Non-equilibrium models for flow in porous media

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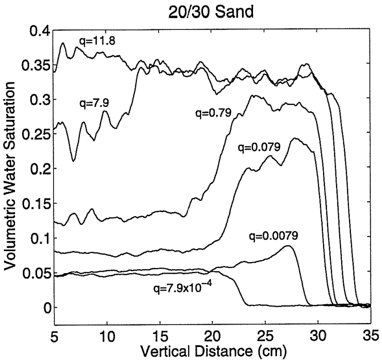
Outlook

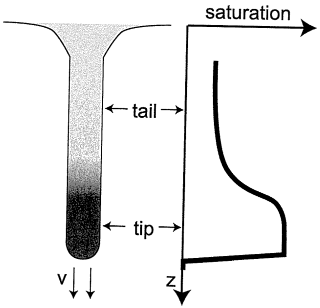
- Motivation
- Porous media flow models: standard vs. non-standard models
- A bit of maths/numerics
- More numerics

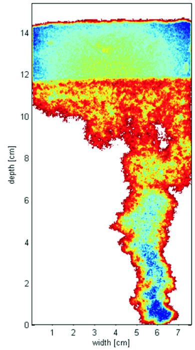
1. Motivation

David A. DiCarlo, Experimental measurements of saturation overshoot on infiltration, Water Resources Research, Vol. 40, W04215, doi:10.1029/2003WR002670, 2004

See also: F. Stauffer, S.M. Hassanizadeh & S. Bottero, Neuweiler







2. Porous media flow models

Variables $(\alpha = w, o)$:

$S_\alpha \in [0,1]$ - phase saturation

q_α - phase velocity

p_α - phase pressure

Equations:

$$\begin{aligned} \frac{\partial S_\alpha}{\partial t} + \nabla \cdot q_\alpha &= 0 \\ -q_\alpha &= \lambda_\alpha(S_\alpha)\nabla p_\alpha \end{aligned}$$

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Constitutive relationships:

$$\begin{aligned} S_o + S_w &= 1 \\ p_o - p_w &= P_c \end{aligned}$$

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P_c - capillary pressure

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Equilibrium:

$$P_c = P_c^e(S_o)$$

2. Porous media flow models: dynamic effects

Variables $(\alpha = w, o)$:

$S_\alpha \in [0,1]$ - phase saturation

q_α - phase velocity

p_α - phase pressure

Quantities:

λ_α - phase mobility

P_c - capillary pressure

τ - dynamic factor

Equations:

$$\begin{aligned} \frac{\partial S_\alpha}{\partial t} + \nabla \cdot q_\alpha &= 0 \\ -q_\alpha &= \lambda_\alpha(S_\alpha)\nabla p_\alpha \end{aligned}$$

Constitutive relationships:

$$\begin{aligned} S_o + S_w &= 1 \\ p_o - p_w &= P_c \end{aligned}$$

Non-equilibrium*:

$$P_c = P_c^e(S_o) + \tau \partial_t S_o$$

*Hassanizadeh & Gray, Water Resour. Res. '93, Belyaev & Hassanizadeh, Transp. Porous Med. '11

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2. Porous media flow models: hysteresis, interfacial area, ...

Variables $(\alpha = w, o)$:

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Constitutive relationships:

$$\begin{aligned} S_o + S_w &= 1 \\ p_o - p_w &= P_c \end{aligned}$$

Non-equilibrium*:

$$P_c \in P_c^e(S_o) + \tau \partial_t S_o + \gamma \,\mathbf{sign}(\partial_t S_o)$$

and/or add interfacial area effects: $\partial_t a + \nabla \mathbf{j}_a = E$, with $a = f(S_w, p_w, p_o)$.

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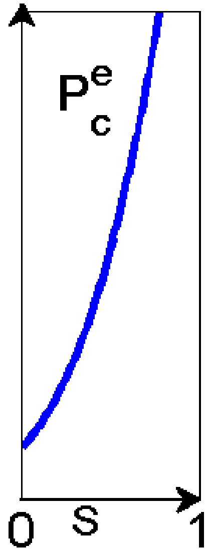
2.1. Modelling porous media flows: why τ?

Capillary pressure in two-phase flow/oil and water:
Equilibrium/classical approach (slow processes)

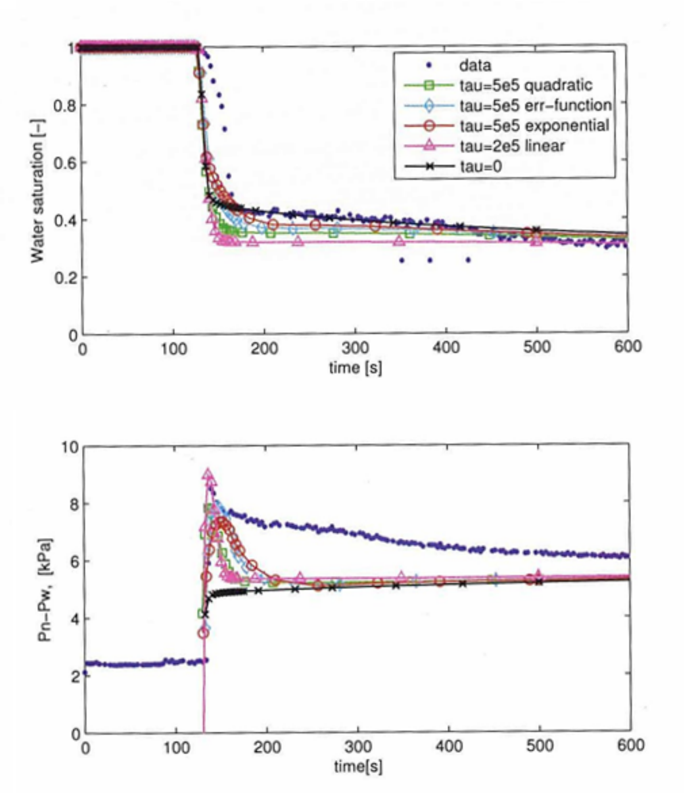
$$p_{\mathsf{o}} - p_{\mathsf{w}} = P_c^e(S_{\mathsf{o}})$$

Dynamic effects involve relaxation:
(Gray & Hassanizadeh)

$$p_{\mathsf{o}} - p_{\mathsf{w}} = P_c^e(S_{\mathsf{o}}) + \tau \frac{\partial S_{\mathsf{o}}}{\partial t}$$



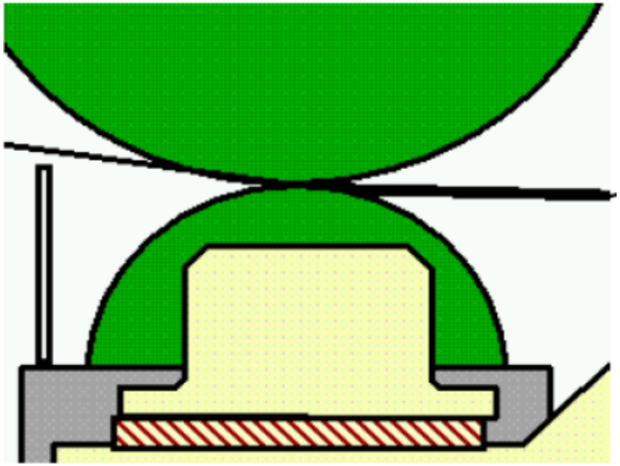
S. Bottero, Advances in the Theory of Capillarity in Porous Media

