Splitting methods for multi-dimensional advection-diffusion equations arising in financial mathematics

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Contents

Bates PIDE and semidiscretization

European put option gives holder the right to sell a given asset at a prescribed maturity date T for a prescribed strike price K.

S_τ : asset price at time $\tau \geq 0$.

Payoff is $\phi(S_T) = \max(K - S_T\,, 0)$.

For the evolution of S_τ consider the Bates model (1996):

$$\begin{cases} dS_\tau = (r - \lambda\zeta)\, S_\tau\, d\tau + \sqrt{V_\tau}\, S_\tau\, dW_\tau^1 + (Y-1)\, S_\tau\, dN_\tau\,, \\ dV_\tau = \kappa(\eta - V_\tau)\, d\tau + \sigma\sqrt{V_\tau}\, dW_\tau^2 \end{cases}$$

with real parameters κ, η, σ, r, λ, ζ.

W_τ^1, W_τ^2 : Brownian motions with correlation factor $\rho \in [-1, 1]$.

N_τ : independent Poisson process with intensity $\lambda > 0$.

Y : lognormal distribution with parameters γ and $\delta > 0$,

probability density function

$$f(y) = \frac{1}{y\delta\sqrt{2\pi}} \exp\left(-\frac{(\log y - \gamma)^2}{2\delta^2}\right) \quad (y > 0),$$

mean relative jump-size $\zeta = \exp(\gamma + \delta^2/2) - 1$.

Option value $u(s, v, t)$ satisfies partial integro-differential equation (PIDE)

$$\begin{aligned} \frac{\partial u}{\partial t} = \; & \tfrac{1}{2}s^2 v \frac{\partial^2 u}{\partial s^2} + \rho\sigma s v \frac{\partial^2 u}{\partial s \partial v} + \tfrac{1}{2}\sigma^2 v \frac{\partial^2 u}{\partial v^2} + (r - \lambda\zeta)s\frac{\partial u}{\partial s} \\ & + \kappa(\eta - v)\frac{\partial u}{\partial v} - (r + \lambda)u + \lambda \int_0^\infty u(sy, v, t) f(y) dy \end{aligned}$$

for $s > 0$, $v > 0$, $0 < t \leq T$.

Truncated spatial domain $[0, S_{\max}] \times [0, V_{\max}]$.

Initial and boundary conditions:

$$\begin{aligned} u(s,v,t) &= \phi(s) && \text{whenever} \quad t = 0, \\ u(s,v,t) &= e^{-rt}K && \text{whenever} \quad s = 0, \\ u(s,v,t) &= 0 && \text{whenever} \quad s = S_{\max}, \\ u_v(s,v,t) &= 0 && \text{whenever} \quad v = V_{\max} \end{aligned}$$

with $\phi(s) = \max(K - s\,,\, 0)$.

Bates PIDE is assumed to hold for $v = 0$.

Semidiscretization on smooth, nonuniform Cartesian grid with large fraction of grid points (s_i, v_j) near $(s, v) = (K, 0)$.

Second-order central finite differences for spatial derivatives.

At $v = 0$, second-order forward finite difference used for u_v.

Discretization of integral term $J(s, v, t) = \int_0^\infty u(sy, v, t) f(y) dy$:

Let $(s, v) = (s_i, v_j)$ and $f_i(x) = f(x/s)/s$ for $x > 0$. Then

$$J(s, v, t) = \int_0^\infty u(x, v, t) f_i(x) dx \approx \int_0^{S_{\max}} u(x, v, t) f_i(x) dx.$$

and using piecewise linear interpolation

$$\begin{aligned} J(s, v, t) &\approx \sum_{k=1}^{m_1} \int_{s_{k-1}}^{s_k} \left[\frac{s_k - x}{h_k} u_{k-1,j}(t) + \frac{x - s_{k-1}}{h_k} u_{k,j}(t) \right] f_i(x) dx \\ &= \sum_{k=1}^{m_1} \left[\alpha_{i,k} u_{k-1,j}(t) + \beta_{i,k} u_{k,j}(t) \right] \end{aligned}$$

with certain known coefficients $\alpha_{i,k}$ and $\beta_{i,k}$.

