Reduced basis method for efficient model order reduction of optimal control problems

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joint work with Giulia Fabrini[2] and Stefan Volkwein[2]

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Technische Universiteit Eindhoven University of Technology

Outline

- Introduction on the Reduced Basis method

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- Controllability of Parametrized Dynamical Systems

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- Greedy Controllability of Parametrized Dynamical Systems

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- Introduction on the Reduced Basis method
- Controllability of Parametrized Dynamical Systems
  - ⋆ Numerical results
- Greedy Controllability of Parametrized Dynamical Systems
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- Greedy Controllability of Reduced-Order Parametrized Dynamical Systems
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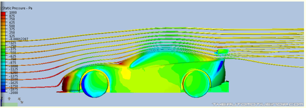
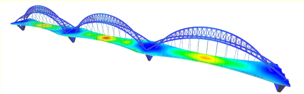
- Conclusions

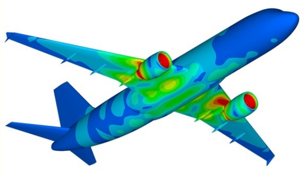
Why Model Order Reduction?

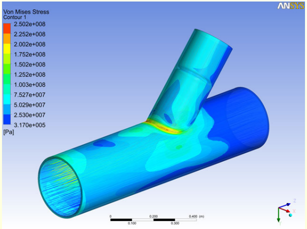
In mechanical field and in applied sciences, simulations (often based on PDEs models) are used to understand system behaviours before actually building or operating on it.

These **systems depend on parameters**: several simulations needed for each change of the parameter value.









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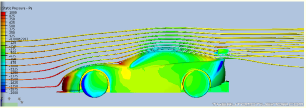

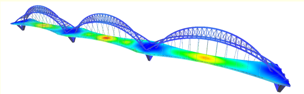

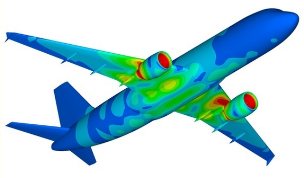

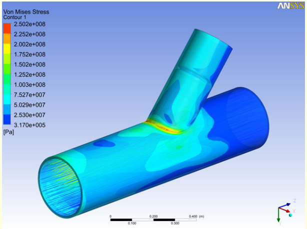

Main idea of the Reduced Basis (RB) method

Instead of restarting from scratch for every new simulation, we can evaluate the behaviour of a system exploiting the knowledge of the solution for some already computed solutions.

Parametric Partial Differential Equations

Many physical phenomena can be described by a PDE.

$$\begin{cases} -\nu \Delta \mathbf{u} + \nabla p = 0 & \text{in } \Omega, \\ \nabla \cdot \mathbf{u} = 0 & \text{in } \Omega, \\ \mathbf{u} = 0 & \text{on } \Gamma_D \\ \nu \dfrac{\partial \mathbf{u}}{\partial \mathbf{n}} - p\mathbf{n} = -\mathbf{n} & \text{on } \Gamma_{in}, \\ \nu \dfrac{\partial \mathbf{u}}{\partial \mathbf{n}} - p\mathbf{n} = 0 & \text{on } \Gamma_{out}, \end{cases}$$

Parametric Partial Differential Equations

Many physical phenomena can be described by a PDE. Some features of the model can be addressed to a parameter $\mu = (\mu_p, \mu_G)$, so that we can describe a set of μPDEs in the same problem.

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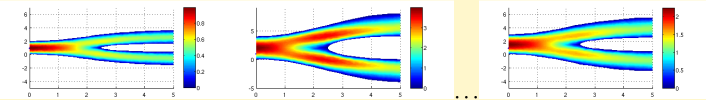
RB Offline Stage - Computationally expensive

RB Online Stage - Real-time evaluable

RB Offline Stage - Computationally expensive

FEM solutions for a representative set of parameter values $\mu_1, \ldots, \mu_N$ with $N \ll \mathcal{N}$



$u_1^{\mathcal{N}} = u^{\mathcal{N}}(\mu_1)$ $\quad$ $u_2^{\mathcal{N}} = u^{\mathcal{N}}(\mu_2)$ $\quad \ldots \quad$ $u_N^{\mathcal{N}} = u^{\mathcal{N}}(\mu_N)$

$\mu = (l, s, \delta), l =$ length, s = thickness, $\delta =$ bifurcation span

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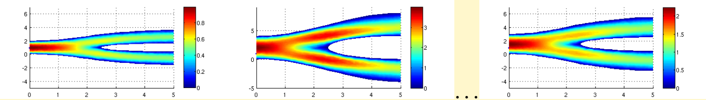

