Efficient Multigrid based solvers for Isogeometric Analysis

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Numerical Analysis

June 1, 2018

Isogeometric Analysis (IgA)

- Extension of the Finite Element Method (FEM)
- Same basis functions (**B-Splines**) are used for approximate geometry Ω_h and solution u_h
- Global mapping from Ω_h to parametric domain $\hat{\Omega}_h$
- Description of the geometry that is highly accurate ('$\Omega = \Omega_h$') throughout all computation steps

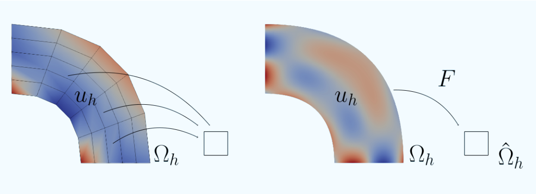


Figure: Poisson problem solved by FEM (left) and IgA (right).

Construction of B-spline basis functions

A *knot vector* is a sequence of non-decreasing points $\xi_i \in \mathbb{R}$ with the following structure:

$$\Xi = (\xi_1, \xi_2, \dots, \xi_i, \dots, \xi_{n+p}, \xi_{n+p+1})$$

where

- n is the number of B-spline basis functions
- p is the degree of the basis functions

Ξ is called *open* and *uniform* if:

- The first and last knots are repeated $p+1$ times
- All $\xi_{p+1}, \dots, \xi_{n+1}$ are equally spaced

Construction of B-spline basis functions

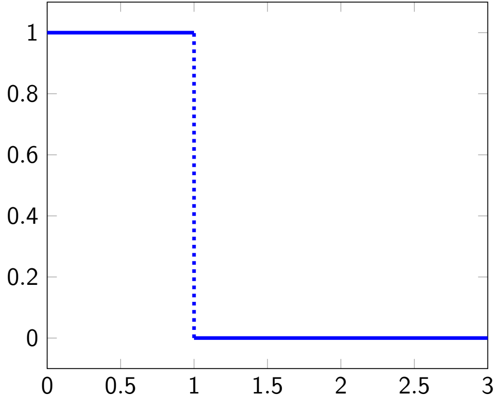
Cox-de Boor recursion formula

$$\phi_{i,0}(\xi) = \begin{cases} 1 & \text{if } \xi_i \leq \xi < \xi_{i+1} \\ 0 & \text{otherwise.} \end{cases}$$

$$\phi_{i,p}(\xi) = \frac{\xi - \xi_i}{\xi_{i+p} - \xi_i}\phi_{i,p-1}(\xi) + \frac{\xi_{i+p+1} - \xi}{\xi_{i+p+1} - \xi_{i+1}}\phi_{i+1,p-1}(\xi)$$

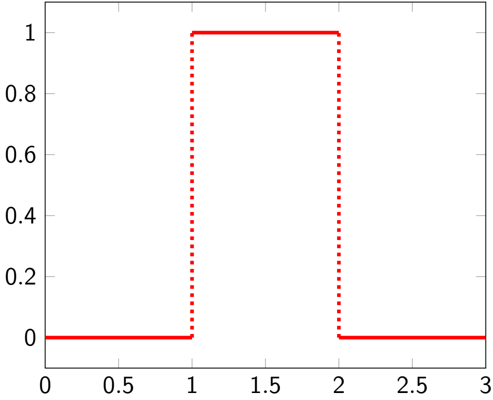
for $p \geq 1$, where $\xi \in [\xi_1, \xi_{n+p+1}]$.

Examples of B-spline basis functions ($p = 0$)



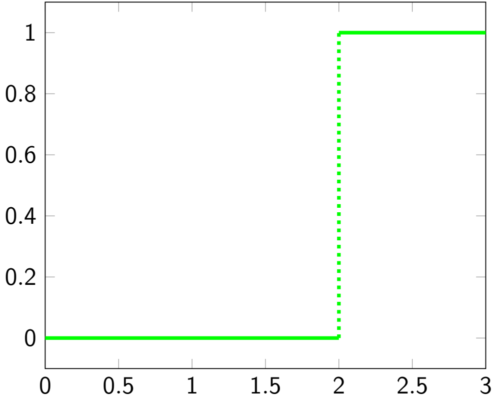
basis functions for $\Xi = \{0, 1, 2, 3\}$

Examples of B-spline basis functions ($p = 0$)



