Methods for studying critical transitions in stochastic climate models

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Introduction

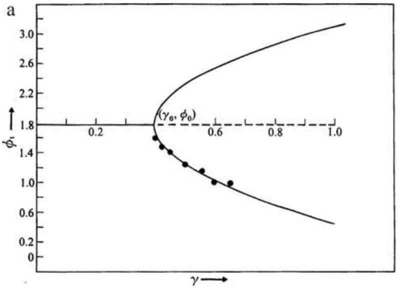
- Groningen-Utrecht collaboration (Fred Wubs, Henk Dijkstra).
- Numerical bifurcation analysis of large scale systems.
- We use techniques from numerical bifurcation theory to study transitions between steady fluid flow patterns and the instabilities involved.
- To capture uncertainty in flows we add stochasticity to the equations and develop solution techniques.

The climate as a dynamical system

Transient runs with full-scale models.

Poor man's continuation.

Compute steady states by (forward) time integration and fit a parabola.



Quon and Ghil [1992]

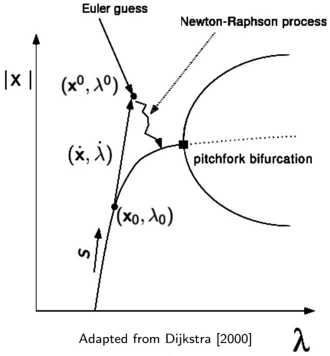
Continuation

Consider a model: $M(\lambda)\,\dfrac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{f}(\mathbf{x};\lambda)$.

Steady states form a branch $(\mathbf{x},\lambda)$, such that $\mathbf{f}(\mathbf{x};\lambda) = 0$.

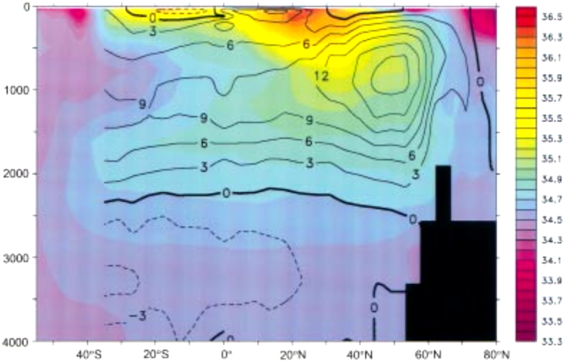
Parameterize the branch: $\gamma(s) = (\mathbf{x}(s),\lambda(s))$.

Travel the branch using a **predictor-corrector** procedure.



Adapted from Dijkstra [2000]

Meridional Overturning Circulation (MOC)



Rahmstorf [1996]

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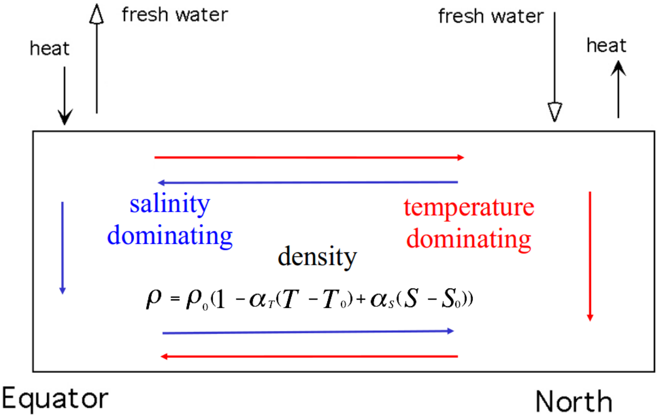


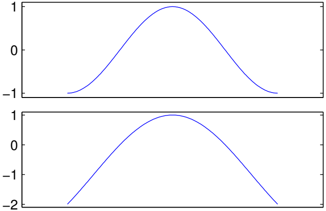
Adapted from Dijkstra [2000]

Idealized 2D MOC

- 2D **primitive equation** ocean model, bounded by latitudes $[-\theta_N, \theta_N]$.
- Coarse grid: $4 \times 32 \times 16$ points with 6 unknowns: (u, v, w, p, T, S).
- Equatorially symmetric **temperature** and **salinity** surface forcing:

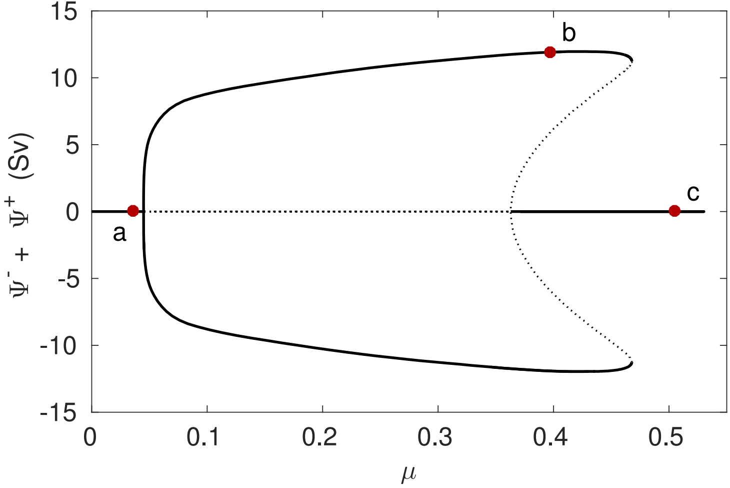
$$\bar{T}(\theta) = T_0 \cos(\pi\theta/\theta_N)$$

$$\bar{F}_s(\theta) = \mu F_0 \frac{\cos(\pi\theta/\theta_N)}{\cos(\theta)}$$

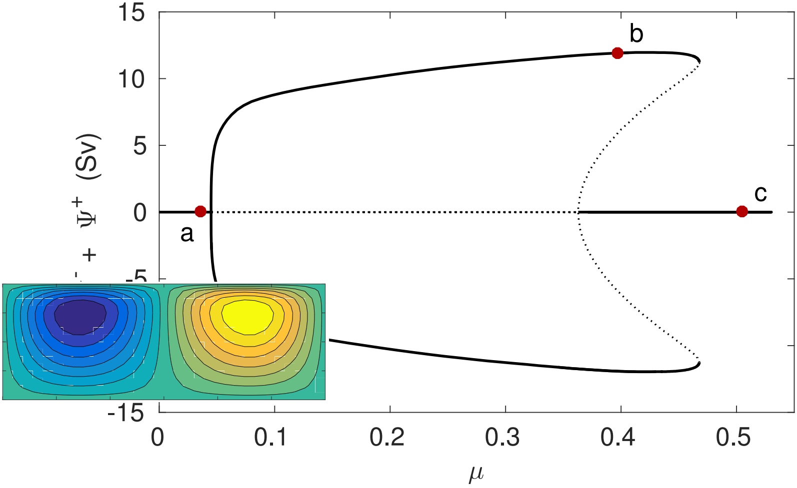


- Bifurcation parameter μ: strength of freshwater forcing.

Idealized 2D MOC



Idealized 2D MOC



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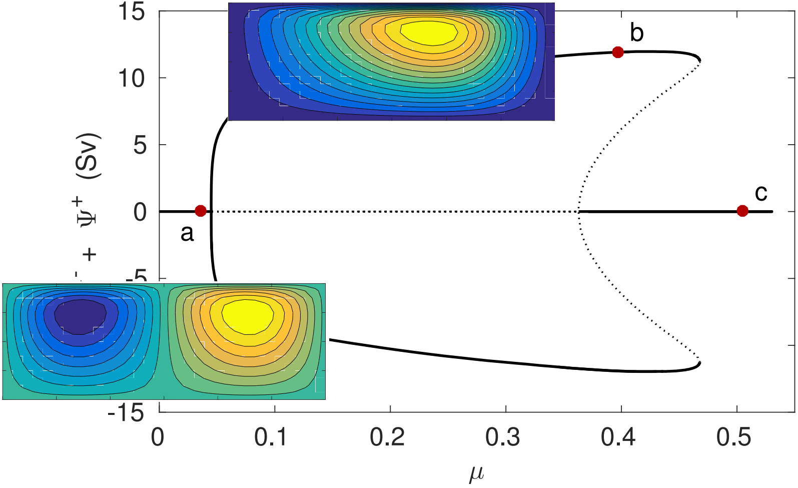

