A promising DPG method for the transport problem

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joint work with
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Convection-Diffusion equations

Let $\varepsilon > 0$. We are interested in approximating the solution to

$$\begin{cases} -\varepsilon u'' + u' = f & \text{on } (0,1) =: \Omega, \\ u(0) = u(1) = 0, \end{cases} \tag{1}$$

by a finite element method (FEM).

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Let Ω_h be a partitioning of Ω into subintervals, say of equal length h. Define $U_h := H_0^1(\Omega) \cap \prod_{K \in \Omega_h} P_1(K)$. The Galerkin finite element approximation is given by $u_h \in U_h$ that solves

$$b(u_h, v_h) = \int_0^1 f v_h \, dx \quad (v_h \in U_h). \tag{3}$$

Convection-Diffusion equations

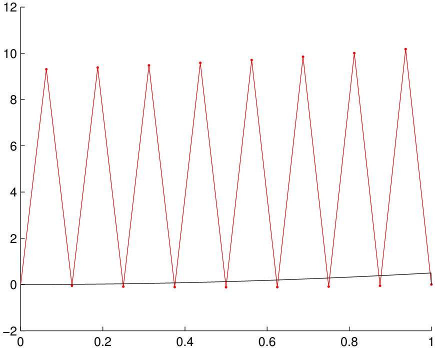

Figure : Exact solution of (1) and Galerkin approximation, for $f(x) = x$ with $h = \frac{1}{16}$.

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Petrov-Galerkin FEM with optimal test functions

For some real Hilbert spaces U and V, let $b : U \times V \mapsto \mathbb{R}$ be a bilinear form. Given $f \in V'$ (the dual of V), consider the variational problem of finding $u \in U$ such that

$$b(u, v) = f(v) \quad (v \in V). \tag{4}$$

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and suppose B is *boundedly invertible*. In a finite element method, we replace U and V by finite-dimensional subspaces $U_h \subset U$ and $V_h \subset V$ and find $u_h \in U_h$ such that

$$b(u_h, v_h) = f(v_h) \quad (v_h \in V_h). \tag{5}$$

We wish to find a test space V_h that guarantees that u_h is the best approximation from the trial space U_h.

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Given a closed linear *trial space* $U_h \subset U$, we set the *optimal test space* (cf. [DG11]) as

$$V_h := \Im(T|_{U_h}).$$

Given $f \in V'$, consider the following *Petrov-Galerkin* problem:
Find $u_h \in U_h$ such that

$$b(u_h, v_h) = f(v_h) \quad (v_h \in V_h = \Im(T|_{U_h})).$$

Proposition

It holds that $u_h = \operatorname{argmin}_{\bar{u}_h \in U_h} \|f - B\bar{u}_h\|_{V'}$.

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Proof.

For any $u_h, w_h \in U_h$,

$$\begin{aligned}\langle f - Bu_h, Bw_h\rangle_{V'} &= \langle R^{-1}(f - Bu_h), R^{-1}Bw_h\rangle_V \\ &= (f - Bu_h)(R^{-1}Bw_h) = f(v_h) - b(u_h, v_h).\end{aligned}$$

where $v_h := R^{-1}Bw_h$. $\square$

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Corollary

Equipping U with the energy norm,

$$\|\cdot\|_E := \|B\cdot\|_{V'},$$

we find that

$$u_h = \underset{\bar{u}_h \in U_h}{\operatorname{argmin}} \|u - \bar{u}_h\|_E.$$

- For the convection-diffusion problem, $\|\cdot\|_E$ can be some ε-dependent norm and may not be the norm of interest.