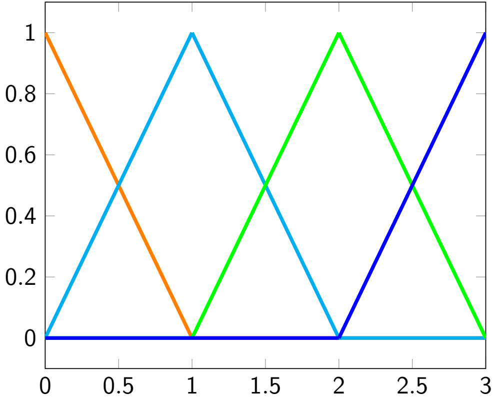
basis functions for $\Xi = \{0, 1, 2, 3\}$

Examples of B-spline basis functions ($p = 0$)



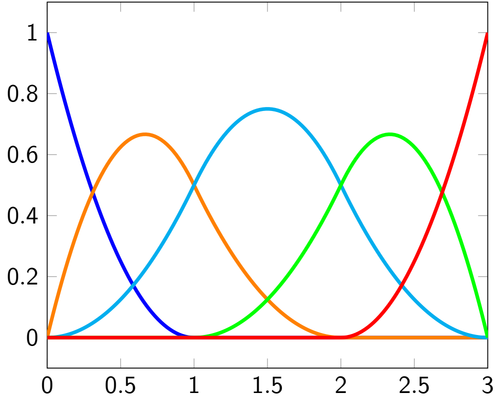
basis functions for $\Xi = \{0, 1, 2, 3\}$

Examples of B-spline basis functions ($p = 1$)



Linear basis functions for $\Xi = \{0, 0, 1, 2, 3, 3\}$

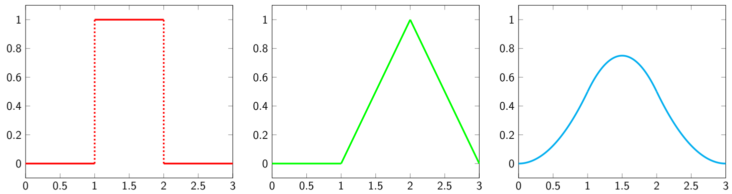
Examples of B-spline basis functions ($p = 2$)



Quadratic basis functions for $\Xi = \{0, 0, 0, 1, 2, 3, 3, 3\}$

Properties of B-spline basis functions

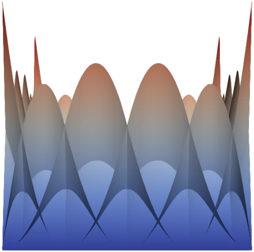
- Compact support $\Rightarrow$ Sparse system matrices
- Strictly positive $\Rightarrow$ Mass matrix positive
- Partition of unity $\Rightarrow$ Direct mass lumping

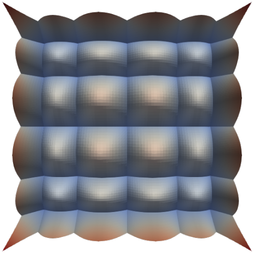


B-spline basis functions in 2D

Extension 2D

Tensor product of the 1D B-spline basis functions

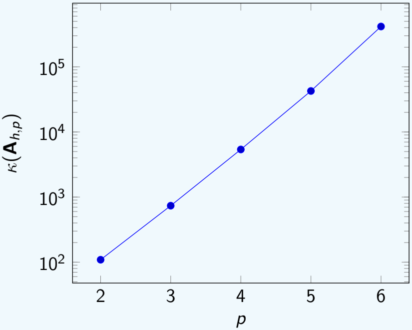




Need for efficient solvers

Condition number of stiffness/mass matrix

$\kappa(\mathbf{A}_{h,p})$ scales exponentially with approximation order p



Observation

The linear system $\mathbf{A}_{h,p}\mathbf{x}_{h,p} = \mathbf{b}_{h,p}$

- reduces to standard FEM for $p = 1$;
- becomes more difficult to solve for increasing p.