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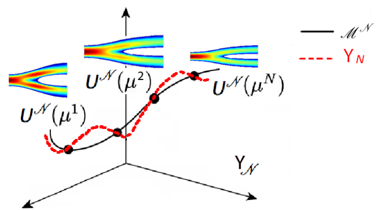
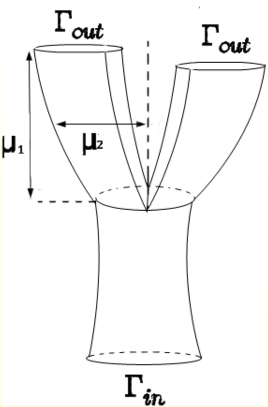
RB Online Stage - Real-time evaluable

For each new parameter vector μ the RB solution is a weighted combinations of the precomputed solutions (Galerkin projection)

$$u^N(\mu) = \sum_{i=1}^{N} \alpha_i(\mu) u_i^{\mathcal{N}}$$



RB - Stokes problem on a 3D parametrized bifurcation



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$\mu = [\mu_1, \mu_2]$ Geometrical parameters

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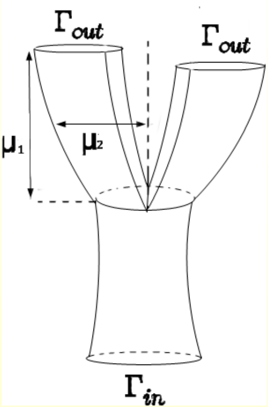

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$\mu_1 \in [6, 13]$: length of left branch

$\mu_2 \in [-0.5, 0.5]$: bending of left branch

$\mathbb{P}_2$-$\mathbb{P}_1$ Taylor-Hood elements $\mathcal{N} = 107803$

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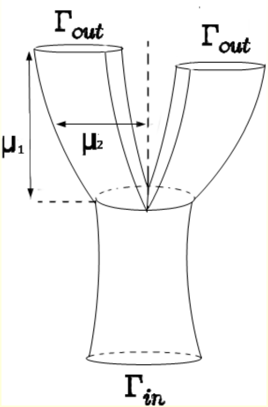

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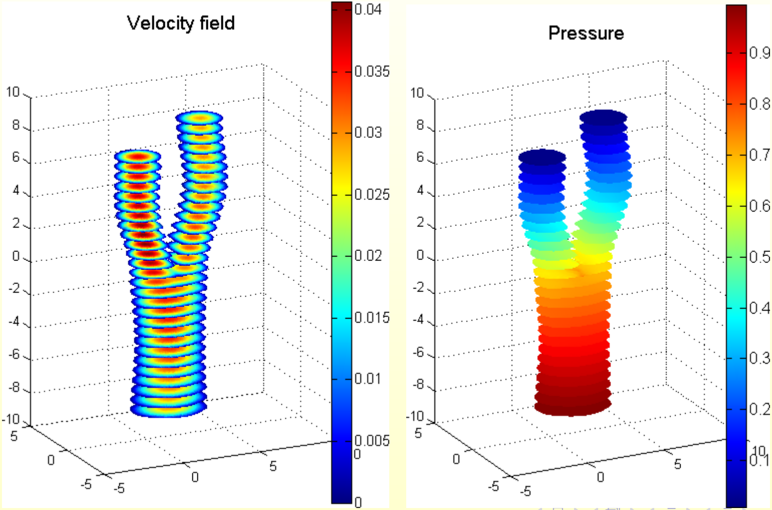
⋆ C. Jaeggli, L. Iapichino and G. Rozza.*An improvement on geometrical parametrizations by transfinite maps*, Comptes Rendus Mathematique, Volume 352, Issue 3, Pages 263- 268, 2014.

⋆ A. Quarteroni, G. Rozza *Numerical Solution of Parametrized Navier-Stokes Equations by Reduced Basis Methods.*Numerical Methods for PDEs, Vol. 23, No.4, pp. 923-948, 2007.

