsymbol{C}\mathrm{D}\boldsymbol{m} = \boldsymbol{P}, \quad \det(\boldsymbol{P}) = \det(\boldsymbol{C})F, \quad \boldsymbol{m}(\partial\mathcal{S}) = \partial\mathcal{T}$$

compute optical surface from $\quad \nabla_{\boldsymbol{x}} c(\cdot, \boldsymbol{m}) - \nabla u_1 = \boldsymbol{0}$

- functionals $(0 < \alpha < 1)$

$$J_{\mathrm{I}}[\boldsymbol{m}, \boldsymbol{P}] = \frac{1}{2} \int_{\mathcal{S}} \|\boldsymbol{C}\mathrm{D}\boldsymbol{m} - \boldsymbol{P}\|_{\mathrm{F}}^2 \, \mathrm{d}\boldsymbol{x}$$

$$I[u_1] = \frac{1}{2} \int_{\mathcal{S}} |\nabla_{\boldsymbol{x}} c(\cdot, \boldsymbol{m}) - \nabla u_1|_2^2 \, \mathrm{d}\boldsymbol{x}$$

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Numerical solution methods

- function space for $\boldsymbol{P}$

$$\mathcal{P}(\boldsymbol{m}) = \{\boldsymbol{P} \in C^1(\mathcal{S})^{2\times 2} \big| \boldsymbol{P}^{\mathrm{T}} = \boldsymbol{P}, \det(\boldsymbol{P}) = \det(\boldsymbol{C})F\}$$

- computation $\boldsymbol{m}$, coupled BVP for $\boldsymbol{m}$

$$\nabla \cdot (\boldsymbol{C}^{\mathrm{T}}\boldsymbol{C}\mathrm{D}\boldsymbol{m}) = \nabla \cdot (\boldsymbol{C}^{\mathrm{T}}\boldsymbol{P}), \quad \boldsymbol{x} \in \mathcal{S}$$

$$(1-\alpha)\boldsymbol{m} + \alpha(\boldsymbol{C}^{\mathrm{T}}\boldsymbol{C}\mathrm{D}\boldsymbol{m})\hat{\boldsymbol{n}} = (1-\alpha)\boldsymbol{b} + \alpha\boldsymbol{C}^{\mathrm{T}}\boldsymbol{P}\hat{\boldsymbol{n}}, \quad \boldsymbol{x} \in \partial\mathcal{S}$$

- space discretization: FVM
- computation u_1, Neumann BVP for u_1

$$\nabla^2 u_1 = \nabla \cdot \nabla_{\boldsymbol{x}} c(\cdot, \boldsymbol{m}), \quad \boldsymbol{x} \in \mathcal{S}$$

$$\nabla u_1 \cdot \hat{\boldsymbol{n}} = \nabla_{\boldsymbol{x}} c(\cdot, \boldsymbol{m}) \cdot \hat{\boldsymbol{n}}, \quad \boldsymbol{x} \in \partial\mathcal{S}$$

Numerical solution methods

- least-squares algorithm, modification for generating function
- algorithm

compute optical map from

$$\boldsymbol{C}\mathrm{D}\boldsymbol{m} = \boldsymbol{P}, \quad \det(\boldsymbol{P}) = \det(\boldsymbol{C})F, \quad \boldsymbol{m}(\partial\mathcal{S}) = \partial\mathcal{T}$$

note: $\boldsymbol{C} = \boldsymbol{C}(\boldsymbol{x}, \boldsymbol{m}, u_1), F = F(\boldsymbol{x}, \boldsymbol{m}, u_1)$

compute optical surface from

$$\nabla_{\boldsymbol{x}} H(\cdot, \boldsymbol{m}, u_1) + H_z(\cdot, \boldsymbol{m}, u_1)\nabla u_1 = \boldsymbol{0}$$