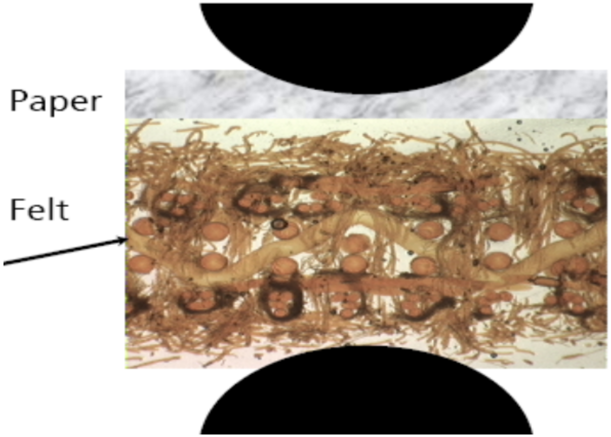


Classical models:

Pressure difference depends monotonically on saturation

2.2. Motivation: dewatering of paper pulp





2.3. Two-phase flow model (in terms of $S = S_o, p_w, p_o$)

Mass balance for oil:

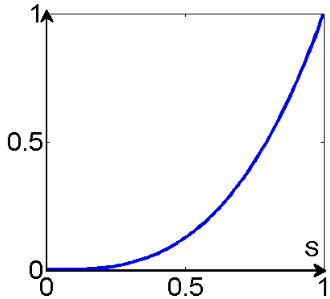
$$\partial_t S - \nabla \cdot \big(\lambda_o(S)\nabla p_o\big) = 0,$$

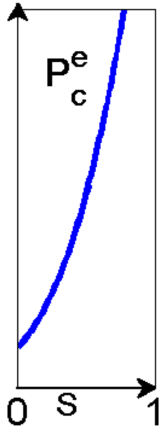
Mass balance for water:

$$-\partial_t S - \nabla \cdot \big(\lambda_w(1-S)\nabla p_w\big) = 0,$$

Pressure-saturaton relationship

$$p_o - p_w = P_c^e(S) + \tau\partial_t S.$$



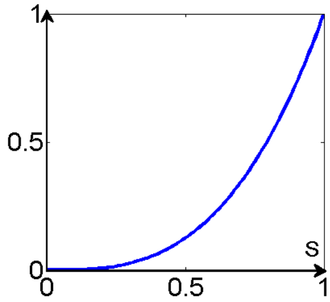


Typical choices:

$\lambda_o(S) = S^p$, $\lambda_w(S) = (1-S)^q$, where $p, q > 1$.

$P_c^e(S) = (1-S)^{-\frac{1}{\lambda}} \quad (\lambda > 1)$

2.3. Two-phase flow model (in terms of $S = S_o, p_w, p_o$)



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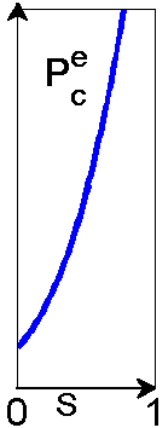

Alternative to one mass balance equation:

$$\nabla \cdot \mathbf{q} = 0, \qquad \text{with} \qquad \mathbf{q} = \big(\lambda_o(S)\nabla p_o + \lambda_w(1-S)\nabla p_w\big).$$

2.4. Porous media: mathematical questions

- Mathematical modelling:
  - Which equations?
  - Correct parameters?
- Mathematical analysis:
  - Do solutions exist? In which sense (strong, weak, . . .)?
  - How many solutions?
- Numerical methods:
  - Which discretization scheme (FEM, MFEM, DG, FV, . . .)? Convergence, efficiency?
  - Nonlinear models: iterative methods?
- Upscaling (complex geometries, highly oscillatory coefficients):
  - How are different scales interconnected (e.g. pore scale to Darcy scale)?
  - How to deal with free/moving interfaces?

3. Mathematics

Model:

$$
\text{(P1)} \begin{cases}
\partial_t S = \nabla \cdot \big(\lambda_o(S)\nabla p_o\big), & \text{in} \quad Q = \{(x,t) : x \in \Omega, t > 0\}, \\
-\partial_t S = \nabla \cdot \big(\lambda_w(1-S)\nabla p_w\big), & \text{in} \quad Q, \text{ or, alternatively} \\
0 = \nabla \cdot \big(\lambda_o(S)\nabla p_o + \lambda_w(1-S)\nabla p_w\big), & \text{in} \quad Q, \\
p_o - p_w = P_c^e(S) + \tau f(S)\partial_t S, & \text{in} \quad Q, \\
p(x,t) = 0, & \text{on} \quad \partial\Omega, t > 0, \\
S(x,0) = S_0, & \text{in} \quad \Omega.
\end{cases}
$$

Scaling

Characteristic values: $L,\ Q,\ P := \sigma\sqrt{\frac{\Phi}{K}},\ T = \frac{\Phi L}{Q}$

Rem: capillary number $N_c = \dfrac{KP}{\mu Q L} = \dfrac{\sigma\sqrt{K\Phi}}{\mu Q L}$

$$\tau = \tilde{\tau}\frac{\mu Q^2}{\Phi^2\sigma^2}$$

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p(x,t) = 0, & \text{on } \partial\Omega, t>0,\\
S(x,0) = S_0, & \text{in } \Omega.
\end{cases}
$$

- Existence/uniqueness of weak solutions: R. Showalter '75, M. Ptashnyk '06, A. Mikelič '10, Fan, P. '11, Cancès, Choquet, Fan, P '12, B. Schweizer '13, Cao, P '15, '16
- Numerical methods: D. Arnold et al. '81, Helmig et al. '07, Peszynska, Yi '08, C. Cuesta, P. '09, Kissling et al. '12, Fan '13, Zhang & Zegeling '16, Karpinski, P., Radu '17
- Travelling wave analysis, relation to non-classical shocks: LeFloch '02, Cuesta, Hulshof, van Duijn '00, Rohde '05, van Duijn, Peletier, P '07, Kissling et al '09, Corli, Rohde '12, Nieber et al '05, Spayd, Shearer '11, van Duijn, Fan, Peletier, P '13
- Heterogeneous media: Helmig, Weiss, Wohlmuth '07, van Duijn, Cao, P. '16

3.1. Analysis (existence, uniqueness)

Existence, degenerate case*

Problem P Find $(S,p) \in L^2(0,T;S_D + W_0^{1,2}(\Omega)) \times L^2(0,T;W_0^{1,r}(\Omega))$ s.t. $\partial_t S \in L^2(0,T;L^2(\Omega))$, $S(\cdot,0) = S^0$, $\sqrt{\lambda_w(1-S)}\nabla(\tau(S)\partial_t S) \in (L^2(Q))^d$, and

$$\int_0^T \int_\Omega \partial_t S \phi dx dt - \int_0^T \int_\Omega \lambda_o(S)\nabla p \cdot \nabla \phi dx dt + \int_0^T \int_\Omega \nabla\Theta(S) \cdot \nabla \phi dx dt = 0,$$
$$\int_0^T \int_\Omega (\lambda_o(S) + \lambda_w(1-S))\nabla p \cdot \nabla \psi dx dt + \int_0^T \int_\Omega \lambda_w(1-S)\nabla(\tau f(S)\partial_t S) \cdot \nabla \psi dx dt = 0,$$

for all $\phi, \psi \in L^2(0,T;W_0^{1,2}(\Omega))$.