Choosing the norm on V

We can equip V with the *optimal test norm* $\|B' \cdot\|_{U'}$, so that for $w \in U$:

$$\|w\|_E = \|Bw\|_{V'} = \sup_{0 \neq v \in V} \frac{|(Bw)(v)|}{\|B'v\|_{U'}} = \sup_{0 \neq v \in V} \frac{|(B'v)(w)|}{\|B'v\|_{U'}} = \|w\|_U.$$

PG with optimal test space

Given a trial space $U_h \subset U$, the optimal test space $V_h = \Im(T|_{U_h})$, is determined by solving for each $u_h \in U_h$:

$$\langle Tu_h, v\rangle_V = b(u_h, v) \quad (v \in V).$$

- It is rarely possible to find an exact expression for Tu_h.

PG with projected optimal test space

Given a trial space $U_h \subset U$, the projected optimal test space $V_h = \Im(T_h|_{U_h})$, is determined by

$$\langle T_h u_h, \tilde{v}_h \rangle_V = b(u_h, \tilde{v}_h) \quad (u_h \in U_h,\ \tilde{v}_h \in \tilde{V}_h).$$

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$$\gamma_h := \inf_{0 \neq w_h \in U_h} \sup_{0 \neq \tilde{v}_h \in \tilde{V}_h} \frac{b(w_h, \tilde{v}_h)}{\|w_h\|_U \|\tilde{v}_h\|_V} > 0.$$

(Indeed, if $\|\cdot\|_U = \|\cdot\|_E$ and $\tilde{V}_h = V$, then $\gamma_h = 1$.)

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(Indeed, if $\|\cdot\|_U = \|\cdot\|_E$ and $\tilde{V}_h = V$, then $\gamma_h = 1$.) This method gives a near-best approximation in the energy norm:

$$\|u - u_h\|_E \leq \frac{1}{\gamma_h} \inf_{w_h \in U_h} \|u - w_h\|_E.$$

PG for the convection-diffusion problem

On a domain $\Omega \subset \mathbb{R}^n$, for $\varepsilon > 0$ consider the boundary value problem

$$\left\{ \begin{aligned} -\varepsilon \Delta u + \mathbf{b} \cdot \nabla u &= f \quad \text{on } \Omega, \\ u &= g \quad \text{on } \partial\Omega, \end{aligned} \right.$$

which can be written in mixed form:

$$\left\{ \begin{aligned} \boldsymbol{\sigma} - \sqrt{\varepsilon} \nabla u &= 0 \quad \text{on } \Omega, \\ -\sqrt{\varepsilon}\, \operatorname{div} \boldsymbol{\sigma} + \mathbf{b} \cdot \nabla u &= f \quad \text{on } \Omega, \\ u &= g \quad \text{on } \partial\Omega. \end{aligned} \right.$$

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The *ultra-weak* formulation of this problem reads as :
Find $(\boldsymbol{\sigma}, u, \theta) \in L_2(\Omega)^2 \times L_2(\Omega) \times H_{00}^{\frac{1}{2}}(\partial\Omega \setminus \Gamma_+)'$ such that

$$\begin{cases} \int_\Omega \boldsymbol{\sigma} \cdot \boldsymbol{\tau} + \sqrt{\varepsilon} u \operatorname{div} \boldsymbol{\tau} = \int_{\partial\Omega} \sqrt{\varepsilon} g \boldsymbol{\tau} \cdot \mathbf{n}, \\ \int_\Omega \sqrt{\varepsilon} \boldsymbol{\sigma} \cdot \nabla v - u\mathbf{b} \cdot \nabla v + \sqrt{\varepsilon} \int_{\partial\Omega \setminus \Gamma_+} \theta v = \int_\Omega f v - \int_{\Gamma_-} g v \mathbf{b} \cdot \mathbf{n}, \end{cases}$$

for all $v \in H_{0,\Gamma_+}(\Omega)$ and $\boldsymbol{\tau} \in H(\operatorname{div}, \Omega)$.

