Stability of finite difference schemes on non–uniform grids for the Black–Scholes equation

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We consider the well-known Black–Scholes equation from financial option pricing theory,

$$\frac{\partial u}{\partial t} = \frac{1}{2}\sigma^2 s^2 \frac{\partial^2 u}{\partial s^2} + rs\frac{\partial u}{\partial s} - ru \quad (s > 0,\, t > 0)$$

with given real constants $r, \sigma > 0$. The Black–Scholes equation is a time-dependent advection-diffusion-reaction equation and is supplemented with initial and boundary conditions.

A popular approach for the numerical solution of time-dependent partial differential equations is the method-of-lines. It consists of two steps:

1. Spatial discretization: the partial derivatives $\partial u/\partial s, \partial^2 u/\partial s^2$ are discretized on a finite spatial grid, yielding a (large) system of ordinary differential equations
$$U'(t) = AU(t) + b(t) \quad (t > 0) \tag{1}$$
with given fixed matrix A and vectors $b(t)$.
2. Temporal discretization: the above system of ordinary differential equations is numerically integrated in time.

Our research focuses on the stability analysis of second-order finite difference methods for the spatial discretization of the Black–Scholes equation. We first present practical upper bounds for

$$||e^{tA}||_2 \quad (t > 0)$$

where $||\cdot||_2$ denotes a scaled version of the standard spectral norm. We subsequently present sufficient conditions for contractivity in the maximum-norm,

$$||e^{tA}||_\infty \leq 1 \quad (t > 0).$$

A virtue of our stability analysis is that it applies to spatial grids that are not uniform. Such grids are often used in actual applications. Numerical experiments are provided which support our theoretical results.

Finally, we briefly discuss the stability of temporal discretization schemes for (1) w.r.t. $||\cdot||_2$ and $||\cdot||_\infty$