3D Stokes problem solved by RB

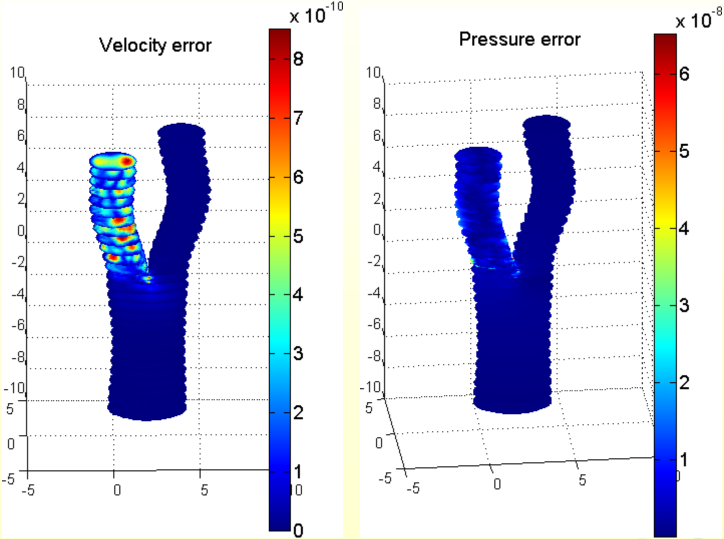
Reduced Basis solutions by using 30 basis functions

$(\mu_1, \mu_2) = (6, -0.48)$,N=30 $\mathcal{N} = 107803$



3D Stokes problem solved by RB

Errors between the RB solutions and FE solutions



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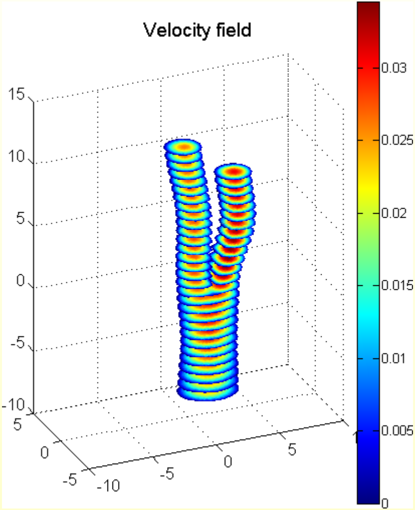
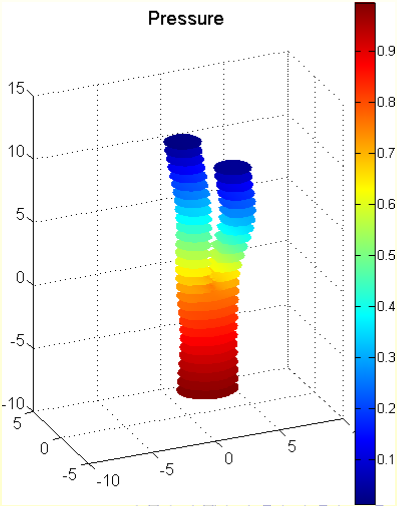
$(\mu_1, \mu_2) = (11, 0.5)$, N=30 $\mathcal{N} = 107803$





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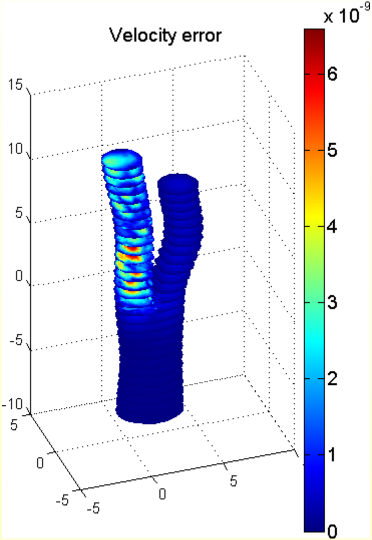

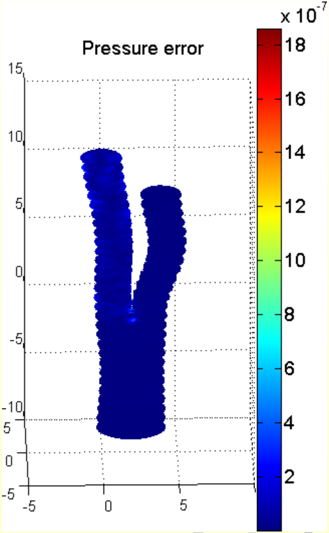

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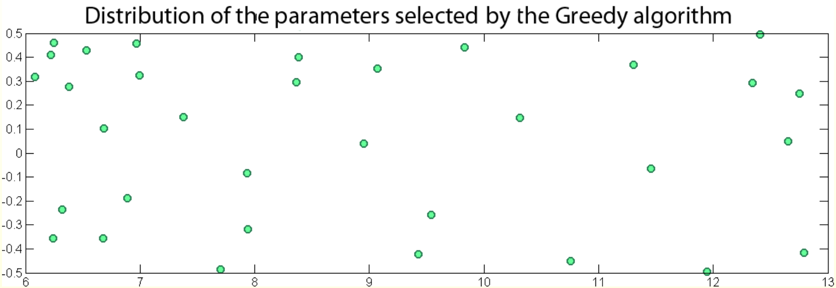

