Optimization in Chaotic Systems : A Least Squares Shadowing Approach

Korneel Dumon

KU Leuven
CS Department

korneel.dumon@kuleuven.be

May 13, 2016

Overview

Chaos in Engineering

Chaos appears frequently in nature

Chaos in Engineering

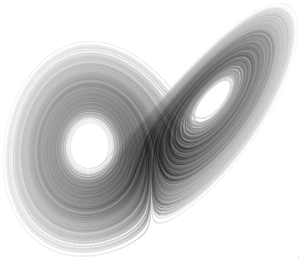
Is there anything to be gained?

Speedo LZR Racer

1987 America's Cup

Ergodicity

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T J(u)dt = \frac{1}{\mu(X)} \int_X J(u)d\mu$$



$\implies$ Well-defined properties for optimization

Random Process $\iff$ Chaotic Process

Calculating Sensitivities

$$\dot{u}(t;s) = f(u;s)$$

Tangent equation:

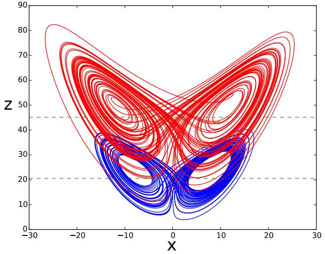
$$\frac{dv(t)}{dt} = \left.\frac{\partial f}{\partial u}\right|_{u_r(t)} v(t) + \left.\frac{\partial f}{\partial s}\right|_{u_r(t)}$$

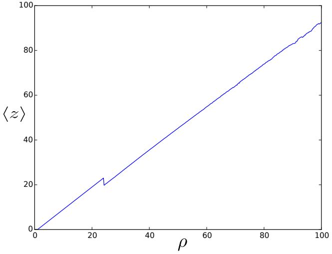
where $v(t) = \frac{du(t)}{ds}$ and $v(0) = 0$.

- Calculate reference trajectory $u_r(t)$
- Calculate state sensitivity $v(t)$

Sensitivity in Chaotic systems

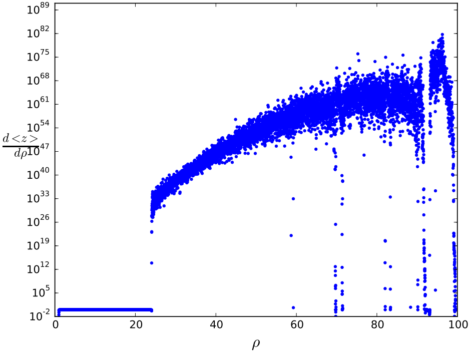
Ergodic average Lorenz system: $\langle J(t) \rangle = \lim_{T \to \infty} \frac{1}{T} \int_0^T z(t) dt$





$\Longrightarrow$ Average value of $z(t) \sim \rho$

Sensitivity in Chaotic systems



In chaotic systems:

$$v(t) \sim e^{\lambda_1 t}$$

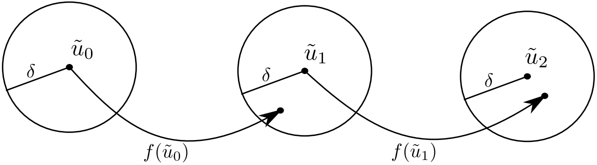
with λ_1 the largest positive Lyapunov exponent.

Pseudo-trajectory

Consider a discrete time dynamic system: $u_{k+1} = f(u_k)$.

$\Longrightarrow$ The sequence $\{\tilde{u}_k\}$ is a δ-pseudo-trajectory if

$$\|\tilde{u}_{k+1} - f(\tilde{u}_k)\| < \delta, \quad \forall k \in \mathbb{N}$$



- Local errors in numerical simulation
- Perturbed system dynamics: $\tilde{u}_{k+1} = f(\tilde{u}_k; s + \Delta s) \approx f(\tilde{u}_k; s) + \frac{\partial f}{\partial s}\Delta s$

Shadowing Lemma

$\Longrightarrow$ True trajectories exist near pseudo-trajectories

Lemma

For all pseudo-trajectories $\tilde{u}(t)$ in the neighbourhood of a hyperbolic set, there exists a true trajectory $u(t)$ for which

$$|\tilde{u}(t) - u(\tau(t))| < \epsilon.$$

Note: free initial condition

Least Squares Shadowing

Numerically compute a shadow trajectory
$\Longrightarrow$ **least squares shadowing** [Wang]

$$\bar{u}, \eta = \arg\min \quad \frac{1}{2}\int_0^T \left(\|\bar{u}(t) - u_r(t)\|^2 + \eta(t)^2\right) dt$$

$$s.t. \qquad (1+\eta)\frac{d\bar{u}(t)}{dt} = f(\bar{u}(t); s + \delta s), \quad 0 \le \tau \le T.$$