Choosing the norm on V

We can equip V with the *optimal test norm* $\|B' \cdot \|_{U'}$, so that for $w \in U$:

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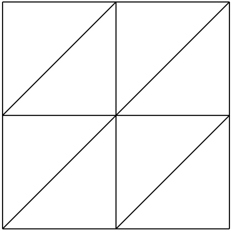
For the current problem, the (squared) optimal test norm reads as

$$\|\boldsymbol{\tau} + \sqrt{\varepsilon}\nabla v\|^2_{L_2(\Omega)^2} + \|\sqrt{\varepsilon}\,\mathrm{div}\,\boldsymbol{\tau} - \mathrm{div}\,\mathbf{b}v\|^2_{L_2(\Omega)} + \|\sqrt{\varepsilon}v|_{\partial\Omega\setminus\Gamma_+}\|^2_{H^{\frac{1}{2}}_{00}(\partial\Omega\setminus\Gamma_+)} .$$

Using this norm in a finite element discretizaton, we expect near-best approximations from U_h in the L_2-norm.

Numerical experiments

We look at the domain $\Omega = [0,1]^2$ and use uniform partitions Ω_h as in the figure.



$$U_h = \{(\boldsymbol{\sigma}_h, u_h, -\boldsymbol{\sigma}_h|_{\partial\Omega\setminus\Gamma_+} \cdot \mathbf{n}) : u_h \in P_h, \boldsymbol{\sigma}_h \in \Sigma_h\},$$

- $P_h = \{u_h \in L_2(\Omega) : u_h|_K \text{ is linear on each } K \in \Omega_h\}$.
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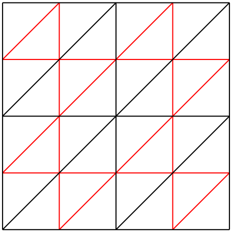

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For the test search space we used $\tilde{V}_h = \mathrm{RT1}_{h/2} \times \tilde{P}^3_{h/2}$:

- $\mathrm{RT1}_{h/2}$ the first order RT-space on an extra refinement $\Omega_{h/2}$.
- $\tilde{P}^3_{h/2}$ the space of continuous piecewise cubics on $\Omega_{h/2}$.

Experiment 1: a solution with boundary layer

Here $\mathbf{b} = [2,1]^\top$ and f is prescribed such that the exact solution is

$$u(x,y) = [x + (e^{b_1 x/\varepsilon} - 1)/(1 - e^{b_1/\varepsilon})] \cdot [y + (e^{b_1 y/\varepsilon} - 1)/(1 - e^{b_2/\varepsilon})].$$

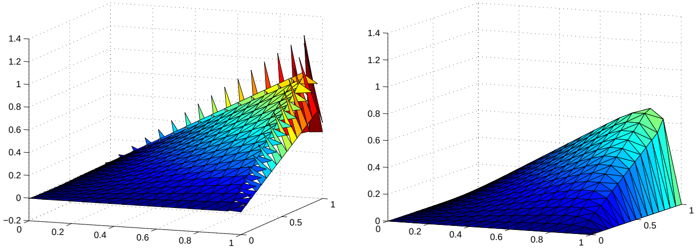

Figure : Our method (left) compared to SUPG (right), $h = \frac{1}{16}, \varepsilon = 10^{-2}$

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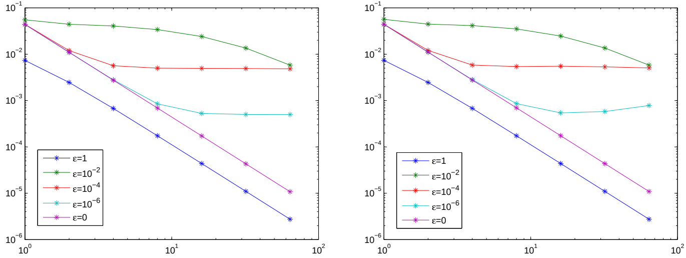

Figure : L_2-vs. $\frac{1}{h}$ error in u_h of the best L_2-approximation (left) and our method (right)

Experiment 2: a solution with internal layer that is not aligned with the mesh

Here $\mathbf{b} = [2, 1]^\top$ and

$$f = \begin{cases} 1 - x, & \text{if} \quad y - 2x > 1/2 \\ 0, & \text{otherwise.} \end{cases}$$

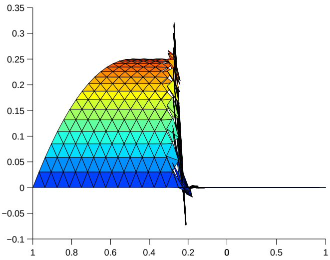

Figure : The approximate solution u_h for our method(left) and SUPG (right) $h = \frac{1}{16}$, $\varepsilon = 10^{-6}$.