Note: alternative form, total flow

Thm: Problem P has a unique solution. The solution component S is essentially bounded by 0 and 1.

Note: Similar results for models involving hysteresis

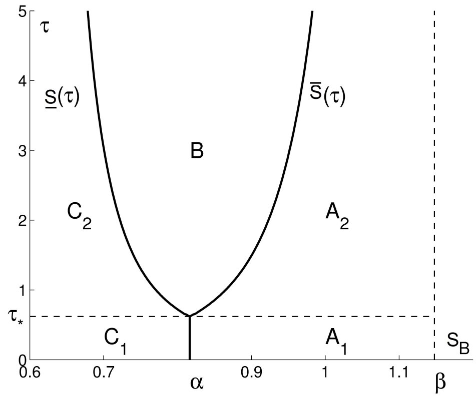
*Mikelič ('10), Koch, Rätz, Schweizer ('13), Cao, P ('15, '16)

3.2. Non-monotonic solutions, saturation overshoot

Results*:

- One dimensional, travelling wave analysis;
- Analysis (given S_{initial}), dependence on τ:
  - Occurence of nonmonotonic wave profiles;
  - Magnitude of the overshoot;

  Given the "inflow" saturation S_B, profiles displaying oscillations or "plateau" values $\bar{S}$ can be obtained depending on τ
- Limit cases, hyperbolic model: new criteria for selecting the entropy solution;
- Similar results for models involving hysteresis.



*van Duijn, Peletier, P ’07, ’13

4. Numerical examples*

$$\begin{cases} \dfrac{\partial S}{\partial t} + \dfrac{\partial f(S)}{\partial x} = \dfrac{\partial}{\partial x}\left(H(S)\dfrac{\partial}{\partial x}\left(S + \tau\dfrac{\partial S}{\partial t}\right)\right) & \text{in} \quad \mathbb{R} \times \mathbb{R}^+, \\ S(x,0) = u_B \tilde{H}(-x) & \text{for} \quad x \in \mathbb{R}, \end{cases}$$

with $\tilde{H}$ - smooth approximation of the Heaviside graph.

Numerical scheme:

Implicit for higher order terms, first order in time & finite differences;

Explicit for convection, *minmod* flux limiting scheme, upwind & Richtmyer.

Rem: The τ - $\bar{S}(\tau)$ diagram is not involved in the scheme!

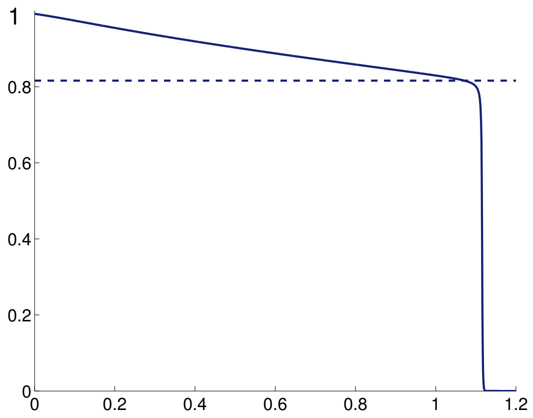
*van Duijn, Peletier, P (SIAM J. Math. Anal., 2007)

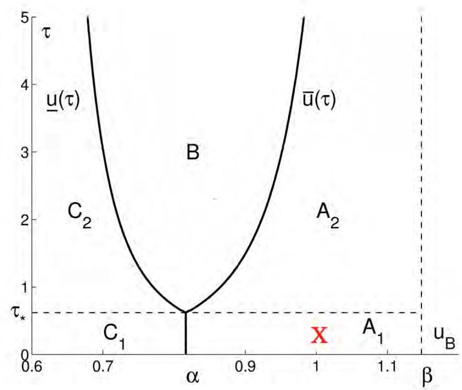
Cuesta, P. (J. Comput. Appl. Math., 2009)

van Duijn, Fan, Peletier, P (2013)

Examples (1D, $H \equiv 1$):

"Standard" case: $\tau = 0.2$, $u_B = 1.0$, case $\mathcal{A}_1$:

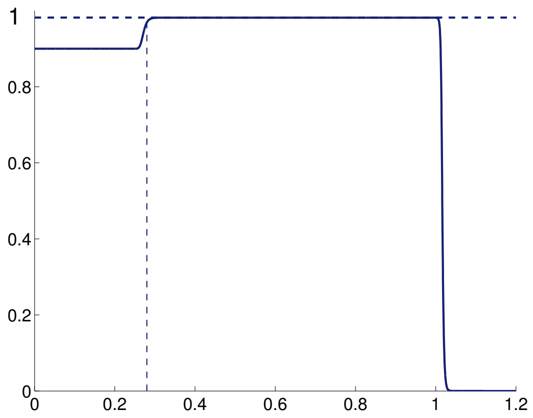


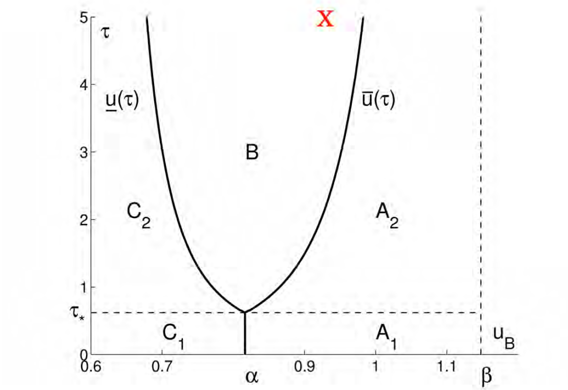


Rem: Since $\tau < \tau_* \approx 0.61$, the solution first decays to $\alpha = \bar{S}(\tau)$.