- Calculate reference trajectory $u_r(t)$ at parameter value s
- Solve LSS problem to find shadow $\bar{u}(t)$ at parameter value $s + \delta s$

LSS Sensitivity Calculation

Linearising around the parameter value s

$$\bar{u}(t;s+\delta s) \approx u_r(t;s) + \mathbf{v(t)}\delta s$$
$$\frac{d\tau(t;s+\delta s)}{dt} \approx 1 + \boldsymbol{\sigma}\mathbf{(t)}\delta s$$

LSS problem is now parameterized by state sensitivity $v(t)$ and $\sigma(t)$

$$v,\sigma = \text{argmin} \quad \frac{1}{2}\int_0^T \left(\|v(t)\|^2 + \alpha^2\sigma^2(t)\right) dt$$
$$s.t. \quad \frac{dv}{dt} = \sigma(t) f(u_r;s) + \left.\frac{\partial f}{\partial u}\right|_{u_r} v(t) + \left.\frac{\partial f}{\partial s}\right|_{u_r}$$

$\Longrightarrow$ Equivalent of the tangent equation!

LSS Sensitivity Calculation

Optimality conditions for the QP

$$\begin{cases} \dot{v}(t) = \left.\frac{\partial f}{\partial u}\right|_{u_r} v(t) + \left.\frac{\partial f}{\partial s}\right|_{u_r} + \sigma(t) f(u_r; s), \\ \dot{w}(t) = -\frac{\partial f}{\partial u}^T w(t) + v(t), \qquad w(0) = w(T) = 0, \\ \alpha^2 \sigma(t) - w^T(t) f(u_r; s) = 0. \end{cases}$$

Can be reduced to

$$AW = b$$

where A is block-tridiagonal.

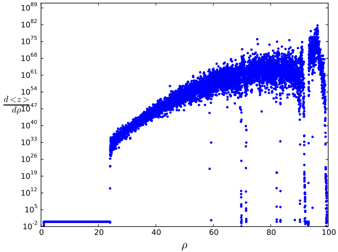
$$A = \begin{bmatrix} T_1 & & & \\ R_1 & T_2 & & \\ & R_2 & T_3 & \\ & & \ddots & \ddots \end{bmatrix} \begin{bmatrix} I & U_1 & & \\ & I & U_2 & \\ & & I & \ddots \\ & & & \ddots \end{bmatrix}$$

Lorenz Sensitivity Calculation

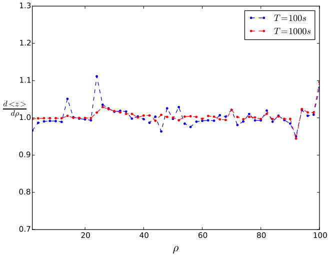
Lorenz system example:

$\longrightarrow$ Finding $\frac{d<z>}{d\rho}$.

Tangent equation



LSS

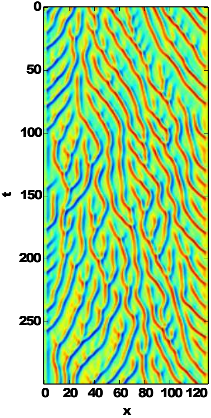


Kuramoto-Sivashinsky equation

$$\dot{u} = -u_{xx} - u_{xxxx} - (u + c)u_x$$

$\Longrightarrow$ simple PDE which exhibits chaos

- Thin film flow
- Flame front

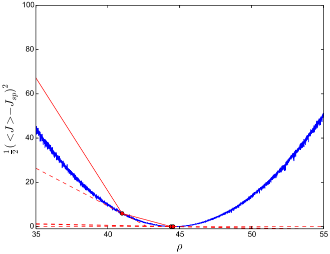


Optimization Results

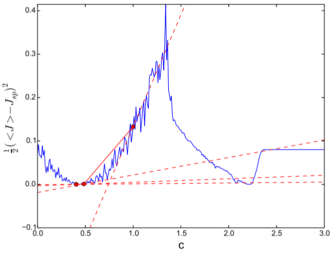
Setpoint Optimization with

- Lorenz objective: $J_L = < z(t) >$
- KS objective: $J_{KS} = < \frac{1}{L} \int_0^L u(x,t)dx >$

Lorenz



KS



Conclusion

Future work:

- Extension to PDEs
  - 2D Kuramoto Sivashinsky
  - turbulent Navier-Stokes
- Large-scale numerical linear algebra
  - multilevel iterative methods
  - double loop optimization
  - efficient optimization methods
- Parallelization
  - parallel numerical algorithms
  - multiple shooting and time-parallelism
  - storage bottleneck: checkpointing
- Applications
  - turbulent flow optimization
  - shape optimization for drag reduction