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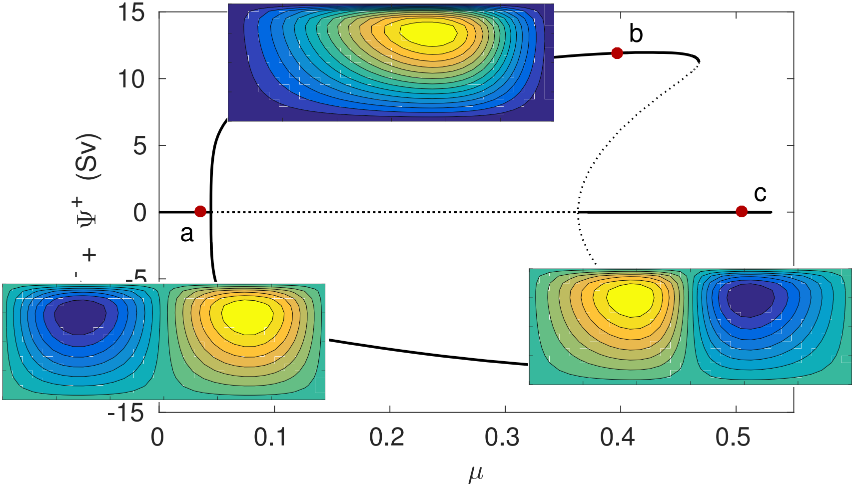


Stochastic formulation

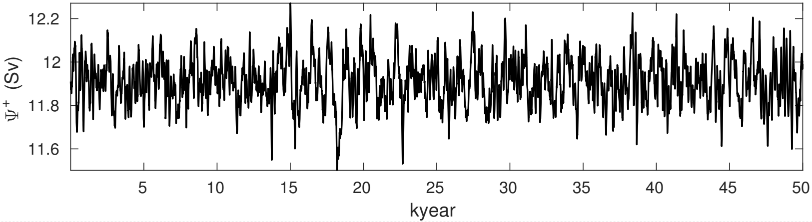
- Zero-mean white noise $\zeta(\theta, t)$ is added to the forcing:

$$F_s(\theta, t) = (1 + \sigma\ \zeta(\theta, t))\bar{F}_s(\theta)$$

- Stochastic problem formulation:

$$M(\mathbf{p})\ \mathrm{d}\mathbf{x}_t = \mathbf{f}(\mathbf{x}_t; \mathbf{p})\ \mathrm{d}t + \mathbf{g}(\mathbf{x}_t; \mathbf{p})\ \mathrm{d}\mathbf{W}_t$$

- Traditional approach: time series



Covariance matrix

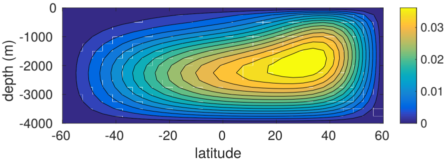
- From the time series we construct a covariance matrix.
- Its eigenvectors are the **empirical orthogonal functions** (EOF).
- The eigenvalues give the **explained variability**:

Method	λ_1	λ_2	λ_3	λ_4
Time series	0.679	0.170	0.082	0.033

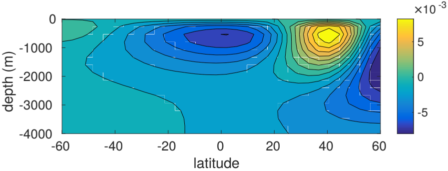
First four weighted eigenvalues of the covariance matrix.