Examples (1D, $H \equiv 1$):

Non-standard case: $\tau = 5 > \tau_*$, case $\mathcal{B}$:

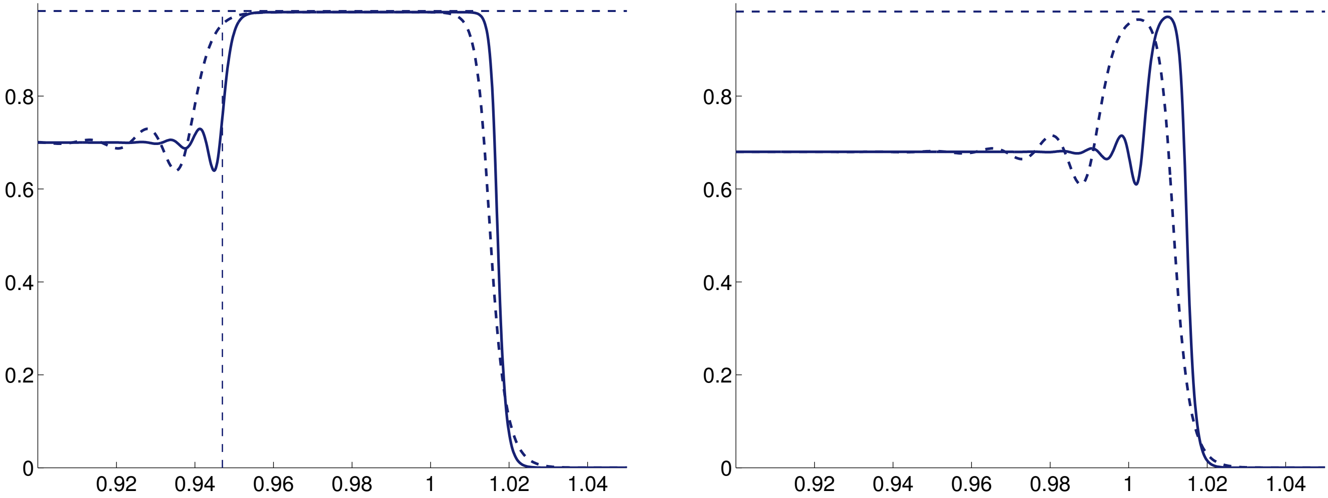




Rem: Plateau value ($S \approx 0.98$) agrees excellently with the diagram: $\bar{S}(\tau = 5) \approx 0.98$!

Examples (1D, $H \equiv 1$):

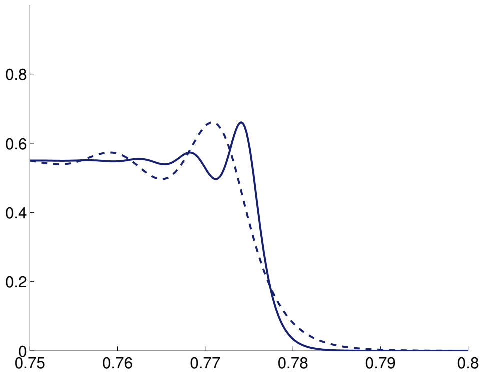
"Non-standard": $\tau = 5$ (large), u_B close to a critical value $\underline{S}(\tau = 5) \approx 0.68$

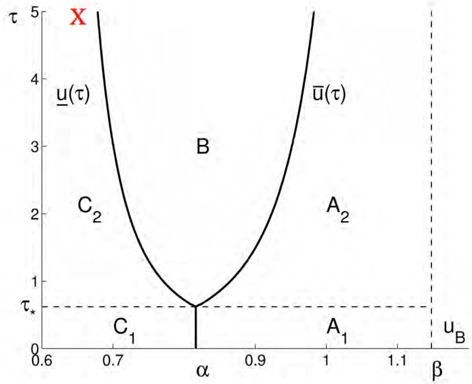


Rem: As S_B decays, the plateau vanishes and the solution transforms into an (oscillatory) front!

Examples (1D, $H \equiv 1$):

"Nearly-standard" case: $\tau = 5 > \tau_*$, case $\mathcal{C}_2$:





4.1. Higher dimensional examples

$$\begin{cases} \partial_t S = \nabla \cdot \big(\lambda_o(S) K \nabla p_o\big), , \\ -\partial_t S = \nabla \cdot \big(\lambda_w(1-S) K \nabla p_w\big), , \\ p_o - p_w = P_c^e(S) + \tau(S)\partial_t S, . \end{cases}$$

Approaches:

> O-type multipoint flux approximation finite volume scheme: higher order, convergence by compactness

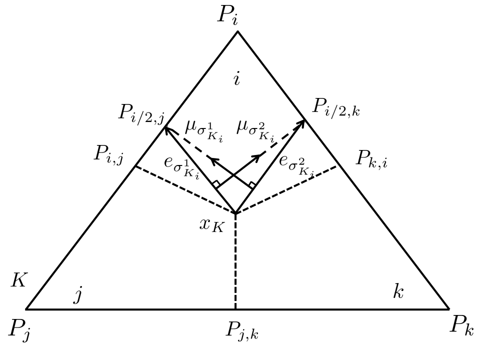
> DG scheme, symmetric/non-symmetric/incomplete interior penalty: error estimates, linearization schemes

$$\frac{s_w^{n+1} - s_w^n}{\Delta t}\phi + \nabla \cdot \big(\lambda_n(s_w^{n+1}) K \nabla p_n^{n+1}\big) = 0$$
$$\frac{s_w^{n+1} - s_w^n}{\Delta t}\phi + \nabla \cdot \big(\lambda_w(s_w^{n+1}) K \nabla (p_n^{n+1} - p_c^{n+1})\big) = 0$$
$$p_c^{n+1} = p_{c,eq}(s_w^{n+1}) - \tau \frac{s_w^{n+1} - s_w^n}{\Delta t}$$

(or variants...)

* Cao, P. (2015), Karpinski, Radu, P. (2016)

MPFV scheme (X. Cao)



$$\begin{cases} \mu_{\sigma^1_{K_i}} \cdot e_{\sigma^1_{K_i}} = 1, \\ \mu_{\sigma^1_{K_i}} \cdot e_{\sigma^2_{K_i}} = 0, \\ \mu_{\sigma^2_{K_i}} \cdot e_{\sigma^1_{K_i}} = 0, \\ \mu_{\sigma^2_{K_i}} \cdot e_{\sigma^2_{K_i}} = 1. \end{cases}$$

Vector: $\mathbf{v} = (\mathbf{v} \cdot e_{\sigma^1_{K_r}})\, \mu_{\sigma^1_{K_r}} + (\mathbf{v} \cdot e_{\sigma^2_{K_r}})\, \mu_{\sigma^2_{K_r}}.$

Gradient: $\nabla_{K_r} v_K := (v_{\sigma^1_{K_r}} - v_K) \cdot \mu_{\sigma^1_{K_r}} + (v_{\sigma^2_{K_r}} - v_K) \cdot \mu_{\sigma^2_{K_r}}.$

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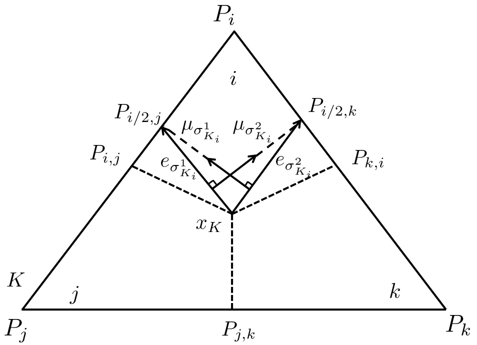

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Vector: $\mathbf{v} = (\mathbf{v} \cdot e_{\sigma^1_{K_r}})\, \mu_{\sigma^1_{K_r}} + (\mathbf{v} \cdot e_{\sigma^2_{K_r}})\, \mu_{\sigma^2_{K_r}}.$