FD discretization of Bates problem yields initial value problem for large system of stiff ordinary differential equations (ODEs)

$$U'(t) = \boldsymbol{A}U(t) + G(t) \quad (0 < t \leq T), \quad U(0) = U_0$$

with given matrix $\boldsymbol{A}$ and vectors $G(t)$, U_0.

Cell averaging applied to smooth initial data at $s = K$.

Matrix $\boldsymbol{A}$ can be written as

$$\boldsymbol{A} = \boldsymbol{D} - (r + \lambda)\boldsymbol{I} + \lambda \boldsymbol{J}$$

where $\boldsymbol{D}$ represents the convection-diffusion part, $\boldsymbol{J}$ represents the integral and $\boldsymbol{I}$ is the identity matrix.

$\boldsymbol{D}$ is sparse, but $\boldsymbol{J}$ has large dense diagonal blocks.

Spectrum of $\boldsymbol{J}$ lies in complex unit disk.

Adaptation of ADI schemes

Operator splitting methods for time discretization of ODE systems stemming from jump-diffusion models considered by for example

- Andersen & Andreasen ('00)
- Tavella & Randall ('00)
- Almendral & Oosterlee ('05)
- Cont & Voltchkova ('05)
- d'Halluin, Forsyth & Vetzal ('05)
- Briani, Natalini & Russo ('07)
- Itkin & Carr ('11)
- Kwon & Lee ('11)
- Salmi & Toivanen ('14)
- Salmi, Toivanen & Von Sydow ('14)
- Von Sydow, Toivanen & Zhang ('15)
- Kaushansky, Lipton & Reisinger ('17)
- In 't H. & Toivanen ('16, '18)

Splitting

$$\boldsymbol{A} = \boldsymbol{A}_0^{(\mathrm{J})} + \boldsymbol{A}_0^{(\mathrm{D})} + \boldsymbol{A}_1 + \boldsymbol{A}_2$$

where

$\lambda \boldsymbol{J} = \boldsymbol{A}_0^{(\mathrm{J})}$	represents integral term,
$\boldsymbol{A}_0^{(\mathrm{D})}$	represents $\partial^2 u/\partial s \partial v$ term,
$\boldsymbol{A}_1$	represents $\partial u/\partial s$, $\partial^2 u/\partial s^2$ terms,
$\boldsymbol{A}_2$	represents $\partial u/\partial v$, $\partial^2 u/\partial v^2$ terms

and $(r+\lambda)\boldsymbol{I}$ is distributed evenly over $\boldsymbol{A}_1$ and $\boldsymbol{A}_2$. Write

$$\mathbf{F}_0(t, V) = \left[\boldsymbol{A}_0^{(\mathrm{J})} + \boldsymbol{A}_0^{(\mathrm{D})}\right] V + G_0^{(\mathrm{J})}(t) + G_0^{(\mathrm{D})}(t),$$

$$\mathbf{F}_1(t, V) = \boldsymbol{A}_1 V + G_1(t),$$

$$\mathbf{F}_2(t, V) = \boldsymbol{A}_2 V + G_2(t),$$

$$\mathbf{F} = \mathbf{F}_0 + \mathbf{F}_1 + \mathbf{F}_2.$$

Parameter $\theta > 0$.

Step-size $\Delta t > 0$ and temporal grid points $t_n = n \cdot \Delta t$.

Adaptation of three alternating direction implicit (ADI) schemes defining $U_n \approx U(t_n)$ for $n = 1, 2, 3, \ldots$

Adaptation of Douglas (Do) scheme:

$$\begin{cases} Y_0 = U_{n-1} + \Delta t\, \mathbf{F}(t_{n-1}, U_{n-1}), \\ Y_j = Y_{j-1} + \theta \Delta t\, (\mathbf{F}_j(t_n, Y_j) - \mathbf{F}_j(t_{n-1}, U_{n-1})) \quad (j = 1, 2), \\ U_n = Y_2\,. \end{cases} \tag{1}$$

First-order consistent in ODE sense for any given θ.