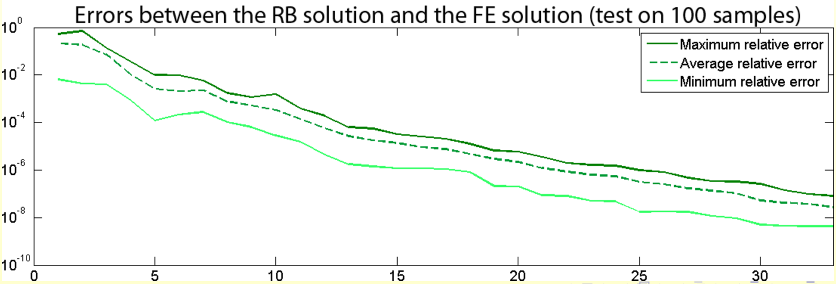

Controllability of Parametrized Dynamical Systems

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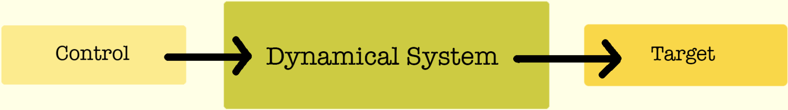

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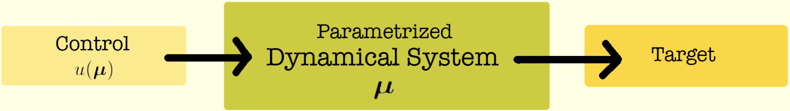

Controllability of Parametrized Dynamical Systems

Consider the finite dimensional linear control system (possibly obtained from a PDE control problem after space discretization, i.e. FE scheme $x \in V_{\mathcal{N}}$)

$$\begin{cases} x'(t) = A(\nu)x(t) + Bu(t), & t \in (0,T); \\ x(0) = x_0. \end{cases} \tag{1}$$

- The (column) vector valued function $x(t,\nu) = [x_1(t,\nu), \dots, x_{\mathcal{N}}(t,\nu)] \in \mathbb{R}^{\mathcal{N}}$ is the state of the system. $A(\nu) \in \mathbb{R}^{\mathcal{N} \times \mathcal{N}}$,$B \in \mathbb{R}^{\mathcal{N} \times M}$.
- ν is a multi-parameter living in a compact set K of $\mathbb{R}^d$.
- $u = u(t,\nu)$ is a M-component control vector in $\mathbb{R}^M, M \leq \mathcal{N}$.

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Controllability problem

Given a control time $T > 0$ and a final target $x_1 \in \mathbb{R}^{\mathcal{N}}$ we look for a control $u(t,\nu)$ such that the solution of (1) satifies the controllability condition: $x(T,\nu) = x_1$.

We assume that (1) is controllable for all values of ν (E. Zuazua, Automatica, 2014).

Solution of the controllability problem (classical approach)

If $\tilde{\varphi}^\circ$ is a minimizer of the following quadratic functional in $\mathbb{R}^{\mathscr{N}}$:

$$J_\nu(\varphi^\circ) = \frac{1}{2}\int_0^T |B^\star \varphi(t,\nu)|^2 dt + \langle x_1, \varphi^\circ\rangle + \langle x_0, \varphi(0,\nu)\rangle,$$

then $\tilde{u}(t,\nu) = B^\star \tilde{\varphi}(t,\nu)$, where $\tilde{\varphi}(t,\nu)$ is the solution of the adjoint system (5) associated to $\tilde{\varphi}^\circ$:

$$\begin{cases} \varphi'(t,\nu) = A(\nu)^\star \varphi(t,\nu); \qquad t\in(0,T); \\ \varphi'(T,\nu) = \varphi^\circ, \end{cases} \tag{2}$$

is the control that steers the solution of (1) to x_1: $\tilde{x}(T,\nu) = x_1$, where $\tilde{x}(T,\nu)$ is the solution of:

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$\star$ S. Micu, E. Zuazua, "An introduction to the controllability of linear PDE", T. Sari (Ed.), Contrôle non linéaire et applications, Collection Travaux en Cours Hermann (2005), pp. 67-150.