Experiment 2: the local L_2- error in the case $\varepsilon = 0$

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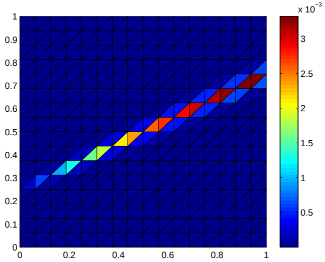

Figure : The local L_2-error in u_h for $h = \frac{1}{16}$ and $\varepsilon = 0$, for our method (left) and the best approximation (right).

Experiment 3: example from [DH13]

Here $\mathbf{b} = [1, 0]^\top$ and right-hand side f such that

$$u(x, y) = \left(\frac{e^{r_1(x-1)} - e^{r_2(x-1)}}{e^{-r_1} - e^{-r_2}} + x - 1 \right) \sin \pi y,$$

where $r_{1,2} = \frac{-1 \pm \sqrt{1 + 4\varepsilon^2 \pi^2}}{-2\varepsilon}$.

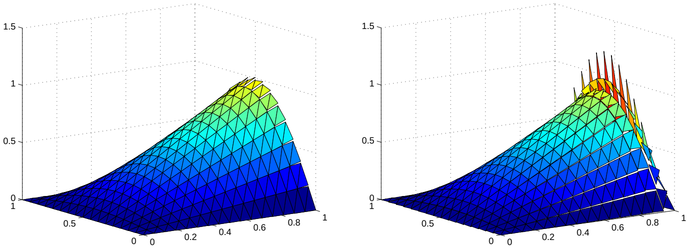

Figure : The approximate solution u_h for our method(left) and Demkowicz/Heuer ([DH13]) (right) $h = \frac{1}{16}$, $\varepsilon = 10^{-4}$.

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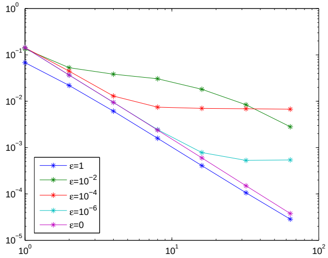

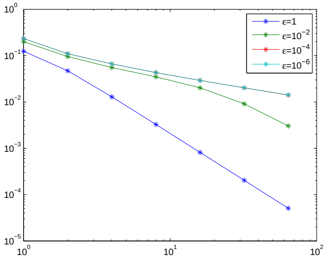

Figure : L_2-vs. $\frac{1}{h}$ error in u_h of our method (left) and Demkowicz/Heuer (right)

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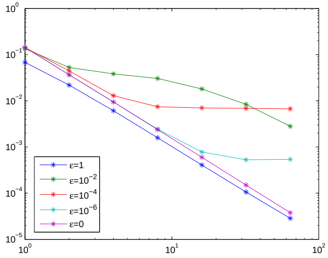


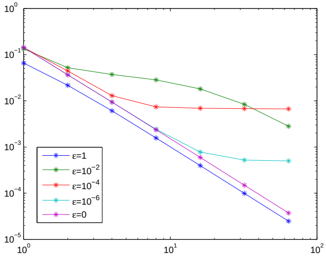


Figure : L_2-vs. $\frac{1}{h}$ error in u_h of our method (left) the best L_2-approximation (right)

Experiment 4: Peterson's mesh

Here $\mathbf{b} = [0, 1]^\top$ and

$$\begin{cases} f = 0, \\ u|_{\Gamma_-} = x^2. \end{cases}$$

With Peterson's mesh, Discontinuous Galerkin (DG) has an error rate of $O(h^{\frac{3}{2}})$ (cf. [Pet1991]).