Adaptation of modified Craig–Sneyd (MCS) scheme:

$$
\begin{cases}
Y_0 = U_{n-1} + \Delta t\, \mathbf{F}(t_{n-1}, U_{n-1}), \\
Y_j = Y_{j-1} + \theta \Delta t\, (\mathbf{F}_j(t_n, Y_j) - \mathbf{F}_j(t_{n-1}, U_{n-1})) \quad (j = 1, 2), \\
\widehat{Y}_0 = Y_0 + \theta \Delta t\, (\mathbf{F}_0(t_n, Y_2) - \mathbf{F}_0(t_{n-1}, U_{n-1})), \\
\widetilde{Y}_0 = \widehat{Y}_0 + (\tfrac{1}{2} - \theta) \Delta t\, (\mathbf{F}(t_n, Y_2) - \mathbf{F}(t_{n-1}, U_{n-1})), \\
\widetilde{Y}_j = \widetilde{Y}_{j-1} + \theta \Delta t\, (\mathbf{F}_j(t_n, \widetilde{Y}_j) - \mathbf{F}_j(t_{n-1}, U_{n-1})) \quad (j = 1, 2), \\
U_n = \widetilde{Y}_2\,.
\end{cases}
\tag{2}
$$

Second-order consistent in ODE sense for any given θ.

MCS scheme for PDEs by In 't H. & Welfert ('09).

Adaptation of stabilizing correction Adams2-type (SC2A) scheme:

$$\begin{cases} Y_0 = U_{n-1} + \Delta t \sum_{i=1}^{2} \big(\hat{b}_i \, \mathbf{F}_0(t_{n-i}, U_{n-i}) + \check{b}_i \sum_{j=1}^{2} \mathbf{F}_j(t_{n-i}, U_{n-i})\big), \\ Y_j = Y_{j-1} + \theta \Delta t \, (\mathbf{F}_j(t_n, Y_j) - \mathbf{F}_j(t_{n-1}, U_{n-1})) \quad (j = 1, 2), \\ U_n = Y_2 \end{cases} \tag{3}$$

with $(\hat{b}_1, \hat{b}_2) = \big(\frac{3}{2}, -\frac{1}{2}\big)$ and $(\check{b}_1, \check{b}_2) = \big(\frac{3}{2} - \theta, \, \theta - \frac{1}{2}\big)$.

Second-order consistent in ODE sense for any given θ.

Stabilizing correction linear multistep (SCLM) schemes for PDEs studied by Bruno & Cubillos ('16) and Hundsdorfer & In 't H. ('18).

Generalization of implicit-explicit (IMEX) linear multistep methods by adding dimension splitting.

Summary main characteristics of three ADI-type schemes:

All treat the integral and mixed derivative terms jointly and explicitly.

(1): one-step method, order 1, per time step two tridiagonal solves and one multiplication by $\boldsymbol{J}$

(2): one-step method, order 2, per time step four tridiagonal solves and two multiplications by $\boldsymbol{J}$

(3): two-step method, order 2, per time step two tridiagonal solves and one multiplication by $\boldsymbol{J}$

Stability analysis

Linear scalar test equation

$$U'(t) = (\lambda_0 + \mu_0 + \mu_1 + \mu_2)\, U(t). \tag{4}$$

Let $w_0 = \lambda_0\,\Delta t$ and $z_j = \mu_j\,\Delta t$ for $j = 0, 1, 2$.

Application of (1), (2), (3) to (4) yields

$$U_n = R(w_0, z_0, z_1, z_2)\, U_{n-1}\,,$$

$$U_n = S(w_0, z_0, z_1, z_2)\, U_{n-1}\,,$$

$$U_n = T_1(w_0, z_0, z_1, z_2)\, U_{n-1} + T_0(w_0, z_0, z_1, z_2)\, U_{n-2}$$

with rational functions R, S, T_1, T_0.