Gradient: $\nabla_{K_r} v_K := (v_{\sigma^1_{K_r}} - v_K) \cdot \mu_{\sigma^1_{K_r}} + (v_{\sigma^2_{K_r}} - v_K) \cdot \mu_{\sigma^2_{K_r}}.$

With $S = S_w, \bar{p} = p_w$ and $p = p_n$ solve for every time step:

$$\mathsf{m}(K)\frac{S_K^{n+1} - S_K^n}{h} = k_o(S_K^{n+1}) \sum_{r=i,j,k} \Bigg(\mathsf{m}(\sigma^1_{K_r}) \Big((\bar{p}^{n+1}_{\sigma^1_{K_r}} - \bar{p}^{n+1}_K)\, \mu_{\sigma^1_{K_r}} + (\bar{p}^{n+1}_{\sigma^2_{K_r}} - \bar{p}^{n+1}_K)\, \mu_{\sigma^2_{K_r}} \Big) \cdot \mathbf{n}_{\sigma^1_{K_r}}$$
$$+\mathsf{m}(\sigma^2_{K_r}) \Big((\bar{p}^{n+1}_{\sigma^1_{K_r}} - \bar{p}^{n+1}_K)\, \mu_{\sigma^1_{K_r}} + (\bar{p}^{n+1}_{\sigma^2_{K_r}} - \bar{p}^{n+1}_K)\, \mu_{\sigma^2_{K_r}} \Big) \cdot \mathbf{n}_{\sigma^2_{K_r}} \Bigg),$$

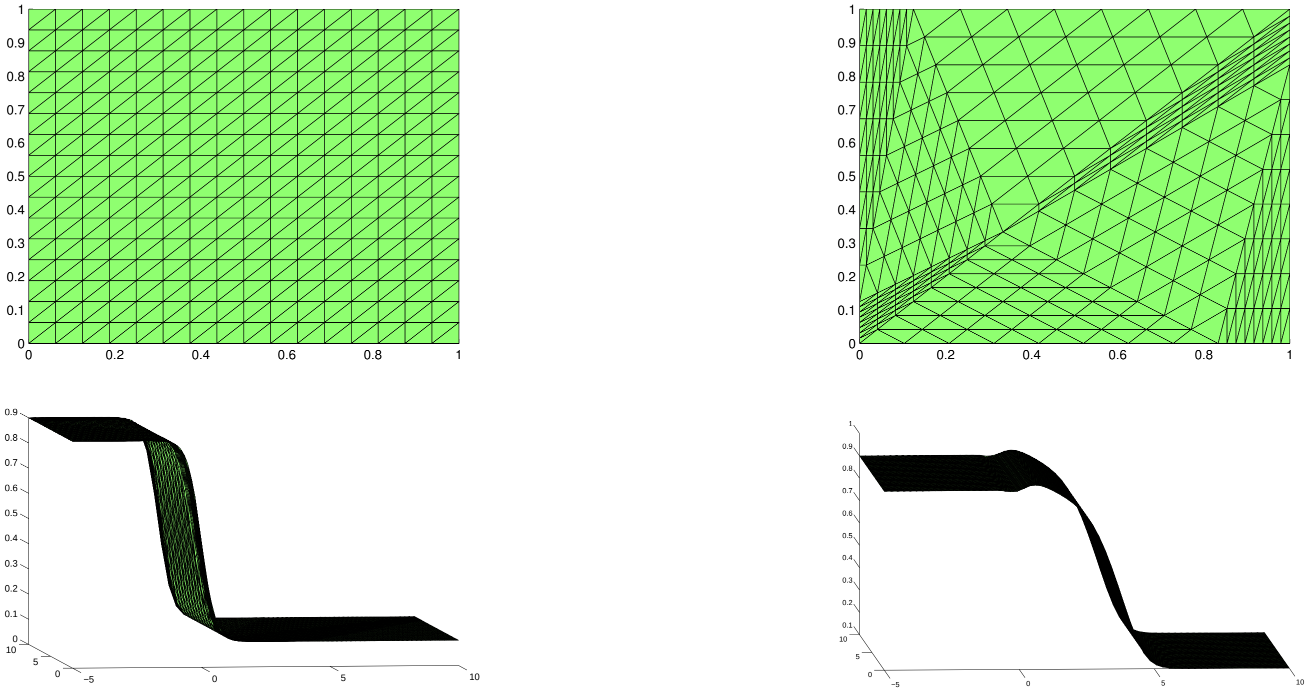
$$-\mathsf{m}(K)\frac{S_K^{n+1} - S_K^n}{h} = k_w(S_K^{n+1}) \sum_{r=i,j,k} \Bigg(\mathsf{m}(\sigma^1_{K_r}) \Big((p^{n+1}_{\sigma^1_{K_r}} - p^{n+1}_K)\, \mu_{\sigma^1_{K_r}} + (p^{n+1}_{\sigma^2_{K_r}} - p^{n+1}_K)\, \mu_{\sigma^2_{K_r}} \Big) \cdot \mathbf{n}_{\sigma^1_{K_r}}$$
$$+\mathsf{m}(\sigma^2_{K_r}) \Big((p^{n+1}_{\sigma^1_{K_r}} - p^{n+1}_K)\, \mu_{\sigma^1_{K_r}} + (p^{n+1}_{\sigma^2_{K_r}} - p^{n+1}_K)\, \mu_{\sigma^2_{K_r}} \Big) \cdot \mathbf{n}_{\sigma^2_{K_r}} \Bigg),$$

$$\bar{p}^{n+1}_K - p^{n+1}_K = p_c(S^{n+1}_K) + \tau \frac{S_K^{n+1} - S_K^n}{h}$$

At edge points $P_{i/2,j}$: normal flux continuity and pressure difference condition

Rem: Rigorous convergence proof for the numerical scheme (stability, energy estimates, compactness)

MPFV scheme (X. Cao)



MPFV scheme, anisotropic tensor $(1/1000)$

No. of cells	$E^{u}_{\mathcal{T},h}$	α	$E^{pre}_{\mathcal{T},h}$	β	$E^{flux}_{\mathcal{T},h}$	γ
512	4.1225×10^{-19}	–	6.2198×10^{-10}	–	1.6581×10^{-7}	–
2048	1.0170×10^{-19}	2.0192	1.5364×10^{-10}	2.0173	8.2410×10^{-8}	1.0086
8192	2.5304×10^{-20}	2.0069	3.8292×10^{-11}	2.0044	4.1142×10^{-8}	1.0022
32768	6.3925×10^{-21}	1.9849	9.5654×10^{-12}	2.0011	2.0563×10^{-8}	1.0006

Convergence results for uniform mesh, $\tau = 1$ and in the anisotropic case.