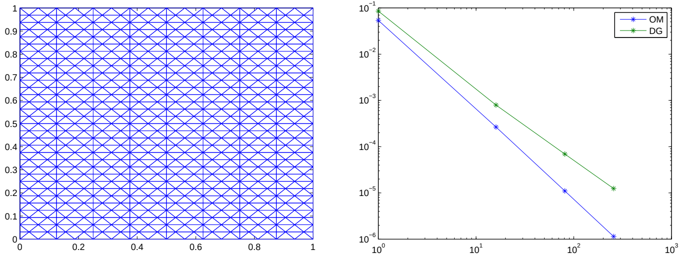


Figure : Example of Peterson's mesh (left) and L_2-error vs. $\frac{1}{h}$ in u_h for OM and DG (right).

Test search space

- Although the above results are promising, we have not yet been able to prove that the test search space is large enough to satisfy the inf-sup condition.
- Most recent work: a method for the transport problem for which we can prove stability.
- In fact: if the convection term is constant, then the optimal test functions for this problem can be determined exactly.

The transport problem

Let Ω be a domain. The transport problem reads as:

$$\begin{cases} \mathbf{b} \cdot \nabla u + cu = f & \text{on } \Omega, \\ u = g & \text{on } \Gamma_-. \end{cases}$$

To get to the variational form, this time we apply integration by parts on each element separately (earlier proposed in [DG11]). The variational problem reads as follows: Let
$U := L_2(\Omega) \times H_{0,\Gamma_-}(\mathbf{b}; \partial\Omega_h)$, $V := H(\mathbf{b}; \Omega_h)$. Given $f \in H(\mathbf{b}; \Omega_h)'$, find $(u, \theta) \in U$ such that

$$b_h(u, \theta; v) := \int_\Omega (cv - \mathbf{b} \cdot \nabla_h v - v \operatorname{div} \mathbf{b}) u \, d\mathbf{x} + \int_{\partial\Omega_h} [\![v\mathbf{b}]\!] \theta \, d\mathbf{s} = f(v).$$

for all $v \in V$.

The case where $\mathbf{b}$ is constant

Let $(u,\theta) \in U$. On each $K \in \Omega_h$, the restriction $t_K = T(u,\theta)|_K$ of the optimal test function is the unique function that satisfies

$$\langle t_K, v \rangle_{H(\mathbf{b},K)} = \int_K (cv - \mathbf{b} \cdot \nabla_h v)\, d\mathbf{x} + \int_{\partial K} v\theta \mathbf{b} \cdot \mathbf{n}\, ds$$

for all $v \in H(\mathbf{b}, K)$. Suppose u is a polynomial and $\theta = w|_{\partial K}$ for some polynomial w. The above is then equivalent to:

$$\begin{cases} -\partial_{\mathbf{b}}^2 t_K + t_K &= \partial_{\mathbf{b}} u + cu \quad \text{on } K, \\ \partial_{\mathbf{b}} t_K &= \theta - u \quad \text{on } \partial K_+ \cup \partial K_-, \end{cases}$$

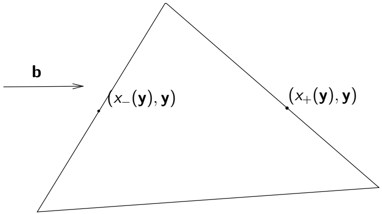
- Solving this often leads to non-polynomial optimal test functions that are difficult to use in implementations.