- functional

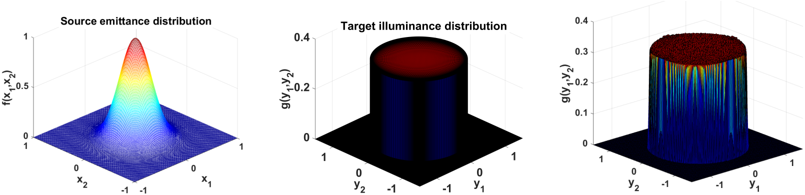
$$I[u_1, \boldsymbol{m}] = \frac{1}{2}\int_{\mathcal{S}} \left|\nabla_{\boldsymbol{x}} H(\cdot, \boldsymbol{m}, u_1) + H_z(\cdot, \boldsymbol{m}, u_1)\nabla u_1\right|_2^2 \mathrm{d}\boldsymbol{x}$$

Numerical solution methods

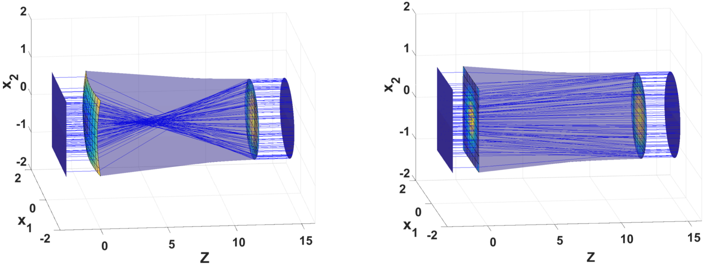
- iteration scheme

$$
\begin{aligned}
\boldsymbol{P}^{k+1} &= \mathrm{argmin}_{\boldsymbol{P}\in\mathcal{P}(\boldsymbol{m}^k)} J_{\mathrm{I}}[\boldsymbol{m}^k, \boldsymbol{P}] \\
\boldsymbol{b}^{k+1} &= \mathrm{argmin}_{\boldsymbol{b}\in\mathcal{B}} J_{\mathrm{B}}[\boldsymbol{m}^k, \boldsymbol{b}] \\
\boldsymbol{m}^{k+1} &= \mathrm{argmin}_{\boldsymbol{m}\in\mathcal{M}} J[\boldsymbol{m}, \boldsymbol{P}^{k+1}, \boldsymbol{b}^{k+1}] \\
u_1^{k+1} &= \mathrm{argmin}_{v\in\mathcal{U}} I[v, \boldsymbol{m}^{k+1}]
\end{aligned}
$$

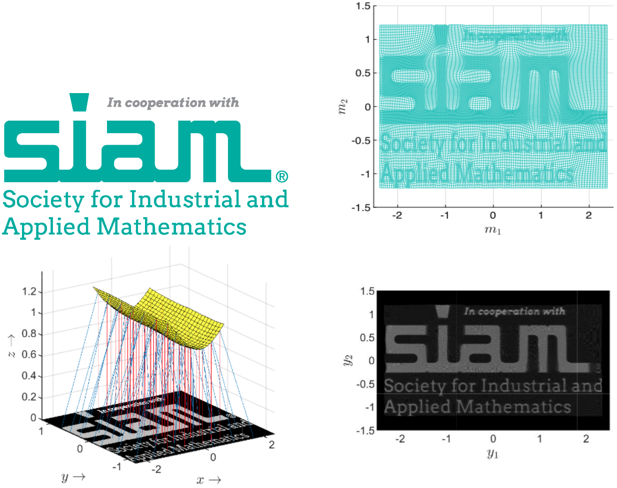
- computation u_1 included in iteration double freeform lens for laser beam shaping



source emittance (left), target illuminance (middle) and ray-traced target illuminance (right) double freeform lens for laser beam shaping



convex (left) and concave entrance surface (right) uniform source and SIAM logo target distribution



Conclusions

- mathematical/numerical methods for freeform optical design
- based on Hamilton's characteristic functions
- combine optical map with conservation of luminous flux
- solution method: iteration scheme with least-squares solvers
- complicated source/target distributions possible, high contrast
- code rewritten to production code

Computational illumination Optics group

- 4 vacancies for PhD students this year
- 5 more in 2025
- see: `https://www.win.tue.nl/~martijna/Optics`