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Lagrangian approach

CG method

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Greedy Controllability

Greedy controllability: Main idea

Controls $u(t,\nu)$ are chosen to be of minimal norm satisfying the controllability condition: $x(T,\nu) = x_1$; and lead to a manifold of dimension d in $[L^2(0,T)]^M$:

$$\nu \in K \to u(t,\nu) \in [L^2(0,T)]^M.$$

This manifold inherits the regularity of the mapping $\nu \to A(\nu)$.

Idea of the Greedy controllability*

To diminish the computational cost we look for the very distinguished realisations of the parameter of that yield the best possible approximation of this manifold.

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Given an error ε the goal is to find $\nu_1,\ldots,\nu_{n(\varepsilon)}$ so that
for all parameter values ν the corresponding control $u(t,\nu)$ can be
approximated by a linear combination of $u(t,\nu_1),\ldots,u(t,\nu_{n(\varepsilon)})$ with an error $\leq \varepsilon$.
And of course to do it with a minimum number $n(\varepsilon)$.

Greedy controllability: The approach

Greedy controllability - Offline

Select (with a greedy approach) $v_1, \ldots, v_{n(\varepsilon)} \in K$ and compute $\varphi_1^\circ, \ldots, \varphi_{n(\varepsilon)}^\circ$

Greedy controllability - Online

$\forall v \in K$, $\bar{\varphi}_v^\circ \in \text{span}\{\varphi_1^\circ, \ldots, \varphi_{n(\varepsilon)}^\circ\}$ and $\bar{u}_v(t,v) = B^\star \bar{\varphi}_v(t,v)$ is such that $|\bar{x}(T,v) - x_1| \leq \varepsilon$.

$\star$M. Lazar E. Zuazua, Greedy controllability of finite dimensional linear systems, Automatica, Dic 2016

Selection of the parameter values

Greedy algorithm (offline)

Select $v_1 \in K$, compute φ_1°, $\varphi_1(t,v)$, $u_1(t,v)$, $\Phi_1^\circ = \{\varphi_1^\circ\}$.

Find $v_2 = \arg\max_{v\in K} dist(\varphi_v^\circ, \Phi_1^\circ)$.

Compute φ_2°, $\varphi_2(t,v)$, $u_2(t,v)$, $\Phi_2^\circ = \mathrm{span}\{\varphi_1^\circ, \varphi_2^\circ\}$.

Find $v_3 = \arg\max_{v\in K} dist(\varphi_v^\circ, \Phi_2^\circ)$.

...

until $\forall v \in K, dist(\varphi_v^\circ, \Phi_n^\circ) \leq toll$

$\star$M. Lazar E. Zuazua, Greedy controllability of finite dimensional linear systems, Automatica, Dic 2016

Selection of the parameter values

Greedy algorithm (offline)

Select $\nu_1 \in K$, compute φ_1°, $\varphi_1(t,\nu)$, $u_1(t,\nu)$. $\Phi_1^\circ = \{\varphi_1^\circ\}$.

Find $\nu_2 = \arg\max_{\nu \in K} \mathit{dist}(\varphi_\nu^\circ, \Phi_1^\circ)$ (without computing φ_ν°!!).

Compute φ_2°, $\varphi_2(t,\nu)$, $u_2(t,\nu)$, $\Phi_2^\circ = \mathrm{span}\{\varphi_1^\circ, \varphi_2^\circ\}$.

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$\star$M. Lazar E. Zuazua, Greedy controllability of finite dimensional linear systems, Automatica, Dic 2016

Controllability distance $dist(\varphi_2^\circ, \varphi_1^\circ)$

φ_1° as been computed in step 1, φ_2° is unknown, a surrogate of $dist(\varphi_2^\circ, \varphi_1^\circ)$ can be computed.

Adjoint solution

$$\begin{cases} \varphi'(t, \nu_2) = A^\star(\nu_2)\varphi(t, \nu); t \in (0, T); \\ \varphi'(T, \nu_2) = \varphi_1^\circ, \end{cases}$$

$$\tilde{u}(t, \nu_2) = B^\star \varphi(t, \nu_2)$$

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State solution

$$\begin{cases} \bar{x}'(t,\nu_2) = A(\nu_2)\bar{x}(t,\nu) + B\tilde{u}(t,\nu_2), t \in (0,T); \\ \bar{x}(0,\nu_2) = x_0. \end{cases}$$

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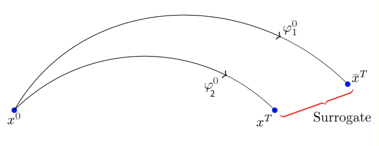

Surrogate:
$dist(\bar{x}(T, v_2) - x(T, v_2)) = dist(\bar{x}(T, v_2) - x_1)$

⋆M. Lazar E. Zuazua, Greedy controllability of finite dimensional linear systems, Automatica, Dic 2016

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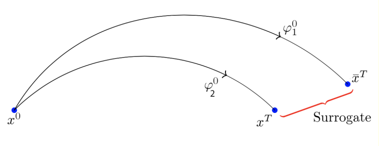

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The exploration of the parameter domain requires **repetitive evaluations** of the Adjoint and State systems.