Efficient solvers for high-order B-spline-based discretizations are needed

Solution strategy

Use the error of low-order discretizations to update the solution of high-order discretizations $\Rightarrow$ **p-multigrid**

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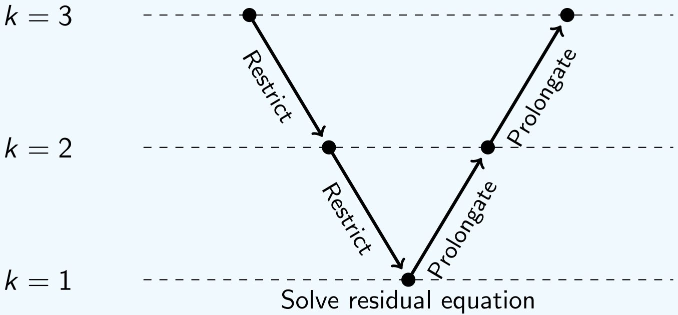
p-multigrid

- Hierarchy of discretizations with different orders k
- Low-order error is used to update high-order solution
- Smoothing steps are applied at each k-level ($\bullet$)



Prolongation/Restriction

Restrict residual $\mathbf{r}_k$ from level k to level $k-1$:

$$I_k^{k-1} := (\mathbf{M}_{k-1}^{k-1})^{-1}\mathbf{M}_k^{k-1}$$

Prolongate error $\mathbf{e}_{k-1}$ from level $k-1$ to level k:

$$I_{k-1}^{k} := (\mathbf{M}_{k}^{k})^{-1}\mathbf{M}_{k-1}^{k}$$

Where:

- $(\mathbf{M}_k^l)_{(i,j)} := \int_{\hat{\Omega}_h} \phi_i^k(\xi)\ \phi_j^l(\xi)\ c(\xi)\ \mathrm{d}\hat{\Omega}$
- $\mathbf{M}_k^k$ is in practice replaced by its lumped counterpart

V-cycle p-multigrid

Solution procedure

- Start with initial guess $\mathbf{u}_{h,p}^{(0)}$
- Obtain correction $\tilde{\mathbf{e}}_{h,p}^{(n)}$ with single V-cycle
- Solution update:

$$\mathbf{u}_{h,p}^{(n+1)} \leftarrow \mathbf{u}_{h,p}^{(n)} + \tilde{\mathbf{e}}_{h,p}^{(n)}$$

- Stopping criterion:

$$\frac{||\mathbf{r}_{h,p}^{(n)}||}{||\mathbf{r}_{h,p}^{(0)}||} < \epsilon$$

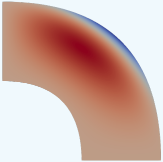
Numerical Results

Consider

$$\begin{aligned} -\Delta u &= f && \text{on } \Omega \\ u &= u_{exact} && \text{on } \partial\Omega \end{aligned}$$

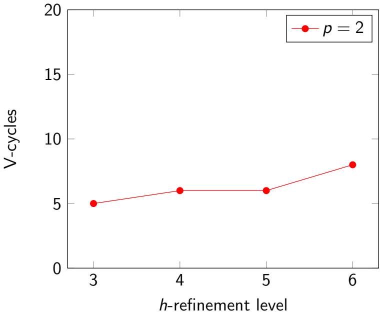
where

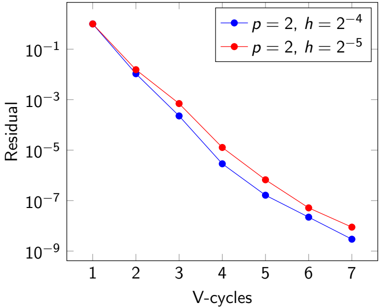
$$u_{exact}(x, y) = -(x^2 + y^2 - 1)(x^2 + y^2 - 4)xy^2$$



p-multigrid as a solver

- SOR ($\tau = \frac{4}{3}$) for pre/post-smoothing ($\nu = 4$)
- Conjugate Gradient at level $k = 1$ ($\epsilon = 10^{-4}$)





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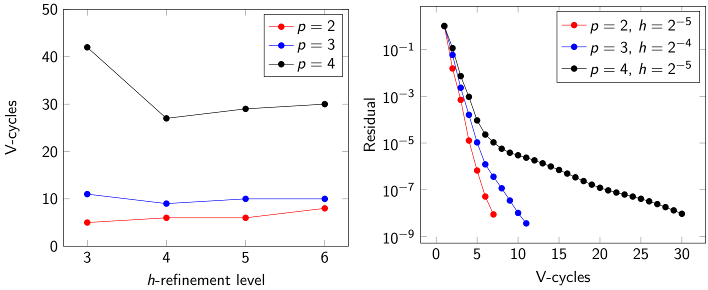