First Empirical Orthogonal Function (EOF)

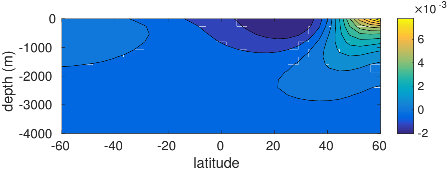
Streamfunction



Temperature



Salinity



Stochastic formulation (2)

- Stochastic problem formulation

$$M(\mathbf{p})\ \mathrm{d}\mathbf{x}_t = \mathbf{f}(\mathbf{x}_t; \mathbf{p})\ \mathrm{d}t + \mathbf{g}(\mathbf{x}_t; \mathbf{p})\ \mathrm{d}\mathbf{W}_t$$

- After linearization

$$M(\mathbf{p})\ \mathrm{d}\mathbf{X}_t = A(\mathbf{x}^*; \mathbf{p})\mathbf{X}_t\ \mathrm{d}t + B(\mathbf{x}^*; \mathbf{p})\ \mathrm{d}\mathbf{W}_t$$

- **Generalized Lyapunov equation**

$$ACM^T + MCA^T + BB^T = 0$$

Stochastic formulation (2)

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- **Generalized Lyapunov equation**

$$ACM^T + MCA^T + BB^T = 0$$

Only for non-singular M!

Reduction to a non-singular problem

- Linearized stochastic formulation for the MOC

$$\begin{pmatrix} 0 & 0 \\ 0 & M_{22} \end{pmatrix} \begin{pmatrix} d\mathbf{X}_{t,1} \\ d\mathbf{X}_{t,2} \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} \mathbf{X}_{t,1} \\ \mathbf{X}_{t,2} \end{pmatrix} dt + \begin{pmatrix} 0 \\ B_2 \end{pmatrix} d\mathbf{W}_t$$

- Deterministic and stochastic part

$$\begin{aligned} A_{11}\mathbf{X}_{t,1} + A_{12}\mathbf{X}_{t,2} &= 0 \\ M_{22}\, d\mathbf{X}_{t,2} &= S\mathbf{X}_{t,2}\, dt + B_2\, d\mathbf{W}_t \end{aligned}$$

where $S = A_{22} - A_{21}A_{11}^{-1}A_{12}$

- We have to solve

$$SC_{22}M_{22}^T + M_{22}C_{22}S^T + B_2B_2^T = 0$$

Iterative solution method

$$ACM^T + MCA^T + BB^T = 0$$

- Solve the projected system

$$\begin{aligned} V^TAVTV^TM^TV \quad &+ \quad V^TMVTV^TA^TV \quad + \quad V^TBB^TV \quad = \\ \tilde{A}T\tilde{M}^T \quad &+ \quad \tilde{M}T\tilde{A}^T \quad + \quad \tilde{B}\tilde{B}^T \quad = 0 \end{aligned}$$

- VTV^T is a **low-rank approximation** of the covariance matrix C.

Computing the residual

$$\tilde{A}T\tilde{M}^T + \tilde{M}T\tilde{A}^T + \tilde{B}\tilde{B}^T = 0$$

with $C \approx VTV^T$

- The residual is given by

$$R = A\ VTV^T\ M^T + M\ VTV^T\ A^T + BB^T$$

- It is generally a full dense $n \times n$ matrix
- 2-norm can be determined by computing the largest eigenvalues
- We use Lanczos for this

Iterative solution method (2)