$\star$M. Lazar E. Zuazua, Greedy controllability of finite dimensional linear systems, Automatica, Dic 2016

Reduced Basis Method in the Greedy Controllability approach

Reduced order Controllability distance $dist(\varphi_2^\circ, \varphi_1^\circ)$

φ_1° as been computed in step 1, φ_2° is unknown, we propose a **reduced order surrogate** of $dist(\varphi_2^\circ, \varphi_1^\circ)$.

RB Adjoint solution

$$\begin{cases} \varphi'(t, \nu_2) = A^\star(\nu_2)\varphi(t,\nu); t \in (0,T); \\ \varphi'(T,\nu_2) = \varphi_1^\circ, \end{cases}$$

RB State solution

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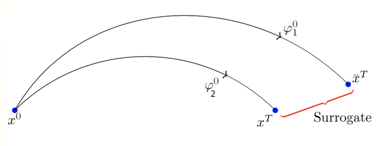

RB Surrogate:

$$dist(\bar{x}(T,\nu_2) - x(T,\nu_2)) = dist(\bar{x}(T,\nu_2) - x_1)$$

$\star$ L. Iapichino, S. Volkwein, G. Fabrini. Reduced-Order Greedy Controllability of Finite Dimensional Linear Systems, Proceeding of 9th Vienna International Conference on Mathematical Modelling, 2018.

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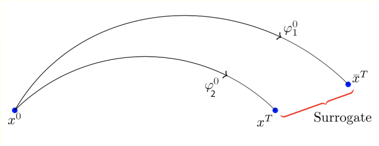

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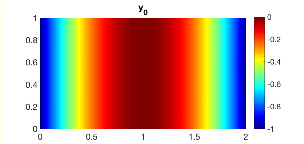
Reduced Basis computation and parameter selection

- **STEP 1**: The initial basis is composed by the target function x_1.
- **STEP 2**: Run the Greedy algorithm (with repetitive evaluations of the REDUCED systems) and select ν_{next}.
- **STEP 3**: Find the optimal control $u(t, \nu_{next})$ and the state solution $x(t, \nu_{next})$.
- **STEP 4**: Enrich the existing basis with the POD of $x(t, \nu_{next})$.
- **STEP 5**: Repeat STEP 2 for the selection of the remaining parameter values until the surrogate distance is smaller than the desired tolerance.

Numerical Results

Numerical results

$$\begin{cases} y_t(t,\boldsymbol{x},\nu) - \nu\Delta y(t,\boldsymbol{x},\nu) + \boldsymbol{v}\cdot\nabla y(t,\boldsymbol{x},\nu) = 0, & \text{a.e. in } Q, \\ \nu\,\frac{\partial y}{\partial \boldsymbol{n}}(t,\boldsymbol{s},\nu) = \sum_{i=1}^{M} u_i(t) b_i(\boldsymbol{s}), & \text{a.e. on } \Sigma, \\ y(0,\boldsymbol{x},\nu) = y_\circ(\boldsymbol{x}), & \text{a.e. in } \Omega \end{cases}$$



- $\mathscr{D} = \{\nu \in \mathbb{R} \,|\, 0.5 \le \nu \le 4\} \subset \mathbb{R}$, $\boldsymbol{v} = 0.1$;
- $\Omega \subset \mathbb{R}^2$, bounded domain with Lipschitz-continuous boundary $\Gamma = \partial\Omega$, $\Sigma = (0,1)\times\Gamma$, $Q = (0,1)\times\Omega$;
- $b_i : \Gamma \to \mathbb{R}$, $1 \le i \le M$, denote given control shape functions:

$$\Gamma = \bigcup_{i=1}^{M} \Gamma^i, \quad b_i(\boldsymbol{s}) = \chi_{\Gamma^i}(\boldsymbol{s}),\ 1 \le i \le M, \quad \|b_i\|^2_{L^2(\Gamma)} = \int_{\Gamma^i} 1^2\,\mathrm{d}\boldsymbol{s} = |\Gamma^i|, \|b_i\|_{L^2(\Gamma)} = |\Gamma^i|^{1/2}$$

- $y_1(\boldsymbol{x}) = 10$.