No. of cells	$E^{u}_{\mathcal{T},h}$	α	$E^{pre}_{\mathcal{T},h}$	β	$E^{flux}_{\mathcal{T},h}$	γ
512	1.0717×10^{-18}	–	1.6150×10^{-9}	–	2.2817×10^{-7}	–
2048	2.5510×10^{-19}	2.0708	3.8511×10^{-10}	2.0682	1.1271×10^{-7}	1.0175
8192	6.2565×10^{-20}	2.0276	9.4870×10^{-11}	2.0213	5.6171×10^{-8}	1.0047
32768	1.5440×10^{-20}	2.0187	2.3603×10^{-11}	2.0069	2.8058×10^{-8}	1.0014

Convergence results for nonuniform mesh, $\tau = 1$ and in the anisotropic case.

DG scheme (S. Karpinski)

Interior penalty discontinuous Galerkin approximation*:

Given $P_n^n \in V_h^p(\Omega)$, $P_c^n \in V_h^p(\Omega)$, and $S_w^n \in V_h^s(\Omega)$, find $P_n^{n+1} \in V_h^p(\Omega)$, $P_c^{n+1} \in V_h^p(\Omega)$, and $S_w^{n+1} \in V_h^s(\Omega)$, s.t. for all $\psi_s \in V_h^s(\Omega)$, $\psi_n \in V_h^p(\Omega)$, and $\psi_w \in V_h^p(\Omega)$:

$$\begin{aligned}
&\sum_{T_i\in\mathcal{T}} \int_{T_i} (-1)^\alpha \partial^- S_w^{n+1} \phi \psi_\alpha + \sum_{T_i\in\mathcal{T}} \int_{T_i} \lambda_\alpha(S_w^{n+1}) K \nabla \left(P_\alpha^{n+1} - gz\rho_\alpha\right) \nabla\psi_\alpha \\
&\quad - \sum_{F_i\in\mathcal{F}} \int_{F_i} \{\lambda_\alpha(S_w^{n+1}) K\nabla \left(P_\alpha^{n+1} - gz\rho_\alpha\right) \cdot \mathbf{n}\} [\![\psi_\alpha]\!] + \theta \sum_{F_i\in\mathcal{F}} \int_{F_i} [\![P_\alpha^{n+1}]\!] \{\lambda_\alpha(S_w^{n+1}) K \nabla\psi_\alpha \cdot \mathbf{n}\} \\
&\quad + \sigma_\alpha \sum_{F_i\in\mathcal{F}} \int_{F_i} \frac{f(k_p)}{|F_i|} [\![P_\alpha^{n+1}]\!] [\![\psi_\alpha]\!] \\
&= \theta \sum_{F_i\in\Gamma} \int_{F_i} [\![p_\alpha^D]\!] \{\lambda_\alpha(s^D) K\nabla\psi_\alpha \cdot \mathbf{n}\} + \sigma_\alpha \sum_{F_i\in\Gamma} \int_{F_i} \frac{f(k_p)}{|F_i|} [\![p_\alpha^D]\!] [\![\psi_\alpha]\!] \quad \text{for } \alpha = n, w
\end{aligned}$$

$$\sum_{T_i\in\mathcal{T}} \int_{T_i} P_c^{n+1} \psi_s = \sum_{T_i\in\mathcal{T}} \int_{T_i} p_{c,eq}(S_w^{n+1}) \psi_s - \sum_{T_i\in\mathcal{T}} \int_{T_i} \tau \partial^- S_w^{n+1} \psi_s$$

*Karpinski, P ’17

DG scheme (S. Karpinski)

Interior penalty discontinuous Galerkin approximation*:

Given $P_n^n \in V_h^p(\Omega)$, $P_c^n \in V_h^p(\Omega)$, and $S_w^n \in V_h^s(\Omega)$, find $P_n^{n+1} \in V_h^p(\Omega)$, $P_c^{n+1} \in V_h^p(\Omega)$, and $S_w^{n+1} \in V_h^s(\Omega)$, s.t. for all $\psi_s \in V_h^s(\Omega)$, $\psi_n \in V_h^p(\Omega)$, and $\psi_w \in V_h^p(\Omega)$:

$$
\begin{aligned}
&\sum_{T_i\in\mathcal{T}}\int_{T_i}(-1)^\alpha\partial^- S_w^{n+1}\phi\psi_\alpha + \sum_{T_i\in\mathcal{T}}\int_{T_i}\lambda_\alpha(S_w^{n+1})K\nabla\left(P_\alpha^{n+1}-gz\rho_\alpha\right)\nabla\psi_\alpha\\
&\quad -\sum_{F_i\in\mathcal{F}}\int_{F_i}\{\lambda_\alpha(S_w^{n+1})K\nabla\left(P_\alpha^{n+1}-gz\rho_\alpha\right)\cdot\mathbf{n}\}[\![\psi_\alpha]\!] + \theta\sum_{F_i\in\mathcal{F}}\int_{F_i}[\![P_\alpha^{n+1}]\!]\{\lambda_\alpha(S_w^{n+1})K\nabla\psi_\alpha\cdot\mathbf{n}\}\\
&\quad +\sigma_\alpha\sum_{F_i\in\mathcal{F}}\int_{F_i}\frac{f(k_p)}{|F_i|}[\![P_\alpha^{n+1}]\!][\![\psi_\alpha]\!]\\
&= \theta\sum_{F_i\in\Gamma}\int_{F_i}[\![p_\alpha^D]\!]\{\lambda_\alpha(s^D)K\nabla\psi_\alpha\cdot\mathbf{n}\} + \sigma_\alpha\sum_{F_i\in\Gamma}\int_{F_i}\frac{f(k_p)}{|F_i|}[\![p_\alpha^D]\!][\![\psi_\alpha]\!] \quad \text{for } \alpha = n, w
\end{aligned}
$$

$$
\sum_{T_i\in\mathcal{T}}\int_{T_i}P_c^{n+1}\psi_s = \sum_{T_i\in\mathcal{T}}\int_{T_i}p_{c,eq}(S_w^{n+1})\psi_s - \sum_{T_i\in\mathcal{T}}\int_{T_i}\tau\partial^- S_w^{n+1}\psi_s
$$

Theorem. If $p_n \in L^2(0,T;H^{k_p+1}(\Omega))$, $p_c \in L^2(0,T;H^{k_p+1}(\Omega))$ and $s_w \in H^2(0,T;H^{k_s+1}(\Omega))$ and with σ_n, σ_w large enough there exists $C > 0$ s.t

$$
\|e_{s,h}^{N+1}\|_{\Omega,0}^2 + \Delta t\sum_{n=0}^{N}\|\partial^- e_{s,h}^{n+1}\|_{\Omega,0}^2 + \Delta t\sum_{n=0}^{N}\left(\|e_{p_c,h}^{n+1}\|_{\Omega,DG}^2 + \|e_{p_n,h}^{n+1}\|_{\Omega,DG}^2\right) \le C\Delta t^2 + C\frac{h^{2k_s}}{k_s^{2k_s}} + C\frac{h^{2k_p}}{k_p^{2k_p-2}}
$$

*Karpinski, P '17

Linearization of the DG scheme (S. Karpinski)*

$$-\frac{s_w^{n+1,i}-s_w^n}{\Delta t}\phi+\nabla\cdot\left(\lambda_n(s_w^{n+1,i-1})K\nabla p_n^{n+1,i}\right)=0$$