Expressions are

$$R(w_0, z_0, z_1, z_2) \; = 1 + \tfrac{z}{p} \, ,$$

$$S(w_0, z_0, z_1, z_2) \; = 1 + \tfrac{z}{p} + \theta \tfrac{(w_0+z_0)z}{p^2} + (\tfrac{1}{2} - \theta) \tfrac{z^2}{p^2} \, ,$$

$$T_1(w_0, z_0, z_1, z_2) = 1 + \tfrac{3}{2} \tfrac{w_0+z_0}{p} + (\tfrac{3}{2} - \theta) \tfrac{z_1+z_2}{p},$$

$$T_0(w_0, z_0, z_1, z_2) = -\tfrac{1}{2} \tfrac{w_0+z_0}{p} + (\theta - \tfrac{1}{2}) \tfrac{z_1+z_2}{p}$$

with $z = w_0 + z_0 + z_1 + z_2$ and $p = (1 - \theta z_1)(1 - \theta z_2)$.

Natural requirement:

$$|w_0| + |z_0| \leq 2\sqrt{\mathrm{Re}(z_1)\mathrm{Re}(z_2)} \; , \; \mathrm{Re}(z_j) \leq -\tfrac{1}{2}|w_0| \; (j = 1, 2) \, . \qquad (5)$$

Unconditional stability results following In 't H. & Toivanen ('18).

Theorem 1

For the adaptation (1) *there holds:*

If $\theta \geq \frac{1}{2}$*, then* $|R| \leq 1$ *whenever* $w_0, z_0, z_1, z_2 \in \mathbb{C}$ *satisfy* (5)*.*

Theorem 2

For the adaptation (2) *there holds:*

(a) If $\theta \geq \frac{1}{3}$*, then* $|S| \leq 1$ *whenever* $w_0, z_0, z_1, z_2 \in \mathbb{R}$ *satisfy* (5)*.*

(b) If $\frac{1}{2} \leq \theta \leq 1$*, then* $|S| \leq 1$ *whenever* $w_0, z_0, z_1, z_2 \in \mathbb{C}$ *satisfy* (5)*.*

Proofs employ In 't H. & Welfert ('07, '09), In 't H. & Mishra ('11).

Consider the characteristic polynomial

$$P(\zeta; w_0, z_0, z_1, z_2) = \zeta^2 - T_1(w_0, z_0, z_1, z_2)\zeta - T_0(w_0, z_0, z_1, z_2).$$

For any given point (w_0, z_0, z_1, z_2) the adaptation (3) is *stable* iff the root condition holds: both roots ζ of P have a modulus at most one, and those with modulus equal to one are simple.

Theorem 3

If $\theta \geq \frac{2}{3}$, then (3) *is stable whenever $w_0, z_0, z_1, z_2 \in \mathbb{R}$ satisfy* (5).

Proof employs Hundsdorfer & In 't H. ('18).

Numerical experiments

Consider following adaptations (1), (2), (3):

- Do scheme with $\theta = \frac{1}{2}$
- MCS scheme with $\theta = \frac{1}{2}$
- MCS scheme with $\theta = \frac{1}{3}$
- SC2A scheme with $\theta = \frac{3}{4}$

CS and MCS with step-size $\Delta t = T/N$

Do and SC2A with step-size $\Delta t = T/(2N)$

Global *temporal* errors are computed in maximum norm at $t = T$ on region of interest $\frac{1}{2}K < s < \frac{3}{2}K$ and $0 < v < 1$.