Control problem

For $\nu \in \mathscr{D}$, find $u(t,\nu)$ such that $y_t(T,\boldsymbol{x},\nu) = y_1(\boldsymbol{x})$

Finite element model

$$\tilde{\mathrm{A}}(\nu) = \big(\big(a(\varphi_j,\varphi_i;\nu)\big)\big)_{1\le i,j\le\mathcal{N}} \qquad \tilde{\mathrm{B}} = \big(\big(\langle b_j,\varphi_i\rangle_{L^2(\Gamma)}\big)\big)_{1\le i\le\mathcal{N},1\le j\le M}$$

$$\mathrm{x}(t,\nu) = \big(\mathrm{y}_i^{\mathcal{N}}(t)\big)_{1\le i\le\mathcal{N}} \qquad \mathrm{x}_\circ = \big(\mathrm{y}_{\circ i}^{\mathcal{N}}\big)_{1\le i\le\mathcal{N}} \quad x_1 = \big(\mathrm{y}_{1i}^{\mathcal{N}}\big)_{1\le i\le\mathcal{N}}$$

Then, (21) leads to the $\mathcal{N}$-dimensional dynamical system

$$\mathrm{M}x'(t,\nu) = \tilde{\mathrm{A}}(\nu)x(t,\nu) + \tilde{\mathrm{B}}u(t,\nu) \text{ for } t\in(0,T], \quad x(0) = \mathrm{x}_\circ. \tag{6}$$

Setting $\mathrm{A}(\nu) = \mathrm{M}^{-1}\tilde{\mathrm{A}}(\nu)$ and $\mathrm{B} = \mathrm{M}^{-1}\tilde{\mathrm{B}}$ problem (6) can be expressed as

$$x'(t,\nu) = \mathrm{A}(\nu)x(t,\nu) + \mathrm{B}u(t,\nu) \text{ for } t\in(0,T], \quad x(0,\nu) = \mathrm{x}_\circ.$$

Linear quadratic optimization problem

$$\min J(\varphi_\circ) = \frac{1}{2}\int_0^T \|\mathrm{B}^\top\varphi(t,\nu)\|_{\mathbb{R}^M}^2\,\mathrm{d}t - \langle \mathrm{x}_1,\varphi_\circ\rangle_{\mathbb{R}^{\mathcal{N}}} + \langle \mathrm{x}_\circ,\varphi(0,\nu)\rangle_{\mathbb{R}^{\mathcal{N}}} \tag{7a}$$

subject to the differential equation

$$-\varphi'(t,\nu) = \mathrm{A}(\nu)^\top\varphi(t,\nu) \text{ for } t\in[0,T), \quad \varphi(T,\nu) = \varphi_\circ. \tag{7b}$$

Numerical results - greedy controllability

Results with the classical optimization approach ($\mathcal{N} = 881$)

ν	0.5	1	1.5	2	2.5	3	3.5	4
iterations	59	34	28	29	28	24	20	20
cpu time	62	38.23	32	32.9	31.88	25.71	21.68	21.57
$\parallel x(T,\nu) - x_1 \parallel$	8.6e-3	2.8e-3	9.6e-3	7.4e-3	8.1e-3	6.9e-3	7.7e-3	6.5e-3

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Greedy controllability - offline

N=9, $\Xi_{train} \subset K, dim(\Xi_{train}) = 1000$, cpu time =1h10m.
Each step (parameter exploration) requires 7.5 minutes.

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Greedy controllability - offline

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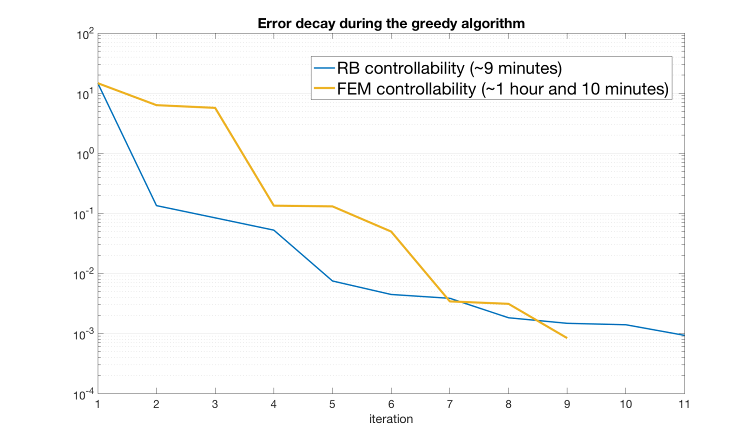
Each step (parameter exploration) requires 7.5 minutes.