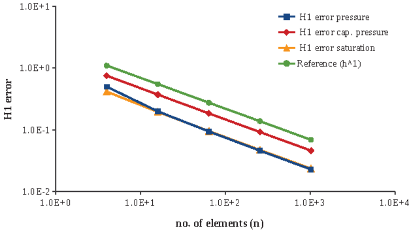
$$\frac{s_w^{n+1,i}-s_w^n}{\Delta t}\phi+\nabla\cdot\left(\lambda_w(s_w^{n+1,i-1})K\nabla(p_n^{n+1,i}-p_c^{n+1,i})\right)=0$$

$$L_s(s_w^{n+1,i}-s_w^{n+1,i-1})+p_c^{n+1,i}-p_{c,eq}(s_w^{n+1,i-1})+\tau\frac{s_w^{n+1,i}-s_w^n}{\Delta t}=0$$

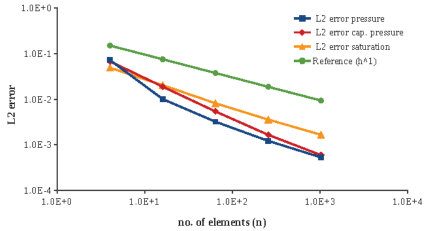
with $L_s \geq L_{p_{c,eq}}$.

Rem: Rigorous H^1 convergence proof, $(s_w^{n+1,i}, p_n^{n+1,i}, p_c^{n+1,i}) \to (s_w^{n+1}, p_n^{n+1}, p_c^{n+1})$ as $i \to \infty$.

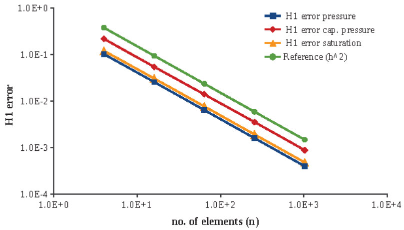
*Karpinski, P, Radu '17



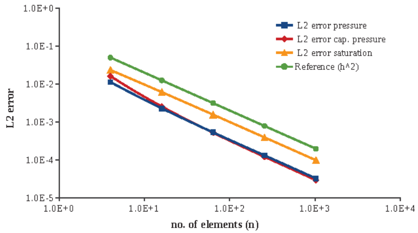
(a) H^1 error for piecewise linear polynomials



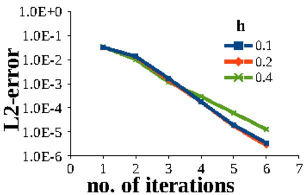
(b) L^2 error for piecewise linear polynomials



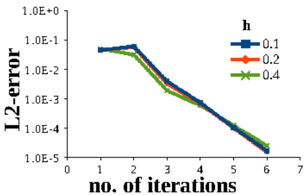
(c) H^1 error for piecewise quadratic polynomials



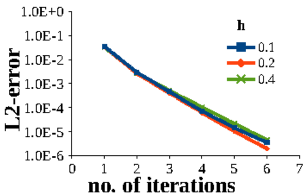
(d) L^2 error for piecewise quadratic polynomials



(a) L^2 error in p_n.

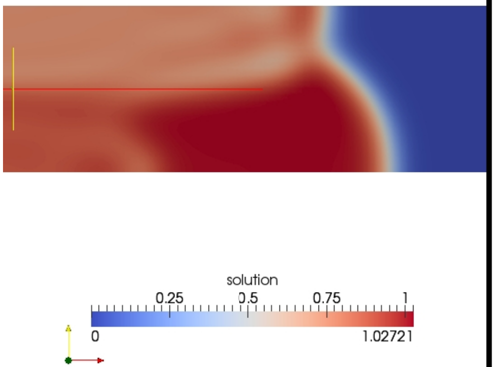


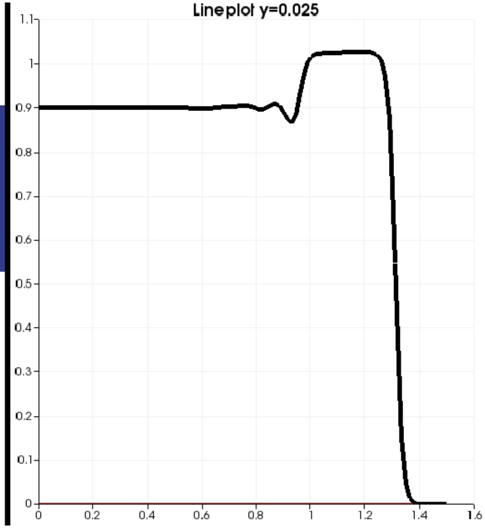
(b) L^2 error in p_c.

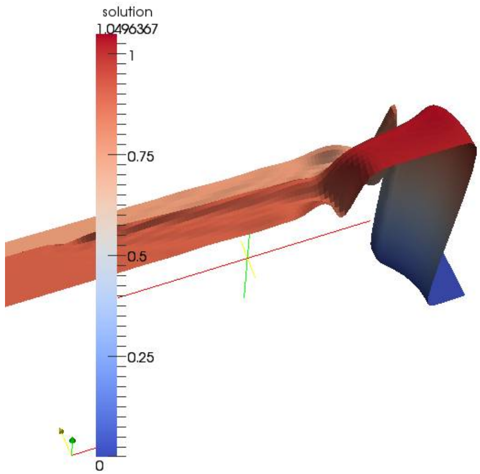


(c) L^2 error in s_w.

DG scheme (S. Karpinski)







Homogeneous media

Heterogeneous media

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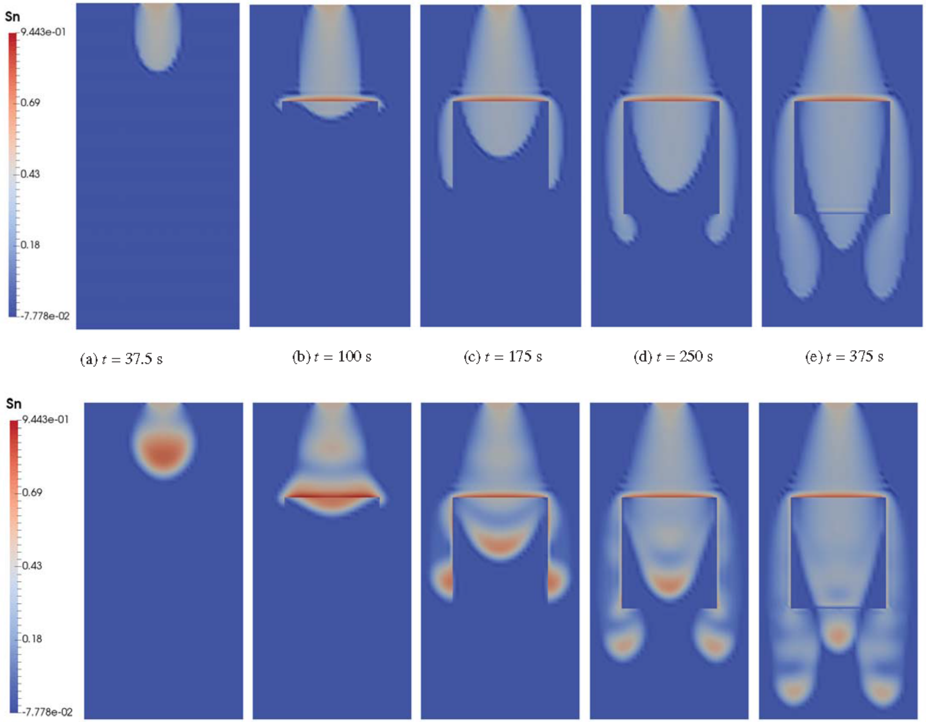
DG calculations, heterogeneous media (S. Karpinski)