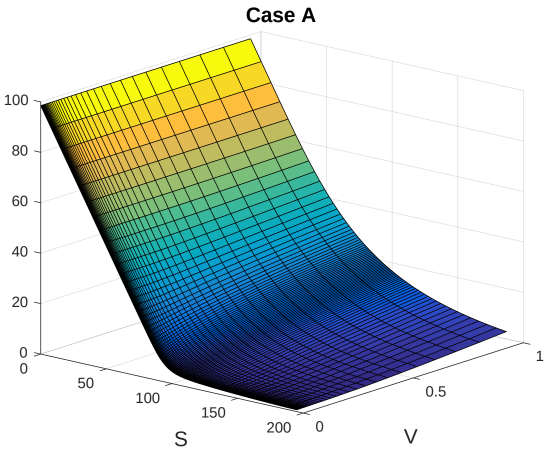
Spatial discretization on 200×100 nonuniform grid.

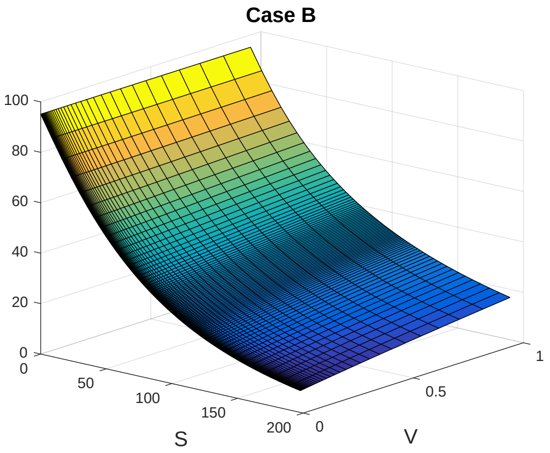
Three cases of Bates parameter sets:

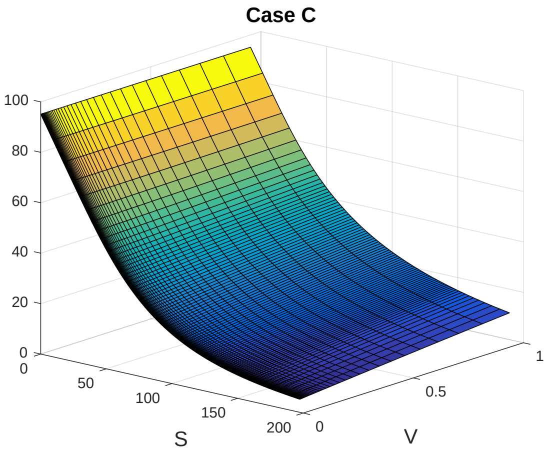
	Case A	Case B	Case C
κ	2	1.5	2.5
η	0.04	0.1	0.05
σ	0.25	0.3	0.6
ρ	-0.5	-0.5	-0.8
r	0.03	0.05	0.01
λ	0.2	5	10
γ	-0.5	0.3	-0.05
δ	0.4	0.1	0.01
T	0.5	1	5
K	100	100	100

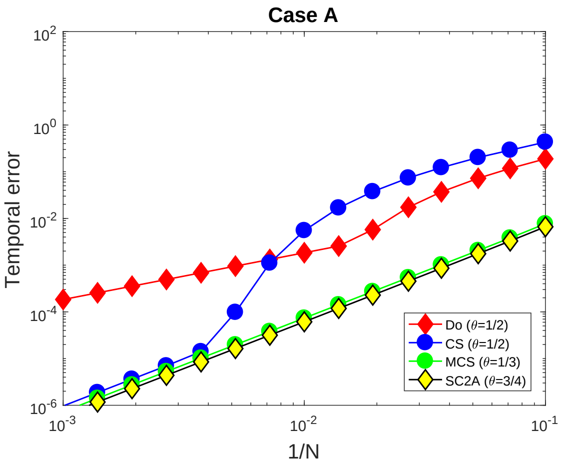
Cases A and B from literature.

Case C new (constructed).