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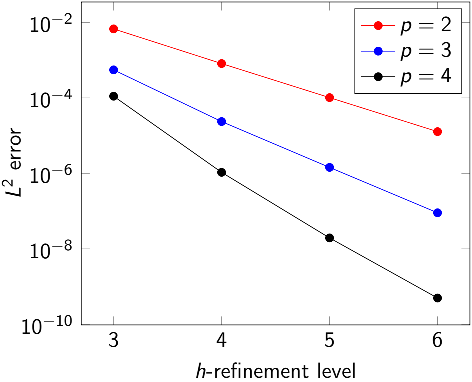
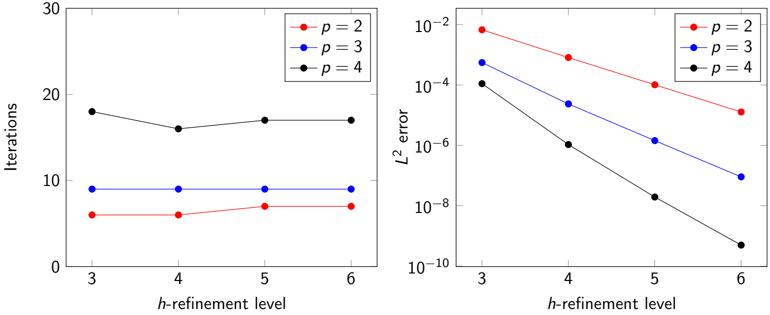


Optimal spatial convergence $\mathcal{O}(h^{p+1})$

p-multigrid as a preconditioner

- Conjugate Gradient as outer solver ($\epsilon = 10^{-8}$)
- 1 V-cycle as preconditioner in every iteration



Observations

Numerical results indicate:

- Number of V-cycles/iterations is relatively low ✓
- Number of V-cycles/iterations is h-independent ✓
- Optimal $\mathcal{O}(h^{p+1})$ spatial convergence is achieved ✓
- Number of V-cycles/iterations is p-dependent ✗

Spectral analysis

Error reduction factors:

$$r^{\mathcal{S}}(\mathbf{v}) = \frac{|\mathcal{S}(\mathbf{v})|}{|\mathbf{v}|} \qquad r^{CGC}(\mathbf{v}) = \frac{|CGC(\mathbf{v})|}{|\mathbf{v}|}$$

where $\mathcal{S}(\cdot)$ and $CGC(\cdot)$ denote a smoothing step and coarse grid correction applied on $\mathbf{v}$, respectively.

Here $(\mathbf{v_i})$ are the generalized eigenvectors which satisfy:

$$\mathbf{A}_{h,p}\mathbf{v}_i = \lambda_i {\mathbf{M}^{\mathbf{C}}}_{h,p}\mathbf{v}_i, \qquad i = 1, \ldots, N_{dof}$$

Spectral analysis

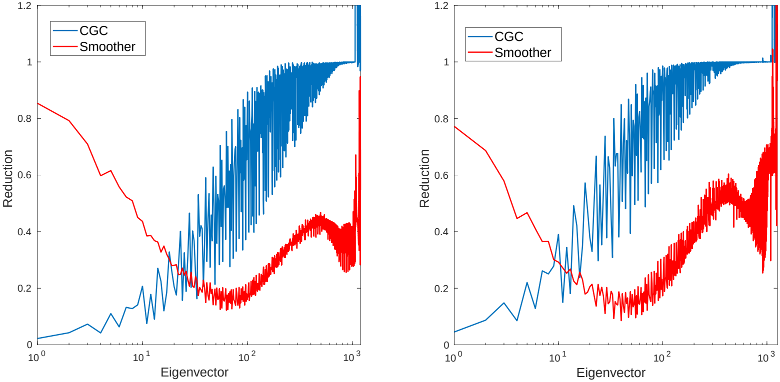


Figure: Reduction factors $(\mathbf{v}_i)$ for $p = 2$ (left) and $p = 3$ (right).

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Forthcoming Work

- Obtain p-independence by alternative smoothers $(*)$
- Explore flexibility of coarsening in both h and p
  - N_{dof} at 'coarsest' level is relatively high

$(*)$ C. Hofreither and S. Takacs. *Robust Multigrid for IgA Based on Stable Splittings of Spline Spaces* SIAM Journal on Numerical Analysis, 55(4): 2004-2024, 2017