- Solve the projected system

$$\tilde{A}T\tilde{M}^T + \tilde{M}T\tilde{A}^T + \tilde{B}\tilde{B}^T = 0$$

- Compute an approximation of the residual

$$R = A\ VTV^T\ M^T + M\ VTV^T\ A^T + BB^T$$

- Expand V with eigenvectors of R, or A^{-1} times eigenvectors of R

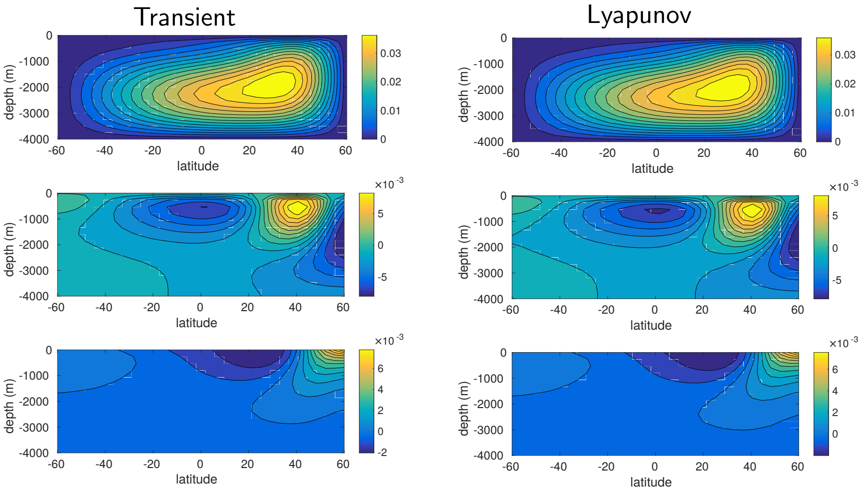
Residual Approximation-based Iterative Lyapunov Solver

Properties

Unlike other iterative procedures it...

- can be restarted
  - By keeping only the most important directions in V
- can be used in continuation
  - By using the V from the previous continuation step as initial guess [Kuehn, 2012]
- can do without a matrix inverse

Comparison of the first EOF



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Comparison of the eigenvalues

Method	λ_1	λ_2	λ_3	λ_4
RAILS	0.677	0.176	0.078	0.033
Time series	0.679	0.170	0.082	0.033

First four weighted eigenvalues of the covariance matrix.

Comparison with other methods

- RAILS (our method)
  - Uses eigenvectors of R.
- Extended Krylov [Simoncini, 2007]
  - Uses $K(A, B) + K(A^{-1}, B)$.
- LRCF-ADI [Penzl, 2000]
  - Uses $(A - \sigma M)^{-1}$ for many different σ.

Performance

Method	Rank	MVPs	IMVPs	t_c (s)	t_s (s)
RAILS	56	709	0	0	8
RAILS (S^{-1})	63	146	146	28	4
Extended Krylov	73	224	256	28	4
LRCF-ADI	244	793	818	1104	181

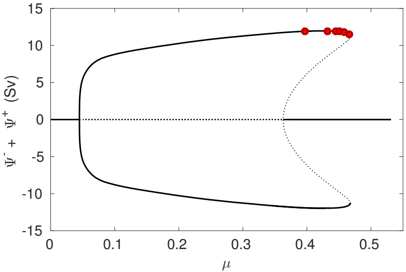
On a $4 \times 32 \times 16$ grid, 10^{-3} relative residual tolerance.

Performance on different grid sizes

Size	Rank	MVPs	t_s (s)
32	56	727	9
64	83	1048	33
128	136	1609	152

On a $4 \times n \times 16$ grid, 10^{-3} relative residual tolerance.

Performance in continuation



MVPs when reusing the solution space

μ_k	0.397	0.433	0.445	0.451	0.458	0.466
Initial $V_k = \emptyset$	709	829	856	931	961	1006
Initial $V_k = V_{k-1}$	709	474	448	448	475	502

Towards 3D

- Taking 4 grid points in the zonal direction

Method	Size	Rank	MVPs	IMVPs	t_c (s)	t_s (s)
Extended Krylov	32	177	762	889	28	60
	64	286	1275	1530	350	340
RAILS	32	164	595	0	0	12
	64	269	865	0	0	57
	128	425	1336	0	0	295

On a $4 \times n \times 16$ grid, 10^{-3} relative residual tolerance.

Summary

- We presented a framework for performing **bifurcation analysis** in **stochastic** models using **continuation**.
- We showed how the **variance** in the system can be studied by solving **Lyapunov equations**.
- Solving Lyapunov equations is much faster than the transient approach.
- Our method is very well suited for solving these problems.
- Especially in a continuation context.