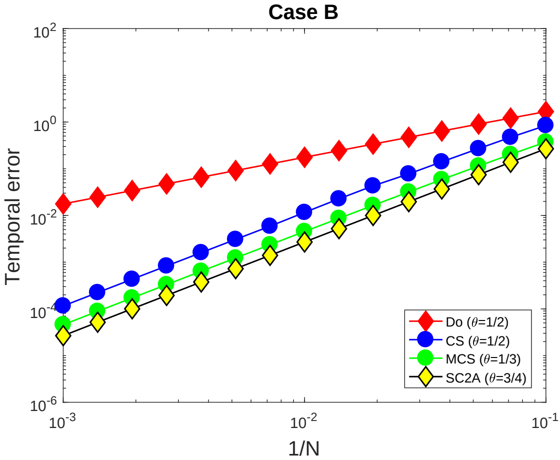

Case A
100
80
60
40
20
0
0
50
100
150
200
S
0
0.5
1
V

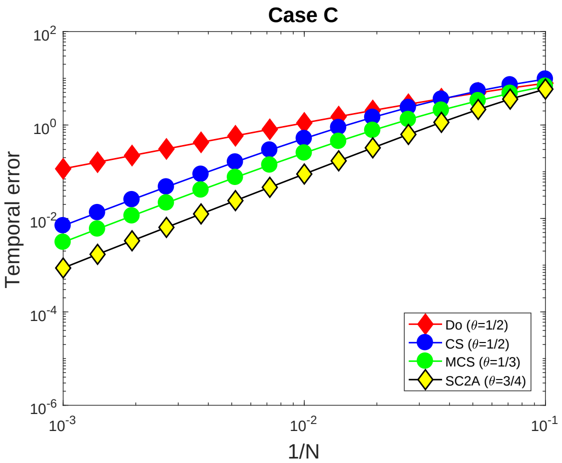

Case B
100
80
60
40
20
0
0
50
100
150
200
S
0
0.5
1
V


Case C
100
80
60
40
20
0
0
50
100
150
200
S
0
0.5
1
V



Case A
Temporal error
1/N
10^{2}
10^{0}
10^{-2}
10^{-4}
10^{-6}
10^{-3}
10^{-2}
10^{-1}
Do (θ=1/2)
CS (θ=1/2)
MCS (θ=1/3)
SC2A (θ=3/4)


Case B
Temporal error
1/N
10^{2}
10^{0}
10^{-2}
10^{-4}
10^{-6}
10^{-3}
10^{-2}
10^{-1}
Do (θ=1/2)
CS (θ=1/2)
MCS (θ=1/3)
SC2A (θ=3/4)


Case C
Temporal error
1/N
10^{2}
10^{0}
10^{-2}
10^{-4}
10^{-6}
10^{-3}
10^{-2}
10^{-1}
Do (θ=1/2)
CS (θ=1/2)
MCS (θ=1/3)
SC2A (θ=3/4)

Conclusions and future research

Conclusions

- SC2A ($\theta = \frac{3}{4}$) and MCS ($\theta = \frac{1}{3}$) preferable
- highly efficient
- unconditionally stable
- smooth, second-order convergence
- uniform in number of spatial grid points

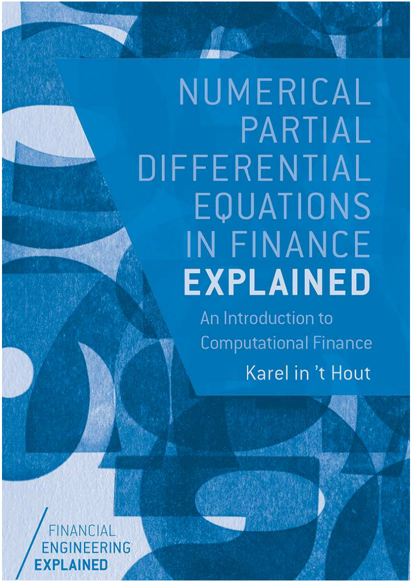
Future research

- further analysis SCLM schemes
- application to e.g. American options

References

Main references

- W. Hundsdorfer and K.J. in ’t Hout: On multistep stabilizing correction splitting methods with applications to the Heston model. SIAM J. Sci. Comput. 40, A1408–A1429 (2018).
- K.J. in ’t Hout and J. Toivanen: ADI schemes for valuing European options under the Bates model. Appl. Numer. Math. 130, 143–156 (2018).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