RB Greedy controllability - offline

N=11, $\Xi_{train} \subset K, dim(\Xi_{train}) = 1000$, cpu time = 9 minutes.
Each step (parameter exploration) requires from 0.5 seconds to 0.98 seconds.
15 RB functions selected.

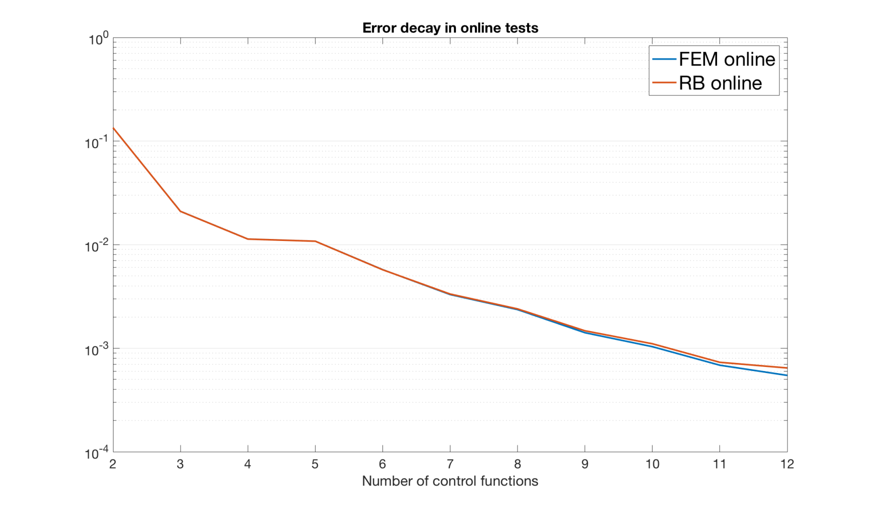
Numerical results - RB Greedy controllability

RB and FEM Greedy controllability - offline



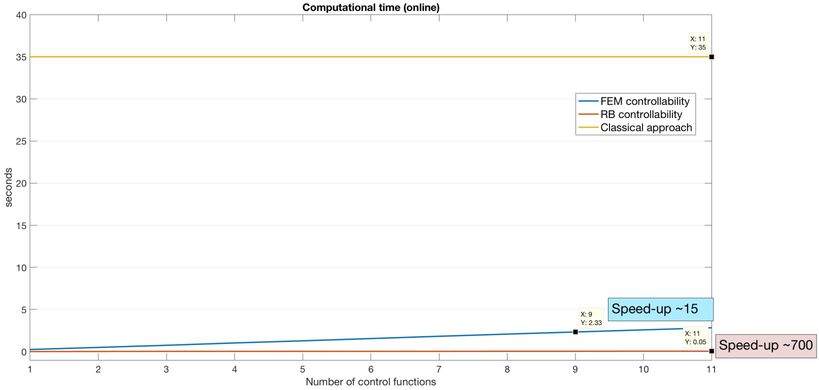
Numerical results - RB Greedy controllability

RB and FEM Greedy controllability - online



Numerical results - RB Greedy controllability

RB and FEM Greedy controllability - online



Conclusions

- The proposed Reduced Greedy controllability approach has been used to solve **controllability problems for parameter dependent dynamical systems**, the problem is not solved entirely by the RB method, but the latter is used only locally into a greedy controllability technique.
- The use of RB method in the greedy controllability approach allows to further **speedup the computational times** required for the solution retaining the same level of accuracy.

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4TU-AMI Big Data symposium

The goal of the event is to build bridges between Model Order Reduction and Big Data and discuss its relevance to industrial applications. The symposium will feature talks by speakers from academia and industry, as well as a panel discussion.

Title: Reducing dimensions in Big Data: Model Order Reduction in action.
Date: June 8, 2018
Location: Utrecht BCN (Central Station, Catherijnesingel 48).

Please register here before June 4, 2018.

Programme:
13:30 hrs.: Wil Schilders
"Combining traditional modelling, model order reduction and big data"
14:00 hrs.: Zoi Tokoutsi
"Real-time Optimization of Thermal Ablation Cancer Treatments"
14:30 hrs.: Break
14:45 hrs.: Kathrin Smetana
"Randomized model order reduction"
15:15 hrs.: Jacquelien Scherpen
"Structure preserving order reduction of networked linear systems"
15:45 hrs.: Break
16:00 hrs.: Panel discussion
16:45 hrs.: Closing and drinks