The case where $\mathbf{b}$ is constant

We can replace the inner product on $H(\mathbf{b}, K)$ by the equivalent inner product (see also [DG11])

$$\langle\!\langle v, z \rangle\!\rangle_{K,\mathbf{b}} := \langle \partial_{\mathbf{b}} v, \partial_{\mathbf{b}} z \rangle_{L_2(K)} + \int_{\partial K_+} v(\mathbf{s}) z(\mathbf{s}) |(\tfrac{\mathbf{b}}{|\mathbf{b}|} \cdot \mathbf{n}_K)(\mathbf{s})| r_+(\mathbf{s}) d\mathbf{s},$$

where $r_+(\mathbf{s})$ denotes the distance of $x_+(\mathbf{s})$ to ∂K_- along $\mathbf{b}$.



The case where $\mathbf{b}$ is constant

Using this inner product we can rewrite the equations for the optimal test functions as:

$$\left\{\begin{array}{rcll} -\partial_{\mathbf{b}}^2 t_K & = & \partial_{\mathbf{b}} u + cu & \text{on } K, \\ \partial_{\mathbf{b}} t_K + t_K \frac{|\mathbf{b}\cdot\mathbf{n}_K|}{|\mathbf{b}|\mathbf{b}\cdot\mathbf{n}_K} r_+ & = & \theta - u & \text{on } \partial K_+, \\ \partial_{\mathbf{b}} t_K & = & \theta - u & \text{on } \partial K_-. \end{array}\right.$$

The optimal test function is then a (piecewise) polynomial of degree $k+2$:

$$|\mathbf{b}|\, t_K(x,\mathbf{y}) = -|\mathbf{b}|^{-1} \int_{x_+(\mathbf{b})}^{x} \int_{x_+(\mathbf{y})}^{z} (\partial_{\mathbf{b}} u + cu)(q,\mathbf{y})dqdz$$
$$+ \left(\theta(x_-(\mathbf{y}),\mathbf{y}) - u(x_+(\mathbf{y}),\mathbf{y}) + |\mathbf{b}|^{-1} \int_{x_+(\mathbf{y})}^{x_-(\mathbf{y})} cu(q,\mathbf{y})dq\right)\Big(x - x_+(\mathbf{y})\Big)$$
$$+ |\mathbf{b}|^2 \frac{\theta(x_+(\mathbf{y}),\mathbf{y}) - \theta(x_-(\mathbf{y}),\mathbf{y}) + |\mathbf{b}|^{-1} \int_{x_-(\mathbf{y})}^{x_+(\mathbf{y})} cu(q,\mathbf{y})dq}{x_+(\mathbf{y}) - x_-(\mathbf{y})}.$$

The case of variable $\mathbf{b}$

- Approximate $\mathbf{b}$ and c by piecewise constants on Ω_h or on a uniform refinement $\Omega_{\tilde{h}}$.
- For a trial space U_h determine the corresponding exact optimal test space with respect to $\Omega_{\tilde{h}}$.
- Using (a slight modification of) this space as test search space allows us to prove stability.
- In practice we can replace the test search space by its containing space $\prod_{K \in \Omega_{\tilde{h}}} P_{k+2}(K)$

Numerical experiments

In the experiments presented below we took $U_h = P_{h,0} \times Q_{h,1}$ as trial space, with

- $P_{h,0} = \{u_h \in L_2(\Omega) : u_h|_K \in P_0(K) \ \forall \ K \in \Omega_h\}$.
- $Q_{h,1} = \{w_h|_{\partial\Omega_h} : w_h \in C_{0,\Gamma_-}(\Omega)$ and $w_h|_K \in P_1(K) \ \forall \ K \in \Omega_h\}$.

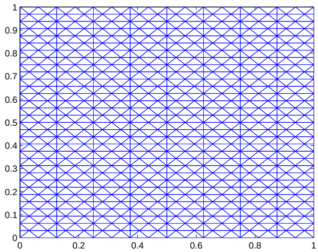
As test search space we took

- $V_{\tilde{h}} = \{v_h \in H(\mathbf{b}, \Omega_{\tilde{h}}) : v_h|_{\tilde{K}} \in P_2(\tilde{K}) \ \forall \ \tilde{K} \in \Omega_{\tilde{h}}\}$.