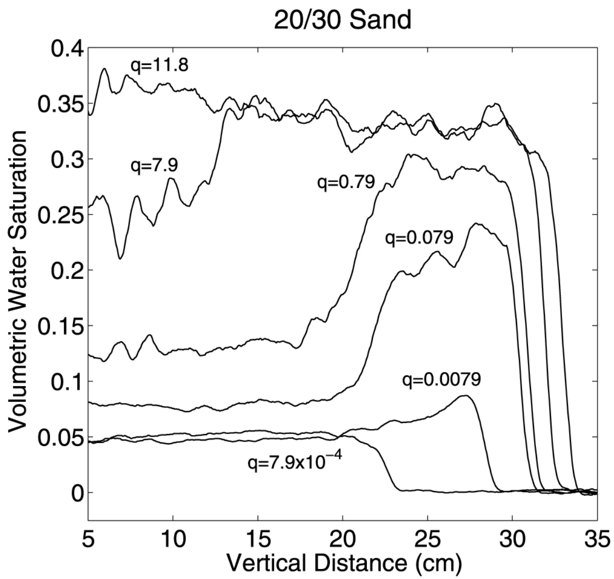
(a) $t = 37.5$ s (b) $t = 100$ s (c) $t = 175$ s (d) $t = 250$ s (e) $t = 375$ s

(f) $t = 125$ s (g) $t = 200$ s (h) $t = 300$ s (i) $t = 400$ s (j) $t = 500$ s

Equilibrium vs. dynamic

4.2. Saturation overshoot

Infiltration problem: $S(t, 0) = u_\ell$, $S(t, "+\infty") = u_r$



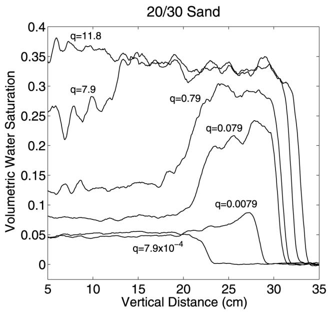
4.2. Saturation overshoot

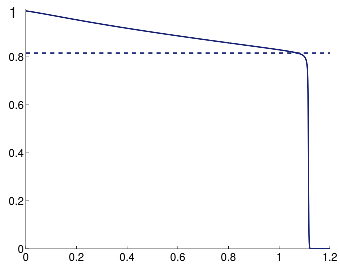
Infiltration problem: $u(t,0) = u_\ell$, $u(t, "+\infty") = u_r$

- The standard two-phase model ($P_c = P_c^e(u)$)

$$\partial_t u + \nabla \cdot \big(f(u)\big) = \nabla \cdot \big(H(u)\nabla P_c(u)\big)$$

provides *monotone* saturation profiles!





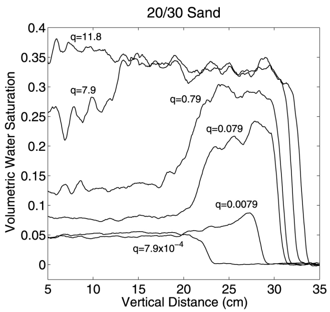
4.2. Saturation overshoot

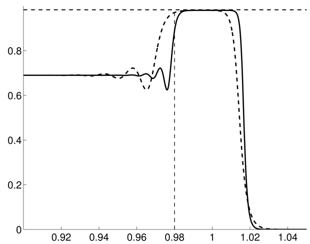
Infiltration problem: $u(t,0) = u_\ell$, $u(t, " + \infty") = u_r$

- The non-equilibrium model ($P_c = P_c^e(u) + \tau\partial_t u$)

$$\partial_t u + \nabla \cdot \big(K(u)\mathbf{g}\big) = \nabla \cdot \big(H(u)\nabla(P_c^e(u) + \tau\partial_t u)\big)$$

  allows for non-monotone profiles.





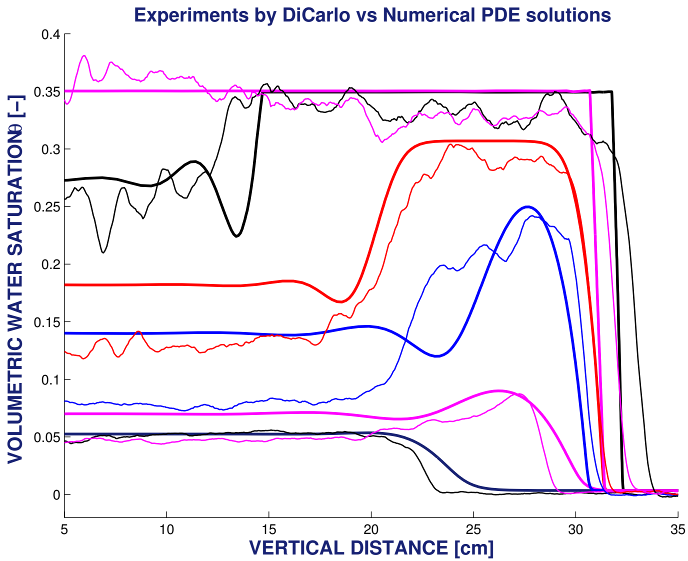
4.2. Saturation overshoot

Infiltration problem: $S(t,0) = u_B$, $S(t, \text{"}\infty\text{"}) = u_r$

Two-phase (water-air) porous media flow model, fully nonlinear, degenerate,

parameters for 20/30 sand, total velocities as in the experiments,

τ is fitted (e.g. $\tau \approx 2850 kg m^{-1} s^{-1}$ for the red profile)



Rem: Standard ($\tau = 0$) model provides *monotone* profiles

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Conclusions and perspectives

- Non-standard two-phase flow models: explain effects ruled out by standard models
- Mathematical analysis (existence/uniqueness) of weak solutions
- Numerical analysis (convergent numerical schemes, iterations)
- Heterogeneous media (interfaces between homogeneous layers), fractures
- Domain decomposition, adaptive modelling and discretization
- Model derivation (pore to core upscaling, account for evolving interfaces)
- Other effects: reactive flows, mechanics

Conclusions and perspectives

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Summer school 'Upscaling techniques for mathematical models involving multiple scales' (June 26-29, UHasselt)

`http://www.uhasselt.be/multiscalemethods_summer-school`

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fwo Research Foundation Flanders Opening new horizons

ODYSSEUS Topwetenschap in Vlaanderen

FOM

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