It turned out that for the test search space no additional refinements of Ω_h were needed.

Numerical experiment: Peterson's mesh again

Here $\mathbf{b} = [0,1]^\top$, $c = 0$, $f = 0$ and $u|_{\Gamma_-} = x^2$.
With Peterson's mesh, Discontinuous Galerkin (DG) has an error rate of $O(h^{0.75})$.



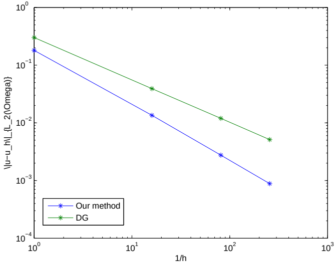


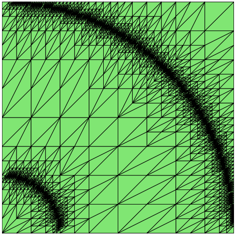
Figure : Example of Peterson's mesh (left) and L_2-error vs. $\frac{1}{h}$ in u_h for OM and DG (right).

An adaptive experiment

Here $\mathbf{b} = [y, -x]^\top$, $c = 0$, $f = 0$,

$$u(0, y) = \begin{cases} 0, & y \leq 1/4 \\ 1, & y > 1/4 \end{cases}$$

and an approximation of $\|f - B_h(u_h, \theta_h)\|_{V'}$ is being used as an error estimator.



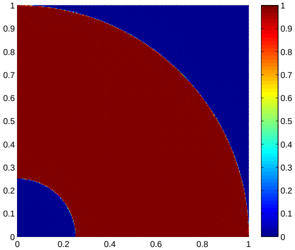


Figure : The mesh after some adaptive refinements (left) and the approximated solution at the last iteration (right)

An adaptive experiment

Here $\mathbf{b} = [y, -x]^\top$, $c = 0$, $f = 0$,

$$u(0, y) = \begin{cases} 0, y \leq 1/4 \\ 1, y > 1/4 \end{cases}$$

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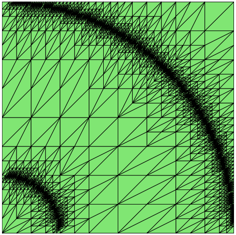

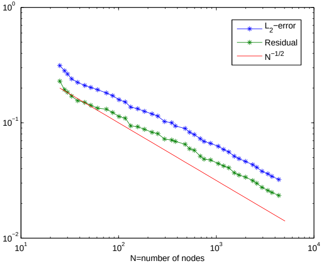

Figure : The mesh after some adaptive refinements (left) and plot of the true and estimated error (right)

Postprocessing

Notice that, for the best L_2-approximation $\tilde{u} \in \prod_{K \in \Omega_h} P_1(K)$ of u, for any $K \in \Omega_h$:

- $\int_K \tilde{u}\, d\mathbf{x} = \int_K u\, d\mathbf{x}$.
- $\tilde{u}(\mathbf{b_K}) = \frac{1}{|K|} \int_K u\, d\mathbf{x}$, where $\mathbf{b_K}$ denotes the barycenter of K.
- In particular $\min_{\mathbf{x} \in K} u(\mathbf{x}) \leq \tilde{u}(\mathbf{b_K}) \leq \max_{\mathbf{x} \in K} u(\mathbf{x})$.

Locate the elements K where the overshoots show up and replace $u_h|_K$ by the constant function $u_h(b_K)$.

Postprocessing

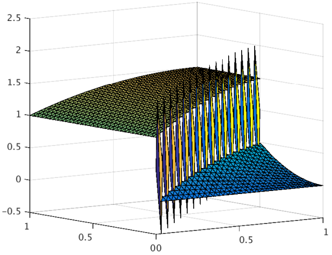

Figure : The approximation of u by our method using piecewise linear functions, before postprocessing (left) and after (right), for $f = 1 - x$, with $u(x,0) \equiv 0$, $u(0,y) \equiv 1$.

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